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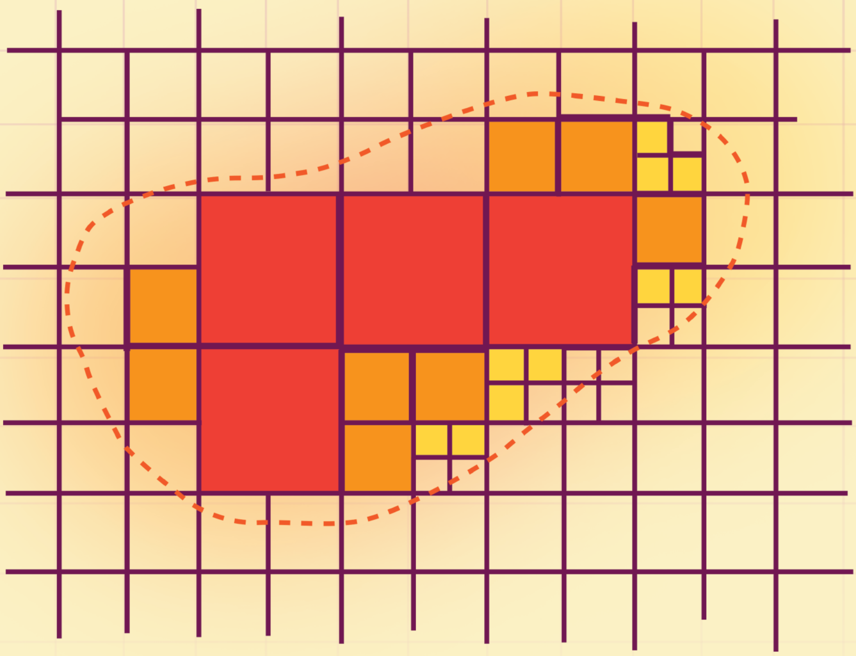
A User-Friendly Introduction to Lebesgue Measure and Integration

Gail S. Nelson



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http://dx.doi.org/10.1090/stml/078

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Providence, Rhode Island

2010 *Mathematics Subject Classification*. Primary 26-XX, 28-XX.

For additional information and updates on this book, visit
www.ams.org/bookpages/stml-78

Library of Congress Cataloging-in-Publication Data

Nelson, Gail Susan.
A user-friendly introduction to Lebesgue measure and integration / Gail S. Nelson.
pages cm. – (Student mathematical library ; volume 78)
Includes bibliographical references and index.
ISBN 978-1-4704-2199-1 (alk. paper)
1. Measure theory. 2. Lebesgue integral. 3. Integration, Functional. I. Title.

QC20.7.M43N45 2015
515′.83–dc23 2015022834

10 9 8 7 6 5 4 3 2 1   20 19 18 17 16 15

To all of the students I've had. Thanks for teaching me. Most of all, thanks to my parents, the best teachers I have had.

Contents

Preface

When I first had the chance to teach a second course in real analysis I did the usual thing; I searched for a textbook that would fit the course as I envisioned it. I wanted to show my students that real analysis was more than just ϵ-δ proofs. I also hoped this course would provide a bit of a head start to those students heading off to graduate school. However, I could not find any text that suited the needs of my target audience, undergraduate students who had seen the basics of sequences and series up through and including the introduction to Riemann integration. So I started teaching the course from scratch, creating my own notes as the course progressed. The feedback from my students was that, while they liked the course in general, they missed having a textbook. Based on this feedback, each of the next couple of times I taught the course, part of the work assigned to the students was to "write" their book. The students took turns carefully rewriting their notes which were then collected in a binder in a central location. The resulting "textbooks" generated in this fashion formed the skeleton of this book.

