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A User-Friendly Introduction to Lebesgue Measure and Integration

Gail S. Nelson





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 $10 \ 9 \ 8 \ 7 \ 6 \ 5 \ 4 \ 3 \ 2 \ 1 \qquad 20 \ 19 \ 18 \ 17 \ 16 \ 15$

To all of the students I've had. Thanks for teaching me. Most of all, thanks to my parents, the best teachers I have had.

Contents

Preface	vii
Chapter 0. Review of Riemann Integration	1
§0.1. Basic Definitions	1
$\S 0.2.$ Criteria for Riemann Integrability	5
0.3. Properties of the Riemann Integral	10
§0.4. Exercises	11
Chapter 1. Lebesgue Measure	15
§1.1. Lebesgue Outer Measure	15
§1.2. Lebesgue Measure	29
§1.3. A Nonmeasurable Set	46
§1.4. Exercises	52
Chapter 2. Lebesgue Integration	57
§2.1. Measurable Functions	57
§2.2. The Lebesgue Integral	67
$\S2.3.$ Properties of the Lebesgue Integral	80
$\S 2.4.$ The Lebesgue Dominated Convergence Theorem	88
$\S2.5.$ Further Notes on Integration	99
§2.6. Exercises	102

Chapter	3. L^p spaces	107
$\S{3.1.}$	$L^1[a,b]$	107
$\S{3.2.}$	L^p Spaces	120
$\S{3.3.}$	Approximations in $L^p[a, b]$	131
§3.4.	$L^2[a,b]$	134
$\S{3.5.}$	L^2 Theory of Fourier Series	139
$\S{3.6.}$	Exercises	149
Chapter	4. General Measure Theory	153
§4.1.	Measure Spaces	153
$\S4.2.$	Measurable Functions	165
$\S4.3.$	Integration	173
$\S4.4.$	Measures from Outer Measures	185
$\S4.5.$	Signed Measures	196
$\S4.6.$	Exercises	203
Ideas for	r Projects	209
References		217
Index		219

Preface

When I first had the chance to teach a second course in real analysis I did the usual thing; I searched for a textbook that would fit the course as I envisioned it. I wanted to show my students that real analysis was more than just ϵ - δ proofs. I also hoped this course would provide a bit of a head start to those students heading off to graduate school. However, I could not find any text that suited the needs of my target audience, undergraduate students who had seen the basics of sequences and series up through and including the introduction to Riemann integration. So I started teaching the course from scratch, creating my own notes as the course progressed. The feedback from my students was that, while they liked the course in general, they missed having a textbook. Based on this feedback, each of the next couple of times I taught the course, part of the work assigned to the students was to "write" their book. The students took turns carefully rewriting their notes which were then collected in a binder in a central location. The resulting "textbooks" generated in this fashion formed the skeleton of this book.

The prerequisite for this course is a standard undergraduate first course in real analysis. Students need to be familiar with basic limit definitions, and how these definitions are used in sequences and in defining continuity and differentiation. The properties of a supremum (or least upper bound) and infimum (or greatest lower bound) are used repeatedly. The definition of compactness via open coverings is used in this text, but primarily for \mathbb{R}^n . I also assume students have seen sequences and series of functions and understand pointwise and uniform convergence. Since a major focus of this text is Lebesgue integration, it is also assumed that students have studied Riemann integration in their first real analysis course. Chapter 0 briefly covers Riemann integration with the approach that is later mimicked in defining the Lebesgue integral, that is, the use of upper and lower sums. (I do realize there are other approaches to the Riemann integral. The approach which uses step functions is the one used in Chapter 4 when the subjects of general measure and integration are introduced.) However, Chapter 0 exists primarily as a source of review and can be omitted.

One of the standard topics in the first analysis course that I teach is the completeness of the set of real numbers. The students often see this first in terms of every nonempty bounded set having a least upper bound. Later they are introduced to the Cauchy criterion and shown that in the real number system all Cauchy sequences converge. My experience has been that this Cauchy criterion is not fully appreciated by my students. In this second course in real analysis completeness via Cauchy sequences is a recurring theme; we first revisit the completeness of \mathbb{R} , then L^1 ; and more generally L^p .

I want to keep my course as "real" as possible. Instead of introducing measure via the Carathéodory definition, I opt to introduce Lebesgue measure through the more "concrete" definition using outer measure. In this way, Lebesgue measure is a natural extension of the concept of length, or area, or volume, depending on dimension.

So, here is my course. I start with a review of Riemann integration. I tend to keep this review to a minimum since most of the main theorems have their Lebesgue counterparts later in Chapter 2. As soon as possible, we move into Chapter 1 which covers Lebesgue outer measure and Lebesgue measure. It should be noted that Section 1.3 contains the classic construction of a nonmeasurable set which assumes knowledge of countability and familiarity with the Axiom of Choice. This section is not needed for the later chapters and can be omitted, although it justifies the difference between measure and

Preface

outer measure and is referenced in Remark 4.1.14. The Lebesgue integral is defined in Chapter 2. Chapter 3 is where I introduce L^p spaces and use these as examples of Banach spaces. Later in the chapter L^2 is shown to be an example of a Hilbert space. My goal for a onesemester course is to end somewhere in Chapter 4, usually around Section 4.3. Sections 4.4 and 4.5 are independent of each other and at various time I have ended with one or the other.

I have also included an appendix entitled "Ideas for Projects". Most of these are topics that I had at one time considered including as part of my course. Instead, I reserved them for student presentations. My students typically work on these in pairs. In the past I just assigned the topic with a pointer to a possible source. However, here I have included sketches of how one might proceed.

