Ramsey Theory on the Integers

Second Edition
To
Eleanor
Emma and Sarah

– Bruce

To
Elisa, Quinn, Ava
Pearl, Doug, Jason

– aaron
# Contents

List of Tables xi

Preface to the Second Edition xiii

Acknowledgements xv

Preface to the First Edition xvii

Chapter 1. Preliminaries 1

§1.1. The Pigeonhole Principle 3

§1.2. Ramsey’s Theorem 6

§1.3. Some Notation 9

§1.4. Three Classical Theorems 11

§1.5. A Little More Notation 14

§1.6. Exercises 17

§1.7. Research Problems 20

§1.8. References 21

Chapter 2. Van der Waerden’s Theorem 23

§2.1. The Compactness Principle 27

§2.2. Alternate Forms of van der Waerden’s Theorem 29

§2.3. Computing van der Waerden Numbers 31
<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Sections</th>
<th>Pages</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>Bounds on van der Waerden Numbers</td>
<td>§2.4-§2.10</td>
<td>38-62</td>
</tr>
<tr>
<td>3</td>
<td>Supersets of $AP$</td>
<td>§3.1-§3.9</td>
<td>67-111</td>
</tr>
<tr>
<td>4</td>
<td>Subsets of $AP$</td>
<td>§4.1-§4.5</td>
<td>113-145</td>
</tr>
<tr>
<td>5</td>
<td>Other Generalizations of $w(k;r)$</td>
<td>§5.1-§5.5</td>
<td>147-177</td>
</tr>
</tbody>
</table>
List of Tables

Table 2.1: Mixed van der Waerden values 37
Table 2.2: Values and lower bounds for $w(k; r)$ 43
Table 3.1: Values and lower bounds for $Q_{k-i}(k)$ 77
Table 3.2: Values and lower bounds for $GQ_f(x)(k)$ 80
Table 3.3: Values and lower bounds for $SP_m(k)$ 92
Table 3.4: Values of $R(S_n, k; r)$ and $R(AP \cup P_n, k)$ 102
Table 4.1: Values of $w'(f(x), 3; 2)$ 132
Table 5.1: Values and lower bounds for $T(a, b; 2)$ 162
Table 5.2: Degree of regularity of $(a, b)$-triples 163
Table 5.3: Values of $R(c, AUG_b, 3)$ 174
Table 6.1: Values and lower bounds for $R(AP_{(m)}, k; 2)$ 186
Table 6.2: Degree of regularity of families of type $AP_{(m)}$ 196
Table 10.1: Values of $\Delta(D, k)$ 318
Table 10.2: Number of squarefree and cubefree colorings 328
Preface to the Second Edition

As noted in the preface of the first edition, one of the book’s primary purposes is to serve as a source of unsolved problems in the flourishing and relatively “wide-open” area of Ramsey theory on the integers. We especially hoped the book would provide beginning researchers, including graduate students and undergraduate students, with a range of accessible topics in which to delve. It seems that we have already been at least somewhat successful because, since the publication of the first edition, a good number of the research problems have been solved or partially solved, and several of the theorems have been improved upon. As a consequence, the new edition includes many substantial revisions and additions.

Various new sections have been added and others have been significantly updated. Among the newly introduced topics are: rainbow Ramsey theory, an “inequality” version of Schur’s theorem, monochromatic solutions of recurrence relations, Ramsey results involving both sums and products, monochromatic sets avoiding certain differences, Ramsey properties for polynomial progressions, generalizations of the Erdős-Ginzberg-Ziv theorem, and the number of arithmetic progressions under arbitrary colorings. We also offer many new results and proofs among the topics that are not new to this edition, most of
which were not known when the first edition was published. Furthermore, the book’s tables, exercises, lists of open research problems, and bibliography have all been significantly updated. Finally, we have repaired numerous misprints.
We wish to thank a number of individuals for finding errors in the first edition and for helping to improve the exposition of the text, including: Charles Baker, Arie Bialostocki, Srashti Dwivedi, Sohail Farhangi, Paul Frigge, Andraes Jönsson, Jonathan Leprince, Ryan Matzke, James Henry Sanders, and Hunter Snevily. Special thanks go to Sarah Landman for her careful proofreading. We are grateful to all the readers of the first edition who contacted us with insightful comments and suggestions. We also express our gratitude to our fellow Ramsey theorists for enlightening us on some recent advances in the field and for pointing us to a number of intriguing problems. We appreciate the kind and professional support given to us throughout the process by Edward Dunne, Christine Thivierge, and the rest of the AMS production staff.

Bruce Landman adds a special debt of gratitude to his wife, Eleanor, for her support, patience, and sense of humor throughout this endeavor, and to his two wonderful daughters, Emma and Sarah, who add so much meaning to everything he does.

Aaron would like to thank: his parents, Pearl and Doug, for providing him with a happy and supportive upbringing; his brother Jason (sorry for waking you with the snare drum, hiding in the closet to scare you, etc.); his advisor, Doron Zeilberger, without whom he
probably would have dropped out of graduate school; and the love-of-his-life, Elisa, for just being fantastic and for giving him two absolutely wonderful children, Quinn and Ava (and one other child . . . kidding, readers, we only have two).
Preface to the First Edition

Ramsey Theory on the Integers covers a variety of topics from the field of Ramsey theory, limiting its focus to the set of integers – an area that has seen a remarkable burst of research activity during the past twenty years.

