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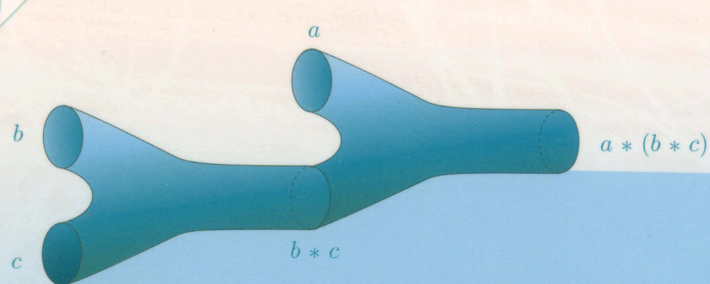
Enumerative Geometry and String Theory

Sheldon Katz



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Dedicated to the memory of Isadore Glaubiger

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IAS/Park City Mathematics Institute

The IAS/Park City Mathematics Institute (PCMI) was founded in 1991 as part of the "Regional Geometry Institute" initiative of the National Science Foundation. In mid-1993 the program found an institutional home at the Institute for Advanced Study (IAS) in Princeton, New Jersey. The PCMI continues to hold summer programs in Park City, Utah.

The IAS/Park City Mathematics Institute encourages both research and education in mathematics and fosters interaction between the two. The three-week summer institute offers programs for researchers and postdoctoral scholars, graduate students, undergraduate students, high school teachers, mathematics education researchers, and undergraduate faculty. One of PCMI's main goals is to make all of the participants aware of the total spectrum of activities that occur in mathematics education and research: we wish to involve professional mathematicians in education and to bring modern concepts in mathematics to the attention of educators. To that end the summer institute features general sessions designed to encourage interaction among the various groups. In-year activities at sites around the country form an integral part of the High School Teacher Program.

Each summer a different topic is chosen as the focus of the Research Program and Graduate Summer School. Activities in the Undergraduate Program deal with this topic as well. Lecture notes from the Graduate Summer School are published each year in the IAS/Park City Mathematics Series. Course materials from the Undergraduate Program, such as the current volume, are now being published as part of the IAS/Park City Mathematical Subseries in the Student Mathematical Library. We are happy to make available more of the excellent resources which have been developed as part of the PCMI.

John Polking, Series Editor
February 20, 2006

Preface

This book is based on a series of fifteen advanced undergraduate lectures I gave at the Park City Mathematics Institute (PCMI) during the summer of 2001. A separate book based on the program of graduate lectures is being published in the Park City Mathematics Series.

My charge was to give students exposure to some of the important ideas related to modern enumerative geometry as it has evolved in recent years. My goals were to provide some background in and inspiration from classical enumerative geometry, to explain the rudiments of stable maps and Gromov-Witten theory, and to explain connections to string theory in physics.

The students were quite good, as I had been told to expect. Students were selected from a competitive application process. Furthermore, PCMI runs two series of undergraduate lectures: an "elementary" lecture series and an "advanced" lecture series. The students self-select which lecture series they will attend, so my lectures were populated by the strongest of this already talented group of undergraduates. The elementary lectures were given by Ruth Gornet on the subject of differential geometry.

While my experience with the undergraduate students was close to my expectations, there was however a big surprise: my lectures were heavily attended by graduate students. Graduate students repeatedly told me that my lectures helped them solidify their knowledge of Gromov-Witten theory, as I presented a more leisurely introduction than they had seen before, made clearer than they had seen before how Gromov-Witten theory builds upon classical enumerative geometry, gave many examples, and connected to physics more than they had seen before. For this reason, while writing this book primarily for advanced undergraduates, I am keeping graduate students in mind as an important secondary audience.

Designing the course was a bit of a challenge. I could assume that the students were smart and willing to work hard and that they had "mathematical maturity", but I could not assume exposure to any specific area of mathematics beyond a standard undergraduate course in linear algebra. For that reason, the lectures contained introductory material on abstract algebra, geometry, analysis, and topology. Most of the participating students already knew some of these topics, but few knew all of them. Thus the lectures served both as a review and as an introduction to a range of areas in undergraduate mathematics.

The incorporation of physics presented another challenge, as I could not assume anything more than exposure to a first undergraduate physics course. Here, I did not even pretend to be pedagogical or complete. I cut corners by explaining a range of relevant ideas of physics via the simplest examples, emphasizing connections to enumerative geometry throughout. While the physics lectures were undoubtedly the most difficult part of the course for the students, I hoped that they would get a firm impression of the myriad connections between geometry and physics through this very brief introduction. I have similar hopes for the reader of this book.

