Mathematical Surveys and Monographs

Volume 6

Introduction to the Theory of Algebraic Functions of One Variable

Claude Chevalley

American Mathematical Society

http://dx.doi.org/10.1090/surv/006

Other Titles in This Series

44 **J. Amorós, M. Burger, K. Corlette, D. Kotschick, and D. Toledo,** Fundamental groups of compact Kähler manifolds, 1996

43 **James E. Humphreys,** Conjugacy classes in semisimple algebraic groups, 1995

42 **Ralph Freese, Jaroslav Ježek, and J. B. Nation,** Free lattices, 1995

41 **Hal L. Smith,** Monotone dynamical systems: an introduction to the theory of competitive and cooperative systems, 1995

40.2 **Daniel Gorenstein, Richard Lyons, and Ronald Solomon,** The classification of the finite simple groups, number 2, 1995

40.1 **Daniel Gorenstein, Richard Lyons, and Ronald Solomon,** The classification of the finite simple groups, number 1, 1994

39 **Sigurdur Helgason,** Geometric analysis on symmetric spaces, 1993

38 **Guy David and Stephen Semmes,** Analysis of and on uniformly rectifiable sets, 1993

37 **Leonard Lewin, Editor,** Structural properties of polylogarithms, 1991

36 **John B. Conway,** The theory of subnormal operators, 1991

35 **Shreeram S. Abhyankar,** Algebraic geometry for scientists and engineers, 1990

34 **Victor Isakov,** Inverse source problems, 1990

33 **Vladimir G. Berkovich,** Spectral theory and analytic geometry over non-Archimedean fields, 1990

32 **Howard Jacobowitz,** An introduction to CR structures, 1990

31 **Paul J. Sally, Jr. and David A. Vogan, Jr., Editors,** Representation theory and harmonic analysis on semisimple Lie groups, 1989

30 **Thomas W. Cusick and Mary E. Flahive,** The Markoff and Lagrange spectra, 1989

29 **Alan L. T. Paterson,** Amenability, 1988

28 **Richard Beals, Percy Deift, and Carlos Tomei,** Direct and inverse scattering on the line, 1988

27 **Nathan J. Fine,** Basic hypergeometric series and applications, 1988

26 **Hari Bercovici,** Operator theory and arithmetic in H^∞, 1988

25 **Jack K. Hale,** Asymptotic behavior of dissipative systems, 1988

24 **Lance W. Small, Editor,** Noetherian rings and their applications, 1987

23 **E. H. Rothe,** Introduction to various aspects of degree theory in Banach spaces, 1986

22 **Michael E. Taylor,** Noncommutative harmonic analysis, 1986

21 **Albert Baernstein, David Drasin, Peter Duren, and Albert Marden, Editors,** The Bieberbach conjecture: Proceedings of the symposium on the occasion of the proof, 1986

20 **Kenneth R. Goodearl,** Partially ordered abelian groups with interpolation, 1986

19 **Gregory V. Chudnovsky,** Contributions to the theory of transcendental numbers, 1984

18 **Frank B. Knight,** Essentials of Brownian motion and diffusion, 1981

17 **Le Baron O. Ferguson,** Approximation by polynomials with integral coefficients, 1980

16 **O. Timothy O'Meara,** Symplectic groups, 1978

15 **J. Diestel and J. J. Uhl, Jr.,** Vector measures, 1977

14 **V. Guillemin and S. Sternberg,** Geometric asymptotics, 1977

13 **C. Pearcy, Editor,** Topics in operator theory, 1974

12 **J. R. Isbell,** Uniform spaces, 1964

11 **J. Cronin,** Fixed points and topological degree in nonlinear analysis, 1964

(See the AMS catalog for earlier titles)

MATHEMATICAL
Surveys and Monographs

Volume 6

Introduction to the Theory of Algebraic Functions of One Variable

Claude Chevalley

American Mathematical Society
Providence, Rhode Island

2000 *Mathematics Subject Classification.* Primary 12F05, 14C40, 12J12;
Secondary 13P99.

International Standard Serical Number 0076-5376
International Standard Book Number 0-8218-1506-7
Library of Congress Catalog Card Number 51-4717

Copying and reprinting. Individual readers of this publication, and nonprofit libraries acting for them, are permitted to make fair use of the material, such as to copy a chapter for use in teaching or research. Permission is granted to quote brief passages from this publication in reviews, provided the customary acknowledgment of the source is given.

