Mathematical Surveys and Monographs

Number 7
Part 1

The Algebraic Theory of Semigroups

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http://dx.doi.org/10.1090/surv/007.1

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2000 *Mathematics Subject Classification.* Primary 20-XX.

International Standard Serial Number 0076-5376

International Standard Book Number 0-8218-0271-2

Library of Congress Catalog Card Number: 61-15686

Second Edition, 1964
Reprinted with corrections, 1977.

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10 9 8 7 6 5 06 05 04 03 02 01

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PREFACE

So far as we know, the term "semigroup" first appeared in mathematical literature on page 8 of J.-A. de Séguier's book, *Élements de la Théorie des Groupes Abstraits* (Paris, 1904), and the first paper about semigroups was a brief one by L. E. Dickson in 1905. But the theory really began in 1928 with the publication of a paper of fundamental importance by A. K. Suschkewitsch. In current terminology, he showed that every finite semigroup contains a "kernel" (a simple ideal), and he completely determined the structure of finite simple semigroups. A brief account of this paper is given in Appendix A.

Unfortunately, this result of Suschkewitsch is not in a readily usable form. This defect was removed by D. Rees in 1940 with the introduction of the notion of a matrix over a group with zero, and, moreover, the domain of validity was extended to infinite simple semigroups containing primitive idempotents. The Rees Theorem is seen to be the analogue of Wedderburn's Theorem on simple algebras, and it has had a dominating influence on the later development of the theory of semigroups. Since 1940, the number of papers appearing each year has grown fairly steadily to a little more than thirty on the average.

It is in response to this developing interest that this book has been written. Only one book has so far been published which deals predominantly with the algebraic theory of semigroups, namely one by Suschkewitsch, *The Theory of Generalized Groups* (Kharkow, 1937); this is in Russian, and is now out of print. A chapter of R. H. Bruck's *A Survey of Binary Systems* (Ergebnisse der Math., Berlin, 1958) is devoted to semigroups. There is, of course, E. Hille's book, *Functional Analysis and Semi-groups* (Amer. Math. Soc. Colloq. Publ., 1948), and the 1957 revision thereof by Hille and R. S. Phillips; but this deals with the analytic theory of semigroups and its application to analysis. The time seems ripe for a systematic exposition of the algebraic theory. (Since the above words were written, there has appeared such an exposition, in Russian: *Semigroups*, by E. S. Lyapin, Moscow, 1960.)

The chief difficulty with such an exposition is that the literature is scattered over extremely diverse topics. We have met this situation by confining ourselves to a portion of the existing theory which has proved to be capable of a well-knit and coherent development. All of Volume 1 and the first half of Volume 2 center around the structure of semigroups of certain types (such as simple semigroups, inverse semigroups, unions of groups, semigroups with minimal conditions, etc.) and their representation by mappings or by matrices. The second half of Volume 2 treats the theory of congruences and the embedding of semigroups in groups, including a modest account of the active French school of semigroups (which they call "demi-groupes") founded in 1941 by P. Dubreil.

In order to keep our book within reasonable bounds, moreover, we have construed the term "algebraic" in a somewhat narrow sense: the semigroups under consideration are not endowed with any further structure. This has the effect of excluding not only topological semigroups, but ordered semigroups as well. Fortunately, a good account of lattice-ordered semigroups and groups is to be found in G. Birkhoff's *Lattice Theory* (Amer. Math. Soc. Colloq. Publ., 1940; revised 1948). It also excludes P. Lorenzen's generalization of multiplicative ideal theory (see, for example, §5 of W. Krull's *Idealtheorie*, Ergebnisse der Math., Berlin, 1935) to any commutative semigroup S with cancellation, in which S (or its quotient group) is endowed with a family of subsets called r-ideals, satisfying certain conditions analogous to those for closed sets in topology.

The book aims at being largely self-contained, but it is assumed that the reader has some familiarity with sets, mappings, groups, and lattices. The material on these topics in an introductory text such as Birkhoff and MacLane, *A Survey of Modern Algebra* (New York, Revised Edition, 1953) should suffice. Only in Chapter 5 will more preliminary knowledge be required, and even there the classical definitions and theorems on the matrix representations of algebras and groups are summarized.