The book has two primary purposes: (1) to provide students with a gentle, but meaningful, introduction to mathematical research – to give them an appreciation for the essence of mathematical research and its inescapable allure and also to get them started on their own research work; (2) to be a resource for all mathematicians who are interested in combinatorial or number theoretical problems, particularly “Erdős-type” problems.

Many results in Ramsey theory sound rather complicated and can be hard to follow; they tend to have a lot of quantifiers and may well involve objects whose elements are sets whose elements are sets (that is not a misprint). However, when the objects under consideration are sets of integers, the situation is much simpler. The student need not be intimidated by the words “Ramsey theory,” thinking that the subject matter is too deep or complex – it is not! The material in this book is, in fact, quite accessible. This accessibility, together with the fact that scores of questions in the subject are still to be answered, makes Ramsey theory on the integers an ideal subject for
a student’s first research experience. To help students find suitable projects for their own research, every chapter includes a section of “Research Problems,” where we present a variety of unsolved problems, along with a list of suggested readings for each problem.

*Ramsey Theory on the Integers* has several unique features. No other book currently available on Ramsey theory offers a cohesive study of Ramsey theory on the integers. Among several excellent books on Ramsey theory, probably the most well-known, and what may be considered the Ramsey theory book, is by Graham, Rothschild, and Spencer (*Ramsey Theory, 2nd Edition* [175]). Other important books are by Graham (*Rudiments of Ramsey Theory* [168]), McCutcheon (*Elemental Methods in Ergodic Ramsey Theory* [279]), Nešetřil and Rödl (*Mathematics of Ramsey Theory* [295]), Prömel and Voigt (*Aspects of Ramsey Theory* [305]), Furstenberg (*Dynamical Methods in Ramsey Theory* [156]), and Winn (*Asymptotic Bounds for Classical Ramsey Numbers* [401]). These books, however, generally cover a broad range of subject matter of which Ramsey theory on the integers is a relatively small part. Furthermore, the vast majority of the material in the present book is not found in any other book. In addition, to the best of our knowledge, ours is the only Ramsey theory book that is accessible to the typical undergraduate mathematics major. It is structured as a textbook, with numerous (over 150) exercises, and the background needed to read the book is rather minimal: a course in elementary linear algebra and a 1-semester junior-level course in abstract algebra would be sufficient; an undergraduate course in elementary number theory or combinatorics would be helpful, but not necessary. Finally, *Ramsey Theory on the Integers* offers something new in terms of its potential appeal to the research community in general. Books offering a survey of solved and unsolved problems in combinatorics or number theory have been quite popular among researchers; they have also proven beneficial by serving as catalysts for new research in these fields. Examples include *Old and New Problems and Results in Combinatorial Number Theory* [128] by Erdős and Graham, *Unsolved Problems in Number Theory* [190] by Guy, and *The New Book of Prime Number Records* [320] by Ribenboim. With our text we hope to offer mathematicians an additional resource for intriguing unsolved problems. Although not
nearly exhaustive, the present book contains perhaps the most substantial account of solved and unsolved problems in Ramsey theory on the integers.

This text may be used in a variety of ways:
- as an undergraduate or graduate textbook for a second course in combinatorics or number theory;
- in an undergraduate or graduate seminar, a capstone course for undergraduates, or an independent study course;
- by students working under an REU program, or who are engaged in some other type of research experience;
- by graduate students looking for potential thesis topics;
- by the established researcher seeking a worthwhile resource in its material, its list of open research problems, and its somewhat enormous (often a fitting word when discussing Ramsey theory) bibliography.

Chapter 1 provides preliminary material (for example, the pigeonhole principle) and a brief introduction to the subject, including statements of three classical theorems of Ramsey theory: van der Waerden’s theorem, Schur’s theorem, and Rado’s theorem. Chapter 2 covers van der Waerden’s theorem; Chapters 3–7 deal with various topics related to van der Waerden’s theorem; Chapter 8 is devoted to Schur’s theorem and a generalization; Chapter 9 explores Rado’s theorem; and Chapter 10 presents several other topics involving Ramsey theory on the integers.

The text provides significant latitude for those designing a syllabus for a course. The only material in the book on which other chapters depend is that through Section 2.2. Thus, other chapters or sections may be included or omitted as desired, since they are essentially independent of one another (except for an occasional reference to a previous definition or theorem). We do, however, recommend that all sections included in a course be studied in the same order in which they appear in the book.

Each chapter concludes with a section of exercises, a section of research problems, and a reference section. Since the questions contained in the Research Problem sections are still open, we cannot say
with certainty how difficult a particular one will be to solve; some may actually be quite simple and inconsequential. The problems that we deem most difficult, however, are labeled with the symbol ∗. The reference section of each chapter is organized by section numbers (including the exercise section). The specifics of each reference are provided in the bibliography at the end of the book.

The material covered in this book represents only a portion of the subject area indicated by the book’s title. Many additional topics have been investigated, and we have attempted to include at least references for these in the reference sections. Yet, for every problem that has been thought of in Ramsey theory, there are many more which that problem will generate and, given the great variety of combinatorial structures and patterns that lie in the set of integers, countless new problems wait to be explored.