Republication, systematic copying, or multiple reproduction of any material in this publication (including abstracts) is permitted only under license from the American Mathematical Society. Requests for such permission should be addressed to the Assistant to the Publisher, American Mathematical Society, P.O. Box 6248, Providence, Rhode Island 02940-6248. Requests can also be made by e-mail to `reprint-permission@ams.org`.

© Copyright 1951 by the American Mathematical Society. All rights reserved.
Printed in the United States of America.

The American Mathematical Society retains all rights
except those granted to the United States Government.
♾ The paper used in this book is acid-free and falls within the guidelines
established to ensure permanence and durability.

Visit the AMS home page at URL: `http://www.ams.org/`

12 11 10 9 8 04 03 02 01 00

To

SYLVIE CHEVALLEY

TABLE OF CONTENTS

INTRODUCTION ix
TABLE OF CONTENTS vii
NOTATIONS FREQUENTLY USED xiii
Chapter I. PLACES AND DIVISORS 1
1. Fields of algebraic functions of one variable 1
2. Places 1
3. Places of the field $K\langle x\rangle$ 2
4. Existence of places 6
5. The order function. The degree of a place 9
6. The theorem of independence 11
7. Divisors 13
8. The divisor of a function 15
Chapter II. THE THEOREM OF RIEMANN-ROCH 20
1. The genus 20
2. Fields of genus zero 23
3. Fields of genus one 24
4. Repartitions 25
5. Differentials 28
6. The canonical class 31
7. The local components of a differential 33
8. Fields of elliptic functions 34
Chapter III. THE $\mathfrak{p}$-ADIC COMPLETIONS 39
1. Definition of the $\mathfrak{p}$-adic completion 39
2. Hensel's lemma 43
3. Structure of $\mathfrak{p}$-adic completions 44
4. Generalization of the notion of repartition 46
5. Residues of a differential 48
Chapter IV. EXTENSIONS OF FIELDS OF ALGEBRAIC FUNCTIONS OF ONE VARIABLE 51
1. The relative degree and the ramification index 51
2. The case of normal algebraic extensions 53
3. Integral bases 54
4. Kronecker products of commutative algebras 57
5. Extension of the $\mathfrak{p}$-adic completion 59
6. The Puiseux expansions 64
7. Norm and conorm; trace and cotrace 65
8. The different 69
9. Structure of hyperelliptic fields 74
Chapter V. EXTENSIONS OF THE FIELD OF CONSTANTS 79
1. Separable transcendental extensions 79
2. Relatively algebraically closed subfields 82
3. Commutative algebras 85
4. Definition of the extended field 88
5. The effect on a place 92
6. The effect on the genus 96
Chapter VI. EXACT DIFFERENTIALS 101
1. The differential dx in $K\langle x\rangle$ 101
2. Trace and cotrace of differentials 103
3. The differential dx in an arbitrary field 108

4. Derivations of fields 111
5. Derivations and differentials 116
6. Extension of the notion of cotrace 118
7. Derivations of the field of constants 125
8. Differentials of the second kind 127
Chapter VII. The Riemann Surface 133
1. Definition of the Riemann surface 133
2. Meromorphic functions on the Riemann surface 136
3. On singular homology theory 141
4. Periods of differentials 145
5. The bilinear function $j(\omega, \omega')$ 153
6. Definition of the intersection numbers 156
7. Geometric lemmas 162
8. The homology groups of the Riemann surface 166
9. The theorem of Abel 172
10. Fields of genus one 176
11. The Riemann surface as an analytic manifold 178
12. The bilinear inequalities of Riemann 182
Index 187

INTRODUCTION

An algebraic function y of a complex variable x is a function which satisfies an equation of the form $F(x, y) = 0$, where F is a polynomial with complex coefficients; i.e., y is a root of an algebraic equation whose coefficients are rational functions of x. This very definition exhibits a strong similarity between the notions of algebraic function and algebraic number, the rational functions of x playing a role similar to that played by the rational numbers. On the other hand, the equation $F(x, y) = 0$ may be construed to represent a curve in a plane in which x and y are the coordinates, and this establishes an intimate link between the theory of algebraic functions of one variable and algebraic geometry.

