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Hilbert's Tenth Problem

An Introduction to Logic, Number Theory, and Computability

M. Ram Murty
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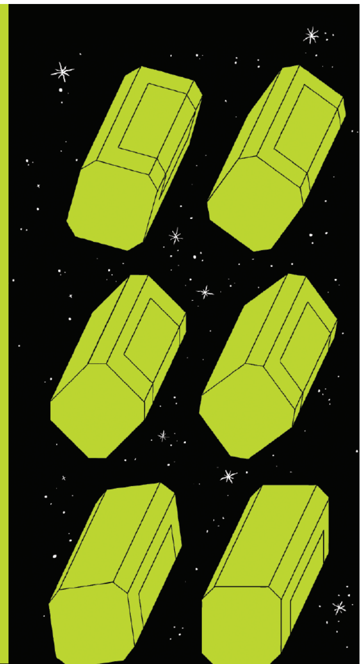

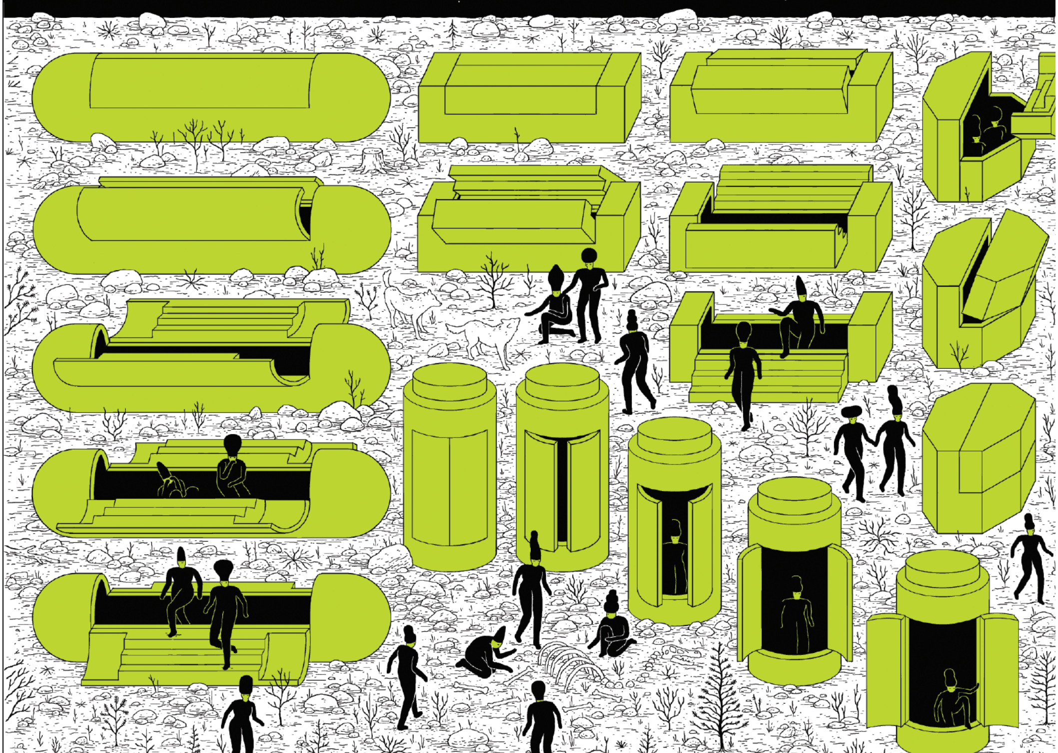

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An algebra of mind, a scheme of sense,
A symbol language without depth or wings,
A power to handle deftly outward things
Are our scant earnings of intelligence.
The Truth is greater and asks deeper ways.