We would like to thank Dr. Edward Dunne and the members of the AMS production staff for their assistance in producing this book. We also thank Tom Brown, Scott Gordon, Jane Hill, Dan Saracino, Dan Schaal, Ralph Sizer, and the AMS reviewers for their helpful comments and advice, which greatly improved the manuscript. We also express our gratitude to Ron Graham and Doron Zeilberger for their support of this project. We owe a big debt to the pioneers and masters of the field, especially Ron Graham, Jarik Nešetril, Joel Spencer, Neil Hindman, Tom Brown, Timothy Gowers, Hillel Furstenberg, Vitaly Bergelson, Vojtěch Rödl, Endre Szemerédi, László Lovász (we had to stop somewhere), and of course Bartel van der Waerden, Issai Schur, Richard Rado, and Frank Ramsey. To all of the others who have contributed to the field of Ramsey theory on the integers, we extend our sincere appreciation. Finally, we want to acknowledge that this book would not exist without the essential contributions of the late Paul Erdős. But beyond the content of his achievements, he has personally inspired the authors as mathematicians. Our professional lives would have had far less meaning and fulfillment without his work and his presence in our field. For that pervasive, though perhaps indirect, contribution to this text, we are in his debt.
### Notation

<table>
<thead>
<tr>
<th>Notation</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>⌈⋅⌉</td>
<td>Ceiling function</td>
<td>111</td>
</tr>
<tr>
<td>⌊⋅⌋</td>
<td>Floor function</td>
<td>111</td>
</tr>
<tr>
<td>⊕</td>
<td>Modular addition</td>
<td>117</td>
</tr>
<tr>
<td>⊖</td>
<td>$A \ominus B = {a - b : a \in A, b \in B}$</td>
<td>313</td>
</tr>
<tr>
<td>[a, b]</td>
<td>${a, a + 1, \ldots, b}$</td>
<td>9</td>
</tr>
<tr>
<td>A - B</td>
<td>${x \in A : x \not\in B}$</td>
<td>9</td>
</tr>
<tr>
<td>A ≺ B</td>
<td>Nonoverlapping sets: $\max(a \in A) &lt; \min(b \in B)$</td>
<td>342</td>
</tr>
<tr>
<td>A</td>
<td>Family of arithmetic progressions with gaps in $D$</td>
<td>114</td>
</tr>
<tr>
<td>$ADW(k, \ell)$</td>
<td>Ascending/descending waves numbers</td>
<td>112</td>
</tr>
<tr>
<td>$AP$</td>
<td>Family of arithmetic progressions</td>
<td>19</td>
</tr>
<tr>
<td>$AP_{a(m)}$</td>
<td>Family of arithmetic progressions with gaps congruent to $a \pmod{m}$</td>
<td>183</td>
</tr>
<tr>
<td>$AP^*_{a(m)}$</td>
<td>$AP_{a(m)} \cup A_{{m}}$</td>
<td>187</td>
</tr>
<tr>
<td>$AP_{(m)}$</td>
<td>Set of arithmetic progressions (mod $m$)</td>
<td>184</td>
</tr>
<tr>
<td>AUG$_b$</td>
<td>Family of augmented progressions with tail $b$</td>
<td>168</td>
</tr>
<tr>
<td>AW($k; r$)</td>
<td>Ascending waves number</td>
<td>110</td>
</tr>
<tr>
<td>B</td>
<td>“Brown” number</td>
<td>320</td>
</tr>
<tr>
<td>$b(m, k; r)$</td>
<td>Ramsey-type number for nonoverlapping $m$-sets</td>
<td>343</td>
</tr>
<tr>
<td>C</td>
<td>$c(R; r)$</td>
<td>286</td>
</tr>
<tr>
<td>cul$_j$</td>
<td>Culprit of color $j$</td>
<td>34</td>
</tr>
<tr>
<td>C($k$)</td>
<td>$k$-nonrepetitive number</td>
<td>329</td>
</tr>
<tr>
<td>D</td>
<td>$\Delta_{(a,b,c)}$</td>
<td>224</td>
</tr>
<tr>
<td>$\Delta(D, k; r)$</td>
<td>Ramsey-type number for diffsequences</td>
<td>303</td>
</tr>
<tr>
<td>$D_S(k; r)$</td>
<td>Difference $S$-free number</td>
<td>322</td>
</tr>
<tr>
<td>diam</td>
<td>Diameter of a set</td>
<td>342</td>
</tr>
<tr>
<td>doa</td>
<td>Degree of accessibility</td>
<td>306</td>
</tr>
<tr>
<td>dor</td>
<td>Degree of regularity</td>
<td>150</td>
</tr>
<tr>
<td>dor$_k$</td>
<td>Degree of regularity for $T_{k-1}(a)$</td>
<td>160</td>
</tr>
<tr>
<td>DW($k$)</td>
<td>2-color descending wave number</td>
<td>81</td>
</tr>
<tr>
<td>Notation</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>----------</td>
<td>-------------</td>
<td>------</td>
</tr>
<tr>
<td>F</td>
<td>$f_n$</td>
<td>139</td>
</tr>
<tr>
<td></td>
<td>$F(k; r)$</td>
<td>298</td>
</tr>
<tr>
<td></td>
<td>$FS(T)$</td>
<td>297</td>
</tr>
<tr>
<td>G</td>
<td>$\Gamma_m(k)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$GQ_\delta(k)$</td>
<td>85</td>
</tr>
<tr>
<td></td>
<td>$g(r, k)$</td>
<td>78</td>
</tr>
<tr>
<td></td>
<td>$gs$</td>
<td>338</td>
</tr>
<tr>
<td>K</td>
<td>$(k, n, d)$-progression</td>
<td>69</td>
</tr>
<tr>
<td>H</td>
<td>$H(s_1, \ldots, s_k)$</td>
<td>69</td>
</tr>