Whoever wants to give an exposition of the theory of algebraic functions of one variable is more or less bound to lay more emphasis either on the algebraico-arithmetic aspect of this branch of mathematics or on its geometric aspect. Both points of view are acceptable and have been in fact held by various mathematicians. The algebraic attitude was first distinctly asserted in the paper *Theorie der algebraischen Funktionen einer Veränderlichen*, by R. Dedekind and H. Weber (Journ. für Math., 92, 1882, pp. 181–290), and inspires the book *Theorie der algebraischen Funktionen einer Variabeln*, by Hensel and Landsberg (Leipzig, 1902). The geometric approach was followed by Max Noether, Clebsch, Gordan, and, after them, by the geometers of the Italian school (cf. in particular the book *Lezione di Geometria algebrica*, by F. Severi, Padova, 1908). Whichever method is adopted, the main results to be established are of course essentially the same; but this common material is made to reflect a different light when treated by differently minded mathematicians. Familiar as we are with the idea that the pair "observed fact—observer" is probably a more real being than the inert fact or theorem by itself, we shall not neglect the diversity of these various angles under which a theory may be photographed. Such a neglect should be particularly avoided in the case of the theory of algebraic functions, as either mode of approach seems liable to provoke strong emotional reactions in mathematical minds, ranging from devout enthusiasm to unconditional rejection. However, this does not mean that the ideal should consist in a mixture or synthesis of the two attitudes in the writing of any one book: the only result of trying to obtain two interesting photographs of the same object on the same plate is a blurred and dull image. Thus, without attacking in any way the validity *per se* of the geometric approach, we have not tried to hide our partiality to the algebraic attitude, which has been ours in writing this book.

The main difference between the present treatment of the theory and the one to be found in Dedekind-Weber or in Hensel-Landsberg lies in the fact that the constants of the fields of algebraic functions to be considered are not necessarily the complex numbers, but the elements of a completely arbitrary field. There

are several reasons which make such a generalization necessary. First, the analogy between algebraic functions and algebraic numbers becomes even closer if one considers algebraic functions over finite fields of constants. In that case, on the one hand class field theory has been extended to the case of fields of functions, and, on the other hand, the transcendental theory (zeta function, L-series) may also be generalized (cf. the paper of F. K. Schmidt, *Analytische Zahlentheorie in Körpern der Charakteristik p*, Math. Zeits., 33, 1931). Moreover, A. Weil has succeeded in proving the Riemann hypothesis for fields of algebraic functions over finite fields, thereby throwing an entirely new light on the classical, i.e., number theoretic, case (cf. the book of A. Weil, *Sur les courbes algébriques et les variétés qui s'en déduisent*, Paris, Hermann, 1948; this book contains an exposition of the theory from a geometric point of view, although this point of view is rather different from that of the Italian geometers). Secondly, if S is an algebraic surface, and R the field of rational functions on S, then R is a field of algebraic functions of one variable over $K\langle x\rangle$, where K is the basic field and x a non constant element of R. E. Picard, among others, has very successfully used the method of investigation of S which amounts to studying the relationship between R and various fields of the form $K\langle x\rangle$ (cf. E. Picard and G. Simart, *Théorie des fonctions algébriques de deux variables independantes*, Paris, Gauthier-Villars, 1897). Now, even when K is the field of complex numbers, $K\langle x\rangle$ is not algebraically closed, which makes it necessary to have a theory of fields of algebraic functions of one variable over fields which are not algebraically closed.

The theory of algebraic functions of one variable over non algebraically closed fields of arbitrary characteristic has been first developed by H. Hasse, who defined for these fields the notion of a differential (H. Hasse, *Theorie der Differentiale in algebraischen Funktionenkörpern mit volkommenen Konstantenkörper*, Journ. für Math., 172, 1934, pp. 55–64), and by F. K. Schmidt, who proved the Riemann-Roch theorem (F. K. Schmidt, *Zur arithmetischen Theorie der algebraischen Funktionen*, I, Math. Zeits., 41, 1936, p. 415). In this book, we have used the definition of differentials and the proof of the Riemann-Roch theorem which were given by A. Weil (A. Weil, *Zur algebraischen Theorie der algebraischen Funktionen*, Journ. für Math., 179, 1938, pp. 129–133).

