A (Terse) Introduction to Lebesgue Integration

John Franks
A (Terse)
Introduction
to Lebesgue
Integration
The images on the cover are representations of the ergodic transformations in Chapter 7. The figure with the implied cardioid traces iterates of the squaring map on the unit circle. The “spirograph” figures trace iterates of an irrational rotation. The arc of + signs consists of iterates of an irrational rotation. I am grateful to Edward Dunne for providing the figures.

For additional information and updates on this book, visit www.ams.org/bookpages/stml-48
To my family: Judy, Josh, Mark and Alex
# Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Preface</td>
<td>xi</td>
</tr>
<tr>
<td><strong>Chapter 1. The Regulated and Riemann Integrals</strong></td>
<td></td>
</tr>
<tr>
<td>§1.1. Introduction</td>
<td>1</td>
</tr>
<tr>
<td>§1.2. Basic Properties of an Integral</td>
<td>2</td>
</tr>
<tr>
<td>§1.3. Step Functions</td>
<td>4</td>
</tr>
<tr>
<td>§1.4. Uniform and Pointwise Convergence</td>
<td>7</td>
</tr>
<tr>
<td>§1.5. Regulated Integral</td>
<td>8</td>
</tr>
<tr>
<td>§1.6. The Fundamental Theorem of Calculus</td>
<td>13</td>
</tr>
<tr>
<td>§1.7. The Riemann Integral</td>
<td>16</td>
</tr>
<tr>
<td><strong>Chapter 2. Lebesgue Measure</strong></td>
<td>25</td>
</tr>
<tr>
<td>§2.1. Introduction</td>
<td>25</td>
</tr>
<tr>
<td>§2.2. Null Sets</td>
<td>27</td>
</tr>
<tr>
<td>§2.3. Sigma Algebras</td>
<td>29</td>
</tr>
<tr>
<td>§2.4. Lebesgue Measure</td>
<td>31</td>
</tr>
<tr>
<td>§2.5. The Lebesgue Density Theorem</td>
<td>35</td>
</tr>
<tr>
<td>§2.6. Lebesgue Measurable Sets – Summary</td>
<td>37</td>
</tr>
<tr>
<td><strong>Chapter 3. The Lebesgue Integral</strong></td>
<td>41</td>
</tr>
<tr>
<td>§3.1. Measurable Functions</td>
<td>41</td>
</tr>
<tr>
<td>§3.2. The Lebesgue Integral of Bounded Functions</td>
<td>48</td>
</tr>
<tr>
<td>§3.3. The Bounded Convergence Theorem</td>
<td>56</td>
</tr>
<tr>
<td>Chapter 4. The Integral of Unbounded Functions</td>
<td>63</td>
</tr>
<tr>
<td>§4.1. Non-negative Functions</td>
<td>63</td>
</tr>
<tr>
<td>§4.2. Convergence Theorems</td>
<td>67</td>
</tr>
<tr>
<td>§4.3. Other Measures</td>
<td>72</td>
</tr>
<tr>
<td>§4.4. General Measurable Functions</td>
<td>77</td>
</tr>
<tr>
<td>Chapter 5. The Hilbert Space $L^2$</td>
<td>83</td>
</tr>
<tr>
<td>§5.1. Square Integrable Functions</td>
<td>83</td>
</tr>
<tr>
<td>§5.2. Convergence in $L^2$</td>
<td>89</td>
</tr>
<tr>
<td>§5.3. Hilbert Space</td>
<td>95</td>
</tr>
<tr>
<td>§5.4. Fourier Series</td>
<td>99</td>
</tr>
<tr>
<td>§5.5. Complex Hilbert Space</td>
<td>104</td>
</tr>
<tr>
<td>Chapter 6. Classical Fourier Series</td>
<td>111</td>
</tr>
<tr>
<td>§6.1. Real Fourier Series</td>
<td>111</td>
</tr>
<tr>
<td>§6.2. Integrable Complex-Valued Functions</td>
<td>119</td>
</tr>
<tr>
<td>§6.3. The Complex Hilbert Space $L^2_c[-\pi, \pi]$</td>
<td>122</td>
</tr>
<tr>
<td>§6.4. The Hilbert Space $L^2_c(T)$</td>
<td>125</td>
</tr>
<tr>
<td>Chapter 7. Two Ergodic Transformations</td>
<td>129</td>
</tr>
<tr>
<td>§7.1. Measure Preserving Transformations</td>
<td>130</td>
</tr>
<tr>
<td>§7.2. Ergodicity</td>
<td>134</td>
</tr>
<tr>
<td>§7.3. The Birkhoff Ergodic Theorem</td>
<td>137</td>
</tr>
<tr>
<td>Appendix A. Background and Foundations</td>
<td>141</td>
</tr>
<tr>
<td>§A.1. The Completeness of $\mathbb{R}$</td>
<td>141</td>
</tr>
<tr>
<td>§A.2. Functions and Sequences</td>
<td>143</td>
</tr>
<tr>
<td>§A.3. Limits</td>
<td>145</td>
</tr>
<tr>
<td>§A.4. Complex Limits</td>
<td>148</td>
</tr>
<tr>
<td>§A.5. Set Theory and Countability</td>
<td>151</td>
</tr>
<tr>
<td>§A.6. Open and Closed Sets</td>
<td>156</td>
</tr>
</tbody>
</table>
Contents