<tr>
<td>I</td>
<td>$I(k_1, k_2, \ldots, k_r)$</td>
<td>163</td>
</tr>
<tr>
<td></td>
<td>$I(k, k)$</td>
<td>243</td>
</tr>
<tr>
<td>J</td>
<td>$J_k(n; r)$</td>
<td>245</td>
</tr>
<tr>
<td>L</td>
<td>$\lambda(c, k; r)$</td>
<td>52</td>
</tr>
<tr>
<td></td>
<td>$\mathcal{L}(k)$</td>
<td>234</td>
</tr>
<tr>
<td>M</td>
<td>$\mu(k)$</td>
<td>122</td>
</tr>
<tr>
<td></td>
<td>$\mathcal{M}(k)$</td>
<td>243</td>
</tr>
<tr>
<td></td>
<td>$\mathcal{M}_j(n)$</td>
<td>227</td>
</tr>
<tr>
<td>N</td>
<td>$\nu(k)$</td>
<td>47</td>
</tr>
<tr>
<td>O</td>
<td>$\Omega_m(k)$</td>
<td></td>
</tr>
<tr>
<td>P</td>
<td>$P_n$</td>
<td>204</td>
</tr>
<tr>
<td></td>
<td>$P_{n, k}$</td>
<td>92</td>
</tr>
<tr>
<td></td>
<td>$pp(k, n; r)$</td>
<td>175</td>
</tr>
<tr>
<td>Q</td>
<td>$Q_n(k)$</td>
<td>69</td>
</tr>
<tr>
<td>Notation</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>----------</td>
<td>-------------</td>
<td>------</td>
</tr>
<tr>
<td>$\mathbb{R}$</td>
<td>The set of real numbers</td>
<td>9</td>
</tr>
<tr>
<td>$R(AP_{a(m)}^*, k, l; r)$</td>
<td>Generalization of $R(AP_{a(m)}^*, k; r)$</td>
<td>155</td>
</tr>
<tr>
<td>$r(\mathcal{E}; s)$</td>
<td>Rado number for equation $\mathcal{E}$</td>
<td>259</td>
</tr>
<tr>
<td>$R(\mathcal{F}, k; r)$</td>
<td>Ramsey-type number for family $\mathcal{F}$</td>
<td>15</td>
</tr>
<tr>
<td>$R(k_1, \ldots, k_r)$</td>
<td>$r$-color (classical) Ramsey number</td>
<td>9</td>
</tr>
<tr>
<td>$R_r(k_1, \ldots, k_r)$</td>
<td>$r$-color (classical) Ramsey number</td>
<td>9</td>
</tr>
<tr>
<td>$RR(S; r)$</td>
<td>Reverse $r$-regular number</td>
<td>207</td>
</tr>
<tr>
<td>$S(k_1, \ldots, k_r)$</td>
<td>Generalized Schur number</td>
<td>233</td>
</tr>
<tr>
<td>$S_r(k)$</td>
<td>Generalized Schur number with all $k_i = k$</td>
<td>235</td>
</tr>
<tr>
<td>$\hat{S}(k_1, \ldots, k_r)$</td>
<td>Strict generalized Schur number</td>
<td>242</td>
</tr>
<tr>
<td>$S_n$</td>
<td>Family of sequences generated by iteration of a polynomial of degree $\leq n$</td>
<td>93</td>
</tr>
<tr>
<td>$S_{n,k}$</td>
<td>Family of $k$-term members of $S_n$</td>
<td>93</td>
</tr>
<tr>
<td>$SF_m(k)$</td>
<td>Ramsey-type number for semi-progressions</td>
<td>83</td>
</tr>
<tr>
<td>$s(r)$</td>
<td>Schur number</td>
<td>223</td>
</tr>
<tr>
<td>$\hat{s}(r)$</td>
<td>Strict Schur number</td>
<td>240</td>
</tr>
<tr>
<td>$T_{a,b}$</td>
<td>Family of $(a,b)$-triples</td>
<td>148</td>
</tr>
<tr>
<td>$T(a, b; r)$</td>
<td>Ramsey-type number for $(a, b)$-triples</td>
<td>148</td>
</tr>
<tr>
<td>$T(a_1, \ldots, a_{k-1})$</td>
<td>Ramsey-type function for generalization of $(a, b)$-triples</td>
<td>160</td>
</tr>
<tr>
<td>$T_{k-1}(a)$</td>
<td>Ramsey-type function $T(a_1, \ldots, a_{k-1})$ with $a = a_1 = a_2 = \cdots = a_{k-1}$</td>
<td>160</td>
</tr>
<tr>
<td>$\Theta(n)$</td>
<td>Set of permutations of $[1, n]$ with no 3-term arithmetic progression</td>
<td>215</td>
</tr>
<tr>
<td>$\theta(n)$</td>
<td>$</td>
<td>\Theta(n)</td>
</tr>
<tr>
<td>$V_m$</td>
<td>${x \in \mathbb{Z}^+ : m \nmid x}$</td>
<td>312</td>
</tr>
<tr>
<td>$V_{m,n}$</td>
<td>${x \in \mathbb{Z}^+ : m \nmid x$ and $n \nmid x}$</td>
<td>345</td>
</tr>
<tr>
<td>$W(k)$</td>
<td>$w(k; 2)$</td>
<td>15</td>
</tr>
<tr>
<td>$w(k; r)$</td>
<td>van der Waerden number</td>
<td>12</td>
</tr>
<tr>
<td>$w(k_1, \ldots, k_r; r)$</td>
<td>Mixed van der Waerden number</td>
<td>36</td>
</tr>
<tr>
<td>$w_2(k; r)$</td>
<td>Mixed van der Waerden number $w(k, 2, 2, \ldots, 2; r)$</td>
<td>60</td>
</tr>
<tr>
<td>$\hat{w}(k; r)$</td>
<td>van der Waerden number with $d$ same color</td>
<td>57</td>
</tr>
<tr>
<td>$w'(c, k; r)$</td>
<td>Ramsey-type number for arithmetic progressions with gaps at least $c$</td>
<td>121</td>
</tr>
<tr>
<td>$w'(f(x), k; r)$</td>
<td>Ramsey-type number for $f$-a.p.'s</td>
<td>126</td>
</tr>
<tr>
<td>$w^+(k, j)$</td>
<td>Ramsey-type number for arithmetic progressions with color discrepancy at least $j$</td>
<td>209</td>
</tr>
<tr>
<td>$\overline{w}(k)$</td>
<td>Ramsey-type number for 3-term arithmetic progression of one color or $k$ consecutive integers of the other color</td>
<td>219</td>
</tr>
<tr>
<td>$\mathbb{Z}$</td>
<td>Set of integers</td>
<td>9</td>
</tr>
<tr>
<td>$\mathbb{Z}^+$</td>
<td>Set of positive integers</td>
<td>9</td>
</tr>
</tbody>
</table>
Bibliography


