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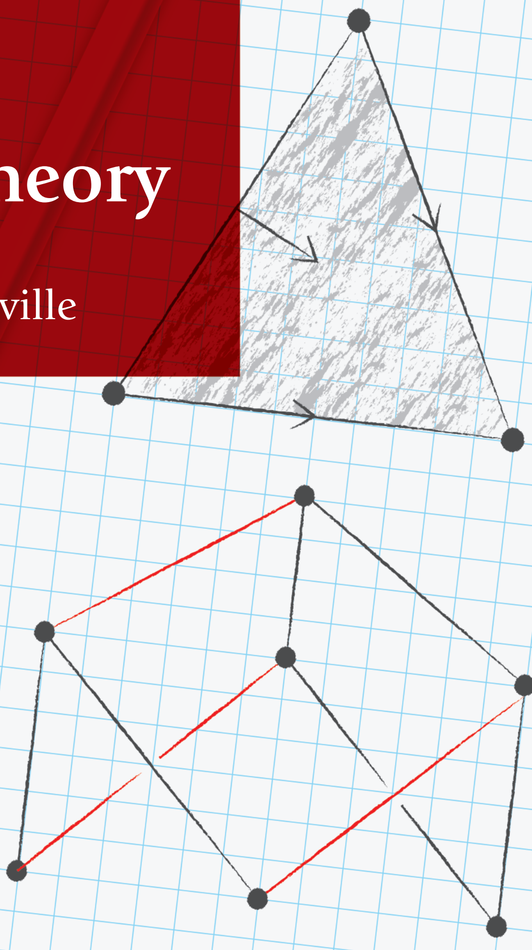
Discrete Morse Theory

Nicholas A. Scoville



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Contents

Preface	ix
Chapter 0. What is discrete Morse theory?	1
§0.1. What is discrete topology?	2
§0.2. What is Morse theory?	9
§0.3. Simplifying with discrete Morse theory	13
Chapter 1. Simplicial complexes	15
§1.1. Basics of simplicial complexes	15
§1.2. Simple homotopy	31
Chapter 2. Discrete Morse theory	41
§2.1. Discrete Morse functions	44
§2.2. Gradient vector fields	56
§2.3. Random discrete Morse theory	73
Chapter 3. Simplicial homology	81
§3.1. Linear algebra	82
§3.2. Betti numbers	86
§3.3. Invariance under collapses	95
Chapter 4. Main theorems of discrete Morse theory	101

§4.1. Discrete Morse inequalities 101
§4.2. The collapse theorem 111

Chapter 5. Discrete Morse theory and persistent homology 117
§5.1. Persistence with discrete Morse functions 117
§5.2. Persistent homology of discrete Morse functions 134

Chapter 6. Boolean functions and evasiveness 149
§6.1. A Boolean function game 149
§6.2. Simplicial complexes are Boolean functions 152
§6.3. Quantifying evasiveness 155
§6.4. Discrete Morse theory and evasiveness 158

Chapter 7. The Morse complex 169
§7.1. Two definitions 169
§7.2. Rooted forests 177
§7.3. The pure Morse complex 179

Chapter 8. Morse homology 187
§8.1. Gradient vector fields revisited 188
§8.2. The flow complex 195
§8.3. Equality of homology 196
§8.4. Explicit formula for homology 199
§8.5. Computation of Betti numbers 205

Chapter 9. Computations with discrete Morse theory 209
§9.1. Discrete Morse functions from point data 209
§9.2. Iterated critical complexes 220

Chapter 10. Strong discrete Morse theory 233
§10.1. Strong homotopy 233
§10.2. Strong discrete Morse theory 242
§10.3. Simplicial Lusternik-Schnirelmann category 249

