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Discontinuous Groups and Automorphic Functions

Joseph Lehner

American Mathematical Society

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DISCONTINUOUS GROUPS AND AUTOMORPHIC FUNCTIONS

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Providence, Rhode Island

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TO

MARY AND JANET

THE SYSTEM OF REFERENCING USED IN THIS BOOK

The book is divided into chapters, sections, and subsections. A reference VIII. 5C is to Subsection C of Section 5 of Chapter VIII. However, if this reference occurs in Chapter VIII, it is abbreviated to 5C. Lemmas, theorems, and corollaries are referred to in the same way: Theorem 3A is the theorem appearing in Section 3, Subsection A of the chapter in which the reference occurs.

The numbered equations run consecutively within each chapter. Equations marked (*), (†), etc., are consecutive within each *subsection*.

An author's name followed by a boldface number enclosed in square brackets—Poincaré [**1**]—is a reference to the LIST OF REFERENCES appearing at the end of the book.

PREFACE

Much has been written on the theory of discontinuous groups and automorphic functions since 1880, when the subject received its first formulation. The purpose of this book is to bring together in one place both the classical and modern aspects of the theory, and to present them clearly and in a modern language and notation. The emphasis in this book is on the fundamental parts of the subject.

In writing the book I had in mind three classes of readers: graduate students approaching the subject for the first time, mature mathematicians who wish to gain some knowledge and understanding of automorphic function theory, and experts. For the first class Chapter II was included; with this chapter the book is almost self-contained. The first chapter, an historical account, is in my opinion essential to an understanding of the whole theory; I hope, in any event, that it will make interesting reading. Chapters III to VII develop the basic and more classical theory; Chapters VIII to XI, the more modern developments. In Chapter VI a connection is made with the theory of Riemann surfaces. A section of NOTES at the end of the book provides additional material and textual comment and criticism.

The book is essentially restricted to functions of a single complex variable. In the last chapter a sketch of the theory for several variables is presented. I hope that this chapter will excite the enthusiasm of readers and make them want to consult the large and rapidly growing literature in this relatively new field, which already has so many accomplishments to its credit. At the present time there seems to be no comprehensive account of this subject.

On the next page I shall detail the many debts incurred in the writing of this book. There is one debt, however, that must be mentioned specially; it is the one I owe to my teacher and friend, Hans Rademacher. I first learned about the subject of this book from him, and his many-sided and continuing interest in mathematics will always be a strong incentive for me to go on.

April, 1963

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In the summer of 1958 the American Mathematical Society concluded an agreement with the U.S. Air Force Office of Scientific Research under which the latter would support financially the preparation of expository works in mathematics. The Society appointed a committee to select suitable manuscripts (Bers, Bochner, Gleason, McShane (chairman), Montgomery). I am grateful to the committee for favoring me with their choice. First drafts of most of the chapters were written under this contract during 1959–60.

The manuscript was completed and considerably improved in 1961–62 when I was a member of the Number Theory Institute at the University of Pennsylvania. There I presented a good deal of the book in seminar. I am greatly indebted to the members of the seminar (Bateman, Chowla, Grosswald, Koppelman, Morris Newman, Pisot, Rademacher, Schoenberg, Straus) for their interest and encouragement, for the many errors they found and the many excellent suggestions they made.

My thanks are due to Michigan State University for granting me leave in 1959–60 and 1961–62.

Many persons helped with suggestions and advice, references, reprints, and critical reading of portions of the manuscript. Besides the above this list includes: Adney, Bers, K. Frederick, Leon Greenberg, Gunning, N. Hills, L. M. Kelly, Larcher, Magnus, L. J. Savage, Seidel, T. Yen, and my student J. R. Smart. To all of them I extend grateful thanks.

It will be clear to all who know the field that much material was borrowed from existing works, usually without acknowledgement. The principal works consulted were:

Ahlfors-Sario [**1**], Fatou [**1**], Ford [**1**], Fricke [**1**], Fricke-Klein [**1**, **2**, **3**, **4**], Gunning [**1**], Magnus [**1**], Siegel [**4**, **7**], Springer [**1**]. A. M. Macbeath, *Discontinuous groups and birational transformations,* Lecture Notes, Queens College, Dundee, 1961.