[12] T. Ahmed, Two new van der Waerden numbers: $w(2; 3, 17)$ and $w(2; 3, 18)$, *Integers* 10 (2010), #A32.


[88] T. C. Brown and A. R. Freedman, Small sets which meet all the \( k(n) \)-term arithmetic progressions in the interval \([1, n]\), *J. Combin. Theory Ser. A* 51 (1989), 244-249.


Bibliography


[201] D. Hilbert, Uber die irreducibilität ganzer rationaler functionen mit
ganzahligen coefficienten, J. Reine Angew. Math. 110 (1892), 104-129.
[202] N. Hindman, Finite sums from sequences within cells of a partition
[203] N. Hindman, Monochromatic sums equal to products in N, Integers
[206] N. Hindman, Partitions and sums and products – two counterexam-
[207] N. Hindman, Partition regularity of matrices, in Combinatorical
[208] N. Hindman, Ultrafilters and combinatorial number theory, Lecture
[210] R. Hirschfeld, On a generalization of the van der Waerden theorem,
[212] R. Irving, An extension of Schur’s theorem on sum-free partitions,
Acta Arith. 25 (1973-74), 55-64.
64-66.
[216] S. Jones and D. Schaal, Two-color Rado numbers for \( x + y + c = kz \),
[217] V. Jungić, Elementary, Topological, and Experimental Approaches
to the Family of Large Sets, Ph.D. thesis, Simon Fraser University,
Burnaby, BC, Canada, 1999.
[218] V. Jungić, On a conjecture of Brown concerning accessible sets, J.
[219] V. Jungić, On a question raised by Brown, Graham and Landman,


[235] W. Kosek and D. Schaal, Rado numbers for the equation $\sum_{i=1}^{m-1} x_i + c = x_m$ for negative values of $c$, *Adv. Appl. Math.* **27** (2001), 805-815.


[261] L. Lane-Harvard and D. Schaal, Disjunctive Rado numbers fo $ax_1 + x_2 = x_3$ and $bx_1 + x_2 = x_3$, *Integers* **13** (2013), #A62.


A. Robertson and D. Zeilberger, A 2-coloring of $[1,n]$ can have $\frac{n^2}{22} + O(n)$ monochromatic Schur triples, but not less!, *Electron. J. Combin.* **5** (1998), R19.


R. Salem and D. Spencer, On sets which do not contain a given number of terms in Arithmetical progression, *Nieuw Arch. Wisk.* **23** (1950), 133-143.


D. Saracino, The 2-color Rado number of $x_1 + x_2 + \cdots + x_{m-1} = ax_m$, *Ars Combin.* **113** (2014), 81-95.

D. Saracino, The 2-color Rado number of $x_1 + x_2 + \cdots + x_{m-1} = ax_m$ II, to appear in *Ars Combin.*


[353] D. Schaal and D. Vestal, Rado numbers for $x_1 + x_2 + \cdots + x_{m-1} = 2x_m$, *Congr. Numer.* 191 (2008), 105-116.