- Sri Aurobindo, “Discoveries of Science II” in *Collected Poems*

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Preface

In 1980, the senior author (MRM) had the grand privilege of meeting Sarvadaman Chowla at the Institute for Advanced Study in Princeton, New Jersey. Chowla had written a small book titled *The Riemann Hypothesis and Hilbert's Tenth Problem* in 1965 and so this was an opportunity to ask him about the seemingly strange title and how it came to be. Was there a connection between the two? Chowla replied, "I don't know. These two problems have always fascinated me and so I chose that as the title." He went on to say that the book was largely an inspired work, written in a single night, and it represents his selection of beautiful pearls from number theory. It was not meant to be a textbook but more of an invitation for further study and "to stimulate the reader".

But the fact of the matter is that the two problems *are* related as we discovered only much later in the work of Martin Davis, Hilary Putnam, Julia Robinson and Yuri Matiyasevič. In fact, many of the famous Hilbert problems are interconnected. This interconnectedness can be used as the focus for mathematical instruction. And it can be done with very few prerequisites. This is the *raison d'être* of this book.

Some of the Hilbert problems such as the Riemann hypothesis (the eighth problem) are still open. The others that have been solved required formidable background and preparation. Hilbert's tenth problem is different in that a basic introduction to elementary number theory and mathematical logic suffices to understand the proof, and this can be done in a relatively short time. In addition to the grand arrangement of mathematical ideas, Hilbert's tenth problem has a colourful cast of characters, many of them tragic heroes, who pondered deeply regarding the enigma of the human being and the nature of mathematical truth.

Hilbert's tenth problem and its solution represent in microcosm the riddle of human life itself and its meaning. This mélange of philosophical and mathematical conundrums are the mysteries that confront us. In many ways, this book is not meant to be a textbook, but rather an invitation to explore further. As Chowla would say, we hope "to stimulate the reader".

M. Ram Murty and Brandon Fodden
Kingston and Ottawa, Ontario
July 2018

Acknowledgments

This book is based on an upper-level undergraduate course given at Queen's University in Ontario in the winter semester of 2007 by the senior author (MRM). The class consisted of primarily undergraduates, several graduate students, and a few post-doctoral fellows. There were also students from the philosophy department. Given the diverse backgrounds of the students, the mathematical prerequisites were kept to a bare minimum requiring only familiarity with basic calculus and linear algebra. The course covered the contents of the first five chapters by first introducing students to logical notation, then elementary number theory, and gradually to notions of computability and decidability and, finally, the proof of Hilbert's tenth problem. The last two chapters were added later and were culled from graduate seminars conducted since the time the course was first given. They require more advanced background, especially the last chapter. If the student is willing to take some of the background material in those chapters on faith, they will acquire a panoramic view of some recent discoveries and new directions. We feel that this assemblage of subject matter can make an excellent introduction to this fascinating topic and can take the student to the frontiers of current research. We thank Kumar Murty, Hector Pasten, and the referees for their comments on an earlier version of this book. We are grateful to Ina Mette and the American Mathematical Society for taking interest in publishing this book, and to Marcia Almeida at the AMS for much help with preparing the manuscript.

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Hilbert's tenth problem is one of 23 problems proposed by David Hilbert in 1900 at the International Congress of Mathematicians in Paris. These problems gave focus for the exponential development of mathematical thought over the following century. The tenth problem asked for a general algorithm to determine if a given Diophantine equation has a solution in integers. It was finally resolved in a series of papers written by Julia Robinson, Martin Davis, Hilary Putnam, and finally Yuri Matiyasevich in 1970. They showed that no such algorithm exists.

This book is an exposition of this remarkable achievement. Often, the solution to a famous problem involves formidable background. Surprisingly, the solution of Hilbert's tenth problem does not. What is needed is only some elementary number theory and rudimentary logic. In this book, the authors present the complete proof along with the romantic history that goes with it. Along the way, the reader is introduced to Cantor's transfinite numbers, axiomatic set theory, Turing machines, and Gödel's incompleteness theorems.

Copious exercises are included at the end of each chapter to guide the student gently on this ascent. For the advanced student, the final chapter highlights recent developments and suggests future directions. The book is suitable for undergraduates and graduate students. It is essentially self-contained.

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