The manuscript was typed by Mrs. L. Steadman, by my wife, and by my daughter. My wife, a former proofreader, read galley and page proof and made the index. They all did a fine job under trying circumstances.

Finally, I must thank Dr. John H. Curtiss and Dr. Gordon L. Walker, former and present Executive Directors of the American Mathematical Society, and Miss Margaret Kellar, who supervised the Air Force contract, and Miss Ellen E. Swanson, Chief Editor, and Mrs. Dorothy A. Honig, Editorial Assistant, who handled with painstaking care the many details involved in seeing the book through the press.

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CHAPTER II

(p. 56) Replace p. 56, ll. 1^*-8^* and p. 57, ll. $1-4$ by the following: Define the *normal closure* of a subset A of a group H to be the smallest normal subgroup (intersection of all normal subgroups) of H that contains A. Choose in N a subset $\mathfrak{N}$ such that N is the normal closure of $\mathfrak{N}$ in F. (For example we could take $\mathfrak{N} = N$.) The set of relations corresponding to the words in $\mathfrak{N}$ is called a *system of defining relations* $\mathfrak{S}$ in G. Each relation in G is a *consequence* of the defining relations in the sense that the left member of the relation is a product of conjugates of powers of the left members of elements of $\mathfrak{S}$. Obviously a system of defining relations is not unique, since any set of relations containing $\mathfrak{S}$ is still a system of defining relations.

More special is the following concept. A set of relations $\mathfrak{R}$ is called a *basis for the relations in G* if

(1) every relation in G is a consequence of relations in $\mathfrak{R}$, and

(2) no relation in $\mathfrak{R}$ is a consequence of the remaining relations in $\mathfrak{R}$.

CHAPTER III

1 (p. 97). The following equation is valid only for $a_i c_i \neq 0$. (Note that $b_i/d_i = U_i(0)$ is finite.) Suppose $a_i = 0$ on an infinite set of indices $\{i\}$; then $c_i \neq 0$ since U_i is nonsingular. We write

$$U_i(z) = \frac{b_i}{d_i}\frac{d_i/c_i}{z + d_i/c_i},$$

and the proof goes through as before. Similarly, if $c_i = 0$.

2 (p. 98). If there is no exceptional value w_0, the set of values $\{T(N_2) \mid T \in \Omega_1\}$ may omit w, but in that case it must assume each $w_1 \neq w$. Then $\{T(N_3) \mid T \in \Omega_2\}$ cannot omit w_1, etc.

3 (Ex. 5, p. 102). If Γ is a discrete group of 2×2 matrices with real

entries, the normalizer of every element in Γ is cyclic. Cf. Hall [1], p. 14. [T belongs to the normalizer of $A \in \Gamma$ if and only if T and A commute. By II. 9F, Theorem 2, either T and A have the same fixed point set or they are both elliptic of period two. Use III. Theorem 2E and the discreteness of Γ; the case in which T and A are both elliptic of period two is easily handled by transforming the fixed points of A to 0 and ∞ and noticing that the fixed points of T and A lie on one straight line and separate each other.]

CHAPTER IV

4 (p. 119). It is not necessary to use isometric circles for the construction of the Schottky groups. It is sufficient that T_j be any linear transformation mapping the interior of C_{2j-1} on the exterior of C_{2j}. The free product of $T_1, \cdots, T_n$ is discontinuous and has the other properties mentioned.

5 (p. 126). *If α is an ordinary point and an elliptic fixed point, Γ_α is finite cyclic.* The subgroup Γ_α contains only elliptic elements and is therefore finite (III. Theorem 2A). Two elliptic elements which have one fixed point in common have the other fixed point in common also, otherwise their commutator would be parabolic. The multipliers of the elliptic elements are roots of unity and finite in number, and the result follows.

5a (p. 129). Each point of a parabolic cycle is called a *parabolic vertex.*

6 (p. 138). This argument is not correct, for the reasoning of 5B cannot be applied to Γ_∞, which has ∞ as a fixed point. However, $\Gamma' = A\Gamma_\infty A^{-1}$ does not have ∞ as a fixed point, and every element of Γ' fixes $A\infty$. Therefore, by the argument of the text, R', the standard fundamental region of Γ', has sides. Hence so does $R_\infty = A^{-1}R'$.