We have included a number of exercises at the end of each section. These are intended to illuminate and supplement the text, and to call attention to papers not cited in the text. They can all be solved by applying the methods and results of the text, and often more simply than in the paper cited.

Each volume has a separate bibliography listing those papers referred to in that volume. No attempt has been made to list those papers on semigroups to which no reference has been made in the text or exercises. The combined bibliography contains about half of the papers which have appeared in the (strictly) algebraic theory of semigroups. (The bibliography in Lyapin's book appears to be complete.) Whenever possible, the reference to the review of each paper in the Mathematical Reviews has been given, (MR x, y) denoting page y of volume x. English translations of Russian titles are those given in the Mathematical Reviews.

The material in Volume 1 (more or less) was presented in a second-year graduate course at Tulane University during the academic year 1958–1959, and this volume has benefited greatly from the students' criticisms. The authors would also like to express their gratitude to Professors A. D. Wallace, D. D. Miller, and P. F. Conrad for many useful suggestions; and, above all, to Dr. W. D. Munn for his very valuable criticisms, especially of Chapter 5, and for his permission to draw on unpublished material from his dissertation (Cambridge University, 1955) for Sections 3.4 and 4.5. We deeply appreciate the thoughtful kindness of Professor Š. Schwarz and the Central Library of the Slovakian Academy of Sciences for sending us (unsolicited) a photostat print of Suschkewitsch's book. Our thanks go also to Mrs. Anna L. McGinity for typing all of Volume 1. Finally, the authors gratefully acknowledge partial support for this work from the National Science Foundation (U.S.A.).

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July 28, 1960

THE TULANE UNIVERSITY OF LOUISIANA

THE ROYAL MILITARY COLLEGE OF SCIENCE

NOTATION USED IN VOLUME ONE

Square brackets are used for alternative readings and for reference to the bibliography.

Let A and B be sets.

$A \subset B$ (or $B \supset A$) means A is properly contained in B.

$A \subseteq B$ (or $B \supseteq A$) means $A \subset B$ or $A = B$.

$A \setminus B$ means the set of elements of A which are not in B.

$A \times B$ means the set of all ordered pairs (a, b) with a in A, b in B.

The signs $\cup$ and $\cap$ are reserved for union and intersection, respectively, of sets and relations. The signs $\vee$ and $\wedge$ will be used for join and meet in [semi]lattices.

$|A|$ means the cardinal number of the set A.

The sign $\circ$ is used for composition of relations (§1.4), but is usually omitted for composition of mappings.

$\square$ denotes the empty set, mapping, or relation.

ι [ι_A] denotes the identity mapping or relation [on the set A].

If ϕ is a mapping whose domain includes A, then $\phi|A$ means ϕ restricted to A.

$\{a_1, \cdots, a_n\}$ means the set whose members are $a_1, \cdots, a_n$. Braces are sometimes omitted on single elements, for example $A \cup b$ instead of $A \cup \{b\}$.

If $P(x)$ is a proposition for each element x of a set X, then the set of all x in X for which $P(x)$ is true is denoted by either $\{x \in X : P(x)\}$ or $\{x : P(x), x \in X\}$.

If $M(x)$ is a set for each x in a set X, then the union of all the sets $M(x)$ with x in X is denoted by either $\bigcup_{x \in X} M(x)$ or $\bigcup\{M(x) : x \in X\}$.

If $F(x)$ is a member of a set C for each x in a set X, then the subset of C consisting of all $F(x)$ with x in X is denoted by $\{F(x) : x \in X\}$. If $X = A \times B$, we may write $\{F(a, b) : a \in A, b \in B\}$ instead of $\{F(a, b) : (a, b) \in A \times B\}$.

If A is a subset of a semigroup S, then $\langle A \rangle$ denotes the subsemigroup of S generated by A. If S is a group, then the subgroup of S generated by A is $\langle A \cup A^{-1} \rangle$, where $A^{-1} = \{a^{-1} : a \in A\}$.

