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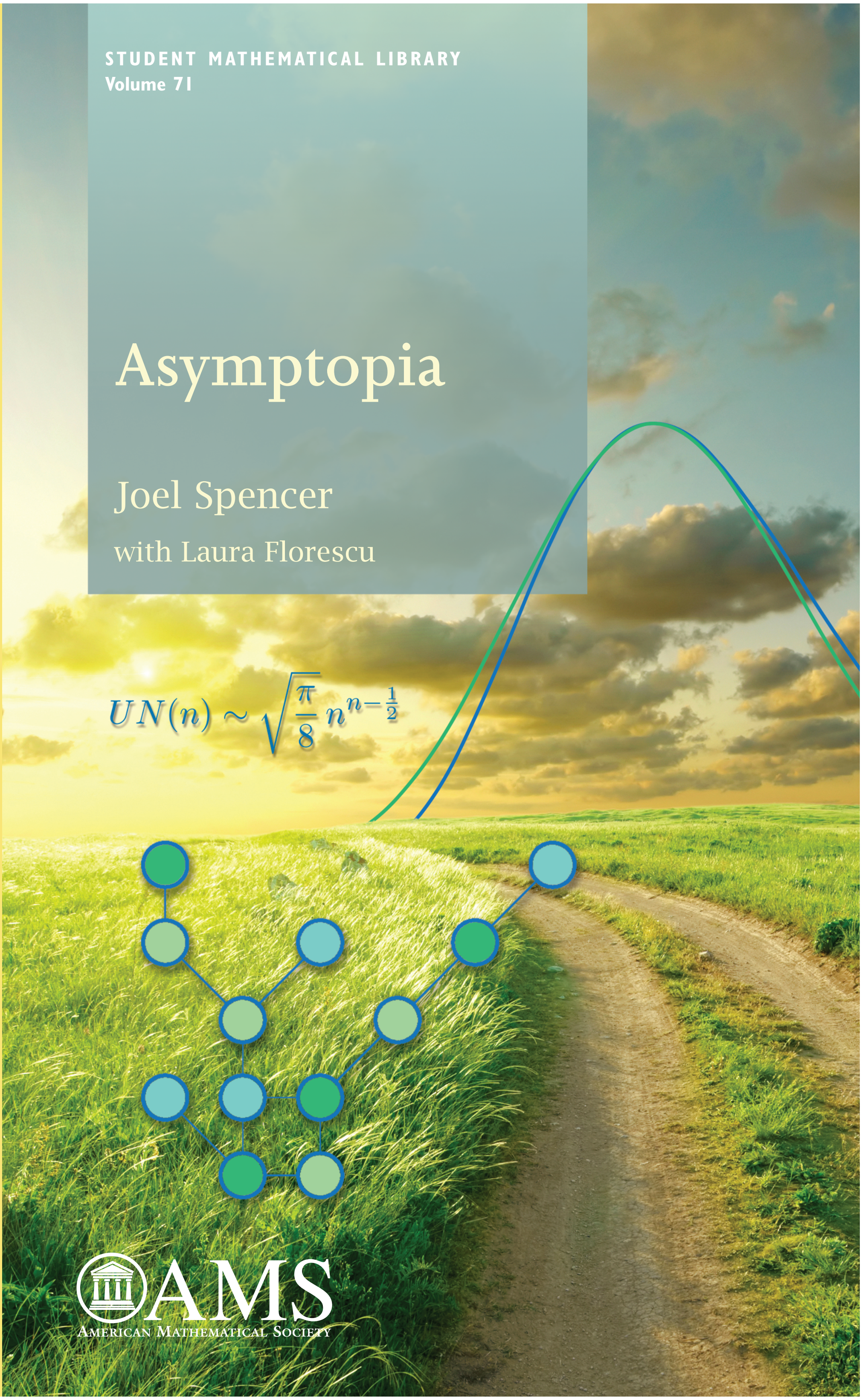
Asymptopia

Joel Spencer
with Laura Florescu



$$UN(n) \sim \sqrt{\frac{\pi}{8}}\, n^{n-\frac{1}{2}}$$

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Contents

Preface

I was 21 years when I wrote this song
I'm 22 now, but I won't be for long
Time hurries on
And the leaves that are green turn to brown
– Paul Simon, *Leaves that Are Green*

1968 was a tumultuous year. America was convulsed by the Vietnam War, nowhere more than on college campuses. The assassinations of Martin Luther King and of Robert Kennedy tore at the nation's heart. The Democratic convention in Chicago was marked by violent riots. America, for many, had become Amerika, the villain. "Do your own thing" was the admonition that resonated so powerfully. Resist authority. Nonconformity was the supreme virtue. For this fledgling mathematician it was a critical juncture. I had left graduate school without a degree. Would my talents find a focus in this chaotic world? My mind swirled with mathematical ideas, but I seemed unable to turn these ideas into a cohesive whole.