Bibliography 257

Notation and symbol index 265

Index 267

Preface

This book serves as both an introduction to discrete Morse theory and a general introduction to concepts in topology. I have tried to present the material in a way accessible to undergraduates with no more than a course in mathematical proof writing. Although some books such as [**102**, **132**] include a single chapter on discrete Morse theory, and one [**99**] treats both smooth and discrete Morse theory together, no book-length treatment is dedicated solely to discrete Morse theory. Discrete Morse theory deserves better: It serves as a tool in applications as varied as combinatorics [**16**, **41**, **106**, **108**], probability [**57**], and biology [**136**]. More than that, it is fascinating and beautiful in its own right. Discrete Morse theory is a discrete analogue of the "smooth" Morse theory developed in Marston Morse's 1925 paper [**124**], but it is most popularly known via John Milnor [**116**]. Fields medalist Stephen Smale went so far as to call smooth Morse theory "the single greatest contribution of American mathematics" [**144**]. This beauty and utility carries over to the discrete setting, as many of the results, such as the Morse inequalities, have discrete analogues. Discrete Morse theory not only is topological but also involves ideas from combinatorics and linear algebra. Yet it is easy to understand, requiring no more than familiarity with basic set theory and mathematical proof techniques. Thus we find several online introductions to discrete Morse theory written by undergraduates. For example, see the notes of Alex Zorn for his REU project at the University of Chicago [**158**], Dominic Weiller's bachelor's thesis [**150**], and Rachel Zax's bachelor's thesis [**156**].

From a certain point of view, discrete Morse theory has its foundations in the work of J. H. C. Whitehead [**151, 152**] from the early to mid-20th century, who made the deep connection between simple homotopy and homotopy type. Building upon this work, Robin Forman published the original paper introducing and naming discrete Morse theory in 1998 [**65**]. His extremely readable *A user's guide to discrete Morse theory* is still the gold standard in the field [**70**]. Forman published several subsequent papers [**66, 68–71**] further developing discrete Morse theory. The field has burgeoned and matured since Forman's seminal work; it is certainly established enough to warrant a book-length treatment.

This book further serves as an introduction, or more precisely a first exposure, to topology, one with a different feel and flavor from other introductory topology books, as it avoids both the point-set approach and the surfaces approach. In this text, discrete Morse theory is applied to simplicial complexes. While restriction to only simplicial complexes does not expose the full generality of discrete Morse theory (it can be defined on regular CW complexes), simplicial complexes are easy enough for any mathematically mature student to understand. A restriction to simplicial complexes is indeed necessary for this book to act as an exposure to topology, as knowledge of point-set topology is required to understand CW complexes. The required background is only a course in mathematical proofs or an equivalent course teaching proof techniques such as mathematical induction and equivalence relations. This is not a book about smooth Morse theory either. For smooth Morse theory, one can consult Milnor's classic work [**116**] or a more modern exposition in [**129**]. A discussion of the relations between the smooth and discrete versions may be found in [**27, 29, 99**].

One of the main lenses through which the text views topology is homology. A foundational result in discrete Morse theory consists of the (weak) discrete Morse inequalities; it says that if K is a simplicial complex and $f\colon K \to \mathbb{R}$ a discrete Morse function with m_i critical simplices of dimension i, then $b_i \leq m_i$ where b_i is the ith Betti number. To prove this theorem and do calculations, we use $\mathbb{F}_2$-simplicial homology and build a brief working understanding of the necessary linear algebra in Chapter 3. Chapter 1 introduces simplicial complexes, collapses, and simple homotopy type, all of which are standard topics in topology.