The prerequisite for this course is a standard undergraduate first course in real analysis. Students need to be familiar with basic limit definitions, and how these definitions are used in sequences and in defining continuity and differentiation. The properties of a supremum (or least upper bound) and infimum (or greatest lower bound) are used repeatedly. The definition of compactness via open coverings is used in this text, but primarily for $\mathbb{R}^n$. I also assume students have seen sequences and series of functions and understand pointwise and uniform convergence. Since a major focus of this text is Lebesgue integration, it is also assumed that students have studied Riemann integration in their first real analysis course. Chapter 0 briefly covers Riemann integration with the approach that is later mimicked in defining the Lebesgue integral, that is, the use of upper and lower sums. (I do realize there are other approaches to the Riemann integral. The approach which uses step functions is the one used in Chapter 4 when the subjects of general measure and integration are introduced.) However, Chapter 0 exists primarily as a source of review and can be omitted.

One of the standard topics in the first analysis course that I teach is the completeness of the set of real numbers. The students often see this first in terms of every nonempty bounded set having a least upper bound. Later they are introduced to the Cauchy criterion and shown that in the real number system all Cauchy sequences converge. My experience has been that this Cauchy criterion is not fully appreciated by my students. In this second course in real analysis completeness via Cauchy sequences is a recurring theme; we first revisit the completeness of $\mathbb{R}$, then L^1; and more generally L^p.

I want to keep my course as "real" as possible. Instead of introducing measure via the Carathéodory definition, I opt to introduce Lebesgue measure through the more "concrete" definition using outer measure. In this way, Lebesgue measure is a natural extension of the concept of length, or area, or volume, depending on dimension.

So, here is my course. I start with a review of Riemann integration. I tend to keep this review to a minimum since most of the main theorems have their Lebesgue counterparts later in Chapter 2. As soon as possible, we move into Chapter 1 which covers Lebesgue outer measure and Lebesgue measure. It should be noted that Section 1.3 contains the classic construction of a nonmeasurable set which assumes knowledge of countability and familiarity with the Axiom of Choice. This section is not needed for the later chapters and can be omitted, although it justifies the difference between measure and outer measure and is referenced in Remark 4.1.14. The Lebesgue integral is defined in Chapter 2. Chapter 3 is where I introduce L^p spaces and use these as examples of Banach spaces. Later in the chapter L^2 is shown to be an example of a Hilbert space. My goal for a one-semester course is to end somewhere in Chapter 4, usually around Section 4.3. Sections 4.4 and 4.5 are independent of each other and at various time I have ended with one or the other.

I have also included an appendix entitled "Ideas for Projects". Most of these are topics that I had at one time considered including as part of my course. Instead, I reserved them for student presentations. My students typically work on these in pairs. In the past I just assigned the topic with a pointer to a possible source. However, here I have included sketches of how one might proceed.

Thanks to the members of the Carleton College mathematics department for their support. I also owe a large debt to all of the students who have been a part of this ongoing project. Without them, this book would never have been created.

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Enjoy exploring the wide world of real analysis!

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Index

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A User-Friendly Introduction to Lebesgue Measure and Integration provides a bridge between an undergraduate course in Real Analysis and a first graduate-level course in Measure Theory and Integration. The main goal of this book is to prepare students for what they may encounter in graduate school, but will be useful for many beginning graduate students as well. The book starts with the fundamentals of measure theory that are gently approached through the very concrete example of Lebesgue measure. With this approach, Lebesgue integration becomes a natural extension of Riemann integration.

Next, L^p-spaces are defined. Then the book turns to a discussion of limits, the basic idea covered in a first analysis course. The book also discusses in detail such questions as: When does a sequence of Lebesgue integrable functions converge to a Lebesgue integrable function? What does that say about the sequence of integrals? Another core idea from a first analysis course is completeness. Are these L^p-spaces complete? What exactly does that mean in this setting?

This book concludes with a brief overview of General Measures. An appendix contains suggested projects suitable for end-of-course papers or presentations.

The book is written in a very reader-friendly manner, which makes it appropriate for students of varying degrees of preparation, and the only prerequisite is an undergraduate course in Real Analysis.

ISBN: 978-1-4704-2199-1

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