§ A.7. Compact Subsets of $\mathbb{R}$ 158
§ A.8. Continuous and Differentiable Functions 160
§ A.9. Real Vector Spaces 162
§ A.10. Complex Vector Spaces 166
§ A.11. Complete Normed Vector Spaces 170

Appendix B. Lebesgue Measure 173
§ B.1. Introduction 173
§ B.2. Outer Measure 174
§ B.3. The $\sigma$-algebra of Lebesgue Measurable Sets 180
§ B.4. The Existence of Lebesgue Measure 189

Appendix C. A Non-measurable Set 193

Bibliography 197

Index 199
Preface

This text is intended to provide a student’s first encounter with the concepts of measure theory and functional analysis. Its structure and content were greatly influenced by my belief that good pedagogy dictates introducing difficult concepts in their simplest and most concrete forms. For example, the study of abstract metric spaces should come after the study of the metric and topological properties of $\mathbb{R}^n$. Multidimensional calculus should not be introduced in Banach spaces even if the proofs are identical to the proofs for $\mathbb{R}^n$. And a course in linear algebra should precede the study of abstract algebra.

Hence, despite the use of the word “terse” in the title, this text might also have been called “A (Gentle) Introduction to Lebesgue Integration”. It is terse in the sense that it treats only a subset of those concepts typically found in a substantive graduate level analysis course. I have emphasized the motivation of these concepts and attempted to treat them in their simplest and most concrete form. In particular, little mention is made of general measures other than Lebesgue until the final chapter. Indeed, we restrict our attention to Lebesgue measure on $\mathbb{R}$ and no treatment of measures on $\mathbb{R}^n$ for $n > 1$ is given. The emphasis is on real-valued functions but complex functions are considered in the chapter on Fourier series and in the final chapter on ergodic transformations. I consider the narrow selection of topics to be an approach at one end of a spectrum whose
other end is represented, for example, by the excellent graduate text [Ru] by Rudin which introduces Lebesgue measure as a corollary of the Riesz representation theorem. That is a sophisticated and elegant approach, but, in my opinion, not one which is suited to a student’s first encounter with Lebesgue integration.

In this text the less elegant, and more technical, classical construction of Lebesgue measure due to Caratheodory is presented, but is relegated to an appendix. The intent is to introduce the Lebesgue integral as a tool. The hope is to present it in a quick and intuitive way, and then go on to investigate the standard convergence theorems and a brief introduction to the Hilbert space of $L^2$ functions on the interval.

This text should provide a good basis for a one semester course at the advanced undergraduate level. It might also be appropriate for the beginning part of a graduate level course if Appendices B and C are covered. It could also serve well as a text for graduate level study in a discipline other than mathematics which has serious mathematical prerequisites.

The text presupposes a background which a student should possess after a standard undergraduate course in real analysis. It is terse in the sense that the density of definition-theorem-proof content is quite high. There is little hand holding and not a great number of examples. Proofs are complete but sometimes tersely written. On the other hand, some effort is made to motivate the definitions and concepts.

Chapter 1 provides a treatment of the “regulated integral” (as found in Dieudonné [D]) and of the Riemann integral. These are treated briefly, but with the intent of drawing parallels between their definition and the presentation of the Lebesgue integral in subsequent chapters.

As mentioned above the actual construction of Lebesgue measure and proofs of its key properties are left for an appendix. Instead the text introduces Lebesgue measure as a generalization of the concept of length and motivates its key properties: monotonicity, countable additivity, and translation invariance. This also motivates the concept
of $\sigma$-algebra. If a generalization of length has these three key properties, then it needs to be defined on a $\sigma$-algebra for these properties to make sense.

In Chapter 2 the text introduces null sets and shows that any generalization of length satisfying monotonicity and countable additivity must assign zero to them. We then define Lebesgue measurable sets to be sets in the $\sigma$-algebra generated by open sets and null sets.

At this point we state a theorem which asserts that Lebesgue measure exists and is unique, i.e., there is a function $\mu$ defined for measurable subsets of a closed interval which satisfies monotonicity, countable additivity, and translation invariance.

The proof of this theorem (Theorem 2.4.2) is included in an appendix where it is also shown that the more common definition of measurable sets (using outer measure) is equivalent to being in the $\sigma$-algebra generated by open sets and null sets.

