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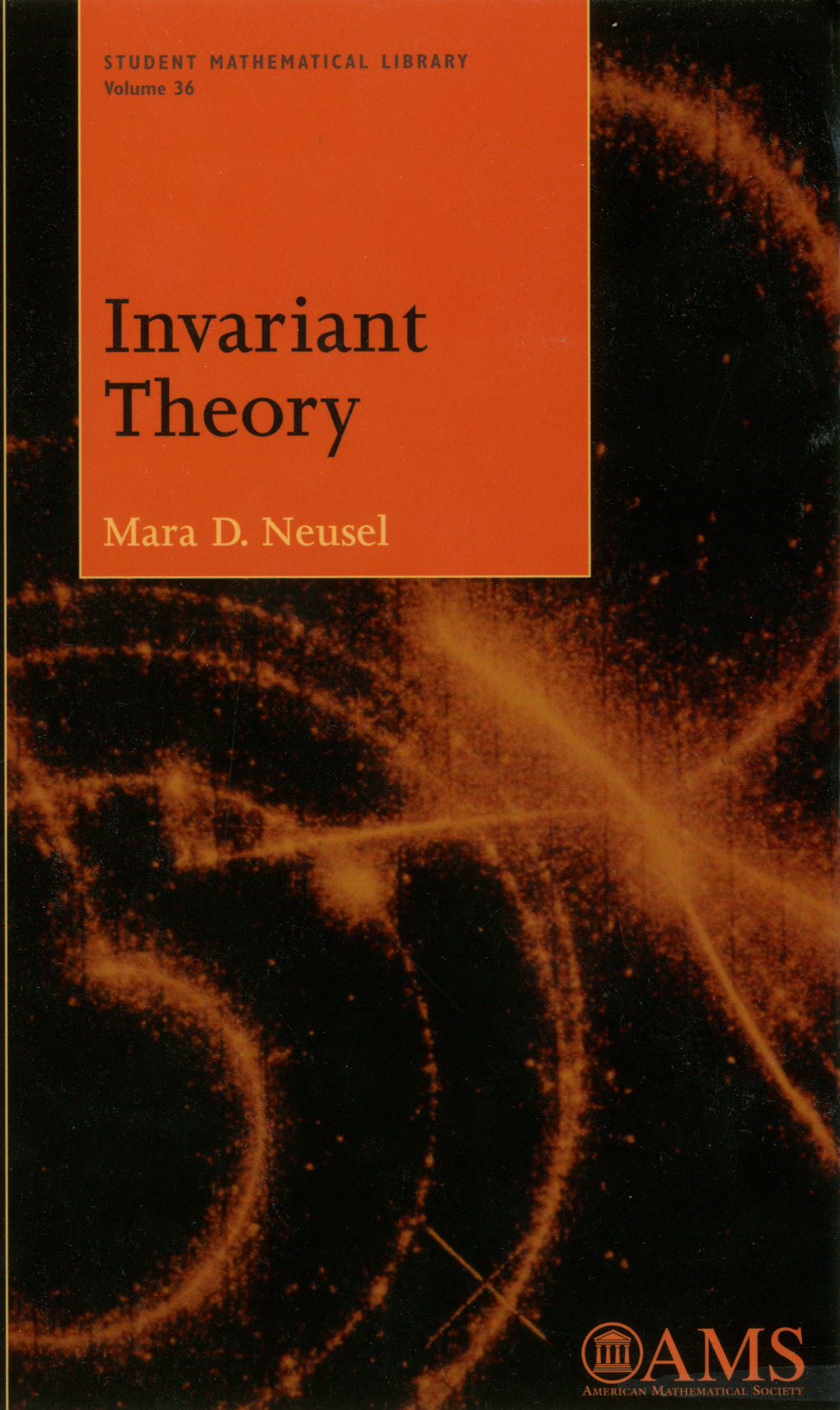
Invariant Theory

Mara D. Neusel



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STUDENT MATHEMATICAL LIBRARY
Volume 36

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Mara D. Neusel

AMS
American Mathematical Society
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2000 *Mathematics Subject Classification.* Primary 13A50;
Secondary 13-XX, 20-XX.