7 (p. 139). By the use of the previous methods the reader may prove Theorems 1F, 1G, 1H, 4C, 4E, 4G for the fundamental region R defined in this subsection. Also Theorem 3B, modified as follows: replace 1) by 1'): Each side is an arc of an isometric circle, the full circle, or a side of R_∞; replace 4) by 4'): s_i and s_i' have equal euclidean length if they are not sides of R_∞.

8 (p. 148). This argument breaks down if N_0 is not relatively compact, for then diameter N_0 is infinite. However, by 7B, Lemma 1, we can assert that K meets only a finite number of λ_i, hence only a finite number of N_i.

CHAPTER V

9 (p. 158). We can prove by reasoning similar to that of the above paragraph that $\hat{F}$ does not have an (isolated) essential singularity at $t = 0$. On

one sequence we have $F(z) \to 0$. Let $t\hat{F}(t) \to 1$ on some sequence; on the corresponding sequence in the z-plane $|F(z)| \to \infty$. Thus F has no unique limit.

10 (p. 172). This argument is not quite adequate. That

$$\int_{C_j'} (w + d_j/c_j)^{-1}\, dw \to 0$$

as $C_j' \to 0$ is clear, for the integrand is bounded. Now

$$\int_{C_j'} (w - z_1)^{-1}\, dw = \log (w_2 - z_1)/(w_1 - z_1),$$

where w_1, w_2 are the endpoints of C_j'. There is a parabolic transformation Q (not belonging to Γ) that fixes z_1 and sends w_2 to w_1; Q has the form $(Qz - z_1)^{-1} = (z - z_1)^{-1} + c$. Hence $(w_2 - z_1)/(w_1 - z_1) \to 1$ as $w_1 \to z_1$, i.e., as $C_j' \to 0$.

11 (p. 174). Let $Az = -(z - z_0)^{-1}$, where z_0 is neither fixed point nor limit point of Γ. Choose a Γ'-image of R', say R_1', which is bounded. Then for $z \in R_1 = A^{-1}R_1'$ we never have $z = z_0$. Hence $A'(z) \neq 0, \infty$ and the displayed equation in the text can be shown to be correct.

12 (p. 179). 1) Since, in the definition of an infinite product, the factors are required never to vanish, the theorem should assert the uniform convergence of (32) in any compact subset of $\mathscr{U}$ that avoids the set $\{z_n\}$. Absolute convergence holds in $\mathscr{U} - \{z_n\}$.

2) In the third line of the proof, after "$F_N \exp i\Phi_N$," add "where $\Phi_N = \sum_{n=1}^{N} \varphi_n$."

3) In the first displayed line on p. 180 we define $\arg u_n(z)$ to be the continuous argument which reduces to 0 at $z = 0$. This determination is unique if $|z| < |z_n|$, i.e., when $|z| \leqq \rho < 1$, for $n \geqq N(\rho)$. Thus $\sum \varphi_n$ converges uniformly in $|z| \leqq \rho$, since $\Phi_{N+p} - \Phi_N$ converges uniformly to zero.

CHAPTER VI

13 (p. 204). The points of $\mathscr{D}^+$ are parabolic vertices and cannot also be elliptic vertices (IV. 4G). Since each elliptic vertex lying in $\mathscr{D}$ is a vertex of some fundamental region (IV. 4E), it follows that there are a finite number of inequivalent elliptic vertices in Γ, that is to say, Γ has a finite number of elliptic fixed point classes. If Γ is a principal circle group, this is true also of the parabolic classes (IV. 7E).

14 (p. 210). We fix the branch of σ_1^{-1} at z_0 so that $L \circ M(z) = z$ in a *whole neighborhood* of z_0. This is possible since σ_1 is a local homeomorphism. Also we can fix the branch of σ_2^{-1} so that $M \circ L(z) = \sigma_2^{-1} \circ \sigma_2(z) = z$ in a neighborhood of z_0. Then $M = L^{-1}$.

15 (p. 215). This note has been eliminated.