Chapter 3 discusses bounded Lebesgue measurable functions and their Lebesgue integral. The last section of this chapter, and some of the exercises following it, focus somewhat pedantically on the concept of “almost everywhere.” The hope is to develop sufficient facility with the concept that it can be treated more glibly in subsequent chapters.

Chapter 4 considers unbounded functions and some of the standard convergence theorems. In Chapter 5 we introduce the Hilbert space of $L^2$ functions on an interval and show several elementary properties leading up to a definition of Fourier series.

Chapter 6 discusses classical real and complex Fourier series for $L^2$ functions on the interval and shows that the Fourier series of an $L^2$ function converges in $L^2$ to that function. The proof is based on the Stone-Weierstrass theorem which is stated but not proved.

Chapter 7 introduces some concepts from measurable dynamics. The Birkhoff ergodic theorem is stated without proof and results on Fourier series from Chapter 6 are used to prove that an irrational rotation of the circle is ergodic and the squaring map $z \mapsto z^2$ on the complex numbers of modulus 1 is ergodic.

Appendix A summarizes the needed prerequisites providing many proofs and some exercises. There is some emphasis in this section
on the concept of countability, to which I would urge students and
instructors to devote some time, as countability plays an very crucial
role in the study of measure theory.

In Appendix B we construct Lebesgue measure and prove it has
the properties cited in Chapter 2. In Appendix C we construct a
non-measurable set.

Finally, at the website http://www.ams.org/bookpages/stml-48
we provide solutions to a few of the more challenging exercises. These
exercises are marked with a (⋆) when they occur in the text.

This text grew out of notes I have used in teaching a one quarter
course on integration at the advanced undergraduate level. With
some selectivity of topics and well prepared students it should be
possible to cover all key concepts in a one semester course.
Bibliography


**Index**

- Complex numbers, 141
- Δ symmetric difference, 35
- ℍ imaginary part, 149
- ℳ Lebesgue measurable sets, 31
- ℳ(ℐ) measurable subsets of ℐ, 31
- ℑ natural numbers, 141
- ℚ rational numbers, 141
- ℝ real numbers, 141
- ℜ real part, 149
- ℤ integers, 141
- $L^2[a, b]$, square integrable functions, 84
- $L^2[a, b]$, square integrable complex functions, 121
- $ℓ^2$, 148
- ≪ absolutely continuous, 74
- ℳ movie, 139
- μ Lebesgue measure, 31, 182
- ν-almost all, 130
- ⊥ perpendicular, 95
- σ-algebra, 29, 37, 76
- σ-algebra generated by a family of sets, 30
- absolute continuity, 65
- absolute convergence, 95, 107, 147, 150
- absolutely continuous, 74
- absolutely continuous measure, 74
- additivity, 54