[354] D. Schaal and M. Zinter, Continuous Rado numbers for the equation $a_1x_1 + a_2x_2 + \cdots + a_{m-1}x_{m-1} + c = x_m$, *Congr. Numer.* 207 (2011), 97-104.


378

Bibliography


[385] X. Sun, New lower bound on the number of ternary square-free words, *J. Integer Seq.* 6 (2003), #03.3.2.


Index

(a, b)-triple, 147
Abbott, H, 351
Accessible set, 303
Ackermann function, 45
Ackermann, W., 45
Alekseev, B., 350
Alon, N., 219
a (mod m)-progression, 183
Arithmetic progression, 12
Arithmetic progression (mod m), 183
Ascending wave, 104
Asymptotic notation, 10
Augmented progression, 168
Baudet’s conjecture, 65
Baumgartner, J., 349
Bergelson, V., 146, 249, 294
Berlekamp, E., 38
Bialostocki, A., xv, 249, 350
Big-O notation, 10
Boundedness conjecture, 208
Brakemeier, W., 350
Brown’s lemma, 319
Brown, T., 111, 146, 349
Burr, S., 294

(C, D)-polynomial, 177
Cantor’s diagonal argument, 27
Ceiling function, 11
Columns condition, 283
Compactness principle, 27
Complete graph, 7
Composition function, 44
Cubefree coloring, 326
Culprit, 33

Datskovsky, B., 248
Degree of accessibility, 306
Degree of regularity, 150, 162, 193
Derived coloring, 53
Descending wave, 81, 196
Deuber, W., 294, 295
Diameter (of a set), 332
Difference T-free, 321
Difference coloring, 19
Diffsequence, 303
Disjunctive Rado number, 293
Doublefree set, 302

