Six Themes on Variation
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Six Themes on Variation

Robert Hardt, Editor

Steven J. Cox
Robin Forman
Frank Jones
Barbara Lee Keyfitz
Frank Morgan
Michael Wolf
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Preface

One November, Rice University hosted a group of thirty undergraduate mathematics majors with the purpose of introducing them to research mathematics and graduate school. The principle part of this introduction was the series of talks and workshops, which all took up some idea or theme from the calculus of variations. These were so successful that the American Mathematical Society encouraged us to present them to a wider audience, in the form you see here.

The calculus of variations is a beautiful subject with a rich history and with origins in the minimization problems of calculus (see Chapter 1). Although, as we will discover in the chapters below, it is now at the core of many modern mathematical fields, it does not have a well-defined place in most undergraduate mathematics courses or curricula. We hope that this small volume will nevertheless give the undergraduate reader a sense of its great character and importance.

An interesting story motivating the calculus of variations comes from Carthage in 900 BC, long before the discovery of calculus by Newton and Leibniz. Queen Dido, as a result of a bargaining negotiation, obtained “as much land as could be enclosed by the skin of an ox.” She had the ox skin cut into strips as thin as practically possible and formed a long cord of fixed length. If her choice of land had been restricted to flat inland territory, then she would presumably have chosen a large circular region. This is because the circle, among
all planar closed curves of fixed length, encloses the maximum area. But she had the choice of territory with a flat coastline and cleverly chose a semi-circular region, with the cord's endpoints on the shore-line. This gives more area and is actually the mathematically optimal solution. A change or \textit{variation} of the shape of the cord cannot give a new region of greater area.

\textit{Calculus of Variations} arises when one \textit{differentiates}, in the sense of the calculus of Newton and Leibniz, a one-parameter family of such variations. This first occurs in the works by P.L.M. de Maupertuis (1698-1759), G.W. Leibniz (1646-1716), Jakob Bernoulli (1654-1705), Johann Bernoulli (1667-1748), L. Euler (1707-1783), and J.L. Lagrange (1736-1813). It has historically largely been the study of optimal paths, for example as a geodesic curve in a space or as a path of least action in space-time. See the nice presentation in Chapter 1 of \textit{The Parsimonious Universe: Shape and Form in the Natural World} by S. Hildebrandt and A. Tromba (Copernicus, New York, 1996).

In modern language, the birth of the calculus of variations occurs in the transition from the study of a critical point of a function on a line (as in calculus) to that of a critical curve or critical surface for a functional, such as length or area, on an infinite-dimensional space of such objects. As discussed in Chapter 1 by Frank Jones, the condition of criticality for these objects leads to the important partial differential equations of Euler and Lagrange. Various physical problems also give rise to natural conditions constraining the space of admissible objects. One such constraint involves a fixed boundary, as with a classical vibrating string or a soap film spanning a wire. Another constraint is seen in Queen Dido's problem. Her problem may be equivalently reformulated as the \textit{isoperimetric problem} of finding a curve of minimum length enclosing a given fixed area. The analogous two-dimensional isoperimetric problem of finding a surface of least area enclosing a given volume (or volumes) occurs in soap bubble models.

In Chapter 2 by Robin Forman, one considers the connection between such critical or equilibrium points and the topology or geometry of the ambient spaces. Here is a quick elegant introduction to the simple, but subtle, ideas of Marston Morse from the 1940s.
Their generalizations involving infinite-dimensional spaces of paths (or solutions of other PDEs) have had a profound influence on 20th century mathematics. One here encounters critical paths that may not be globally or even locally length minimizing. For example, the “ridge trail” over a mountain range is a length-critical path that is *unstable*. Some slight variation may give a (more dangerous) path of shorter length.

Physicists and mathematicians have long been interested in understanding and modeling vibrating strings, as in bowed or plucked instruments. Steve Cox discusses in Chapter 3 the cause of the observed decay of the amplitude. Such decay is usually neglected in introductory treatments in physics courses. His chapter well illustrates the full range and difficulty of scientific inquiry from acquiring experimental data, to synthesizing data, to mathematical modeling, to finding actual or approximate solutions. The discussion here includes a useful introduction and illustration of the classical “Principle of Stationary Action”.

The isoperimetric problem that a surface of least area in space enclosing a single given volume must be an ordinary round sphere was solved rigorously over 100 years ago. It was claimed by Archimedes and Zenodorus in antiquity, but proved by H. Schwarz in 1884. At the Rice undergraduate conference, Frank Morgan discussed the important *Double Bubble Conjecture* that a surface of least area enclosing two fixed volumes consists simply of two adjoined spherical caps joined by a third spherical interface (with radii determined by the given volumes). In 1998, this conjecture was proven by M. Hutchings, F. Morgan, M. Ritoré, and A. Ros. For Chapter 4 of the present volume, Frank Morgan’s original talk has been replaced by a reprint of his excellent 2001 MAA article exposing this result.

Minimal surfaces occur in the calculus of variations as critical points of the area functional and provide models for some soap films. K. Weierstrass (1815-1897) showed that they also enjoy a mathematical representation in terms of complex-valued functions. Chapter 5 by Mike Wolf explains carefully this connection and gives a related, recently discovered representation that allows the construction of several rich new families of minimal surfaces. See the many beautiful
illustrations here. This chapter is a great introduction to some of the many important relationships among the calculus of variations, complex analysis, and differential geometry.

Differential equations for modeling traffic flow are derived and analyzed in the chapter by Barbara Keyfitz. The continuum model derived here is natural, consistent, and leads both to many observed familiar 

discontinuous phenomena such as shock waves and to many important open mathematical problems. It is a great example of the fruitful interplay between pure and applied mathematics. Proper careful modeling not only gives better scientific applications but reveals beautiful often hidden mathematical structures.

On Saturday afternoon of the Rice conference, students also had the opportunity to actively participate in Steve Cox's experiments with vibrating strings (see the illustrations in Chapter 3) or with Frank Morgan's soap films and soap bubbles or to hear from Robin Forman about many recent open problems in mathematics. The format of the Calculus of Variations Conference worked well, and three other similarly structured undergraduate conferences have since been held at Rice: Low Dimensional Geometry and Topology, Geometric Aspects of Combinatorics, and Mathematical Problems in Biology.

The editor appreciates the great patience and help of Ed Dunne of the American Mathematical Society in assembling this book and the suggestion of Carl Pomerance for the title “Six Themes on Variation”.
List of Contributors

Steve Cox, Rice University
Robin Forman, Rice University
Robert Hardt, Rice University
Frank Jones, Rice University
Barbara Lee Keyfitz, University of Houston and The Fields Institute
Frank Morgan, Williams College
Michael Wolf, Rice University
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The calculus of variations is a beautiful subject with a rich history and with origins in the minimization problems of calculus. Although it is now at the core of many modern mathematical fields, it does not have a well-defined place in most undergraduate mathematics courses or curricula. This small volume should nevertheless give the undergraduate reader a sense of its great character and importance.

Interesting functionals, such as area or energy, often give rise to problems whose most natural solution occurs by differentiating a one-parameter family of variations of some function. The critical points of the functional are related to the solutions of the associated Euler-Lagrange equation. These differential equations are at the heart of the calculus of variations. Some of the topics addressed here are Morse theory, wave mechanics, minimal surfaces, soap bubbles, and modeling traffic flow. All are readily accessible to advanced undergraduates.

This book is derived from a workshop that was sponsored by Rice University.