If A and B are subsets of a semigroup S, then AB means $\{ab : a \in A, b \in B\}$.

S^1 [S^0] means the semigroup $S \cup 1$ [$S \cup 0$] arising from a semigroup S by the adjunction of an identity element 1 [a zero element 0], unless S already has an identity [has a zero, and $|S| > 1$], in which case $S^1 = S$ [$S^0 = S$]. (§1.1)

$a|b$ means "a divides b", that is, $b \in aS^1$, where a and b are elements of a commutative semigroup S. (§4.3)

ρ_a [λ_a] denotes the inner right [left] translation $x \to xa$ [$x \to ax$] of a semigroup S, where a is a fixed element of S. (§1.3)

If ρ is an equivalence relation on a set X, and if $(a, b) \in \rho$, then we write $a \rho b$ and say that a and b are *ρ-equivalent*, and that they belong to the same *ρ-class*.

If ρ is a congruence relation on a semigroup S, then S/ρ denotes the factor semigroup of S modulo ρ, and $\rho^\natural$ denotes the natural mapping of S upon S/ρ. (§1.5) S/J denotes the Rees factor semigroup of S modulo an ideal J.

Let S be a semigroup, and let $a \in S$. (Following from §2.1)

$L(a)$ denotes the principal left ideal S^1a.

$R(a)$ denotes the principal right ideal aS^1.

$J(a)$ denotes the principal two-sided ideal S^1aS^1.

$\mathscr{L}$ means $\{(a, b) \in S \times S : L(a) = L(b)\}$.

$\mathscr{R}$ means $\{(a, b) \in S \times S : R(a) = R(b)\}$.

$\mathscr{J}$ means $\{(a, b) \in S \times S : J(a) = J(b)\}$.

$\mathscr{H}$ means $\mathscr{L} \cap \mathscr{R}$.

$\mathscr{D}$ means $\mathscr{L} \circ \mathscr{R}$ $(= \mathscr{R} \circ \mathscr{L})$.

L_a, R_a, J_a, H_a, D_a mean respectively the $\mathscr{L}$, $\mathscr{R}$, $\mathscr{J}$, $\mathscr{H}$, $\mathscr{D}$-class containing a.

$I(a)$ means $J(a) \setminus J_a$. (It is empty or an ideal of S.)

$J(a)/I(a)$ is the principal factor of S corresponding to a. (§2.6)

$\mathscr{T}_X$ means the semigroup of all transformations of a set X. (§1.1)

$\mathscr{G}_X$ means the group of all permutations of a set X. (§1.1)

$\mathscr{I}_X$ means the symmetric inverse semigroup on a set X. (§1.9)

$\mathscr{B}_X$ means the semigroup of all binary relations on X. (§1.4)

$\mathscr{F}_X$ means the free semigroup on X. (§1.12)

$\mathscr{F}\mathscr{G}_X$ means the free group on X. (§1.12)

$\mathscr{C}$ means the bicyclic semigroup. (§1.12)

$\mathscr{M}^0(G; I, \Lambda; P)$ means the Rees $I \times \Lambda$ matrix semigroup over the group with zero G^0, with $\Lambda \times I$ sandwich matrix P.

$\mathscr{M}(G; I, \Lambda; P)$ means the Rees $I \times \Lambda$ matrix semigroup without zero over the group G, with $\Lambda \times I$ sandwich matrix P. (§3.1)

$\mathscr{L}\mathscr{T}(V)$ means the algebra of all linear transformations of a vector space V, or the multiplicative semigroup thereof. (§§2.2, 5.1)

$(\mathfrak{A})_n$ means the algebra of all $n \times n$ matrices over an algebra $\mathfrak{A}$. (§5.1)

$\Phi[S]$ means the algebra of a semigroup S over a field Φ. (§5.2)

$\cong$ means "isomorphic". (§1.3)

$\sim$ means "homomorphic", and sometimes "equivalent". (§1.3)

$\oplus$ is used for the direct sum of algebras, vector spaces, and representations. (§5.1)

The $n \times n$ identity matrix is denoted by:

I_n when it is over a field (§§5.2, 5.3),

U_n when it is over an algebra with identity element u (§5.1),

Δ_n when it is over a group with zero (§3.1).