Any book reflects the interests and point of view of the author. Combining this with space considerations, I have regrettably had to leave much more out than I included. Several exclusions are worth mentioning here. Discrete Morse theory features many interesting computational aspects, only a few of which are touched upon in this book. These include homology and persistent homology computations [**53**, **80**, **82**], matrix factorization [**86**], and cellular sheaf cohomology computations [**48**]. Mathematicians have generalized and adapted discrete Morse theory to various settings. Heeding a call from Forman at the end of *A user's guide to discrete Morse theory* [**70**], several authors have extended discrete Morse theory to certain kinds of infinite objects [**8**, **10**, **12**, **15**, **105**]. Discrete Morse theory has been shown to be a special case [**155**] of Bestvina-Brady discrete Morse theory [**34**, **35**] which has extensive applications in geometric group theory. There is an algebraic version of discrete Morse theory [**87**, **102**, **142**] involving chain complexes, as well as a version for random complexes [**130**]. E. Minian extended discrete Morse theory to include certain collections of posets [**118**], and B. Benedetti developed discrete Morse theory for manifolds with boundary [**28**]. There is also a version of discrete Morse theory suitable for reconstructing homotopy type via a certain classifying space [**128**]. K. Knudson and B. Wang have recently developed a stratified version of discrete Morse theory [**100**]. The use of discrete Morse theory as a tool to study other kinds of mathematics has proved invaluable. It has been applied to study certain problems in combinatorics and graph theory [**16**, **41**, **49**, **88**, **106**, **108**] as well as configuration spaces and subspace arrangements [**60**, **122**, **123**, **139**]. It is also worth noting that before Forman, T. Banchoff also developed a discretized version of Morse theory [**17–19**]. This, however, seemed to have limited utility. E. Bloch found a relationship between Forman's discrete Morse theory and Banchoff's [**36**].

I originally developed these ideas for a course in discrete Morse theory taught at Ursinus College for students whose only prerequisite was a proof-writing course. An introductory course might cover Chapters 1–5 and Chapter 8. For additional material, Chapters 6 and 9 are good choices for a course with students who have an interest in computer science, while Chapters 7 and 10 are better for students interested in pure math. Some of the more technical proofs in these chapters may be skipped. A more advanced course could begin at Chapter 2 and cover the rest of the book, referring back to Chapter 1 when needed. This book could also be used as a supplemental text for a course in algebraic topology or topological combinatorics, an independent study, or a directed study, or as the basis for an undergraduate research project. It is also intended for research mathematicians who need a primer in or reference for discrete Morse theory. This includes researchers in not only topology but also combinatorics who would like to utilize the tools that discrete Morse theory provides.

Exercises and Problems

The structure of the text reflects my philosophy that "mathematics is not a spectator sport" and that the best way to learn mathematics is to actively do mathematics. Scattered throughout each chapter are tasks for the reader to work on, labeled "Exercise" or "Problem." The distinction between the two is somewhat artificial. The intent is that an Exercise is a straightforward application of a definition or a computation of a simple example. A Problem is either integral to understanding, necessary for other parts of the book, or more challenging. The level of difficulty of the Problems can vary substantially.

A note on the words "easy," "obvious," etc.

In today's culture, we often avoid using words such as "easily," "clearly," "obviously," and the like. It is thought that these words can be stumbling blocks for readers who do not find it clear, causing them to become discouraged. For that reason, I have attempted to avoid using these words in the text. However, the text is not completely purged of such words, and I would like to convey what I mean when I use them. I often tell my students that a particular mathematical fact is "easy but it is very difficult to see that it is easy." By this I mean that one may need to spend a significant amount of time struggling to understand the meaning of the claim before it "clicks." So when the reader sees words like "obviously," she should not despair if it is not immediately obvious to her. Rather, the word is an indication that should alert the reader to write down an example, rewrite the argument in her own words, or stare at the definition until she gets it.

Erratum

A list of typos, errors, and corrections for the book will be kept at `http://webpages.ursinus.edu/nscoville/DMTerratum.html`.