As for contents, we have included only the elementary part of the theory, leaving out the more advanced parts such as class field theory or the theory of correspondences. However, we have been guided by the desire of furnishing a suitable base of knowledge for the study of these more advanced chapters. This is why we have placed much emphasis on the theory of extensions of fields of algebraic functions of one variable, and in particular of those extensions which are obtained by adjoining new constants, which may even be transcendental over the field of constants of the original field of functions. That the consideration of such extensions is desirable is evidenced by the paper of M. Deuring, *Arithmetische Theorie der Korrespondenzen algebraischen Funktionenkörper*, Journ. für Math., 177, 1937. The theory of differentials of the second kind has been given only in the case where the field under consideration is of characteristic 0. The

reason for this restriction is that it is not yet clear what the "good" definition of the notion should be in the general case: should one demand only that the residues be all zero, or should one insist that the differential may be approximated as closely as one wants at any given place by exact differentials (or a suitable generalization of these)? Here is a net of problems which, it seems, would deserve some original research. The last chapter of the book is concerned with the theory of fields of algebraic functions of one variable over the field of complex numbers and their Riemann surfaces. The scissor and glue method of approach to the idea of a Riemann surface has been replaced by a more abstract definition, inspired by the one given by H. Weyl in his book on Riemann surfaces, which does not necessitate the artificial selecting of a particular generation of the field by means of an independant variable and a function of this variable. We have also avoided the cumbersome decomposition of the Riemann surface into triangles, this by making use of the singular homology theory, as developed by S. Eilenberg.

I have been greatly helped in the writing of this book by frequent conversations with E. Artin and O. Goldman; I wish to thank both of them sincerely for their valuable contribution in the form of advice and suggestions.

NOTATIONS FREQUENTLY USED

Con $_{R/S}$: conorm from R to S (IV, 7).
Cosp $_{R/S}$: cotrace from R to S (for repartitions, IV, 7; for differentials, VI, 2 and VI, 6).
$d(\mathfrak{a})$: degree of a divisor $\mathfrak{a}$ (I, 7).
$\mathfrak{b}(x)$: divisor of an element x (I, 8).
$\mathfrak{b}(\omega)$: divisor of a differential ω (II, 6).
$\delta(a)$: dimension of the space of differentials which are multiple of a divisor $\mathfrak{a}$ (II, 5).
∂: boundary (VII, 3).
$H_n(X, Y)$: n-dimensional homology group of X modulo Y (VII, 3).
$i(\gamma, \gamma')$: intersection numbers of the 1-chains γ and γ' (VII, 6).
$j(\omega, \omega')$: (VII, 5).
$K\langle\cdots\rangle$: field obtained by adjunction to the field K of the element or elements or set of elements whose symbols are between the sign $\langle$ and the sign $\rangle$; special meaning for fields of algebraic functions of one variable defined in V, 4.
$l(\mathfrak{a})$: dimension of the space of elements which are multiples of a divisor $\mathfrak{a}$ (II, 1).
$\nu_{\mathfrak{p}}$: order function at a place $\mathfrak{p}$ (for elements, I, 5 and III, 1; for repartitions, II, 4; for differentials, II, 6).
$N_{S/R}$: Norm from S to R (IV, 7).
$N_{S/R}^{\mathfrak{P}}$: (IV, 5).
$\omega^{\mathfrak{p}}$: $\mathfrak{p}$-component of a differential ω (II, 7).
res $_{\mathfrak{p}}\omega$: residue of a differential ω at a place $\mathfrak{p}$ (III, 5).
$\mathrm{Sp}_{S/R}$: Trace from S to R (for repartitions, IV, 7; for differentials, VI, 2).
$\mathrm{Sp}_{S/R}^{\mathfrak{P}}$: (IV, 5).
$|\gamma|$: set of points of a chain γ (VII, 3).

INDEX

Roman numbers refer to chapters, arabic numbers refer to sections.

Abel (Theorem of) VII, 9
Above V, 1
Algebraic Functions of one Variable (field of) I, 1
Analytic (function) VII, 1
Analytic (differential form) VII, 11

Barycenter VII, 3
Barycentric Subdivision VII, 3
Below V, 1
Bilinear Equalities VII, 8
Bilinear Inequalities VII, 12
Bounding (cycle) VII, 3
Boundary (of a chain) VII, 3
Boundary Homomorphism VII, 3

Canonical Base (for homology groups) VII, 8
Canonical Class II, 6
Cauchy Sequence III, 1
Cauchy (Theorem of) VII, 4
Circumference (of a disc) VII, 1
Chain VII, 3
Classes of Divisors I, 8
Completion ($\mathfrak{p}$-adic) III, 1
Components (of a repartition) II, 4
$\mathfrak{p}$-Component (of a differential) II, 7 and III, 4
Congruence (for elements) I, 7
Congruence (for repartitions) II, 4 and III, 4
Congruence (for differentials) II, 5
Conorm IV, 7
Constants I, 1
Convergent Sequences III, 1
Cotrace (for repartitions) IV, 7
Cotrace (for differentials) VI, 2 and VI, 6
Cycle VII, 3