Edge, 7
Edge-coloring, 7
End-focused, 53
Equinumerous coloring, 331
Erdős and Turán function, 17, 64
Erdős, P., 19–21, 38, 35, 47, 51, 61
64, 111, 146, 222, 228, 301
Erdős-Ginzburg-Ziv theorem, 336
341, 350
Ergodic theory, 63, 139, 126, 249
<table>
<thead>
<tr>
<th>Author</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Everts, F.</td>
<td>64</td>
</tr>
<tr>
<td>Exoo, G.</td>
<td>249</td>
</tr>
<tr>
<td>Fermat’s last theorem</td>
<td>222</td>
</tr>
<tr>
<td>Fibonacci numbers</td>
<td>139</td>
</tr>
<tr>
<td>Fields medal</td>
<td>44</td>
</tr>
<tr>
<td>Finite sums</td>
<td>297</td>
</tr>
<tr>
<td>Floor function</td>
<td>11</td>
</tr>
<tr>
<td>Folkman, J.</td>
<td>297</td>
</tr>
<tr>
<td>Folkman-Rado-Sanders number</td>
<td>300</td>
</tr>
<tr>
<td>Folkman-Rado-Sanders theorem</td>
<td>298</td>
</tr>
<tr>
<td>Fox, J.</td>
<td>294</td>
</tr>
<tr>
<td>Fredricksen, H.</td>
<td>249</td>
</tr>
<tr>
<td>Freedman, A.</td>
<td>111</td>
</tr>
<tr>
<td>Furstenburg, H.</td>
<td>64</td>
</tr>
<tr>
<td>Gap set</td>
<td>114</td>
</tr>
<tr>
<td>Gap size</td>
<td>320</td>
</tr>
<tr>
<td>Generalized quasi-progression</td>
<td>77</td>
</tr>
<tr>
<td>Generalized Schur inequality</td>
<td>243</td>
</tr>
<tr>
<td>Generalized Schur number</td>
<td>235</td>
</tr>
<tr>
<td>Generating polynomial</td>
<td>92</td>
</tr>
<tr>
<td>Geometric progression</td>
<td>136</td>
</tr>
<tr>
<td>Gleason, A.</td>
<td>21</td>
</tr>
<tr>
<td>Goodman, A.</td>
<td>248</td>
</tr>
<tr>
<td>Gowers, T.</td>
<td>44</td>
</tr>
<tr>
<td>GQ(\delta)-progression</td>
<td>77</td>
</tr>
<tr>
<td>Graham, R.</td>
<td>21</td>
</tr>
<tr>
<td>Graver, J.</td>
<td>21</td>
</tr>
<tr>
<td>Greatest integer function</td>
<td>11</td>
</tr>
<tr>
<td>Greedy algorithm</td>
<td>234</td>
</tr>
<tr>
<td>Greenwood, R.</td>
<td>21</td>
</tr>
<tr>
<td>Grinstead, C.</td>
<td>21</td>
</tr>
<tr>
<td>Guo, S.</td>
<td>294</td>
</tr>
<tr>
<td>Hajnal, A.</td>
<td>64</td>
</tr>
<tr>
<td>Hales-Jewett theorem</td>
<td>152</td>
</tr>
<tr>
<td>Harary, F.</td>
<td>21</td>
</tr>
<tr>
<td>Harborth, H.</td>
<td>294</td>
</tr>
<tr>
<td>Hilbert, D.</td>
<td>11</td>
</tr>
<tr>
<td>Hindman’s theorem</td>
<td>297</td>
</tr>
<tr>
<td>Hindman, N.</td>
<td>294</td>
</tr>
<tr>
<td>Hofstadter G-sequence</td>
<td>349</td>
</tr>
<tr>
<td>Homogeneous equation</td>
<td>251</td>
</tr>
<tr>
<td>Homothetic copy</td>
<td>163</td>
</tr>
<tr>
<td>Hyperedge</td>
<td>41</td>
</tr>
<tr>
<td>Hypergraph</td>
<td>41</td>
</tr>
<tr>
<td>Irving, R.</td>
<td>249</td>
</tr>
<tr>
<td>Isosceles triple</td>
<td>272</td>
</tr>
<tr>
<td>Iterated function</td>
<td>92</td>
</tr>
<tr>
<td>Jungić, V.</td>
<td>349</td>
</tr>
<tr>
<td>Kéry, G.</td>
<td>21</td>
</tr>
<tr>
<td>Kalbfleisch, J.</td>
<td>21</td>
</tr>
<tr>
<td>Kleitman, D.</td>
<td>294</td>
</tr>
<tr>
<td>Kouril, M.</td>
<td>35</td>
</tr>
<tr>
<td>Landman, B.</td>
<td>200</td>
</tr>
<tr>
<td>Large set</td>
<td>134</td>
</tr>
<tr>
<td>(r)-large set</td>
<td>134</td>
</tr>
<tr>
<td>Lattice point</td>
<td>18</td>
</tr>
<tr>
<td>Leader, I.</td>
<td>65</td>
</tr>
<tr>
<td>Least integer function</td>
<td>11</td>
</tr>
<tr>
<td>Lefmann, H.</td>
<td>112</td>
</tr>
<tr>
<td>Leibman, A.</td>
<td>146</td>
</tr>
<tr>
<td>Little-o notation</td>
<td>10</td>
</tr>
<tr>
<td>Liu, A.</td>
<td>351</td>
</tr>
<tr>
<td>Long, A.</td>
<td>210</td>
</tr>
<tr>
<td>Loo, S.</td>
<td>294</td>
</tr>
<tr>
<td>Lovász’s local lemma</td>
<td>64</td>
</tr>
<tr>
<td>Low-difference</td>
<td>69</td>
</tr>
<tr>
<td>m-set</td>
<td>322</td>
</tr>
<tr>
<td>Maasberg, S.</td>
<td>294</td>
</tr>
<tr>
<td>McCutcheon, R.</td>
<td>249</td>
</tr>
<tr>
<td>McKay, B.</td>
<td>21</td>
</tr>
<tr>
<td>Min, Z.</td>
<td>21</td>
</tr>
<tr>
<td>Mixed van der Waerden number</td>
<td>36</td>
</tr>
<tr>
<td>Monochromatic</td>
<td>4</td>
</tr>
<tr>
<td>Monotone arithmetic progression</td>
<td>213</td>
</tr>
<tr>
<td>Monotonic</td>
<td>5</td>
</tr>
<tr>
<td>Morse-Hedlund sequence</td>
<td>349</td>
</tr>
<tr>
<td>Nathanson, M.</td>
<td>219</td>
</tr>
<tr>
<td>Nonhomogeneous equation</td>
<td>259</td>
</tr>
<tr>
<td>Nonoverlapping (sets)</td>
<td>642</td>
</tr>
<tr>
<td>Off-diagonal Rado number</td>
<td>280</td>
</tr>
<tr>
<td>Off-diagonal Ramsey number</td>
<td>280</td>
</tr>
</tbody>
</table>
Index

Party problem, 6 18
Paul, J., 35
Permutation, 213
Piecewise syndetic, 319
Pigeonhole principle, 3 17
\( p_n \)-function, 92
\( p_n \)-sequence, 92
Polynomial progression, 175
Pomerance, C., 64
Probabilistic method, 19 58 65
Property B, 41 64
Prouhet, E., 349
Prouhet-Thue-Morse sequence, 327
Pythagorean triple, 289
Quasi-progression, 68
R"odl, V., 248
Rabung, J., 112 219
Rado numbers, 259 268
Rado’s full theorem, 21 284
Rado’s selection principle, 27
Rado’s single equation theorem, 14
159 248 251 255
Rado, R., 13 63 231 251 291
Radoiˇci´c, R., 350
Rainbow coloring, 329
Ramsey number, 8 21
Ramsey’s theorem, 7 21 222 248
297
Ramsey, F., 11 21
Ramsey-type, 15
Rankin, R., 64
\( \tau \)-coloring, 29
Recurrence relation, 102 285
Regular, 15
Regular equation, 14 252
\( \tau \)-regular, 15
Regular system of equations, 252
Reverse \( \tau \)-regular, 207
Reverse regular, 207
Riddell, J., 64 351
Roberts, S., 21
Robertson, A., 248
Roth, K., 51 64 219
Ruciński, A., 248
S´os, V., 20
Sanders, J., 294 349
Saracino, D., 294
Schaal, D., 249 294 295
Schmidt’s lemma, 42 64
Schmidt, W., 40
Schoen, T., 248
Schur inequality, 243
Schur number, 141 228
Schur triple, 223 227 231
Schur’s theorem, 13 21 223
Schur, I., 11 13 222 249
Scope, 83
Semi-progression, 83
Shelah, S., 46
Shiue, P., 111
Si-Kaddour, H., 295
Sidorenko, A., 219
Spencer, J., 21 65 219 301
Squarefree coloring, 326
Stirling’s formula, 19
Strict generalized Schur number, 242
Strict Schur number, 240
Subgraph, 7
Sun, Z.-W., 294
Superset, 68
Sweet, M., 249
Szekeres, G., 21 112 222 248
Szemerédi’s theorem, 51
Szemerédi, E., 51 64
Tail, 168
Ternary Thue-Morse sequence, 327
Thue, A., 349
Thue-Morse sequence, 327
Tic-tac-toe game, 4 274
Tower function, 44
Turán, P., 47 51
Valid coloring, 16
Valko, B., 219
Van der Waerden number, 12 23 31
Van der Waerden’s theorem, 12 21
25 53
Van der Waerden, B. L., 11 63
Vandermonde determinant, 94
Vertices, 7
Index