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Nick Scoville
Feast of Louis de Montfort

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Notation and symbol index

$(\Bbbk_*, \partial_*)$, 89
$(f_i)_*$, 197
$*$, 22
$B^S(v, e)$, 242
$B_i^f(k)$, 118
CK, 23
D, 25
D_f, 145
F_e, 242
$G = (M, C)$, 228
$H_p^{\sigma,\tau}$, 144
$H_i(K)$, 92
$K(c)$, 112
$K * L$, 35
$K - \{v\}$, 236
$K \approx L$, 240
$K \sim L$, 31
K^i, 19
K_0, 240
K_i, 87
$L_{i,j}(G)$, 183
P^2, 24
P_i^n, 150
S^n, 22
T^2, 23
T_i^n, 152
$V(K)$, 17
$V \cong W$, 199
V_f, 57
$V_p(\sigma)$, 188
$[\alpha, \beta]$, 70
$[v_n]$, 17
$\langle \cdot \rangle$, 21
$\Bbbk^n$, 84
$\Bbbk_*^\Phi(K)$, 195
$\Bbbk_p^\Phi(K)$, 195
Δ^n, 22
$\mathbb{F}_2$, 83
Γ_f, 153
$\mathcal{H}_V$, 66
$\mathcal{M}_p^i$, 201
$\Phi^\infty(c)$, 196
Φ_p, 190
ΣK, 36

$\bar{f}$, 143
$\beta_p^{\sigma,\tau}$, 144
$\overline{\sigma}$, 19
$\chi(K)$, 30
$\dim(K)$, 19
$\mathrm{Im}(A)$, 85
$\ker(A)$, 85
$\leftarrow_V$, 138
$\mathrm{link}_K(V)$, 166
$\mathcal{H}(i)$, 64
$\mathcal{H}_K$, 64
$\mathcal{K}$, 25
$\mathcal{M}$, 26
$\mathcal{M}(K)$, 171
$\mathcal{M}_P(K)$, 179
$\mathcal{P}([v_n])$, 22
$\mathcal{R}(\sigma)$, 178
$\mathcal{S}(\mathcal{R})$, 179
$\mathrm{low}(j)$, 126
$\mathrm{maxf}_0(\sigma)$, 210
$\nearrow$, 31
$\nearrow\nearrow$, 237
$\mathrm{null}(A)$, 85
$\oplus$, 96
$\overline{\mathbb{R}}$, 145
$\partial_K(\sigma)$ (of a simplex), 21
∂_i (linear transformation), 89
$\prec$, 138
$\prec_V$, 138
$\prec_f$, 139
$\mathrm{rank}(A)$, 85
$g|_K$, 74
$\mathrm{scat}(K)$, 250
$\mathrm{scat}(f)$, 249
$\mathrm{scrit}(\mathcal{M})$, 244
$\mathrm{scrit}(f)$, 245
$\searrow$, 31
$\searrow\searrow$, 237
$\sigma_{\_}$, 144
$\sigma^{(i)}$, 20
$\sigma_0 \to \sigma_1 \to \cdots \to \sigma_n$, 60
$\mathrm{star}_K(v)$, 166
$\tau < \sigma$, 19
$\vec{0}$, 83
$\vec{c}_K$, 19
$\vec{f}$, 74
$b_i(K)$, 92
$c(M)$, 157
c_i, 19
$d(f, g)$, 134
$d_B(X, Y)$, 146
$e_p(A)$, 158
$f \sim g$, 239
$f \sim_c g$, 238
$g \leq f$, 175
h_t, 107
$i^{\sigma,\tau}$, 144
m_i^f, 74
$\binom{n}{k}$, 27
$v_0 v_1 \cdots v_n$, 20
2–1, 44

Index

V-path, **60**, 109–111, 159, 217
closed, **61**, 65
extending, 110
maximal, **61**
non-trivial, **60**
n-simplex, **22**, 31, 34, 159, 164, 181
generalized discrete Morse function, 72
is collapsible, 37
not a Morse complex, 176
perfect discrete Morse function, 105
n-sphere, **22**, 31, 34
Betti numbers, 98
perfect discrete Morse function, 105
two critical simplices, 55

algorithm
B-L algorithm
detects collapsibility, 78
B-L algorithm, 76
Cancel, 217
ExtractCancel, 218
ExtractRaw, 216
for discrete Morse matching, 228
Hasse diagram, 224
Argentinean complex, **238**, 246, 250, 255

