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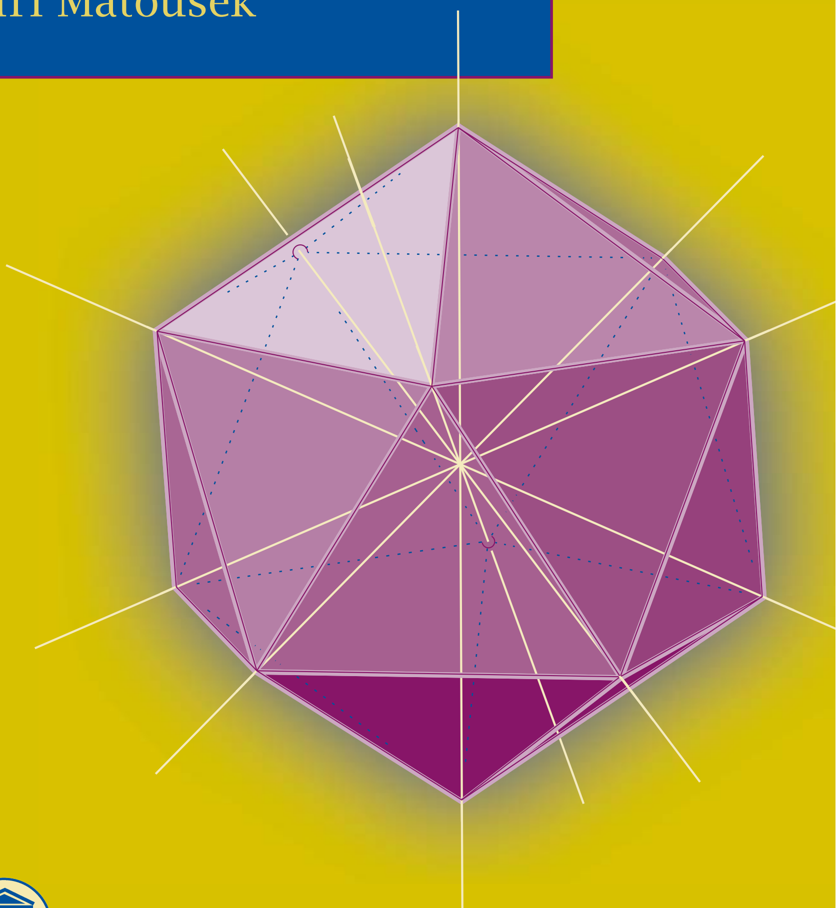
Thirty-three Miniatures

Mathematical and Algorithmic Applications of Linear Algebra

Jiří Matoušek



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Preface

Some years ago I started gathering nice applications of linear algebra, and here is the resulting collection. The applications belong mostly to the main fields of my mathematical interests—combinatorics, geometry, and computer science. Most of them are mathematical, in proving theorems, and some include clever ways of computing things, i.e., algorithms. The appearance of linear-algebraic methods is often unexpected.

At some point I started to call the items in the collection "miniatures". Then I decided that in order to qualify for a miniature, a complete exposition of a result, with background and everything, should not exceed four typeset pages (A4 format). This rule is absolutely arbitrary, as rules often are, but it has some rational core—namely, this extent can usually be covered conveniently in a 90-minute lecture, the standard length at the universities where I happened to teach. Then, of course, there are some exceptions to the rule, such as six-page miniatures that I just couldn't bring myself to omit.

The collection could obviously be extended indefinitely, but I thought thirty-three was a nice enough number and a good point to stop.

The exposition is intended mainly for lecturers (I've taught almost all of the pieces on various occasions) and also for students interested in nice mathematical ideas even when they require some thinking. The material is hopefully class-ready, where all details left to the reader should indeed be devil-free.

I assume a background in basic linear algebra, a bit of familiarity with polynomials, and some graph-theoretical and geometric terminology. The sections have varying levels of difficulty, and generally I have ordered them from what I personally regard as the most accessible to the more demanding.

I wanted each section to be essentially self-contained. With a good undergraduate background you can as well start reading at Section 24. This is kind of opposite to a typical mathematical textbook, where material is developed gradually, and if one wants to make sense of something on page 123, one usually has to understand the previous 122 pages, or with luck, some suitable 38 pages.

Of course, the anti-textbook structure leads to some boring repetitions and, perhaps more seriously, it puts a limit on the degree of achievable sophistication. On the other hand, I believe there are advantages as well: I gave up reading several textbooks well before page 123, after I realized that between the usually short reading sessions I couldn't remember the key definitions (people with small children will know what I'm talking about).

After several sections the reader may spot certain common patterns in the presented proofs, which could be discussed at great length, but I have decided to leave out any general accounts on linear-algebraic methods.

Nothing in this text is original, and some of the examples are rather well known and appear in many publications (including, in a few cases, other books of mine). Several general reference books are listed below. I've also added references to the original sources where I could find them. However, I've kept the historical notes at a minimum, and I've put only a limited effort into tracing the origins of the ideas (apologies to authors whose work is quoted badly or not at all—please let me know about such cases).