Deformation Retract VII, 3
Degree (of a class) I, 8
Degree (of a divisor) I, 7
Degree (of a place) I, 5
Degree (relative, of a place) IV, 1
Derivation VI, 4
Derivation with respect to x, VI, 4
Different IV, 8
Differentiable (simplex) VII, 11
Differential II, 5
Differential Exponent IV, 8
Differential Form VII, 11
Disc VII, 1
(To) Divide (for divisors) I, 7
Divisor I, 7
Divisor (of an element) I, 8
Divisor (of poles) I, 8
Divisor (of zeros) I, 8

Elliptic Functions II, 8
Equivalent (divisors) I, 8
Exact Differential VI, 3
Exactness (Theorem) VII, 3
Excision (Theorem) VII, 3
Expansion (in a $\mathfrak{p}$-adic completion) III, 3
Exponent (of a place in a divisor) I, 7

Field of Constants I, 1
Fixed (place) IV, 1

Generator (positive) VII, 4
Genus II, 1

Highest Common Divisor I, 7
Holomorphic (differential) VII, 4
Holomorphic (function) VII, 2
Homologous VII, 3
Homology Class VII, 3
Homology Group VII, 3
Homology Sequence VII, 3
Hyperelliptic IV, 9

Integral (of a differential) VII, 4 and VII, 11
Integral (at a place) I, 2
Integral Base IV, 3
Integral Divisor I, 7
Integral (in a $\mathfrak{p}$-adic completion) III, 1
Intersection Number VII, 6

Kind (first) II, 5
Kind (second) III, 5

Least Common Multiple I, 7
Local Components (of a differential) II, 7 and III, 4
Logarithmic (periods) VII, 4
Luroth (Theorem of) VI, 2

Meromorphic (function) VII, 2
Meromorphic (differential) VII, 4
Multiple (for divisors) I, 7
Multiple (for differentials) II, 5

Norm (of a divisor) IV, 7

Order (of a differential) II, 6
Order (of an element) I, 3
Order (of a pole or a zero) I, 3
Order (of a repartition) II, 4
Orientation (of the circumference of a disc) VII, 4
Orientation (of the Riemann surface) VII, 11

Periods (of a differential) VII, 4
Periods (of an elliptic function) II, 8
Place I, 2
Pole I, 3
Pole (of a differential) II, 6
Puiseux Expansion IV, 6

Radical (of an algebra) V, 3
Radical Extension (of a field) V, 1
Radius (of a disc) VII, 1
Ramification Index IV, 1
Ramified IV, 1
Relatively Algebraically Closed V, 2
Repartition II, 4 and III, 4
Representatives (system of) III, 3
Residue III, 5 and VII, 4
Residue Field (of a place) I, 2
Riemann (Theorem of) II, 1
Riemann-Roch (Theorem of) II, 5 and II, 6
Riemann Surface VII, 1
Ring (of a place) I, 2

Semi-Simple V, 3
Separable V, 1
Separably Generated V, 1
Separating Variable V, 1
Set of Points (of a chain) VII, 3
Simplex VII, 3

Trace (of a repartition) IV, 7
Trace (of a differential) VI, 2

Uniformizing Variable I, 3
Unit Divisor I, 7
Unramified IV, 1

V-Ring I, 2
Value (taken at a place) I, 3
Variable Place IV, 1

Zero (of an element) I, 3
Zero (of a differential) II, 6

This classical book, written by a famous French mathematician in the early 1950s, presents an approach to algebraic geometry of curves treated as the theory of algebraic functions on the curve. Among other advantages of such an approach, it allowed the author to consider curves over an arbitrary ground field. Among topics discussed in the book are the theory of divisors on a curve, the Riemann–Roch theorem, p-adic completion, extensions of the fields of functions (covering theory) and of the fields of constants, and the theory of differentials on a curve. The last chapter, which is more analytic in flavor, treats the theory of Riemann surfaces.

Prerequisites for reading are minimal and include only an advanced undergraduate algebra course.

ISBN 0-8218-1506-7

9 780821 815069

SURV/6

AMS on the Web
www.ams.org