16 (p. 218). Condition 1) of the lemma is not needed since it is implied by 2). However, 1) is used explicitly in the following construction which forms the basis for the proof of Theorem 5D.

17 (p. 211). Let $C(S)$ be the group of conformal mappings of the Riemann surface S on itself. A. Hurwitz proved:

THEOREM. *If S is a compact surface of genus $g > 1$, $C(S)$ is a finite group of order at most $84(g - 1)$. The upper bound is attained.*

The finiteness of $C(S)$ had been proved earlier by H. A. Schwarz. We shall break up the proof into several theorems.

THEOREM 1. *S is hyperbolic* (i.e., *the universal covering surface $\hat{S}$ is conformally equivalent to a disk*).

To prove the result, we list all surfaces with elliptic and parabolic universal covering surfaces S ($\hat{S}$ conformal with sphere and plane, respectively). Let $\hat{S}$ be equivalent to the domain E on the sphere. The group of covering transformations of $\hat{S}$ is isomorphic with a group Γ of linear transformations of E on itself and no element of Γ has a fixed point in E (cf. VI. 3J; 3K, 1)). If E is the sphere, Γ reduces to the identity and S is of genus zero. If E is the plane, Γ can contain, besides the identity, only elements having ∞ as sole fixed point, that is, translations. Hence Γ is either the identity, the simply periodic group, or the doubly periodic group. The corresponding surfaces S are the plane, the twice punctured sphere, and the torus. Thus a surface of genus > 1 cannot be elliptic or parabolic.

THEOREM 2. *If we write $S = \Gamma\backslash\mathscr{U}$, then Γ contains only the identity and hyperbolic elements.*

This is an immediate consequence of VI. Theorem 3G. For the fundamental region R of Γ is compact in $\mathscr{U}$ and so Γ can have no parabolic elements. But also Γ is free of elliptic elements (VI. Theorem 3K).

THEOREM 3. *$C(S)$ acts discontinuously in S.*

By VI. Corollary 3L, the discontinuity of $C(S)$ in S is equivalent to that of $N_\Omega(\Gamma)/\Gamma$ in $\mathscr{U}$, or what amounts to the same thing, of $N_\Omega(\Gamma)$. Let $N_i \to I$ be a sequence of distinct elements of N_Ω. If $M \in \Gamma$ we have $N_i M N_i^{-1} = M_i'$ with $M_i' \in \Gamma$. Since Γ is discrete, it must be that $M = M_i'$ for $i > i_0$. Thus N_i commutes with M, $i > i_0$, and so has the same fixed point set, since M is hyperbolic (II. 9F, Theorem 2).

Now Γ is not abelian. For all abelian discontinuous groups defined on $\mathscr{U}$ are cyclic and so have genus 0 or 1. Hence let M_1, M_2 be two elements of Γ that do not commute and therefore have different fixed point sets. For all large i the transformation N_i must have the same fixed point set as both M_1 and M_2, which is impossible. The contradiction shows there is no sequence $N_i \to I$; hence N_Ω is discontinuous and so is $C(S)$.

We now prove Hurwitz's theorem. The order of $C(S)$ is equal to the index μ of Γ in $N_\Omega(\Gamma)$. Let R be a fundamental region for Γ, T a fundamental region for N_Ω. Now R consists of μ copies of T, namely, the images of T by representatives of the cosets $N_\Omega(\Gamma)/\Gamma$—cf. VII. 6F. All copies of T have the same hyperbolic area as T, denoted by $|T|$. Hence

$$\mu = |R|/|T|.$$

Since Γ has no elliptic or parabolic elements, $|R| = 4\pi(g - 1)$—cf. V. Ex. 5-2. On the other hand, by Siegel's theorem (Note 19), we have $|T| \geqq \pi/21$.

The lower bound for $|T|$ is attained by the triangle group (0, 3; 2, 3, 7)—cf. VII. 1G and Ex. 2, p. 185.

CHAPTER VII

17a (p. 220). The proof of Poincaré's theorem is not correct: Sections 1C, 1D are completely wrong. For a valid proof, including careful discussion of the hypotheses, cf. B. Maskit, Advances in Mathematics 7 (1971), 219-230; G. de Rham, L'Enseignement Math. 17 (1971), fasc. 1.