Betti number, **92**, 98, 118, 158, 205, 221
bounded by evaders, 166
is an invariant, 96
of S^2, 95
of collapsible complex, 97
of punctured spheres, 99
of spheres, 98
relation to critical simplices, 101
relation to Euler characteristic, 95
binomial coefficient, **27**, 28
Björner's example, **26**
homology, 208
Boolean function, **149**

constant, **152**
evasive, **151**, 152
hider, **149**
induced simplicial complex, **154**
monotone, **152**, 154
projection, **150**, 154, 159
seeker, **149**
threshold, **152**, 153, **155**, 156
boundary
of a simplex, **21**, 87, 89
boundary operator, **89**, 90, 95, 190, 195, 197
commutes with flow, 193
twice is 0, 91

canceling critical simplices, 106, 110, 217
categorical cover, **249**, 251
chain, **88**, **197**
chain complex, **88**, 195, 197, 202
split, **96**, 97
subchain complex, **96**, 97
chain map, **197**, 198
collapse, **31**, 32, 37, 42, 76, 96, 115, 116, 159, 160, 164, 234, 240
strong, **236, 237**
collapse theorem, 115, 121, 141
generalized, 116
strong, 247, 254
collapsible, 74, 78, 105, 181
Betti numbers, 97
does not imply nonevasive, 167
one critical simplex, 55
strongly, **237**, 241, 254
one critical simplex, 248
commutative diagram, **197**, 252
consistent, *see also* gradient vector field, consistent with discrete Morse function
contiguity, 252
contiguity class, **239**
of a point, 239
of the identity, 239
contiguous, **238**
with the identity, 240
critical complex, 188, 201, **204**, 220, 223
homology, 203
CW complex, 103

decision tree algorithm, **155**, 158
complexity, **157**
evader, **157**, 164
induced gradient vector field, 159
discrete Morse function
pseudo-discrete Morse function
linear combination, 137
discrete Morse function, **48**, 77, 108
excellent
homological sequence, 121
all simplices critical, 52
basic, **44**, 47, 50, 52, 55, 106, 123

consistent with a gradient vector field, 137
consistent with a gradient vector field, 135, 137, 141
critical simplex, **46**, **51**, 53, 62, 112, 189
at least one, 53
critical value, **46**, **51**
excellent, **55**, 55, 120
flat, **106**, 120, 137, 245
flat pseudo-discrete Morse function, **141**
flattening, 106, 120, 142, 143
generalized, **71**, 116, 247
sum of, 73
Hasse equivalent, **171**
level subcomplex, **112**, 118
optimal, **74**, 76, 78, 104, 204, 220
not unique, 76
perfect, **104**, 105, 119, 182
non-existence of, 105
primitive, **174**
compatible, **175**
pseudo-discrete Morse function, **135**, 137
pure, **143**
regular simplex, **46**, **51**
regular value, **46**, **51**
strong, **245**, 254
critical object, 254
sum of, 247
weakly increasing, **44**
discrete Morse graph, **228**, 229
discrete Morse inequalities
strong, 103, 104, 205
weak, 101, 104, 165, 187
discrete Morse matching, **170**, 171, 213, 222, 228
critical object, **244**
critical pair, **242**
index, **244**
critical simplex, **244**
strong, **244**
trivial, **228**
discrete Morse spectrum, **77**
discrete Morse vector, **74**, 77, 104
of collapsible complex, 74
discrete vector field, **58**, 68, 159
generalized, **71**, 116
induced partial order, **138**
not a gradient vector field, 59
relation to gradient vector field, 61, 69
relation to Hasse diagram, 68
distance, 134
between discrete Morse functions, 134, 143, 147
Dunce cap, **25**, 38, 105
Betti number, 95
not collapsible, 38

