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Topics in Operator Theory

Edited by
Carl Pearcy

American Mathematical Society

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PREFACE

The articles in this volume are concerned with various aspects of the theory of bounded linear operators on Hilbert space. This area of mathematical research is presently experiencing a period of intense excitement, due, no doubt, to the fact that during the past year several remarkable advances have been made on hard problems in the field. One particular problem on which considerable progress has been made is the "invariant subspace problem." This is the question whether every (bounded, linear) operator T on a separable, infinite-dimensional, complex Hilbert space H maps some (closed) subspace different from (0) and H into itself. It is therefore highly appropriate that the first and last of the five expository articles in this volume deal with invariant subspaces.

The main theme of the first article, by Donald Sarason, may be summarized as follows. If T is a bounded linear operator on H, then the collection of all subspaces M of H such that T maps M into itself forms a complete lattice, denoted by $\mathrm{Lat}(T)$, under the inclusion ordering. It is presently impossible to determine, in general, the lattice-theoretic structure of $\mathrm{Lat}(T)$. (This is not surprising, since we cannot even say whether the possibility $\mathrm{Lat}(T) = \{(0), H\}$ is realizable.) There are a few operators T, however, for which the structure of $\mathrm{Lat}(T)$ is known, and Sarason's article is devoted to a discussion of such operators and their associated invariant subspace lattices. The presentation is extremely lucid, and one of the interesting features of the exposition is the interplay between operator theory and classical analysis that is so often found in Sarason's work.

The second article in this volume is concerned with weighted shift operators. Interest focuses on this class of operators because it is a particularly simple class to define. Let $\{e_n\}_{n=0}^{\infty}$ be an orthonormal basis for H, and let $\alpha = \{\alpha_n\}_{n=0}^{\infty}$ be a bounded sequence of complex numbers (the weight sequence). It is easy to see that there exists a unique bounded operator T_α on H such that $T_\alpha e_n = \alpha_n e_{n+1}$ for every nonnegative integer n, and that the adjoint operator T_α^* is determined by the equations $T_\alpha^* e_0 = 0$ and $T_\alpha^* e_n = \overline{\alpha_{n-1}}\, e_{n-1}$, $n > 0$. The operators T_α and T_α^* are typical examples of *forward* and *backward weighted* *unilateral shifts*, respectively. A different type of weighted shift is obtained by considering an orthonormal bases $\{f_n\}_{n=-\infty}^{\infty}$ for H indexed by the set of all integers, and an associated (bounded) weight sequence $\beta = \{\beta_n\}_{n=-\infty}^{\infty}$. In this case, the unique bounded operator T_β on H that satisfies the equation $T_\beta f_n = \beta_n f_{n+1}$ for all integers n is a typical *forward weighted bilateral shift,* and a typical backward weighted bilateral shift can be defined analogously. Allen Shields has taken essentially all of the information presently known about weighted shift operators (with scalar weights) and incorporated it into his comprehensive article. A central theme of the exposition is the interplay between weighted shift operators and analytic function theory, and, as an added bonus for the reader, the article contains a list of thirty-two interesting research problems.

The third article in this volume is an exposition by Arlen Brown of the theory of spectral multiplicity for normal operators on Hilbert space. The problem treated arises as follows. In general, one wants to know when two operators T_1 and T_2 acting on Hilbert spaces K_1 and K_2, respectively, are unitarily equivalent (i.e., when there is an inner product preserving isomorphism φ of K_1 onto K_2 such that $\varphi T_1 \varphi^{-1} = T_2$). Unitary equivalence is the analog for operators of the concept of isomorphism for groups, rings, etc. The problem is usually attacked by trying to attach to each operator T in a fixed class of operators an indexed family of objects $\{O_\lambda(T)\}_{\lambda\in\Lambda}$ with the property that two operators T_1 and T_2 in the class are unitarily equivalent if and only if $O_\lambda(T_1) = O_\lambda(T_2)$ for all indices λ in Λ. Such a collection $\{O_\lambda(T)\}_{\lambda\in\Lambda}$ is called a *complete set of unitary invariants* for the class of operators. There are a few classes of operators for which a "reasonable" complete set of unitary invariants is known; in this connection see the bibliography of Brown's article. The theory of spectral multiplicity furnishes a reasonable complete set of unitary invariants for normal operators, and this set of invariants can be obtained in several different ways. One may consider the (commutative) C^*-algebra or von Neumann algebra generated by a given normal operator, and study the unitary equivalence problem for such algebras. In this approach, the solution of the unitary equivalence problem for normal operators becomes a corollary of the solution of the problem for commutative C^*-algebras or von Neumann algebras. One may also proceed more directly and focus attention on the concept of a spectral measure. It is this last approach that Brown follows, and his clear presentation of this circle of ideas should lead to a better understanding of multiplicity theory by beginners and experts alike.

The fourth article in this volume, by R. G. Douglas, is concerned with canonical models for operators. The central underlying idea here is that given any contraction operator T on H (i.e., any operator T satifying $\|T\| \leqslant 1$), there is a canonical construction that associates with T an operator M_T that is unitarily equivalent to T, called its "canonical model." Thus, one may study T by studying M_T instead, and this is one of the themes of the book *Harmonic analysis of operators on Hilbert space,* by Sz.-Nagy and Foiaş. Douglas, who has contributed significantly to the geometrization of the theory of canonical models, exposes in his article various important components of this theory, and thereby gives the reader much insight into its successes and failures.

The final article in this volume, written by Allen Shields and myself, is a survey of some invariant-subspace theorems that resulted from the brilliant and elegant method of proof introduced by Victor Lomonosov early in 1973. It would indeed be surprising if further study and refinement of this technique does not lead to additional progress on the invariant subspace problem.

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