18 (Section 2, pp. 230–234). We make a number of comments on the proof of this Section.

2B. We agree that sequences KK^{-1} may be inserted in or deleted from chains; thus $UKK^{-1}V = UV$. This principle is used to get the equation $C = UL_1U^{-1}V$. For $C = ULV_1$ with $V_1 = [i_n, i_{n+1}, \cdots, i_t]$. Hence $C = ULU^{-1}UV_1 = UL_1U^{-1}V$.

2D. The grating (more conveniently rectangular than square) is first drawn in the whole plane (or sphere). It induces a subdivision of the interior of L into a finite number of pieces. A coherent orientation of this complex of pieces induces a consistent orientation of L, and we choose that orientation of the complex which induces the orientation L already has.

Besides the stated requirements on the grating we shall demand that no piece (rectangle or boundary polygon) shall lie in more than one fundamental region unless it encloses a boundary point of the region, in which case it shall lie in those fundamental regions, and only those, that have the point on their boundaries.

Each oriented piece determines several loops, depending on which vertex of the piece is used as a starting point; these loops, however, are all equivalent. If L_i lies entirely in R_j, the loop is defined to be the empty one. If L_i lies in R_j and R_k, the loop is defined to be $i_j i_k i_j$. If L_i contains a vertex, the loop is obtained by writing down in order the fundamental regions crossed by describing L_i in the correct orientation.

What Theorem 2D proves is the following: for any collection of n pieces forming with its boundary M an oriented complex, the path M determines a word $W(M)$ which can be written in the form stated. For our application we have only to observe that any oriented path L of the type we have specified can be regarded as the boundary of a complex of n pieces for a sufficiently large n.

19 (p. 241). Prove the following theorems of Siegel (Ann. of Math. **46** (1945), 708–718), in which Γ always denotes a horocyclic group defined on the unit disk.

THEOREM 1. *The hyperbolic area of every normal polygon of* Γ *is the same.*

[Let N_1, N_2 be two normal polygons; denote hyperbolic area by $|N_1|$. Use the relation $\bar{N}_1 \supset \bigcup_{V\in\Gamma}\{V\bar{N}_2 \cap \bar{N}_1\}$, $\bigcup\{N_2 \cap V^{-1}\bar{N}_1\} \supset N_2$ and the nonoverlapping of images of a normal polygon to get $|\bar{N}_1| = |N_1| \geqq |N_2|$, etc.]

THEOREM 2. *N has finite hyperbolic area if and only if it has a finite number of sides. Γ can be generated by a system of not more than $3|N|/\pi + 6$ generators.*

[Suppose N has $2n$ sides. From the center of the unit disk draw a straight line to each vertex of N dividing it into $2n$ triangles. By the Gauss-Bonnet formula (Ex. 1, p. 185) we obtain $|N| = 2\pi(n-1-\sum l_i^{-1})$, proving that $|N|$ is finite if n is. Suppose conversely that $|N|$ is finite. We select $2n$ consecutive sides arbitrarily and connect an interior point of N to the endpoints of these sides, forming $2n$ triangles. Using the Gauss-Bonnet theorem we get

$$|N| \geqq 2\pi n - \sum_{j=1}^{2n-1} \omega_j - \beta_1 - \gamma_{2n} - \sum_{j=1}^{2n} \alpha_j,$$

where α_j is the vertex angle of the jth triangle and β_j, γ_j are the other angles in counterclockwise order; $\omega_j = \gamma_j + \beta_{j+1}$. Now $\sum \alpha_j \leqq 2\pi$ and $0 \leqq \omega_j < \pi$ – the case $\omega_j = \pi$ can be excluded by combining two adjacent triangles. Hence

$$3\pi + |N| \geqq \sum_{j=1}^{2n-1} (\pi - \omega_j).$$

Assuming the number of sides is infinite, let $n \to \infty$; then $\sum(\pi - \omega_j)$ converges and $\omega_j \to \pi$. Thus $\omega_j = 0$ only finitely often, corresponding to the parabolic vertices. For the vertices inside $\mathcal{U}$ we now have $2\pi/3 < \omega_j < \pi$, $j > j_0$. Summing this equation over the r_j vertices of an ordinary cycle of order l_j, we find $2 < r_j l_j < 3$, a contradiction.

