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To Douglas

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Preface

Graphs measure interactions between objects, from follows on Twitter, to transactions between Bitcoin users, and to the flow of energy in a food chain. Whether we think of graphs as abstract collections of dots and lines, or view them as modeling complex interactions in the real world, there is no question that they play an essential role in understanding nature. Graph theory is a robust topic within mathematics, and sets out to formalize the science of interactions. While graphs statically represent interacting systems, we may also use them to model the dynamic interactions within those systems. For example, imagine an invisible evader loose on a graph, leaving behind only breadcrumb clues to their whereabouts. You set out with pursuers of your own, seeking out the evader's location. Would you be able to detect their location? If so, then how many resources are needed for detection, and how fast can that happen? These basic-seeming questions point towards the broad conceptual framework of the field of pursuit-evasion games.

In pursuit-evasion games, a set of pursuers attempts to locate, eliminate, or contain the threat posed by an evader. Pursuit-evasion has a long history, including the early work by Pierre Bouguer in 1732 on a pirate ship escaping a merchant vessel; see the book *Chases and Escapes: The Mathematics of Pursuit and Evasion* [**157**] for more discussion.

Our focus is on discrete versions of pursuit-evasion games played on graphs. The rules, specified from the outset, greatly determine the difficulty of the questions posed on page xv. For example, the evader may be visible, but the pursuers may have limited movement speed, only moving to nearby vertices adjacent to them. Such a paradigm leads to the game of Cops and Robbers, and deep topics like Meyniel's conjecture on the cop number of a graph. Central to pursuit-evasion games is the optimization of certain parameters, whether they are the cop number, burning number, or localization number, for example. Finding the values, bounds, and algorithms to compute these graph parameters leads to fascinating topics intersecting classical graph theory, geometry, and combinatorial designs.

The book aims to provide a friendly invitation to both pursuit-evasion games and graph theory. An effort is made to make the definitions, examples, and proofs clear and readable. Readers may be undergraduate or graduate students who have taken a previous course in discrete mathematics or graph theory. While our focus is on pursuit-evasion games, we will reveal many fascinating topics in graph theory. Along the way, readers will learn about topics such as treewidth, product graphs, planar graphs, and retracts, to name a few. We summarize the graph theory concepts that will be discussed at the beginning of each chapter. Professional mathematicians and theoretical computer scientists who want to learn about the central pursuit-evasion games topics in one place will find the book a valuable resource, and a trove of conjectures and open problems. Applications of pursuit-evasion games range from robotics [**69**], to mobile computing [**98**], and even to programmable matter [**78**]. Those working in the computing sciences and engineering interested in the mathematical foundations of pursuit-evasion games will find the book a helpful resource.

The book begins with an introduction to pursuit-evasion games and graph theory in Chapter 1. More advanced readers may skip this chapter, or use it as a reference for notation. We then focus on one of the most famous pursuit-evasion models in Chapter 2, the game of Cops and Robbers played on graphs. We consider the search number in Chapter 3, where the robber moves infinitely fast and searchers must be employed to capture them. The search number and its variants tie in closely with parameters such as pathwidth and treewidth. Graph burning is discussed in Chapter 4, where the pursuer is missing and an evader attempts to burn the graph as fast as possible. Graph burning models the spread of contagion in a network, ranging from memes to viruses. The Localization game is studied in Chapter 5, where the robber is invisible but detectable by the cops via distance probes giving partial information about their whereabouts. Firefighter is discussed in Chapter 6, and is a process analogous to graph burning, but the pursuers attempt to contain a fire spreading in a graph. Chapter 7 considers the situation where the robber is invisible when they are close or far from the cop. Our final chapter collects several recent variants of pursuit-evasion games, such as Cops and Eternal Robbers, Angel and Devil, and Lazy Cops and Robbers.

For a typical twelve-week course, the core topics are contained in the first four chapters. With full proofs covered and moving at a leisurely pace, this could take about eight to ten weeks. Topics for the remaining weeks could be taken selectively from the remaining chapters. If certain sections and proofs are skipped at the discretion of the instructor, then the entire book could be covered in either a one- or two-term course. The book may also be used as an adjunct in a graph theory course if the instructor would like to introduce topics in pursuit-evasion games.

While we discuss cutting-edge mathematics, we aim to make the book self-contained, understandable, and accessible to a broad mathematical audience. To aid the reader, we have included dozens of figures explaining concepts and proofs. Each chapter contains several exercises that will be helpful to those wishing to expand or polish their skills, and for devising assignments. There are over 170 exercises of varying degrees of difficulty. The book contains a comprehensive bibliography that includes the relevant references for the topics in each chapter. Theorems without proofs are cited, so that an ambitious reader can read those outside the book.

An innovation we include at the end of each chapter after the first are research projects that are of a larger scope than those found in the exercises. Projects include citations, where the reader is directed to further reading. The reader will likely need to consult a small number of other sources for the additional background to complete these projects. These projects may be done for credit at the end of a course, or used as topics towards an undergraduate thesis. Projects are entirely optional and may be skipped without upsetting the flow of the chapters. The projects may also serve as the basis for an NSF Research Experience for Undergraduates (REU) or NSERC Undergraduate Student Research Award (USRA).

Although pursuit-evasion games might be viewed as an emerging topic, its literature is vast, so we have omitted certain advanced directions. We do not consider the robust literature on graph algorithms in pursuit-evasion games. However, along the way, we will discuss a few algorithmic approaches. Further, we focus on deterministic results. A treatment of pursuit-evasion games using the probabilistic method and stochastic models may be found in [**49**]. Lastly, we restrict the majority of our attention to finite, undirected graphs.

I want to thank the reviewers for their excellent feedback on the book, which truly helped the present book into the form it is now. I want to thank my family, friends, students, and co-authors for their generous support. A warm thank you Ina Mette, Marcia Almeida, Erin Donahue, John F. Brady Jr, and the team at the AMS for making this book a reality. A thank you to Melissa Huggan and Trent Marbach for their edits. I would especially like to thank my husband Douglas, who makes my mathematical practice possible.

One last thank you goes to you, the reader. I hope your experience reading this book will be as fun and inspiring as mine was writing it.

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Graphs measure interactions between objects such as friendship links on Twitter, transactions between Bitcoin users, and the flow of energy in a food chain. While graphs statically represent interacting systems, they may also be used to model dynamic interactions. For example, imagine an invisible evader loose on a graph, leaving only behind breadcrumb clues to their whereabouts. You set out with pursuers of your own, seeking out the evader's location. Would you be able to detect their location? If so, then how many resources are needed for detection, and how fast can that happen? These basic-seeming questions point towards the broad conceptual framework of pursuit-evasion games played on graphs. Central to pursuit-evasion games on graphs is the idea of optimizing certain parameters, whether they are the cop number, burning number, or localization number, for example.

Photo by Jason Gordon.

This book would be excellent for a second course in graph theory at the undergraduate or graduate level. It surveys different areas in graph searching and highlights many fascinating topics intersecting classical graph theory, geometry, and combinatorial designs. Each chapter ends with approximately twenty exercises and five larger scale projects.

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