Walters, M., 146
Wiles, A., 222, 248
Witsenhausen, H., 65
Wow function, 44
Yackel, J., 21
Zaks, A., 219
Zeilberger, D., 248
Zero-sum, 334
Zinter, M., 291
Selected Published Titles in This Series

72 **Mark Kot**, A First Course in the Calculus of Variations, 2014
71 **Joel Spencer**, Asymptopia, 2014
70 **Lasse Rempe-Gillen and Rebecca Waldecker**, Primality Testing for Beginners, 2014
69 **Mark Levi**, Classical Mechanics with Calculus of Variations and Optimal Control, 2014
68 **Samuel S. Wagstaff, Jr.**, The Joy of Factoring, 2013
67 **Emily H. Moore and Harriet S. Pollatsek**, Difference Sets, 2013
65 **Víctor H. Moll**, Numbers and Functions, 2012
64 **A. B. Sossinsky**, Geometries, 2012
63 **María Cristina Pereyra and Lesley A. Ward**, Harmonic Analysis, 2012
62 **Rebecca Weber**, Computability Theory, 2012
60 **Richard Evan Schwartz**, Mostly Surfaces, 2011
59 **Pavel Etingof, Oleg Golberg, Sebastian Hensel, Tiankai Liu, Alex Schwendner, Dmitry Vaintrob, and Elena Yudovina**, Introduction to Representation Theory, 2011
58 **Álvaro Lozano-Robledo**, Elliptic Curves, Modular Forms, and Their L-functions, 2011
57 **Charles M. Grinstead, William P. Peterson, and J. Laurie Snell**, Probability Tales, 2011
56 **Julia Garibaldi, Alex Iosevich, and Steven Senger**, The Erdős Distance Problem, 2011
55 **Gregory F. Lawler**, Random Walk and the Heat Equation, 2010
54 **Alex Kasman**, Glimpses of Soliton Theory, 2010
53 **Jiří Matoušek**, Thirty-three Miniatures, 2010
52 **Yakov Pesin and Vaughn Climenhaga**, Lectures on Fractal Geometry and Dynamical Systems, 2009

For a complete list of titles in this series, visit the AMS Bookstore at [www.ams.org/bookstore/stmlseries/](http://www.ams.org/bookstore/stmlseries/).
Ramsey theory is the study of the structure of mathematical objects that is preserved under partitions. In its full generality, Ramsey theory is quite powerful, but can quickly become complicated. By limiting the focus of this book to Ramsey theory applied to the set of integers, the authors have produced a gentle, but meaningful, introduction to an important and enticing branch of modern mathematics. *Ramsey Theory on the Integers* offers students a glimpse into the world of mathematical research and the opportunity for them to begin pondering unsolved problems.

For this new edition, several sections have been added and others have been significantly updated. Among the newly introduced topics are: rainbow Ramsey theory, an “inequality” version of Schur’s theorem, monochromatic solutions of recurrence relations, Ramsey results involving both sums and products, monochromatic sets avoiding certain differences, Ramsey properties for polynomial progressions, generalizations of the Erdős-Ginzberg-Ziv theorem, and the number of arithmetic progressions under arbitrary colorings. Many new results and proofs have been added, most of which were not known when the first edition was published. Furthermore, the book’s tables, exercises, lists of open research problems, and bibliography have all been significantly updated.

This innovative book also provides the first cohesive study of Ramsey theory on the integers. It contains perhaps the most substantial account of solved and unsolved problems in this blossoming subject. This breakthrough book will engage students, teachers, and researchers alike.

**Reviews of the Previous Edition:**

Students will enjoy it due to the highly accessible exposition of the material provided by the authors.

—MAA Horizons

What a wonderful book! … contains a very “student friendly” approach to one of the richest areas of mathematical research … a very good way of introducing the students to mathematical research … an extensive bibliography … no other book on the subject … which is structured as a textbook for undergraduates. … The book can be used in a variety of ways, either as a textbook for a course, or as a source of research problems. … strongly recommend this book for all researchers in Ramsey theory. … very good book: interesting, accessible and beautifully written. The authors really did a great job!

—MAA Online