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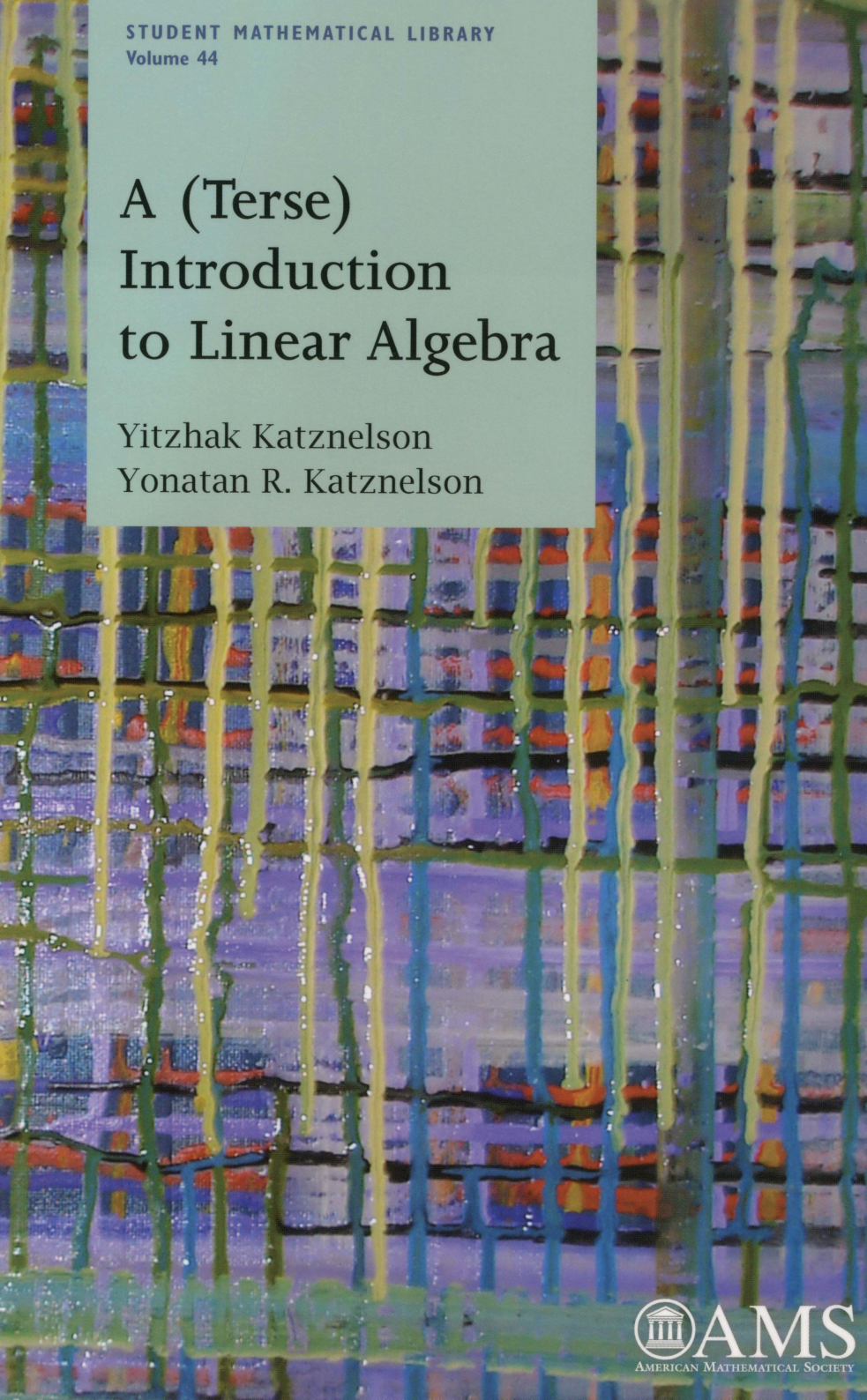
A (Terse) Introduction to Linear Algebra