Cover image © CERN.

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www.ams.org/bookpages/stml-36

Library of Congress Cataloging-in-Publication Data

Neusel, Mara D., 1964-
Invariant theory / Mara D. Neusel.
p. cm. – (Student mathematical library, ISSN 1520-9121 ; v. 36)
Includes bibliographical references and index.
ISBN-13: 978-0-8218-4132-7 (acid-free paper)
ISBN-10: 0-8218-4132-7 (acid-free paper)
1. Invariants. I. Title.

QA201 .N48 2006
512.9′44–dc22 2006049944

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To

Jo Lee

Work hard and stay proud!

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Notation Index

$<_{\text{lex}}$, lexicographic order, 73
$=_{\text{dom}}$, dominance order, 77
$\leq_{\text{dom}}$, dominance order, 77
$\cong$, isomorphic, 14
$\hookrightarrow$ indicates an injective map, 14
$\text{Im}(\phi)$, image of the map ϕ, 14
$\text{Ker}(\phi)$, kernel of the map ϕ, 14
δ_{ij}, Kronecker Delta, 51
$[\frac{n}{m}]$, Gauss bracket, 57

$\emptyset$, empty set, 46
$\cap$, intersection, 35
$\cup$, union, 35
$\sqcup$, disjoint union, 122
$\mathcal{S} \times \mathcal{T}$, cartesian product of $\mathcal{S}$ and $\mathcal{T}$, 13
$\chi_{\mathcal{L}}$, characteristic function of the set $\mathcal{L}$, 79

$\mathbb{N} = \{1, 2, 3, \ldots\}$, the natural numbers, 9
$\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$, the nonnegative integers, 9
$\mathbb{Z}$, the integers, 9
$\mathbb{Z}/n$, integers modulo n, 10
$\mathbb{Q}$, the rational numbers, 9
$\mathbb{R}$, the real numbers, 9
$\mathbb{C}$, the complex numbers, 9
$\mathbb{H}$, real quaternions, 25
$\mathbf{i} = \sqrt{-1}$, 16

$\gcd(a, b)$, the greatest common divisor of a and b, 292

$\mathbb{F}$, field, 11
$\mathbb{F}_2 = \mathbb{Z}/2$, 219
$\mathbb{F}(V)$, field of rational functions, 187
$\mathbb{FF}(R)$, field of fractions of R, 187
$\mathbb{L}/\mathbb{K}$, field extension, 287
$\text{minpol}_{\mathbb{K}}(\lambda)$, minimal polynomial, 289
$\deg_{\mathbb{K}}(\lambda)$, degree of λ over $\mathbb{K}$, 289

I, identity matrix, 18
$\text{diag}(m_{11}, m_{22}, \ldots, m_{nn})$, diagonal matrix, 113
M^{t}, transposed matrix, 47
$\det(\mathsf{M})$, determinant, 12
$\text{tr}(g, \mathbb{F}[V]_{(j)})$, trace of $\rho(g)$ acting on $\mathbb{F}[V]_{(j)}$, 233
$V^* = \text{Hom}(V, \mathbb{F})$, 51
H_{R}, reflecting hyperplane, 142
$\Delta_{\mathsf{R}}(\mathbf{v})$, 267

$1_R \in R$, unity in R, 27
$R^\times$, subset of units in R, 28
$R_{(i)}$, homogeneous piece of the ring R of degree i, 30
$\mathbb{F}G$, group ring, 177
$\mathbb{F}[V]^G$, ring of polynomial invariants, 54