THEOREM 3. *For any* Γ, $|N| \geqq \pi/21$. *The lower bound is attained for the triangle group* (0, 3 ; 2, 3, 7).

[We may assume $|N|$ finite. Apply Ex. 2, p. 185: $|N| = 2\pi\{2g - 2 + \sum(1 - l_i^{-1})\}$. The theorem follows by considering cases. Bear in mind that $|N| > 0$; this excludes the group (1, 0) for example.]

Rankin has established that the corresponding lower bound for horocyclic groups defined in the upper half-plane and having translations is $\pi/3$, and it is attained for the modular group. Cf. R. A. Rankin, *Horocyclic groups*, Proc. London Math. Soc. **4** (1954), 219–234; p. 230.

20 (p. 241). Linda Keen (Acta Math. 115 (1966), 1–16) has proved the following result. Let Γ be a horocyclic group of finite signature (g, n) with $g > 0$, $3g - 3 + n > 0$, and with the usual presentation (cf. VII. (6)). By the translation axis of a hyperbolic transformation A we mean the hyperbolic line joining the fixed points of A and A^{-1}. Let p be the intersection of the translation axes of A_1 and B_1. Apply to p the generating transformations of Γ and join these images in the order of the group relation in the second line of (6). The result will be a strictly convex polygon bounded by hyperbolic lines and which satisfies all the requirements of a canonical polygon as we have defined it.

It is noteworthy that this fundamental region is not only a canonical polygon in the sense of Fricke-Klein but it is even unique.

The proof makes use of Bers' version of the "continuity method" of Klein and Poincaré via the theory of quasiconformal mappings.

21 (p. 242). It is, however, true that every finitely-generated principal circle group admits a fundamental region with a finite number of sides. Cf. M. Heins, *Fundamental polygons of fuchsian and fuchsoid groups*, Ann. Acad. Scient. Fennicae A337 (1964); also L. Greenberg, *Fundamental polygons for fuchsian groups*, J. Analyse (1966). A proof will presumably also be contained in the forthcoming book of Nielsen and Fenchel on discontinuous groups.

22 (p. 254). For example, let F be a free group with free generators $t_1, \cdots, t_r$. We represent F by Γ, a Schottky group (IV. 2C) defined by r pairs of congruent circles that are mated by transformations T_i, $i = 1, \cdots, r$. The circles of one pair must not intersect the circles of another pair, but it is possible to make the circles of a given pair externally tangent. Let there be s pairs of tangent circles. It is easily calculated that the genus of Γ is $r - s$. Since s may be any integer between 0 and r, there is no unique genus associated with the abstract group F.

In the other direction, however, we have the following result. *Let* Γ_1, Γ_2 *be principal-circle groups with relatively compact fundamental region in* $\mathcal{U}$, *and let* Γ_1 *and* Γ_2 *be isomorphic as abstract groups. Then* Γ_1 *has the same signature as* Γ_2.

The hypotheses imply that Γ has a fundamental region with a finite number of sides and no parabolic vertices. The theorem states that in such cases the abstract group determines the genus and the number and orders of the fixed points of the transformation group.

To prove the result we remark first that Γ_i has no parabolic generators. Suppose Γ_1 has signature $(g, n; l_1, \cdots, l_n)$ with elliptic generators E_i. The element E_i generates a maximal finite cyclic subgroup D_i of order l_i. Conversely every maximal finite cyclic subgroup D is conjugate to some D_i. Indeed, let D be generated by E, where E is clearly elliptic. Then E has a fixed point α in $\mathscr{U}$ and α is a vertex of some normal polygon N. Since the fixed point of each generator E_i is conjugate to some vertex in N, it follows that α is conjugate to the fixed point of some E_i and our assertion is established. Under the isomorphism of Γ_1 on Γ_2, a maximal finite cyclic subgroup in Γ_1 is mapped onto a maximal finite cyclic subgroup of the same order in Γ_2, and conversely. The orders of these subgroups are the "periods" l_i in the signature of Γ_i; they are therefore the same in Γ_1 as in Γ_2.