Yitzhak Katznelson
Yonatan R. Katznelson



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Contents

Preface ix

1 Vector Spaces 1

1.1 Groups and fields 1

1.2 Vector spaces 4

1.3 Linear dependence, bases, and dimension 14

1.4 Systems of linear equations 22

*1.5 Normed finite-dimensional linear spaces 32

2 Linear Operators and Matrices 35

2.1 Linear operators 35

2.2 Operator multiplication 39

2.3 Matrix multiplication 41

2.4 Matrices and operators 46

2.5 Kernel, range, nullity, and rank 51

*2.6 Operator norms 56

3 Duality of Vector Spaces 57

3.1 Linear functionals 57

3.2 The adjoint 62

4 Determinants 65

4.1 Permutations 65

4.2 Multilinear maps 69

4.3 Alternating n-forms 74

4.4 Determinant of an operator 76

4.5 Determinant of a matrix 79

5 Invariant Subspaces 85
5.1 The characteristic polynomial 85
5.2 Invariant subspaces 88
5.3 The minimal polynomial 93

6 Inner-Product Spaces 103
6.1 Inner products 103
6.2 Duality and the adjoint 111
6.3 Self-adjoint operators 113
6.4 Normal operators 119
6.5 Unitary and orthogonal operators 121
*6.6 Positive definite operators 127
*6.7 Polar decomposition 128
*6.8 Contractions and unitary dilations 132

7 Structure Theorems 135
7.1 Reducing subspaces 135
7.2 Semisimple systems 142
7.3 Nilpotent operators 147
7.4 The Jordan canonical form 151
*7.5 The cyclic decomposition, general case 152
*7.6 The Jordan canonical form, general case 156

8 Additional Topics 159
8.1 Functions of an operator 159
8.2 Quadratic forms 162
8.3 Perron-Frobenius theory 166
8.4 Stochastic matrices 178
8.5 Representation of finite groups 180

A Appendix 187
A.1 Equivalence relations–partitions 187
A.2 Maps 188
A.3 Groups 189
*A.4 Group actions 194
A.5 Rings and algebras 196

A.6 Polynomials 201

Index **211**

Symbols **215**

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Preface

This book is a presentation of the elements of Linear Algebra that every mathematician, and everyone who uses mathematics, should know. It covers the core material, from the basic notion of a finite-dimensional vector space over a general field, to the canonical forms of linear operators and their matrices, obtained by the decomposition of a general linear system into the direct sum of cyclic systems. Along the way it covers such key topics as: systems of linear equations, linear operators and matrices, determinants, duality, inner products and the spectral theory of operators on inner-product spaces. We conclude with a selection of additional topics, indicating some of the directions in which the core material can be applied and developed.

In its mathematical prerequisites the book is elementary, in the sense that no previous knowledge of linear algebra is assumed. It is self-contained, and includes an appendix that provides all the necessary background material: the *very basic* properties of groups, rings, and of the algebra of polynomials over a field. The book is intended, however, for readers with some mathematical maturity and readiness to deal with abstraction and formal reasoning. It is appropriate for an advanced undergraduate course.

As the title implies, the style of the book is somewhat terse. We mean this in two senses.

First, we focus with few digressions on the principal ideas and results of linear algebra *qua* linear algebra. The book contains fewer routine numerical examples than do many other texts, and offers almost no interspersed applications to other fields; these should be adapted to the readership and, if the book is used in a course, provided by the teacher.

Second, the writing itself tends to be concise and to the point, to the extent that some of the proofs might be better described as detailed lists of hints. This is intentional—we believe that students learn more by having to fill in some details themselves.

Besides its style, this book differs from many other texts on the subject in that we try to present the main ideas, whenever possible, in the context of vector spaces over a general field, $\mathbb{F}$, rather than assuming the underlying field to be $\mathbb{R}$ or $\mathbb{C}$. Inner-product spaces, along with the naturally associated classes of self-adjoint, normal, and unitary (or orthogonal) operators, are introduced later than in many books, and the *spectral theorems* for these operators, besides being fundamentally important on their own, also serve here to pave the way for the notions of *reducing* and *semisimplicity* and, eventually, to the general structure theorems—the Jordan form, when the underlying field is algebraically closed, and the corresponding form over general fields.

The text consists of eight chapters and an appendix. These are divided into sections, and further into subsections. Definitions, propositions, examples, etc., are numbered according to the subsection in which they appear, and no subsection has more than one object (definition, theorem, etc.) of each kind. For example, Lemma 1.3.5 is the lemma appearing in subsection 1.3.5, and Theorem 1.3.5 is the theorem appearing in the same subsection. References to the appendix have the form A.x.y (for subsection y of section x, in the appendix).

