The Erdős Distance Problem
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Foreword

There are several goals for this book. As the title indicates, we certainly hope to familiarize you with some of the major results in the study of the Erdős distance problem. This goal should be easily attainable for most experienced mathematicians. However, if you are not an experienced mathematician, we hope to guide you through many advanced mathematical concepts along the way.

The book is based on the notes that were written for the summer program on the problem, held at the University of Missouri, August 1–5, 2005. This was the second year of the program, and our plan continued to be an introduction for motivated high school students to accessible concepts of higher mathematics.

This book is designed to be enjoyed by readers at different levels of mathematical experience. Keep in mind that some of the notes and remarks are directed at graduate students and professionals in the field. So, if you are relatively inexperienced, and a particular comment or observation uses terminology\(^1\) that you are not familiar with, you may want to skip past it or look up the definitions later. On the other hand, if you are a more experienced mathematician, feel free to skim the introductory portions to glean the necessary notation, and move on to the more specific subject matter.

\(^1\)One example of this is the mention of curvature in the first section of the Introduction.
Our book is heavily problem oriented. Most of the learning is meant to be done by working through the exercises. Many of these exercises are recently published results by mathematicians working in the area. In several places, steps are intentionally left out of proofs and, in the process of working on the exercises, the reader is then asked to fill them in. On a number of occasions, solutions to exercises are used in the book in an essential way. Sometimes the exercises are left till the end of the chapter, but a few times, we intersperse them throughout the chapter to illustrate concepts or to get the reader’s hands dirty, so the ideas really sink in right at that point in the exposition. Also, some exercises are much more complicated than others, and will probably require several hours of concentrated effort for even an advanced student. So please do not get discouraged. Having said that, let us add that you should not rely solely on exercises in these notes. Create your own problems and questions! Modify the lemmas and theorems below, and, whenever possible, improve them! Mathematics is a highly personal experience, and you will find true fulfillment only when you make the concepts in these notes your own in some way. Read this book with a pad of paper handy to really explore these ideas as they come along. Good luck!
Acknowledgements

This book would not have been possible without significant assistance of many people. Any list we write down is guaranteed to be incomplete, but we will give it a try. First, the authors wish to thank Nets Katz for contributing much of the material in Chapters 7 and 8. He also explained to us the importance of this material within the context of the Erdős distance problem and its relatives. We also wish to thank Misha Rudnev, whose collaboration with the second listed author on the finite field variant of the Erdős-Falconer distance problem ultimately led to the last three chapters of the book.

Numerous people have contributed important remarks on various aspects of the book. We are particularly indebted to Bill Banks, Pete Casazza, Jeremy Chapman, David Covert, Lacy Hardcastle, Derrick Hart, Tyler Salisbury-Jones, Doowon Koh, Mihalis Mourgoglou, Laura Poe, Shannon Reed, Krystal Taylor, Ignacio Uriarte-Tuero, Lee Anh Vinh, and Chandra Vaidyanathan.

The authors of the book were profoundly influenced in writing of this book by their conversations with many brilliant mathematicians who contributed to the study of the Erdős distance conjecture and related problems in the past 20 years. We have not had the honor of interacting with nearly all of them, but we did learn much from discussions with Michael Christ, Steve Hofmann, Philippe Jaming, Nets Katz, Mihalis Kolountzakis, Sergei Konyagin, Izabella
Laba, Michael Lacey, Pertti Mattila, Janos Pach, Steen Pedersen, Eric Sawyer, Andreas Seeger, Jozsef Solymosi, Stefan Steinerberger, Endres Szemerédi, Terry Tao, Gabor Tardos, Christoph Thiele, Csaba Tóth, William Trotter, Van Vu, and Yang Wang.

We thank Nancy Brown for the remarkable cover, which captures the central theme of book absolutely beautifully.

Last, but not least, we thank our families. Without their patience and support, nothing truly worthwhile is possible.
Bibliography


Biographical information

The first listed author was born on October 2, 1976 in Seattle and was raised across the water on Bainbridge Island. She graduated from NYU in 1999 and went on to UCLA to get her Ph.D. in December 2004. She spent two years at Georgia Tech as a postdoctoral fellow and has held lecturing positions at Emory University since.

The second listed author was born in Lvov, USSR, on December 14, 1967, emigrated to the United States of America at the age of eleven with his immediate family, and grew up in Chicago, Illinois. He graduated from the University of Chicago in 1989 with a B.S. in Pure Mathematics, and a Ph.D. from UCLA in 1993 under the direction of Christopher Sogge. After appointments at McMaster University, Wright State University, and Georgetown University, the author spent ten years at the University of Missouri, where this book was written. In July of 2010, he moved to the University of Rochester.

The third listed author was born in North Kansas City, Missouri, on May 19, 1982. He graduated from the University of Missouri in 2005 with degrees in Computer Engineering, Electrical Engineering, and Mathematics. Between musical performances and rock climbing excursions, he is working on a Ph.D. in Mathematics under the direction of the second listed author.
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The Erdős problem asks, What is the smallest possible number of distinct distances between points of a large finite subset of the Euclidean space in dimensions two and higher? The main goal of this book is to introduce the reader to the techniques, ideas, and consequences related to the Erdős problem. The authors introduce these concepts in a concrete and elementary way that allows a wide audience—from motivated high school students interested in mathematics to graduate students specializing in combinatorics and geometry—to absorb the content and appreciate its far-reaching implications. In the process, the reader is familiarized with a wide range of techniques from several areas of mathematics and can appreciate the power of the resulting symbiosis.

The book is heavily problem oriented, following the authors’ firm belief that most of the learning in mathematics is done by working through the exercises. Many of these problems are recently published results by mathematicians working in the area. The order of the exercises is designed both to reinforce the material presented in the text and, equally importantly, to entice the reader to leave all worldly concerns behind and launch head first into the multifaceted and rewarding world of Erdős combinatorics.