Let Γ_i have genus g_i, $i = 1, 2$; we have to show that $g_1 = g_2$. Let H_i be the normal closure of the elements of finite order in Γ_i (smallest normal subgroup of Γ_i containing all elements of finite order). There is a theorem of group theory to the effect that a presentation of $K_i = \Gamma_i/H_i$ is obtained from one of Γ_i by setting each generator of finite order equal to 1. Thus K_1 is generated by $2g_1$ elements $A_1, B_1, \cdots, A_{g_1}, B_{g_1}$ with the single defining relation $[A_1, B_1]\cdots[A_{g_1}, B_{g_1}] = 1$, where $[A_i, B_i]$ is the commutator of A_i and B_i. The group K_2 has a similar presentation with g_1 replaced by g_2. Let K_1' be the commutator subgroup of K_1; it is a normal subgroup consisting of all finite products of commutators of pairs of elements in K_1. Then K_1/K_1' is abelian; it is the "free abelian" group of rank $2g_1$. A free abelian group is determined solely by its rank. Since K_2/K_2' is isomorphic to K_1/K_1' (because $\Gamma_1 \simeq \Gamma_2$ and $H_1 \simeq H_2$), the ranks must be the same and so $g_1 = g_2$.

22a (6D, p. 254). A group G is said to be *residually finite* if the intersection of all its *normal* subgroups of finite index is the identity (Hall [**1**], p. 16). Since the subgroups $\bar{G}(\mathfrak{a})$ used in the proof of Theorem 6D are normal, we have proved a stronger property than was claimed, namely, that G and Γ are residually finite.

23 (p. 257). In the above discussion the T_i have no essential significance and should be omitted. Thus we would define

$$P^* = \bigcup A_i R^*.$$

The same remark holds for the further discussion up to and including Theorem 6F.

CHAPTER VIII

24 (p. 270). The fundamental region $\tilde{R}$ thus constructed will be bounded by straight lines and circular arcs, but in general the circular arcs will not be arcs of isometric circles. The fundamental region in $\mathscr{U}$ is a normal polygon with center selected so that it has the required properties. This polygon will coincide with Ford's fundamental region only if its center happens to be the origin (IV. 7).

25 (p. 271). The argument proceeds unchanged to the point where $Y_m = Y_{m+1}(m > N)$. This implies $d'_{m+1} = d'_m$ which, combined with $1 \leqq d'_m < 1 + \lambda_k c$, gives $|d_{m+1} - d_m| < \lambda_k c$. Since $d_{m+1} - d_m = \lambda_k c(t_{m+1} - t_m)$, we have $t_{m+1} = t_m$, hence $d_{m+1} = d_m$ for $m > N$. This is the desired contradiction.

26 (p. 273). It may be wondered why the local variable here is $e(-1/\lambda(\tau - p))$ whereas in V. 1B it was $e(1/c(\tau - p))$. In the former case the transformation fixing p is

$$(\tau' - p)^{-1} = (\tau - p)^{-1} + c;$$

in the present case it is

$$(\tau' - p)^{-1} = (\tau - p)^{-1} - \lambda,$$

as we calculate from the result of VIII. Ex. 2-1.

27 (p. 280). In the proof that the numerators of the terms of $G(\tau, A_j)$ are bounded we use the fact that $\tilde{c}_{j0} > 0$ (cf. 2D). Thus the c's appearing in the matrices $(a\, b\,|\,c\, d)$ of (S) are bounded below in absolute value by a positive constant provided they are not zero.

28 (p. 284). The integral defining the scalar product may be regarded as either a Riemann or a Lebesgue integral. For the purpose of changing variables, which we do several times, it is convenient to use the Riemann integral and to apply the standard theorem (R. Courant, *Differential and Integral Calculus*, vol. 2, Nordemann, New York, 1937, p. 253). The transformation will always be of the form $\tau' = L\tau$, where L is a linear transformation preserving $\mathscr{H}$. Thus $L = (a\; b\,|\,c\; d)$ is continuously differentiable, one-one, and has positive Jacobian $|L'(\tau)| = |c\tau + d|^{-2}$.