Exercises appear at the end of sections, and are numbered accordingly, e.g., exercise **ex3.1.2** is the second exercise of section 3.1.

Starred sections, subsections, and exercises contain material that can be skipped on first reading. Several of these sections, as well as parts of the additional topics (in Chapter 8), require some familiarity with basic analysis, e.g., concepts like convergence and continuity.

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Index

Adjoint
    of a matrix, 112
    of an operator, 63, 112
Algebra, 200
Alternating form, 72
Annihilator, 59
Automorphism, 7
Axiom of choice, 20

Basic partition, 175
Basis, 15
    dual, 58
    standard, 17
Bilinear
    form, 59, 63, 69
    map, 69

Canonical
    prime-power decomposition, 140
Cauchy-Schwarz, 105
Cayley-Hamilton, 95
Character, 124
Characteristic polynomial
    of a matrix, 86
    of an operator, 85
Codimension, 18
Cofactor, 80
Complement, 11
Composition , 39
Congruent
    matrices, 163
Conjugate
    matrices, 43
    operators, 40
Conjugation
    group actions, 196
    in S_n, 67
Connecting chain, 171
Contraction, 132
Coset, 10, 192
Cycle, 65, 66
Cyclic
    decomposition, 152
    system, 94
    vector, 94

Decomposition
    cyclic, 150, 152
    general], 138
    prime power, 140
Degrees of freedom, 28
Determinant
    of a matrix, 79
    of an operator, 76
Diagonal matrix, 44
Diagonal sum, 81, 136
Diagonalizable, 49
Dimension, 17, 18
Direct sum
    formal, 10
    of subspaces, 11

Eigenspace, 86
    generalized, 142
Eigenvalue, 64, 86, 89
Eigenvector, 64, 86, 89
    dominant, 168

Elementary divisors, 155
Equivalence relation, 187
Euclidean space, 103

Factorization
  in $\mathbb{R}[x]$, 206
  prime-power, 140, 141, 204
Field, 2
  extension, 200
Fixed point, 65
Flag, 92
Fourier expansion, 125
Frobenius, 175

Gauss-Jordan elimination, 26
Gaussian elimination, 24
Group, 1, 189
  abelian, 1
  cyclic, 192
  dual, 125
  general linear, 40, 43

Hadamard's inequality, 110
Hamel basis, 20
Hermitian
  form, 103, 163

Ideal, 197
Idempotent, 41, 110
Independent
  subspaces, 11, 13
  vectors, 14
Inertia, law of, 165
Inner product, 103
Irreducible
  polynomial, 203
  system, 135
Isomorphism, 6

Jordan canonical form, 151, 156

k-form, 69
Kernel, 51

Lagrange, 62
Linear
  combination, 9
  operator, 35
  system, 40, 49
Linear equations
  homogeneous, 22
  nonhomogeneous, 22

Markov chain, 178
  reversible, 179
Matrix
  augmented, 25
  companion, 96
  derogatory, 97
  diagonal, 9
  Hermitian, 113
  integral, 82
  nonderogatory, 97
  nonnegative, 166, 170
  orthogonal, 122
  permutation, 44
  positive, 167
  self-adjoint, 113
  skew-symmetric, 9
  stochastic, 178
  strongly transitive, 175
  symmetric, 8
  transitive, 171
  triangular, 9, 81, 92
  unimodular, 82
  unipotent, 151
  unitary , 122
Minimal
  system, 100
Minimal polynomial, 96
  for *(T,v)*, 93

Minmax principle, 118
Monic polynomial, 201
Multilinear
    form, 69
    map, 69

Nilpotent, 147
Nilspace, 142
Norm
    of an operator, 56
    on a vector space, 32
Normal
    operator, 119
Nullity, 51
Nullspace, 52

Operator
    derogatory, 97
    induced on a quotient, 78
    linear, 35
    nonderogatory, 97
    nonnegative definite, 127
    nonsingular, 52
    normal, 119
    orthogonal, 121
    positive definite, 127
    self-adjoint, 113
    singular, 52
    unitary, 121
Order
    element, 192
    group, 189
Orientation, 77
Orthogonal
    operator, 121
    projection, 108
    vectors, 106
Orthogonal equivalence, 122
Orthonormal, 106