I would also appreciate learning about mistakes and hearing suggestions of how to improve the exposition.

Further reading. An excellent textbook is

> L. Babai and P. Frankl, *Linear Algebra Methods in Combinatorics (Preliminary version 2)*, Department of Computer Science, The University of Chicago, 1992.

Unfortunately, it has never been published officially. It can be obtained, with some effort, as lecture notes of the University of Chicago. It contains several of the topics discussed here, a lot of other material in a similar spirit, and a very nice exposition of some parts of linear algebra.

Algebraic graph theory is treated, e.g., in the books

> N. Biggs, *Algebraic Graph Theory*, 2nd edition, Cambridge University Press, Cambridge, 1993

and

> C. Godsil and G. Royle, *Algebraic Graph Theory*, Springer, New York, NY, 2001.

Probabilistic algorithms in the spirit of Sections 11 and 24 are well explained in the book

> R. Motwani and P. Raghavan, *Randomized Algorithms*, Cambridge University Press, Cambridge, 1995.

Acknowledgments. For valuable comments on preliminary versions of this booklet, I would like to thank Otfried Cheong, Esther Ezra, Nati Linial, Jana Maxová, Helena Nyklová, Yoshio Okamoto, Pavel Paták, Oleg Pikhurko, and Zuzana Safernová, as well as all other people whom I may have forgotten to include in this list. Thanks also to David Wilson for permission to use his picture of a random lozenge tiling in Miniature 22, and to Jennifer Wright Sharp for careful copyediting. Finally, I'm grateful to many people at the Department of Applied Mathematics of the Charles University in Prague and at the Institute of Theoretical Computer Science of the ETH Zurich for excellent working environments.

Notation

Most of the notation is defined in each section where it is used. Here are several general items that may not be completely unified in the literature.

The integers are denoted by $\mathbb{Z}$, the rationals by $\mathbb{Q}$, the reals by $\mathbb{R}$, and $\mathbb{F}_q$ stands for the q-element finite field.

The transpose of a matrix A is written as A^T. The elements of that matrix are denoted by a_{ij}, and similarly for all other Latin letters. Vectors are typeset in boldface: $\mathbf{v}, \mathbf{x}, \mathbf{y}$, and so on. If $\mathbf{x}$ is a vector in $\mathbb{K}^n$, where $\mathbb{K}$ is some field, x_i stands for the ith component, so $\mathbf{x} = (x_1, x_2, \ldots, x_n)$.

We write $\langle \mathbf{x}, \mathbf{y} \rangle$ for the standard scalar (or inner) product of vectors $\mathbf{x}, \mathbf{y} \in \mathbb{K}^n$: $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y + x_2 y_2 + \cdots + x_n y_n$. We also interpret such $\mathbf{x}, \mathbf{y}$ as $n \times 1$ (single-column) matrices, and thus $\langle \mathbf{x}, \mathbf{y} \rangle$ could also be written as $\mathbf{x}^T \mathbf{y}$. Further, for $\mathbf{x} \in \mathbb{R}^n$, $\|\mathbf{x}\| = \langle \mathbf{x}, \mathbf{x} \rangle^{1/2}$ is the Euclidean norm (length) of the vector $\mathbf{x}$.

Graphs are simple and undirected unless stated otherwise; i.e., a graph G is regarded as a pair (V, E), where V is the vertex set and E is the edge set, which is a set of unordered pairs of elements of V. For a graph G, we sometimes write $V(G)$ for the vertex set and $E(G)$ for the edge set.

Some conventions. When an important notion is defined in the text, it appears in **boldface**, which should help in looking it up. Less important terms, or general mathematical notions that are only reminded, are marked in *italics*.

In the index, mathematical notation involving a specific letter, such as S_n for the symmetric group or $E(G)$ for the edge set of a graph, is listed at the beginning of the corresponding letter's section. Only notation composed of special symbols or Greek letters appears at the beginning of the index.

Index

This volume contains a collection of clever mathematical applications of linear algebra, mainly in combinatorics, geometry, and algorithms. Each chapter covers a single main result with motivation and full proof in at most ten pages and can be read independently of all other chapters (with minor exceptions), assuming only a modest background in linear algebra.

The topics include a number of well-known mathematical gems, such as Hamming codes, the matrix-tree theorem, the Lovász bound on the Shannon capacity, and a counterexample to Borsuk's conjecture, as well as other, perhaps less popular but similarly beautiful results, e.g., fast associativity testing, a lemma of Steinitz on ordering vectors, a monotonicity result for integer partitions, or a bound for set pairs via exterior products.

The simpler results in the first part of the book provide ample material to liven up an undergraduate course of linear algebra. The more advanced parts can be used for a graduate course of linear-algebraic methods or for seminar presentations.

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