Euler characteristic, 28, **30**, 38, 251
invariance of, 33
of Δ^n, 31
of n-sphere, 31
of a graph, 182
of a point, 35
of suspension, 37
relation to Betti numbers, 95
relation to critical simplices, 101

exclusion lemma, 49, 52, 58, 115, 178, 213
expansion, **31**, 32, 96, 164
 strong, **237**

filtration, 123
flow, **190**
 commutes with boundary, 193
 stabilize, **191**, 192, 193, 195, 199
flow complex, **195**, 198
 homology of, 199
Forman equivalence, **53**, 69, 108, 171
 relation to gradient vector field, 62
 with a 1–1 discrete Morse function, 55, 102
 with a flat discrete Morse function, 106
free pair, **31**, 42

gradient vector field, 43, 56, **57**, 62, 107–109, 111, 135, 170, **188**
 consistent with discrete Morse function, 137
 arrow, **57**, 58
 consistent with discrete Morse function, 135, 137, 141
 counting maximum, 184
 critical simplex, 58
 gradient path, **60**, 217
 head, 42, **57**, 57, 58, 189, 213
 induced by decision tree algorithm, 159
 matching, 57
 maximum, **182**, 183, 184
 minimal, **141**, 141, 142
 tail, 42, **57**, 57, 58, 189, 213
graph, **23**, 30, 49, 56, 69
 adjacent, **183**
 complete graph, 251
 connected, **177**
 counting spanning trees, 184
 degree, **183**
 edge, **177**
 forest, **178**, 251
  root, **178**
 Laplacian, **183**, 185
 Morse complex, 176
 perfect discrete Morse function, 105
 rooted forest, **178**, 178
 spanning tree, 182
 subgraph, 251
 tree, 45, **177**, 181
  leaf, **159**, 181
 vertex, **177**

Hasse diagram, **64**, 71, 139, 170, 223, 228, 242
 algorithm to construct, 224
 directed, **66**, 69, 170, 171, 214, 221
 directed cycle, **67**, 68
 downward, **66**, 228
 Forman vector, **242**
 is poset, 64
 level, **64**, 68
 matching, 169, 171

acyclic, **169**, 171, 228
node, **64**, 228
strong vector, **242**, 244
upward, **66**, **223**, 228
Hasse equivalent, **171**
heresy, 157
homological sequence
of excellent discrete Morse function, 121
homological equivalence, **119**, 123, 143
homological sequence, **118**, 129
does not imply Forman equivalence, 120
homology, 197, 199
critical complex, 203, 205
removing a facet, 98
simplicial, 81, **92**
homotopy
between flat pseudo-discrete Morse functions, 147
of discrete Morse functions, **107**, 109, 135
straight-line, 107, 147

identity map, **198**, 198, 240
inclusion, *see also* linear transformation/simplicial map, inclusion
index of a pair, **157**, 244
interval, **70**, 247
singular, **71**, 116
invariant, **33**, 33, 97
of strong homotopy type, 254
iterated discrete Morse decomposition, **230**

join, *see also* simplicial complex, join

Kirchoff's theorem, 184
Klein bottle, **25**
homology, 205

linear extension, **139**
linear transformation, **84**, 88, 124, 197, 198
image, 95, 197
inclusion, **144**, 198
kernel, **85**, 95, 197
nullity, **85**, 92, 197
rank, **85**, 92, 197
invariant under row operations, 85
well-defined, **198**
lower star filtration, **210**

Möbius band, **26**, 35, 48, 108
matrix
eigenvalue, 184
leading coefficient, **85**
row echelon form, **85**
maximizer, **195**, 196, 202
Morse complex, **171**, **175**
bijection with rooted forest, 178
of a tree is collapsible, 181
of rooted forest, 177
pure, **179**, 181, 183
counting facets, 184
counting facets, 183
Morse matching, *see also* discrete Morse matching
multiset, **145**