Then I met Paul Erdős. Everyone called him *Uncle Paul*.

While others spoke constantly of it, nonconformity was always Uncle Paul's modus operandi. He had no job; he worked constantly. He had no home; the world was his home. Possessions were a nuisance; money a bore. Paul lived on a web of trust, traveling ceaselessly from center to center spreading his mathematical pollen. “Prove and Conjecture!” was his constant refrain.

Were we, in those halcyon days, *students* of Uncle Paul. I think the word inadequate and inaccurate. Better to say that we were *disciples* of Paul Erdős. We (and the list is long indeed) had energy and talent. Paul, through his actions and his theorems and his conjectures and every fibre of his being, showed us the Temple of Mathematics. The Pages of The Book were there, we had only to open them. Does there exist for all sufficiently large n a triangle free graph on n vertices which does not contain an independent set of size $\sqrt{n \ln n}$? We had no doubts—the answer was either yes or no. The answer was in The Book. Pure thought—our thought—would allow its reading.

I would sit with Uncle Paul and discuss an open problem. Paul would have a blank pad of paper on his lap. “Suppose,” he would say in his strong Hungarian accent,[1] “we set

$$p = \sqrt{\frac{\ln n}{n}}.\text{''}$$

He would write the formula for p on the blank page and nothing else. Then his mind sped on, showing how this particular value of p led to the solution. How, I wondered, did Uncle Paul know which value of p to take?

The final form of mathematics, the form that students see in textbooks, was described by Bertrand Russell:

> Mathematics, rightly viewed, possesses not only truth, but supreme beauty—a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show. The true spirit of delight, the exaltation, the sense of being more than Man, which is the touchstone of the highest excellence, is to be found in mathematics as surely as in poetry.

[1]The documentary film “N is a Number” by George Csicsery [**Csi93**], available on the web, shows Uncle Paul in action.

Doing mathematics is anything but austere. As an undergraduate teacher of mine, Gian-Carlo Rota, put it:

> A mathematician's work is mostly a tangle of guesswork, analogy, wishful thinking and frustration, and proof, far from being the core of discovery, is more often than not a way of making sure that our minds are not playing tricks.

That said, the "guesswork" can be finely honed. Uncle Paul's selection of the right p did not come at random. Brilliance, of course, is more than helpful. But we mortals can also sometimes succeed.

Paul Erdős lived[2] in Asymptopia. Primes less than n, graphs with v vertices, random walks of t steps—Erdős was fascinated by the limiting behavior as the variables approached, but never reached, infinity. Asymptotics is very much an art. In his masterwork, *The Periodic Table*, Primo Levi speaks of the personalities of the various elements. A chemist will feel when atoms want or do not want to bind. In asymptotics the various functions $n \ln n$, n^2, $\frac{\ln n}{n}$, $\sqrt{\ln n}$, $\frac{1}{n \ln n}$ all have distinct personalities. Erdős knew these functions as personal friends. This author had the great privilege and joy of learning directly from Paul Erdős. It is my hope that these insights may be passed on, that the reader may similarly feel which function has the right temperament for a given task.

My decision to write this work evolved over many years, and it was my students who opened my eyes. I would teach courses in discrete mathematics, probability, Ramsey theory, graph theory, the probabilistic method, number theory, and other areas. I would carefully give, for example, Erdős's classic result (Theorem 7.1) on Ramsey numbers: If

$$\binom{n}{k} 2^{1-\binom{k}{2}} < 1,$$

then $R(k,k) > n$. I spent much less time on the asymptotic implication (7.6), that $R(k,k) \geq (1+o(1))\frac{k}{e\sqrt{2}}\sqrt{2}^k$. My students showed me

[2]Erdős's breadth was extraordinary. This refers to only one aspect of his oeuvre.

that Asymptopia deserved its own emphasis. A facility with asymptotic calculations could be taught, could be learned, and was a highly pragmatic element of their mathematical education.

Laura Florescu began her graduate studies at the Courant Institute in the fall of 2012. Almost immediately we began our study of mathematical results, old and new, and began work on open questions. This pursuit happily continues to this day. Early on, with this project still in its nascent phase, Laura graciously offered her assistance. We have discussed ideas for the various chapters and sections together. Some of the ideas, such as giving a proof of the Law of the Iterated Logarithm, originated entirely from her and all of the ideas were jointly discussed. She has written early drafts of many sections. (However, all errors in the final copy, what you are reading now, are my responsibility.) I hope that this project has been as much a learning experience for Laura as it has been for me. With her talents and energy, Laura has a bright future ahead of her. Thank you, Laura.