It has certainly been gratifying to see a number of "my" PCMI undergraduate students currently pursuing graduate studies in this research area, near and dear to my own heart. But the PCMI program has a broader purpose—to give students a research experience that will benefit them in their chosen careers. I propose that the model adopted here for undergraduate training—shooting for some reasonably advanced ideas from graduate level mathematics while filling in a broad range of topics en route—is a good way to help advanced undergraduates integrate what they have learned and prepare for their careers. Whether or not they pursue graduate studies in the particular focus areas seems secondary to me, even though I am quite fond of the subject area of this book.

Turning the lectures into a book introduced another set of challenges. Undergraduate students reading this book will probably not have the benefit of lectures at which they can ask questions as they go along. In addition, the PCMI students were helped by an expert teaching assistant, Artur Elezi. Since I assume that the reader will not have direct access to a faculty member, I have added much background material that was left out of the lectures, while trying to keep the informal feel throughout. I added material on differential geometry which was not needed in the original lectures since this topic was covered in Ruth Gornet's lectures. Even so, *the reader is warned that this book is not self-contained.* In particular, working through this book is not a substitute for a more thorough and more pedagogical treatment of any of the background topics reviewed here. I have certainly cut out much important foundational material to streamline the process of getting to the desired results. Even worse, sometimes for the sake of expediency I have given very nonstandard treatments of subjects. Some of these have a somewhat clumsy feel to them. This is especially true of the ad hoc treatment I have given of the beautiful subject of algebraic geometry. An undergraduate student interested in algebraic geometry reading this book is urged to consult the references provided. A graduate student interested in algebraic geometry reading this book is urged to provide the standard definitions of all of the concepts which have been introduced here in an ad hoc fashion.

Each of the chapters of the book corresponds reasonably closely to one of the lectures, the exception being that the material from a lecture entitled "More on bundles" has been divided up and distributed through other chapters, so that there are only fourteen chapters replacing fifteen lectures. Since material has been added beyond the content of the original lectures, each chapter contains substantially more than one lecture's worth of material.

As I have indicated above, this book will be quite challenging for an undergraduate student, and, to be honest, will possibly be too hard to understand completely in a few spots, especially the physics-related material near the end. But I encourage all undergraduate students to "go for it" and persist with the difficult parts. If you reach for the stars, you will at least get to the moon. That said, students would likely get more out of the book if they could consult a faculty member or graduate student when needed, as in a directed reading class.

Acknowledgements

During the more than four years that I have worked on this project, I have been helped in many ways by many people. First and foremost, I would like to thank the students who attended my lectures. They helped with their questions and corrections to the original set of lecture notes, helping me to improve the presentation. More significantly, their energy and enthusiasm convinced me that it would be worthwhile to expand the lectures into a book. They will unfortunately have to remain nameless, as I did not take roll and so would probably miss at least a few dozen if I tried to reconstruct the complete list from PCMI records and my memory. You know who you are, and I want to express my thanks to you here. I also owe a debt of gratitude to Artur Elezi, who both served as a Teaching Assistant at PCMI and suggested many improvements to the original lecture notes.

I would also like to give special thanks to Michael Mulligan and Michael Sommers. These two University of Illinois students read most of the manuscript at different points, reported numerous typos and mistakes, and offered suggestions for improvement. Michael Mulligan worked through Chapters 1–10 during 2004. Michael Sommers read a near-final draft during the summer of 2005 with particular attention to Chapters 10–14 and provided assistance with the bibliography and the index. The book is much better as a result of their efforts.

I also owe a debt of gratitude to Arthur Greenspoon, who volunteered his professional skills and proofread the entire book. Parts of the book were proofread by Daniel Morton, whom I thank as well.

Undoubtedly a number of typos and other errors are present in the final version, for which I alone bear responsibility. I am maintaining a website for the book. It can be reached from the AMS webpage for the book, whose URL is given on the copyright page and on the back cover of the book. The website will contain lists of typographical errors and their corrections.

I thank David Cox, Joachim Koch, and Michael Stone for assistance with figures, and I also thank Randy McCarthy for providing some references.

A number of people involved with PCMI deserve thanks for their assistance: William Barker, Herb Clemens, Catherine Giesbrecht, Roger Howe, and David Morrison.

I also thank the AMS staff who have assisted me directly during this project: Barbara Beeton, Arlene O'Sean, and especially Edward Dunne, as well as the rest of the team working behind the scenes to make this book a reality.

Finally, I want to thank my family for their support and patience during the times when this project made me less available for them than any of us would have liked.