individual elements, **146**
multiplicity, **145**

partially ordered set, *see also* poset
Pascal's rule, 28
persistence, 213
persistent homology, 124, **144**
birth, 127
bar code, **128**
Betti number, **144**
birth, **144**
bottleneck distance, **146**, 147
death, 127, **144**
persistence diagram, **132**, **145**
well-defined, 145
persistence pair, **144**
point at infinity, **132**, 144
poset, **63**, 63, 64, 138, 139
consistent, **139**, 142
projective plane, **24**
homology, 207

rank-nullity theorem, 85, 95, 98
restriction of a function, **74**, 102, 195, 201, 249

simple homotopy type, 34
simple homotopy type, **31**, 32–34, 37, 95, 97, 176
of Δ^n, 37
of a point, **31**, 35
of cone, 37
of spheres, 98
simplex
codimension, **21**
boundary, **21**, 22, 89, 228
codimension, 22, 125
coface, **19**, 31, 49
critical, 55, 62, 71, 101, 115, 193
face, **19**, 31
facet, **21**, 21, 181, 223, 239
join, **212**
regular, 52
simplicial category
of a complex, **250**, 254
of a map, **249**, 252
of Argentinean complex, 255
of suspension, 251
of union, 254
simplicial complex, **17**
c-vector, **19**, 20, 30, 74
collapsible, **35**, 36, 37, 116, 166
cone, **23**, 36
is collapsible, 36
core, **240**, 241
is unique, 241
covering, **249**
dimension, **19**, 20, 33
evasive, **157**, 167
facet, 21, 77, 98, 183, 184, 235
generated by simplex, **19**, 22, 250
generated by simplices, **22**, 22
induced by Boolean function, **153, 154**
isomorphic, 241
isomorphism, **240**, 241
join, **35**
minimal, **239**, 240

  nonevasive, **157**, 166
    implies collapsible, 166
  pure, **179**
  simplex, **19**
  skeleton, **19**
  subcomplex, **19**, 22, 123
    maximal, 210
  suspension, **36**, 37, 251
  vertex, **17**
    dominate, **235**, 236, 239, 240, 255
    link, **166**, 210
    lower link, **210**
    star, **166**, 210
  vertex set, **17**, 19, 33
simplicial map, **234**, 235–237, 239
  inclusion, 234, 240
  null-homotopic, **239**, 250
  retraction, **236**
  strong homotopy equivalence, **240**, 241, 254
stability theorem, 147
strong homotopy type, **237**, 240, 241
  of a point, **237**, 241
strongly homotopic, **239**, 252

topological sorting, **229**
torus, **23**, 30, 47, 58
total ordering, 124, 160, 229
  strict, **138**, 140
triangle inequality, 135, 148

uniform norm, **134**

vector space, 81, 83, 188, 197
  basis element, **84**
  chain, **88**, 89
  dimension, **84**
  direct sum, **96**
  generated by set, **84**, 88
  isomorphic, 199
  isomorphism, **198**, 201, 203
  linear combination, **84**, 193, 195
vertex refinement, **116**

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Discrete Morse theory is a powerful tool combining ideas in both topology and combinatorics. Invented by Robin Forman in the mid 1990s, discrete Morse theory is a combinatorial analogue of Marston Morse's classical Morse theory. Its applications are vast, including applications to topological data analysis, combinatorics, and computer science.

This book, the first one devoted solely to discrete Morse theory, serves as an introduction to the subject. Since the book restricts the study of discrete Morse theory to abstract simplicial complexes, a course in mathematical proof writing is the only prerequisite needed. Topics covered include simplicial complexes, simple homotopy, collapsibility, gradient vector fields, Hasse diagrams, simplicial homology, persistent homology, discrete Morse inequalities, the Morse complex, discrete Morse homology, and strong discrete Morse functions. Students of computer science will also find the book beneficial as it includes topics such as Boolean functions, evasiveness, and has a chapter devoted to some computational aspects of discrete Morse theory. The book is appropriate for a course in discrete Morse theory, a supplemental text to a course in algebraic topology or topological combinatorics, or an independent study.

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