Lectures on Generating Functions

S. K. Lando

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\]
Lectures on Generating Functions
To A. A. Kirillov,
from whom I have first heard the words
“generating function”
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Preface to the English Edition

Modern combinatorics speaks the language of generating functions. The study of this language does not require a bulky knowledge of numerous parts of mathematics; although some preliminary acquaintance with calculus and algebra is more than welcome. On the other hand, generating functions may prove to be extremely useful in further mathematical education because of their deep involvement in various mathematical activities, including computer science. The goal of the present book is to serve as a basis for a one-semester undergraduate course in combinatorics, based on the notion of generating function. It contains many exercises both for class and home work. Of course, it is an introductory book not containing a complete theory. I hope, however, that some of its readers will find in it a good entrance point into the fascinating world of generating functions.

All of the main ideas in the book are introduced on the basis of examples. Sometimes the choice of examples is classical, and in other cases it is justified by my own research experience. This experience concerns first of all graph embeddings into two-surfaces and enumeration of the embeddings. This subject plays a central role in contemporary theoretical physics, and specialists know that it incorporates
many advanced mathematical theories. A variety of generating functions appears naturally in these studies and some of them found their way into this book.

I would like to use this opportunity to express my gratitude to the American Mathematical Society for the suggestion to publish the English translation of the book. In the translation, some minor corrections and changes were made.

Sergei Lando, July 2003
After multiplying by \((2n - 1)!\), the coefficient of \(x^{2n-1}\) in the power expansion of the function \(\tan x\) becomes a positive integer. What is more surprising, this number appears to be equal to the number of up-down permutations of the set \(\{1, \ldots, 2n - 1\}\). This shows that \(\tan x\) is the “exponential generating function” for the sequence of numbers of up-down permutations. This fact can be proved, but we cannot be sure that we understand the phenomenon completely. The function \(\tan x\) is not unique in this sense: coefficients in the expansions of many classical functions have a combinatorial interpretation. Trigonometric, hypergeometric and elliptic functions, elliptic integrals and so on fall into this class. One can even affirm that the coefficients of every function which is interesting by itself and not only as an element of some functional class must have a combinatorial meaning.

Mathematicians of the 18th and 19th centuries knew functions “personally”. I doubt whether there are more specialists nowadays possessing these skills than there were a hundred years ago, in spite of the fact that the roots, the asymptotics, the disk of convergence, the singularities, and the topology of the corresponding Riemann surface can say a lot about the nature of the objects under enumeration.

Generating functions admit a natural splitting into classes. The simplest is the class of rational functions. It is well studied and a huge bunch of problems leading to rational generating functions is known.
Algebraic generating functions also appear frequently. In the beginning of 1960s Schützenberger showed that their non-commutative analogues arise naturally as languages generated by unambiguous formal grammars. However, the class of algebraic functions (in contrast to that of rational ones) is not closed under the natural operation of the Hadamard product. Generally, the Hadamard product of two algebraic functions is an algebro-logarithmic function. And the class of algebro-logarithmic functions, which is closed under the Hadamard product, seems to be natural.

The relationship between algebraic functions and formal grammars indicates that the class of objects under enumeration is essentially one-dimensional: words in languages admit a linear recording. Modern quantum field theory models require enumeration of objects of essentially two-dimensional origin, and the nature of generating functions arising in these problems is far from being understood completely. The elegant method of matrix integration invented by physicists leads to explicit results only in a few cases.

I wanted to write a simple and accessible introduction to generating functions, paying attention first of all to striking examples, not to (often non-existing) general theories. As a result, many important applications of the generating functions method, including Polya’s enumeration theory and $q$-analogues, Poincaré’s generating polynomials and generating families, the theory of ramified coverings and many other important topics are not even mentioned in the book.

My interest in enumerative combinatorics was inspired by a series of problems posed by V. I. Arnold in connection with some problems of the singularity theory as well as his own activities in this field. I was influenced a lot by the combinatorial team of the University Bordeaux I (G. Viennot and others) and by P. Flajolet. The book is based on the series of optional courses I gave for many years to freshmen of the Higher College of Mathematics of the Independent University of Moscow in 1992–99. In giving these courses, I enjoyed substantial help from M. N. Vyalyi, who also helped greatly in preparing the book for publication. The main source of my knowledge in combinatorics is my
friend and long-time coauthor Alexander Zvonkin, whose mastery of creating texts is — alas — beyond my reach.

S. K. Lando
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In combinatorics, one often considers the process of enumerating objects of a certain nature, which results in a sequence of positive integers. With each such sequence, one can associate a generating function, whose properties tell us a lot about the nature of the objects being enumerated. Nowadays, the language of generating functions is the main language of enumerative combinatorics.

This book is based on the course given by the author at the College of Mathematics of the Independent University of Moscow. It starts with definitions, simple properties, and numerous examples of generating functions. It then discusses various topics, such as formal grammars, generating functions in several variables, partitions and decompositions, and the exclusion-inclusion principle. In the final chapter, the author describes applications of generating functions to enumeration of trees, plane graphs, and graphs embedded in two-dimensional surfaces.

Throughout the book, the reader is motivated by interesting examples rather than by general theories. It also contains a lot of exercises to help the reader master the material. Little beyond the standard calculus course is necessary to understand the book. It can serve as a text for a one-semester undergraduate course in combinatorics.

For additional information and updates on this book, visit www.ams.org/bookpages/stml-23