Γ^m denotes the representation of $(\mathfrak{A})_m$ *associated* with the representation Γ of $\mathfrak{A}$. (§5.1)

M_L, M_R, M_J denote the minimal conditions on the set of principal left, right, two-sided ideals, respectively, of a semigroup. (A partially ordered set P is said *to satisfy the minimal condition* if each non-empty subset A of P contains at least one *minimal element*, i.e., an element x of A such that $y < x$ $(y \in P)$ implies $y \notin A$.) (§§5.3, 5.4)

APPENDIX A

A BRIEF ACCOUNT OF THE 1928 PAPER OF SUSCHKEWITSCH

Starting with an arbitrary finite semigroup S, he considers subsets of S of the form Sa having the least possible number of elements. These are evidently just the minimal left ideals of S, and we shall use the current terminology. He shows that each minimal left ideal of S is a left group, and (without using the expression "direct product") shows that it is the direct product of a group and a left zero semigroup. This is, of course, our Theorem 1.27 for finite semigroups. Moreover, any two minimal left ideals of S are isomorphic, and in particular are unions of the same number r of isomorphic groups.

He calls the union K of all the minimal left ideals of S the kernel ("Kerngruppe") of S. If s is the number of distinct minimal left ideals of S, then K is the union of rs mutually isomorphic groups. He shows that these can be arranged in a rectangular array as follows:

K	L_1	L_2	$\cdots L_s$
R_1	H_{11}	$H_{12}\cdots$	H_{1s}
R_2	H_{21}	$H_{22}\cdots$	H_{2s}
$\vdots$	$\vdots$	$\vdots$	$\vdots$
R_r	H_{r1}	$H_{r2}\cdots$	H_{rs}

(This is the source of our "eggbox picture", described in §2.1.) The union of the groups $H_{1\lambda},\cdots,H_{r\lambda}$ in the λth column is the minimal left ideal L_λ $(\lambda = 1,\cdots,s)$. Let $e_{i\lambda}$ be the identity element of the group $H_{i\lambda}$. He shows that the $H_{i\lambda}$ can be arranged so that each $e_{i\lambda}$ acts as a left identity on all the $H_{i\lambda}$ in the same row. When this is done, the union R_i of the groups $H_{i1},\cdots,H_{is}$ in the ith row $(i = 1,\cdots,r)$ is a minimal right ideal of S. Moreover, every minimal right ideal of S is one of the R_i. Hence:

Every finite semigroup has a kernel K which is the union of all the minimal left ideals of S and also of all the minimal right ideals of S. The intersection of a minimal left ideal and a minimal right ideal is a (maximal) subgroup of S. These results were subsequently extended to infinite semigroups having minimal left ideals and minimal right ideals (Exercise 13 of §2.7). We know now that it is simpler to introduce the L's and R's independently, and the H's as intersections thereof.

Suschkewitsch goes on to show by quite an involved argument that K *is uniquely determined by* (1) *the abstract group* H *to which each* $H_{i\lambda}$ *is isomorphic,* (2) *the numbers* r *and* s, *and* (3) *the* $(r-1)(s-1)$ *products* $e_{11}e_{i\lambda}$ $(i = 2, \cdots, r;$ $\lambda = 2, \cdots, s)$. He shows conversely that *the group* H, *the numbers* r *and* s, *and the* $e_{11}e_{i\lambda}$ *can be given arbitrarily.* This is done by means of transformations of a finite set. Thus he succeeds in determining the structure of the most general finite simple semigroup, but yet not (as came later with the Rees Theorem) in a readily usable form.

This theory also occupies the greater part of Chapter 3 of Suschkewitsch's book [1937].

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For pure symbols, see the list of notation on page xiii.

ISBN 978-0-8218-0271-7

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