29 (p. 287). The argument that $T_i R$ lies in $\mathscr{D}$ fails if R is not bounded by isometric circles; in any event a better argument is the following. We know that

$$\bigcup_{T_i \in (\mathbf{D})} T_i R = S_1, \qquad \{0 \leqq u < \lambda_0, v > 0\} = S_2$$

are both (measurable) fundamental regions for the subgroup Γ_∞. Let $\omega\, dxdy$ be an integrable invariant differential on Γ_∞. Using the identities

$$\bar{R}_1 \supset \bigcup_{V \in G} \{ VR_2 \cap \bar{R}_1 \}, \qquad \bigcup_V \{ R_2 \cap V^{-1}\bar{R}_1 \} \supset R_2, \qquad G = \Gamma_\infty$$

and remembering that the boundaries of R_1, R_2 are of measure zero, we

get, by complete additivity,

$$\iint_{R_1} \omega \geqq \sum_V \iint_{VR_2 \cap R_1} \omega = \sum_V \iint_{R_2 \cap V^{-1} R_1} \omega \geqq \iint_{R_2} \omega.$$

By reversing R_1 and R_2 the reverse inequality is obtained. Hence $\iint \omega$ is the same over every measurable fundamental region with boundary of measure zero. This gives the equation on line 3 of page 287.

CHAPTER IX

30 (p. 303). The expansion (3) was developed in Chapter VIII on the assumption that r is an integer, whereas in a few lines we shall permit r to be an arbitrary real number. Therefore it is necessary to reexamine the derivation in VIII. 3A. The difficulty occurs in the second line of (*) where we tacitly assumed that

$$\left(\frac{\tau - p_j}{-\lambda_j \tau + 1 + \lambda_j p_j}\right)^r = \frac{(\tau - p_j)^r}{(-\lambda_j \tau + 1 + \lambda_j p_j)^r}.$$

It is sufficient to prove this for $\tau = p_j + iy, y > 0$. The left member becomes $(iy/(1 - \lambda_j iy))^r$. Since the argument of the quantity inside the parentheses clearly lies between 0 and π, the above equation is justified.

31 (p. 308). To prove $\mathbf{T}_N^0$ is a finite set: let $\tau \in \mathbf{h}$ and let τ_1 be the image of τ in $\mathrm{Cl}\,\tilde{R}$ by $\bar{M} = MA_k^{-1}$, $M \in \Gamma$. If $\tau_1 \in \mathrm{Int}\,\tilde{R}$, cover τ by an interval on $\mathbf{h}$ so small that it is mapped into $\mathrm{Int}\,\tilde{R}$ by $\bar{M}$. If τ_1 is a boundary point of $\tilde{R}$, cover τ by two half-open intervals $(\tau', \tau]$, $[\tau, \tau'')$, $\tau' < \tau < \tau''$, each of which is mapped into $\mathrm{Cl}\,\tilde{R}$ by elements $\bar{M}$. Then τ is covered by the open interval (τ', τ''). Since $\mathrm{Cl}\,\mathbf{h}$ is compact we can select from the above intervals a finite covering of $\mathbf{h}$, each interval of the covering being associated with at most two elements $\bar{M}$. The set $\mathbf{T}_N^0$ consists of exactly those $\bar{M}$.

CHAPTER XI

32 (p. 365). M. Newman has proved (Illinois J. of Math. 8 (1964), 262–265): *Every normal subgroup of Γ is a free group except Γ, Γ^2, and Γ^3. If H is any subgroup of Γ, then H is free if and only if it contains no elements of finite order.* The proof is based on Kuroš's Subgroup Theorem (Hall [1], p. 315) and can be extended to any group of finite signature containing parabolic elements, since such a group is isomorphic to a free product. Cf. M. I. Knopp, J. Lehner, and M. Newman, *Subgroups of F-groups*, Math. Annalen 160 (1965), 312–318.

33 (p. 357). The following argument should be considered *on the* (compact) *Riemann surface* $S = \bar{H}/\mathscr{H}^\tau$. Each $\bar{H}$-equivalence class of points (lines, triangles) determines a single point (line, triangle) on S. The natural triangulation of $R(\bar{H})$ induces a triangulation of S. The statements of the text now become clear.

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