Lectures on Surfaces
(Almost) Everything You Wanted to Know about Them

Anatole Katok
Vaughn Climenhaga
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2000 Mathematics Subject Classification. Primary 51–01, 53–01, 57N05; Secondary 53A05, 57R05.

For additional information and updates on this book, visit www.ams.org/bookpages/stml-46

Library of Congress Cataloging-in-Publication Data
Katok, A. B.
Lectures on surfaces : (almost) everything you wanted to know about them / Anatole Katok, Vaughn Climenhaga.
p. cm. — (Student mathematical library ; v. 46)
Includes bibliographical references and index.
ISBN 978-0-8218-4679-7 (alk. paper)
QA571.K34 2008
516—dc22 2008029299

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Foreword: MASS and REU at Penn State University

This book is part of a collection published jointly by the American Mathematical Society and the MASS (Mathematics Advanced Study Semesters) program as a part of the Student Mathematical Library series. The books in the collection are based on lecture notes for advanced undergraduate topics courses taught at the MASS and/or Penn State summer REU (Research Experiences for Undergraduates). Each book presents a self-contained exposition of a non-standard mathematical topic, often related to current research areas, accessible to undergraduate students familiar with an equivalent of two years of standard college mathematics and suitable as a text for an upper division undergraduate course.

Started in 1996, MASS is a semester-long program for advanced undergraduate students from across the USA. The program’s curriculum amounts to sixteen credit hours. It includes three core courses from the general areas of algebra/number theory, geometry/topology and analysis/dynamical systems, custom designed every year; an interdisciplinary seminar; and a special colloquium. In addition, every participant completes three research projects, one for each core course. The participants are fully immersed into mathematics, and
this, as well as intensive interaction among the students, usually leads to a dramatic increase in their mathematical enthusiasm and achievement. The program is unique for its kind in the United States.

The summer mathematical REU program is formally independent of MASS, but there is a significant interaction between the two: about half of the REU participants stay for the MASS semester in the fall. This makes it possible to offer research projects that require more than seven weeks (the length of the REU program) for completion. The summer program includes the MASS Fest, a two to three day conference at the end of the REU at which the participants present their research and that also serves as a MASS alumni reunion. A non-standard feature of the Penn State REU is that, along with research projects, the participants are taught one or two intense topics courses.

Detailed information about the MASS and REU programs at Penn State can be found on the website www.math.psu.edu/mass.
Preface

This book is a result of the MASS course in geometry in the fall semester of 2007. MASS core courses are traditionally labeled as analysis, algebra, and geometry, but the understanding of each area is broad, e.g. number theory and combinatorics are allowed as algebra courses, topology is considered as a part of geometry, and dynamical systems as a part of analysis. No less importantly, an interaction of ideas and concepts from different areas of mathematics is highly valued.

The topic came to me as very natural under these conditions. Surfaces are among the most common and easily visualized mathematical objects, and their study brings into focus fundamental ideas, concepts, and methods from geometry proper, topology, complex analysis, Morse theory, group theory, and suchlike. At the same time, many of those notions appear in a technically simplified and more graphic form than in their general “natural” settings. So, here was an opportunity to acquaint a group of bright and motivated undergraduates with a wealth of concepts and ideas, many of which would be difficult for them to absorb if presented in a traditional fashion. This is the central idea of the course and the book reflects it closely.

The first, primarily expository, chapter introduces many (but not all) principal actors, such as the round sphere, flat torus, Möbius strip, Klein bottle, elliptic plane, and so on, as well as various methods of
describing surfaces, beginning with the traditional representation by
equations in three-dimensional space, proceeding to parametric rep-
resentation, and introducing the less intuitive, but central for our
purposes, representation as factor spaces. It also includes a prelimi-
nary discussion of the metric geometry of surfaces. Subsequent chap-
ters introduce fundamental mathematical structures: topology, com-
binatorial (piecewise-linear) structure, smooth structure, Riemannian
metric, and complex structure in the specific context of surfaces. The
assumed background is the standard calculus sequence, some linear
algebra, and rudiments of ODE and real analysis. All notions are
introduced and discussed, and virtually all results proved, based on
this background.