Period group, 174
Periodicity, 174
Permutation, 65, 189
Permutation matrix, 44
Perron, 168
Pivot column, 26
Polar decomposition, 128
Polarization, 111
Primary components, 140
Probability vector, 178
Projection, 41
    along a subspace, 36
    orthogonal, 108

Quadratic form, 162
    positive definite, 165
Quotient space, 9

Range, 51
Rank
    column, 29
    of a matrix, 29
    of an operator, 51
    row, 25
Reduced-row-echelon form, 26
Reducing subspace, 135
Regular representation, 185
Representation, 181
    equivalent, 182
    faithful, 181
    reducible, 184
    regular, 185
    unitary, 181
Restriction of an operator, 78
Return times, 171
Ring, 196
Row equivalence, 25

Schur decomposition, 123
Schur's lemma, 100

Self-adjoint
    algebra, 117
    operator, 113
Semisimple
    algebra, 144
    system, 142
Shift
    k-shift, 148
    standard, 148
Similar
    matrices, 49
    operators, 49
Singular value decomposition, 131
Singular values, 130
Solution-set, 8, 23
Span, 9, 14
Spectral mapping theorem, 89
Spectral norm, 162, 167
Spectral Theorems, 114–120
Spectrum, 89, 141
    joint, 117
    of a matrix, 86
    of an operator, 85
Square-free, 143
Steinitz' lemma, 17
Stochastic, 178
Subalgebra, 200
Subgroup
    index, 192
    normal, 193
Submatrix, 30
    principal, 30
Support, 167
Sylvester matrix, 32
Symmetric form, 72
Symmetric group, 65

Tensor product, 12
Trace, 87
Transition matrix, 178
Transposition, 66

Unitary
    group, 183
    operator, 121
    space, 103
Unitary dilation, 132
Unitary equivalence, 122

Vandermonde, 82
Vector space, 4
    complex, 4
    real, 4

Symbols

$\mathbb{C}$, 3
$\mathbb{Q}$, 3
$\mathbb{R}$, 3
$\mathbb{T}^*$, 124
$\mathbb{Z}_2$, 3
$\mathbb{Z}_p$, 3
$\leq_A$, 175

A_σ, 44
A_T, 46
A^{Tr}, 36

$C([0,1])$, 6
$C^\infty([-1,1])$, 6
$C_{\mathbb{R}}([0,1])$, 6
χ_T, 85
$C(X)$, 6
$\mathsf{comm}[T]$, 41
$\mathsf{C}_{\mathbf{v}}$, 38
$\mathsf{C}_{\mathbf{w},\mathbf{v}}$, 48

$\dim \mathcal{V}$, 18

$\mathbb{F}^n$, 4
$\mathbb{F}[x]$, 5
$\mathbb{F}_N[x]$, 8
$\mathbb{F}[x_1,\ldots,x_k]$, 5

$\mathbf{GL}(\mathcal{H})$, 183
$\mathbf{GL}(\mathcal{V})$, 40

$\mathsf{height}[v]$, 147
$HOM(\mathcal{V},\mathcal{W})$, 37

$\mathscr{L}(\mathcal{V})$, 37

$\mathscr{L}(\mathcal{V},\mathcal{W})$, 37

Ш, 170
$\mathscr{M}(n;\mathbb{F})$, 5
$\mathscr{M}(n,m;\mathbb{F})$, 5
$\mathsf{minP}_{\mathrm{T}}$, 96
$\mathsf{minP}_{\mathrm{T,v}}$, 94
$\mathscr{M}(n,\mathbb{Z})$, 82
$\mathscr{ML}(\{\mathcal{V}_j\}_{j=1}^k,\mathcal{W})$, 70
$\mathscr{ML}(\mathcal{V}^{\oplus k})$, 70
$\mathscr{ML}_{sym}(\mathcal{V}^{\oplus k})$, 72
$\mathscr{ML}_{alt}(\mathcal{V}^{\oplus k})$, 72

$\mathscr{O}(n)$, 122

$\pi_{\mathcal{W}}$, 108
$\mathscr{P}(T)$, 40, 98

$\rho(A)$, 29

S_n, 2, 65
$\mathsf{span}[E]$, 9
$\mathsf{span}[T,v]$, 88
$\|\ \|_{sp}$, 167

$T_{\mathcal{W}}$, 78
$T_{\mathcal{V}_1}$, 36

$\mathscr{U}(n)$, 122

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