Thanks to the members of the Carleton College mathematics department for their support. I also owe a large debt to all of the students who have been a part of this ongoing project. Without them, this book would never have been created.

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Enjoy exploring the wide world of real analysis!

Gail S. Nelson Northfield, Minnesota

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Index

B[a,b], 1C[a, b], 149 F_{σ} set, 43, 157 G_{δ} set, 43, 157 L^{1} -norm, 109 $L^1[a,b], 109$ $L^{\infty}[a, b], 130$ L^p -norm, 121 $L^{p}[a, b], 120$ R[a, b], 4 $\mathcal{L}[a,b], 81$ μ -almost everywhere, 169 μ -measurable function, 166 μ^* -measurable, 186 σ -algebra, 42, 155 $||f||_{\infty}, 130$ $\|\cdot\|_{\infty}, 130$ $||f||_1, 109$ $||f||_p, 121$

a.e., 63 algebra σ -algebra, 155 of sets, 154 almost everywhere, 63 arithmetic differences, 46

Banach space, 110, 127 Beppo Levi Theorem, 114 Bessel's inequality, 142 Borel sets, 157 Hausdorff measurable, 194 Borel-Cantelli Lemma, 54

Cantor function, 21 Cantor set, 19, 67 Carathéodory condition, 186 Cauchy-Schwarz Inequality, 136 characteristic function, 58 common refinement, 5, 71 complete measure space, 162 vector space, 110 convergence in measure, 210 convex, 213 covering by intervals, 16 Dirichlet function, 2, 57, 62, 63 Dirichlet kernel, 144 distance between sets, 37

Egorov's Theorem, 209 ess inf, 130 ess sup, 129 essential infimum, 130 essential supremum, 129

Fatou's Lemma, 96, 97, 177, 181 preliminary version, 96 Fejér kernel, 144 Fourier coefficients, 141 Fourier series, 141 Fubini's Theorem, 101, 214 function bounded, 1 Cantor, 21 characteristic, 58 Dirichlet, 2 Lebesgue measurable, 58, 59 measurable, 59, 100 simple, 62, 132 Hahn Decomposition Theorem, 201 Hausdorff dimension, 195 Hausdorff measurable, 194 Hausdorff outer measure, 193 Hilbert space, 138 Hölder's Inequality, 122, 125 hyperplane, 31, 34, 52 induced norm, 136 inner product, 135 inner product space, 135 integrable Lebesgue, 69, 80, 81 Riemann, 4 integral Lebesgue, 69, 80, 81 lower, 3, 69 Riemann, 4 upper, 3, 69 interval, 15 nonoverlapping, 32 volume, 16 Jensen's Inequality, 213 Lebesgue integrable, 69, 80, 81 integral, 69, 80, 81 measurable function, 58 measurable set, 29 measure, 29 outer measure, 17 Lebesgue Dominated Convergence Theorem, 92, 177, 184 lim inf, 64, 168 lim sup, 64, 168 lower integral, 3, 69

lower sum, 3, 68 measurable function general, 166 Lebesgue, 58, 59, 100 measurable partition, 67 measurable set general, 157 Lebesgue, 29 measurable space, 157 measure general, 158 Hausdorff, 194 Hausdorff outer, 193 Lebesgue, 29 point-mass, 175 signed, 196 measure space, 158 complete, 162 Minkowski's Inequality, 126 Monotone Convergence Theorem, 96, 98, 177, 181 negative set, 197 nonoverlapping intervals, 32 norm, 107 $L^{p}, 121$ induced, 136 $L^1, 109$ null, 197 open set union of intervals, 34 outer measure, 17, 185 closed interval, 25 general, 185 Lebesgue, 17 parallelogram law, 139 Parseval's equation, 148 partition, 2 common refinement, 71 measurable, 67 refinement, 71 positive set, 197 Rapidly Cauchy sequence, 211 refinement, 5, 71 common, 5, 71

Index

Riemann integrable, 4 Riesz's Theorem, 210 set of arithmetic differences, 46 signed measure, 196 simple function, 62, 132 general, 171 space measurable, 153 Tchebychev's Inequality, 150 trigonometric polynomial, 142 type F_{σ} set, 43, 157 type G_{δ} set, 43, 157 upper integral, 3, 69 upper sum, 3, 68 Vitali, 48 A User-Friendly Introduction to Lebesgue Measure and Integration provides a bridge between an undergraduate course in Real Analysis and a first graduate-level course in Measure Theory and Integration. The main goal of this book is to prepare students for what they may encounter in graduate school, but will be useful for many beginning grad-



uate students as well. The book starts with the fundamentals of measure theory that are gently approached through the very concrete example of Lebesgue measure. With this approach, Lebesgue integration becomes a natural extension of Riemann integration.

Next, L^p -spaces are defined. Then the book turns to a discussion of limits, the basic idea covered in a first analysis course. The book also discusses in detail such questions as: When does a sequence of Lebesgue integrable functions converge to a Lebesgue integrable function? What does that say about the sequence of integrals? Another core idea from a first analysis course is completeness. Are these L^p-spaces complete? What exactly does that mean in this setting?

This book concludes with a brief overview of General Measures. An appendix contains suggested projects suitable for end-of-course papers or presentations.

The book is written in a very reader-friendly manner, which makes it appropriate for students of varying degrees of preparation, and the only prerequisite is an undergraduate course in Real Analysis.



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