November 2005 *Sheldon Katz*

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Bibliography

We use the following conventions for labeling references in the text and in the bibliography:

- References with one author are labeled by the author's last name.
- References with multiple authors are labeled using the first letter of the authors' last names in alphabetical order.
- If there is more than one reference for the same author or group of authors, they are ordered by publication year.

[Ahlfors] L. Ahlfors, *Complex Analysis*, McGraw-Hill, New York, 1978.

[Arnold] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, Grad. Texts in Math. **60**, Springer-Verlag, New York, 1997.

[Atiyah] M. Atiyah, *The Geometry and Physics of Knots*, Lezioni Lincee, Cambridge Univ. Press, Cambridge, 1990.

[BT] R. Bott and L. W. Tu, *Differential Forms in Algebraic Topology*, Grad. Texts in Math. **82**, Spring-Verlag, New York, 1982.

[CdOGP] P. Candelas, X. de la Ossa, P. Green and L. Parkes, *A pair of Calabi-Yau manifolds as an exactly soluble superconformal field theory*, Nuclear Physics **B359** (1991), 21–74 and in Essays on Mirror Manifolds (S.-T. Yau, ed.), International Press, Hong Kong, 1992, 31–95.

[CGGH] J. Carlson, M. Green, P. Griffiths, and J. Harris, *Infinitesimal variations of Hodge structure, I*, Comp. Math. **50** (1983), 109–205.

[Clemens1] H. Clemens, *Homological equivalence modulo algebraic equivalence is not finitely generated*, Inst. Hautes Études Sci. Publ. Math. **58** (1983), 19–38 (1984).

[Clemens2] H. Clemens, *A Scrapbook of Complex Curve Theory*, Grad. Stud. Math. **55**, AMS, Providence, RI, 2003.

[Conway] J.B. Conway, *Functions of One Complex Variable I*, 2nd ed., Grad. Texts in Math. **11**, Springer-Verlag, New York, 1978.

[CK] D. A. Cox and S. Katz, *Mirror Symmetry and Algebraic Geometry*, Mathematical Surveys and Monographs, AMS, Providence, RI, 1999.

[ES] G. Ellingsrud and S. A. Strømme, *The number of twisted cubics on the general quintic threefold*, Math. Scand. **76** (1995).

[Feynman1] R. P. Feynman, R. Leighton, and M. Sands, *The Feynman lectures on Physics*, Vols. 1, 2, 3, Addison-Wesley Publishing Co., Inc., Reading, MA, 1963, 1964, 1965.

[Feynman2] R. P. Feynman, *QED: The Strange Theory of Light and Matter*, Princeton University Press, Princeton, NJ, 1988.

[FU] *Geometry and Quantum Field Theory* (Freed and Uhlenbeck, eds.), IAS/Park City Mathematics Series **1**, AMS, Providence, RI, 1995.

[Freed] D. Freed, *Five lectures on supersymmetry*, AMS, Providence, RI, 1999.

[Fulton1] W. Fulton, *Introduction to Intersection Theory in Algebraic Geometry*, CBMS Regional Conference Series in Mathematics **54**, AMS, Providence, RI, 1984.

[Fulton2] W. Fulton, *Intersection Theory*, Springer-Verlag, New York-Berlin-Heidelberg, 1984.

[Fulton3] W. Fulton, *Algebraic curves*, Adv. Book Classics, Addison-Wesley, Redwood City, CA, 1989.

[FP] W. Fulton and R. Pandharipande, *Notes on stable maps and quantum cohomology*, in Algebraic Geometry (Santa Cruz, 1995) (J. Kollár, R. Lazarsfeld, and D. Morrison, eds.), Proceedings of Symposia in Pure Mathematics **62**, Part 2, AMS, Providence, RI, 1997, 45–96.

[Givental] A. Givental, *Equivariant Gromov-Witten Invariants*, Internat. Math. Res. Notices (1996), 613–663.

[GSW] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring theory*, Vols. 1 and 2, Cambridge University Press, Cambridge, UK, 1987.

[Greenberg] M. Greenberg, *Lectures on Algebraic Topology*, Benjamin, New York, 1967.

[GH] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1994.

[Harris] J. Harris, *Algebraic Geometry: A First Course*, Grad. Texts in Math. **133**, Springer-Verlag, New York, 1992.

[Hartshorne] R. Hartshorne, *Algebraic Geometry*, Grad. Texts in Math. **52**, Springer-Verlag, New York-Heidelberg, 1977.

[Hilbert] *Mathematical Developments arising from Hilbert problems*, F. Browder (ed.), Proceedings of Symposia in Pure Mathematics **28**, AMS, Providence, RI, 1976.

[HKKPTVVZ] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil, and E. Zaslow. *Mirror symmetry*, Clay Mathematics Monographs, AMS, Providence, RI, 2003.