The focal point of the book is the Euler characteristic, which ap-
pears in many different guises and ties together concepts from com-
binatorics, algebraic topology, Morse theory, ODE, and Riemannian
geometry. The repeated appearance of the Euler characteristic pro-
vides both a unifying theme and a powerful illustration of the notion
of an invariant in all those theories.

A further idea of both the motivations and the material presented
in the book may be found in the Table of Contents, which is quite
detailed.

My plan for teaching the course was somewhat bold and ambi-
tious, and could have easily miscarried had I not been blessed with a
teaching assistant who became the book’s co-author. I decided to use
no text either for my own preparations or as a prop for students. In-
stead, I decided to present the material the way I understand it, with
not only descriptions and examples, but also proofs, coming directly
from my head. A mitigating factor was that, although sufficiently
broadly educated, I am not a professional topologist or geometer.
Hence, the stuff I had ready in my head or could easily reconstruct
should not have been too obscure or overly challenging.

So, this is how the book came about. I prepared each lecture
(usually without or with minimal written notes), and my TA, the
third year Ph.D. student Vaughn Climenhaga, took notes and within
24 hours, usually less, prepared a very faithful and occasionally even
somewhat embellished version typed in TeX. I usually did some very
light editing before posting each installation for the students. Thus, the students had the text growing in front of their eyes in real time.

By the end of the Fall semester the notes were complete: additional work involved further editing and, in a few cases, completing and expanding proofs; a slight reordering of material to make each chapter consist of complete lectures; and in a couple of cases, merging two lectures into one, if in class a considerable repetition appeared. But otherwise the book fully retained the structure of the original one-semester course, and its expansion is due to the addition of a large number of pictures, a number of exercises (some were originally given in separate homework sets, others added later), and some “prose”, i.e. discussions and informal explanations. All results presented in the book appeared in the course, and, as I said before, only in a few cases did proofs need to be polished or completed.

Aside from creating the original notes, my co-author Vaughn Climenhaga participated on equal terms in the editorial process, and, very importantly, he produced practically all of the pictures, including dozens of beautiful 3-dimensional images for which, in many cases, even the concept was solely his. Without him, I am absolutely sure that I would not have been able to turn my course into a book in anything approaching the present timeframe, and even if I did at all, the quality of the final product would have been considerably lower.

Anatole Katok
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4 **W. J. Kaczor and M. T. Nowak**, Problems in mathematical analysis I: Real numbers, sequences and series, 2000
3 **Roger Knobel**, An introduction to the mathematical theory of waves, 2000
2 **Gregory F. Lawler and Lester N. Coyle**, Lectures on contemporary probability, 1999
1 **Charles Radin**, Miles of tiles, 1999
Surfaces are among the most common and easily visualized mathematical objects, and their study brings into focus fundamental ideas, concepts, and methods from geometry, topology, complex analysis, Morse theory, and group theory. At the same time, many of those notions appear in a technically simpler and more graphic form than in their general “natural” settings.

The first, primarily expository, chapter introduces many of the principal actors—the round sphere, flat torus, Möbius strip, Klein bottle, elliptic plane, etc.—as well as various methods of describing surfaces, beginning with the traditional representation by equations in three-dimensional space, proceeding to parametric representation, and also introducing the less intuitive, but central for our purposes, representation as factor spaces. It concludes with a preliminary discussion of the metric geometry of surfaces, and the associated isometry groups. Subsequent chapters introduce fundamental mathematical structures—topological, combinatorial (piecewise-linear), smooth, Riemannian (metric), and complex—in the specific context of surfaces.

The focal point of the book is the Euler characteristic, which appears in many different guises and ties together concepts from combinatorics, algebraic topology, Morse theory, ordinary differential equations, and Riemannian geometry. The repeated appearance of the Euler characteristic provides both a unifying theme and a powerful illustration of the notion of an invariant in all those theories.

The assumed background is the standard calculus sequence, some linear algebra, and rudiments of ODE and real analysis. All notions are introduced and discussed, and virtually all results proved, based on this background.

This book is a result of the MASS course in geometry in the fall semester of 2007.