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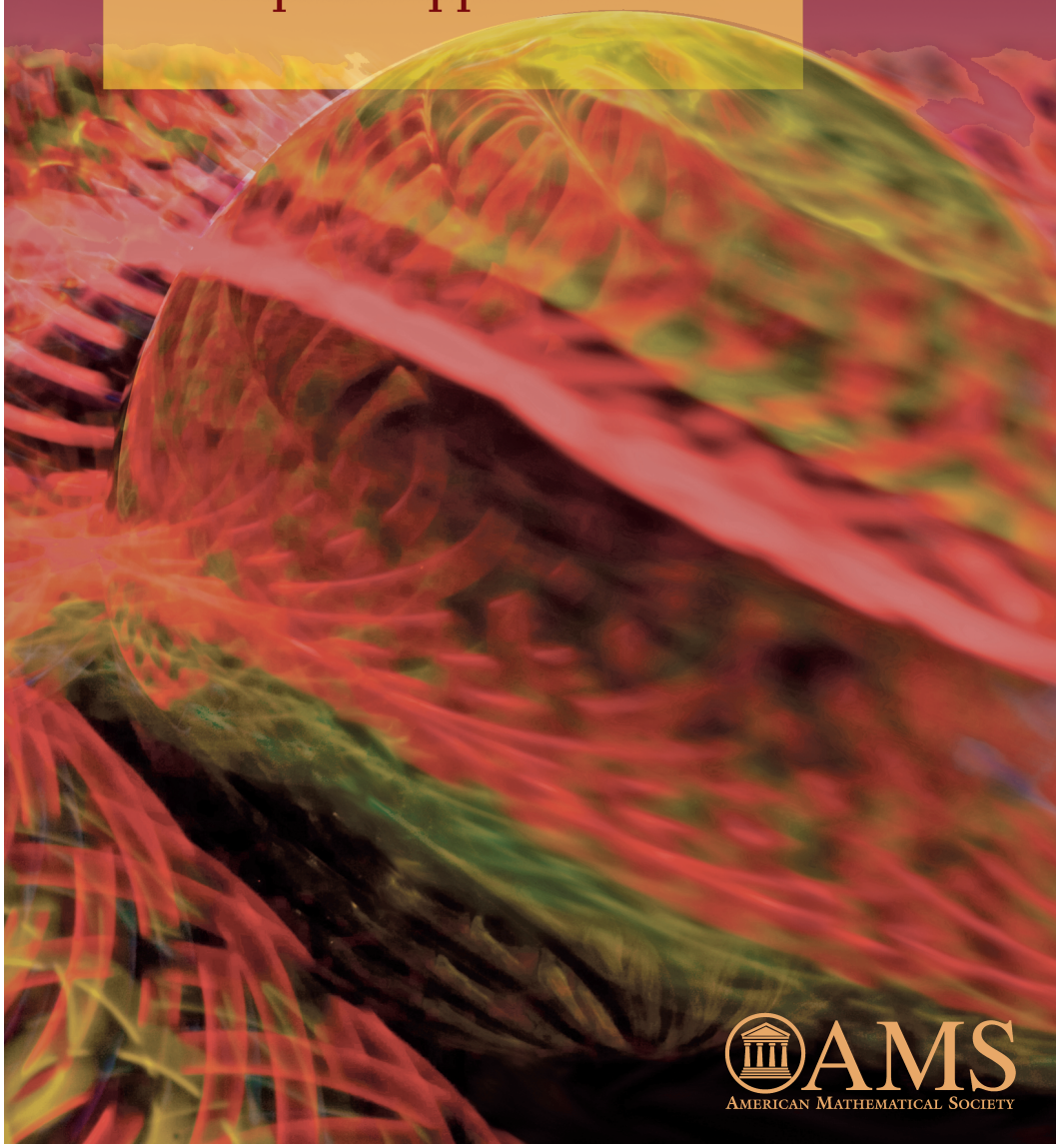
Matrix Groups for Undergraduates

Second Edition

Kristopher Tapp



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Kristopher Tapp

Providence, Rhode Island

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2010 *Mathematics Subject Classification.* Primary 20-02, 20G20; Secondary 20C05, 22E15.

For additional information and updates on this book, visit
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Library of Congress Cataloging-in-Publication Data

Names: Tapp, Kristopher, 1971–

Title: Matrix groups for undergraduates / Kristopher Tapp.

Description: Second edition. — Providence, Rhode Island : American Mathematical Society, [2016] — Series: Student mathematical library ; volume 79 — Includes bibliographical references and index.

Identifiers: LCCN 2015038141 — ISBN 9781470427221 (alk. paper)

Subjects: LCSH: Matrix groups. — Linear algebraic groups. — Compact groups. — Lie groups. — AMS: Group theory and generalizations – Research exposition (monographs, survey articles). msc — Group theory and generalizations – Linear algebraic groups and related topics – Linear algebraic groups over the reals, the complexes, the quaternions. msc — Group theory and generalizations – Representation theory of groups – Group rings of finite groups and their modules. msc — Topological groups, Lie groups – Lie groups – General properties and structure of real Lie groups. msc

Classification: LCC QA184.2 .T37 2016 — DDC 512/.2–dc23 LC record available at http://lccn.loc.gov/2015038141

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ISBN: 978-1-4704-2722-1
9 781470 427221
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