$\overline{R}$, integral closure of R, 187
$\deg(A)$, the degree of A, 262
$(I : J)$, colon ideal, 40
$I + J$, sum of the ideals I and J, 35
I/J, ideal quotient, 38
IJ, product of the ideals I and J, 35
$\sqrt{I}$, radical of the ideal I, 35
I^{e}, extended idea , 37
J^{c}, contracted ideal, 37
$\mathrm{Nil}(R)$, nilradical of R, 36
$\mathfrak{m} \subset R$, maximal ideal in R, 36
$\mathfrak{p} \subset R$, prime ideal in R, 36
$\mathrm{Ann}(M)$, annihilator ideal, 182
$\mathrm{ht}(I)$, height of the ideal I, 209
$V(I)$, affine algebraic set, 250

$\mathbb{R}[x]$, ring of polynomials in x with real coefficients, 29
$\mathbb{R}[x_1, \ldots, x_n]$, ring of polynomials in n variables with real coefficients, 30
$\mathbf{x}^I = x_1^{i_1} \cdots x_n^{i_n}$, a monomial in n variables, 30
$\deg(p)$, total degree of p, 30
$\deg_{x_j}(p)$, x_j-degree of p, 30
x-degree, 30
$\lambda(I)$, associated partition of I, 76
I_{red}, reduced exponent sequence, 77
$\mathbf{x}^{I_{\mathrm{red}}}$, reduced monomial, 77
$\partial/\partial x$, partial derivative w.r.t. x, 110
$\Delta(f)$, Hessian of f, 115
$\mathrm{Jac}(f_1, \ldots, f_n)$, Jacobian determinant of $f_1, \ldots, f_n$, 110

$\mathrm{Hom}_R(M, N)$, R-module homomorphisms $M \to N$, 171
$P(M, t)$, Poincaré series, 226

A_n, alternating group, 12
D_{2n}, dihedral group of order $2n$, 11
$\mathsf{GL}(n, \mathbb{F})$, general linear group, 11
$\mathsf{SL}(n, \mathbb{F})$, special linear group, 12
Σ_n, symmetric group, 10
$O(3, \mathbb{R})$, group of real orthogonal 3×3-matrices, 156
Q_8, quaternion group of order 8, 11
Q_{2^n}, generalized quaternion group, 25
Q_{4m}, dicyclic group, 25
V_4, Klein Four Group, 13
$W(B_n)$, group of signed permutations, 113
$Y_{\mathcal{T}}$, Young subgroup of Σ_n associated to the set $\mathcal{T}$, 89

G/H, quotient group, 13
$H \leq G$, H is a subgroup of G, 11
$H \lhd G$, H is a normal subgroup of G, 12
$s(G)$, number of pseudoreflections in G, 269

G-action, 196
G-map, 45
G-set, 45
G-set, isomorphic, 45
$\mathrm{Stab}_G(s)$, stabilizer subgroup, 46
$o[s]$, orbit of the element s, 46

η_G, Noether map, 118
η_G^G, Noether map, 118
Pol, polarization map, 125
Tr^G, transfer, 103
Tr_H^G, relative transfer, 106
π^G, averaging operator, 105
π_H^G, relative averaging operator, 107
$\Phi_f(X)$, characteristic polynomial of f, 191
$c_i(x)$, ith orbit Chern class of x, 100
$c_{\mathrm{top}}(x)$, top orbit Chern class of x, 100
$c(I)$, Ith polarized Chern class, 124
$o(s)$, orbit sum of the element s, 71
p_i, symmetric power sum, 68
s_i, ith elementary symmetric polynomial, 67
$s(I)$, Ith polarized elementary symmetric polynomial, 124
$\mathrm{N}^G(q)$, the norm of q, 189

$\mathcal{C}$, code, 219
$\mathcal{C}^{\perp}$, dual code, 219
$W_{\mathcal{C}}(x, y)$, weight enumerator, 219

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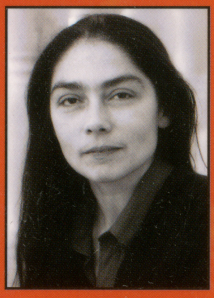

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ISBN 0-8218-4132-7

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