- Algebra of functions, 114
- almost all, 58
- almost everywhere, 58
- Axiom of choice, 155, 193, 194
- Bessel’s inequality, 102, 109
- bijection, 144
- bijective, 143
- bilinearity, 88, 163
- Birkhoff ergodic theorem, 137
- Borel σ-algebra, 30
- Borel sets, 30
- bounded convergence theorem, 56, 58
- Cantor middle third set, 38
- Carleson’s theorem, 117, 125
- Cartesian product, 143, 153
- Cauchy sequence, 91, 147, 150, 171
- Cauchy-Schwarz inequality, 86, 96, 163, 168
- characteristic function, 41
- classical Fourier coefficients, 113
- closed, 156
- closed interval, 156
- closed subspace, 99
- codomain, 143
- commutativity, 88, 163
- compact, 158
- complement, 151
complete, 91
complete normed space, 171
complete orthonormal family, 101, 108
complex conjugate, 148
complex convergent sequence, 149
complex Fourier coefficient, 108
complex Fourier series, 108
complex Hilbert space, 106
complex limit, 149
complex numbers, 141
continuity, 160
continuous, 160
corverge, 147, 150
corverge absolutely, 147, 150
corverge pointwise, 8
corverge uniformly, 7
corvergence in norm, 91
corvergence in the mean, 60, 61
corvergent sequence, 145
corvergent sequence in a normed space, 170
countability, 152
countable, 152
countable additivity, 26, 31, 38, 173, 181, 189, 190
countable additivity of Lebesgue integral, 71
countable subadditivity, 38, 177
dense, 156
density, of step and continuous functions in $L^2$, 90
Dirac $\delta$-measure, 77
distributivity, of $\cup$ and $\cap$, 151
domain, 143
dyadic rationals, 158
Egorov’s theorem, 72
ergodic, 134
ergodic theorem, 137
ergodic transformation, 134
essential bound, 60
essentially bounded, 60
Euler’s formula, 16, 112, 118, 149
even function, 118
extended real numbers, 44
extended real-valued function, 44
Fatou’s lemma, 69, 71
favorite movie, 139
finite, 144
finite intersection property, 160
finite measure, 73
finite measure space, 130
forward orbit, 129
Fourier coefficient, 101, 108
Fourier series, 101, 108, 113, 123, 127
full measure, 28
function, 143
greatest lower bound, 142
Hölder inequality, 86
half open, 156
Heine-Borel theorem, 158, 175
Hermitian form, 104, 167
Hilbert space, 92, 106
image, 143
imaginary part, 149
indicator function, 41
infimum, 142
infinite, 144
infinite series, 70
injection, 144
injective, 143
inner product, 88
inner product on $L^2[a,b]$, 88
inner product space, 163
integers, 141
integrable function, 63, 82
integrable function (complex), 119
integral with respect to $\nu$, 73
interval partition, 5
inverse function, 144
kernell, 99, 104
least upper bound, 142
Lebesgue convergence theorem, 67, 78, 82, 120
Lebesgue density point, 37
Lebesgue density theorem, 37
Lebesgue integrable, 77
Lebesgue integral, 42, 54
Lebesgue integral of a bounded function, 51
Lebesgue measurable, 180
Lebesgue measurable function, 45
Lebesgue measurable set, 31, 182, 183, 189
Lebesgue measure, 31, 174, 190
Lebesgue outer measure, 175
Lebesgue simple, 42
limit, 145
limit in a normed space, 170
limit point, 158
linear function, 98
linear functional, 96, 106
linearity, 42
Littlewood’s three principles, 34, 72
lower bound, 142
Mean value theorem, 161
measurable function, 44, 45
measurable function (complex), 119
measurable partition, 41
measurable set, 31, 174, 182, 189
measurable subset of $T$, 126
measurable with respect to a $\sigma$-algebra, 73
measure, 73
measure preserving transformation, 130
Minkowski inequality, 87
modulus, 149, 168
monotone, 146
monotone convergence theorem, 70
monotone decreasing, 146
monotone increasing, 146
monotonicity, 42, 181
movie, 139
natural numbers, 141
non-measurable set, 193
norm, 84, 163, 168
normed linear space, 164
nowhere dense, 39
null set, 28
open interval, 156
open set, 156
orbit, 129
orthogonal projection, 99, 107
orthonormal, 165
orthonormal family, 100, 106
outer measure, 175
parallelogram law, 164, 169
perfect, 39
perpendicular, 95, 168
Poincaré recurrence theorem, 133
pointwise, 56
positive definite, 88, 105, 163, 167
power set, 154
Pythagorean theorem, 95, 168
Radon-Nikodym derivative, 76
Radon-Nikodym theorem, 76
range, 143
rational numbers, 141
real numbers, 141
real part, 149
recurrent, 129, 132
regulated function, 9
regulated integral, 11
Riemann integrable, 18
Riemann integral, 18
sequence, 144
set difference, 32, 151
set inverse, 144
Sigma algebra, 29
simple, 42
simple functions, 41
square integrable, 84
square integrable complex functions, 121
square summable, 148
standard Hermitian form, 167
step function, 5
subadditivity, 174
supremum, 142
surjection, 144
surjective, 143
symmetric difference, 35
translation invariance, 26, 31, 173, 178, 190
triangle inequality, 164, 169
uncountable, 152
uniform continuity, 160
uniform convergence, 7
uniformly continuous, 12, 160
upper bound, 142

vector space of complex-valued functions, 166
vector space of real-valued functions, 2, 162

weak convergence of measures, 77
This book provides a student’s first encounter with the concepts of measure theory and functional analysis. Its structure and content reflect the belief that difficult concepts should be introduced in their simplest and most concrete forms.

Despite the use of the word “terse” in the title, this text might also have been called A (Gentle) Introduction to Lebesgue Integration. It is terse in the sense that it treats only a subset of those concepts typically found in a substantial graduate-level analysis course. The book emphasizes the motivation of these concepts and attempts to treat them simply and concretely. In particular, little mention is made of general measures other than Lebesgue until the final chapter and attention is limited to $\mathbb{R}$ as opposed to $\mathbb{R}^n$.

After establishing the primary ideas and results, the text moves on to some applications. Chapter 6 discusses classical real and complex Fourier series for $L^2$ functions on the interval and shows that the Fourier series of an $L^2$ function converges in $L^2$ to that function. Chapter 7 introduces some concepts from measurable dynamics. The Birkhoff ergodic theorem is stated without proof and results on Fourier series from Chapter 6 are used to prove that an irrational rotation of the circle is ergodic and that the squaring map on the complex numbers of modulus 1 is ergodic.

This book is suitable for an advanced undergraduate course or for the start of a graduate course. The text presupposes that the student has had a standard undergraduate course in real analysis.