My editor, Ina Mette, deserves special recognition. We have known each other for many years and I had always wanted to write a book under her editorship. Conversations about this current work took place over a long period of time. Through lunches at Lure in New York, at Central Kávéház in Budapest, through numerous emails and phone calls, the outlines of this current work came into focus. Ina has always been insightful in her suggestions and fully supportive of my ofttimes ill-defined ruminations. Thank you, Ina.

1968 was special for me personally as well as professionally. It was the year I married my wife Mary Ann, whom I wish to thank once again for her assistance, encouragement, and understanding. Without her, this enterprise would have had little meaning.

Joel Spencer
New York
Fall, 2013

A Reader's Guide

> I have never let my schooling interfere with my education.
> – Mark Twain

The Student Mathematical Library is aimed at undergraduate students, but our focus is somewhat broader. We may also envision a graduate student looking for a pragmatic view of asymptotic calculations. We may also envision a high school student learning new mathematical relationships. The common denominator is a love of mathematics. To the largest degree possible, we have strived to make this work self-contained. The reader should be aware, however, of certain assumptions.

Calculus. We do assume a knowledge of first year calculus, as taught in U.S. colleges and, often, high schools. Differentiation and integration is done without proof. The definite integral

$$\int_{-\infty}^{\infty} e^{-x^2/2} dx = \sqrt{2\pi}$$

is assumed; this appears with surprising frequency. We do *not* use differential equations, nor partial differential equations, nor algebra, nor topology. We do *not* use material from the course frequently called (in the U.S.) analysis. In particular, all interchanges of $\lim_n \int f_n(x)dx$ and $\int \lim_n f_n(x)dx$ are done from scratch.

Probability. A number of basic distributions are considered in this work. These include the binomial, the Poisson, and the Gaussian distributions. We have defined these when they appear. Still, some prior knowledge of the notions of random variable, expectation, variance, and independence would be helpful to the student.

Graph Theory. We do not assume a knowledge of graph theory. Still, some prior knowledge of what a graph is, as a set of vertices and edges, would be helpful. We examine the random graph $G(n,p)$. Again, a prior familiarity would be helpful but not necessary.

Number Theory. We expect the reader to know what prime numbers are and to know the unique factorization of positive integers into primes. Otherwise, our presentation of number theory is self-contained.

Algorithms. The mathematical analysis of algorithms is a fascinating subject. Here we give some glimpses into the analyses, but our study of algorithms is self-contained. Certainly, no actual programming is needed.

Our final chapter, Really Big Numbers!, is different in flavor. This author has always been fascinated with big numbers. This chapter is basically a paper written for the *American Mathematical Monthly* three decades ago. Some of the material uses ordinal numbers, like ω^ω, which may be new to the reader.

We sometimes skirt a topic, pulling from it only some asymptotic aspects. This is particularly noticeable in Ramsey theory, one of our favorite topics.

Certain sections are technically quite complicated and are labelled as such. They may be skipped without losing the thread of the argument.

Facility with logarithms is assumed throughout. We use $\ln x$ for natural logarithm and $\lg x$ for the logarithm to the base two.

Asymptopia is a beautiful world. Enjoy!

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Index

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Photo by Mary Ann Spencer

Asymptotics in one form or another are part of the landscape for every mathematician. The objective of this book is to present the ideas of how to approach asymptotic problems that arise in discrete mathematics, analysis of algorithms, and number theory. A broad range of topics is covered, including distribution of prime integers, Erdős Magic, random graphs, Ramsey numbers, and asymptotic geometry.

The author is a disciple of Paul Erdős, who taught him about Asymptopia. Primes less than n, graphs with v vertices, random walks of t steps—Erdős was fascinated by the limiting behavior as the variables approached, but never reached, infinity. Asymptotics is very much an art. The various functions $n\ln n$, n^2, $\frac{\ln n}{n}$, $\sqrt{\ln n}$, $\frac{1}{n\ln n}$ all have distinct personalities. Erdős knew these functions as personal friends. It is the author's hope that these insights may be passed on, that the reader may similarly feel which function has the right temperament for a given task. This book is aimed at strong undergraduates, though it is also suitable for particularly good high school students or for graduates wanting to learn some basic techniques.

Asymptopia is a beautiful world. Enjoy!

"This beautiful book is about how to estimate large quantities — and why. Building on nothing more than first-year calculus, it goes all the way into deep asymptotical methods and shows how these can be used to solve problems in number theory, combinatorics, probability, and geometry. The author is a master of exposition: starting from such a simple fact as the infinity of primes, he leads the reader through small steps, each carefully motivated, to many theorems that were cutting-edge when discovered, and teaches the general methods to be learned from these results."

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–Günter M. Ziegler, Freie Universität Berlin, coauthor of "Proofs from THE BOOK"

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