[Hulek] K. Hulek, *Elementary Algebraic Geometry*, Student Mathematical Library, Vol. **20**, AMS, Providence, RI, 2003.

[Jacobson] N. Jacobson, *Basic algebra I*, Freeman and Co., San Francisco, 1974.

[JK] T. Johnsen and S. Kleiman, *Rational curves of degree at most* 9 *on a general quintic threefold*, Comm. Algebra **24** (1996), 2721–2753.

[Katz] S. Katz, *On the finiteness of rational curves on quintic threefolds*, Compositio Math. **60** (1986), 151–162.

[Kontsevich] M. Kontsevich, *Enumeration of rational curves via torus actions*, in The Moduli Space of Curves (Texel Island, 1994) (R. Dijkgraaf, C. Faber, and G. van der Geer, eds.), Progress in Math. **129**, Birkhäuser, Boston-Basel-Berlin, 1995, 335–368.

[KM] M. Kontsevich and Yu. Manin, *Gromov-Witten classes, quantum cohomology, and enumerative geometry*, Comm. Math. Phys. **164** (1994), 525–562.

[Lang] S. Lang, *Algebra*, Addison-Wesley, New York, 1965.

[LLY] B. Lian, K. Liu, and S.-T. Yau, *Mirror principle I*, Asian J. Math. **1** (1997), 729–763.

[MacLane] S. MacLane, *Categories for the working mathematician*, Graduate Texts in Mathematics **5**, Springer-Verlag, New York, 1998.

[MT] I. Madsen and J. Tornehave, *From calculus to cohomology: De Rham cohomology and characteristic classes*, Cambridge University Press, Cambridge, 1997.

[Manin] Yu. I. Manin, *Gauge Field Theory and Complex Geometry*, Springer-Verlag, Berlin, 1988.

[MNOP] D. Maulik, N. Nekrasov, A. Okounkov, and R. Pandharipande, *Gromov-Witten theory and Donaldson-Thomas theory, I*, preprint.

[Miranda] R. Miranda, *Algebraic Curves and Riemann Surfaces*, Graduate Studies in Mathematics **5**, AMS, Providence, RI, 1995.

[Munkres] J. R. Munkres, *Topology: A first course*, Prentice-Hall, Englewood Cliffs, NJ, 1975.

[OV] H. Ooguri and C. Vafa, *Knot Invariants and Topological Strings*, Nucl. Phys. **B577** (2000), 419–438.

[Polchinski] J. Polchinski, *String Theory*, Vols. 1 and 2, Cambridge University Press, Cambridge, 1998.

[Reid] M. Reid, *Undergraduate Algebraic Geometry*, London Math. Society Student Texts **12**, Cambridge University Press, 1988.

[Shankar] R. Shankar, *Principles of Quantum Mechanics*, Plenum Press, New York, 1994.

[Spanier] E. Spanier, *Algebraic Topology*, McGraw-Hill, New York, 1966.

[Spivak] M. Spivak, *Calculus on Manifolds*, Benjamin, Menlo Park, 1965.

[Vick] J. W. Vick, *Homology Theory: An Introduction to Algebraic Topology*, 2nd ed., Grad. Texts in Math. **145**, Springer-Verlag, New York, 1994.

[Warner] F. W. Warner, *Foundations of Differentiable Manifolds and Lie Groups*, Grad. Texts in Math. **94**, Springer-Verlag, New York, 1983.

[Wells] R. O. Wells, *Differential Analysis on Complex Manifolds*, Grad. Texts in Math. **65**, Springer-Verlag, New York, 1980.

[Witten] E. Witten, *Mirror Manifolds and Topological Field Theory*, in Essays on Mirror Manifolds, S.-T. Yau (ed.), International Press, Hong Kong, 1992.

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Perhaps the most famous example of how ideas from modern physics have revolutionized mathematics is the way string theory has led to an overhaul of enumerative geometry, an area of mathematics that started in the eighteen hundreds. Century-old problems of enumerating geometric configurations have now been solved using new and deep mathematical techniques inspired by physics!

The book begins with an insightful introduction to enumerative geometry. From there, the goal becomes explaining the more advanced elements of enumerative algebraic geometry. Along the way, there are some crash courses on intermediate topics which are essential tools for the student of modern mathematics, such as cohomology and other topics in geometry.

The physics content assumes nothing beyond a first undergraduate course. The focus is on explaining the action principle in physics, the idea of string theory, and how these directly lead to questions in geometry. Once these topics are in place, the connection between physics and enumerative geometry is made with the introduction of topological quantum field theory and quantum cohomology.

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