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Geometric Asymptotics

Revised Edition

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GEOMETRIC ASYMPTOTICS

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American Mathematical Society
Providence, Rhode Island

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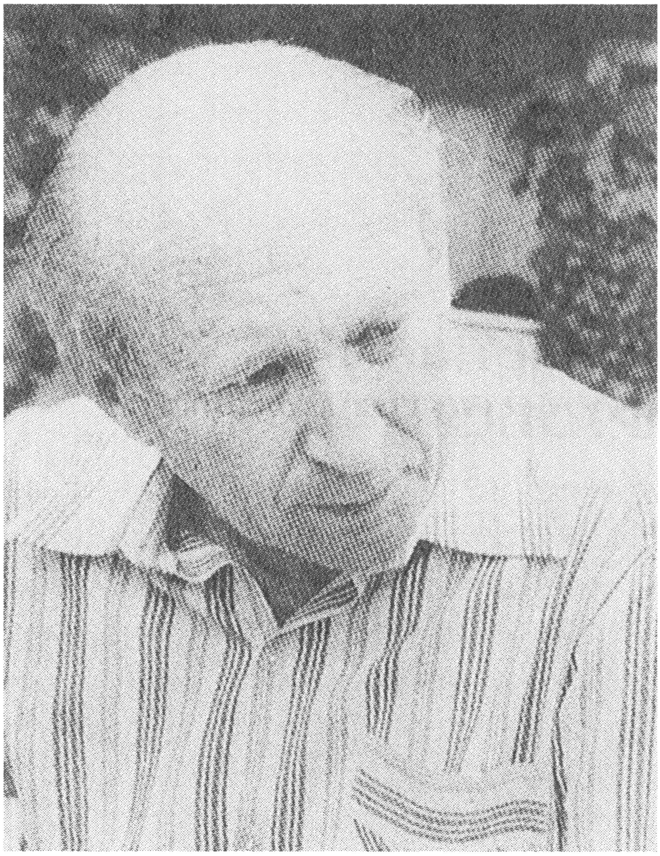
GEOMETRIC ASYMPTOTICS

To our teacher



I. M. GEL′FAND

who has been a guide and inspiration to us we dedicate this book as an expression of our love and admiration.

Preface

There has been a great deal of active research recently in the fields of symplectic geometry and the theory of Fourier integral operators. In symplectic geometry there has been much progress in understanding the geometry of dynamical systems, and the process of "quantization" as applied not only to the theory of dynamical systems, but also as a key tool in the analysis of group representations. Fourier integral operators have made possible a much more systematic analysis of the singularities of solutions of linear partial differential equations than existed heretofore, together with a good deal of geometric information associated with the spectra of such operators. These two subjects are modern manifestations of themes that have occupied a central position in mathematical thought for the past three hundred years—the relations between the wave and the corpuscular theories of light. The purpose of this book is to develop these themes, and present some of the recent advances, using the language of differential geometry as a unifying influence.

We cannot pretend to do justice to the history of our subject, either in the short space of this preface, or in the body of the book. Yet some very brief mention of the early history is in order. The contributions of Newton and Huygens to the theory of light are known to all students. (Huygens' "envelope construction"—wherein the superposition of singularities distributed along a family of surfaces accumulates as singularity along the envelope—makes its appearance in the modern setting as a formula for the behaviour of wave front sets under functorial operations; cf. Chapter VI, equation (3.6).) In Fresnel's prize memoir of 1818, he combined Huygens' envelope construction with Young's "principle of interference" to explain not only the rectilinear propagation of light but also diffraction effects. Thus Fresnel applied Huygens' method to the superposition of oscillations instead of disturbances and was able to calculate the diffraction caused by straight edges, small aperatures and screens. In Chapter VI we study the geometry of the superposition of "disturbances" while in Chapter VII we study the geometry of the superposition of (high frequency) oscillations. In 1828

Hamilton published his fundamental paper on geometrical optics, introducing his "characteristics", as a key tool in the study of optical instruments. It wasn't until substantially later that Hamilton realized that his method applied equally well to the study of mechanics. Hamilton's method was developed by Jacobi and has been a cornerstone of theoretical mechanics ever since. In Chapter IV we discuss the modern version of the Hamilton-Jacobi theory—the geometry of symplectic manifolds. It is interesting to note that although Hamilton was aware of the work of Fresnel, he chose to ignore it completely in his fundamental papers on geometrical optics. Nevertheless, he made a theoretical prediction in the wave theory—the possibility of "conical refraction"—which was experimentally verified soon thereafter. Unfortunately, we shall have nothing to say in this book on the issue of conical refraction (the problem of multiple characteristics in hyperbolic differential equations) but hope that some of the methods that we develop will prove useful in this connection. In 1833, Airy published a paper dealing with the behavior of light near a caustic. (A caustic is a set where geometrical optics predicts an infinite intensity of light. An object placed at a caustic will become quite hot, and hence the name.) In this paper, he introduced the functions now known as Airy functions. In Chapter VII, we show how the theory of singularities of mappings can be applied to understand and extend Airy's results. In 1887, Kelvin introduced the method of stationary phase—a technique for the asymptotic evaluation of certain types of definite integrals—in order to explain the V shaped wake that trails behind a ship as it moves across the water. The method of stationary phase, in a form expounded by Hörmander, will be our principal analytical tool.

We now turn to a chapter by chapter description of the contents of the book.

Chapter I. Introduction. The Method of Stationary Phase

The solution of the reduced wave equation $\Delta\mu + k^2\mu = 0$ with initial data prescribed on a hypersurface $S \subset \mathbf{R}^3$ is given by an integral of the form

$$\int \frac{e^{ik|x-y|}}{|x-y|} a(y)\, dS. \tag{$\star$}$$

In order to see how $(\star)$ behaves in the range of large frequencies we first consider more general integrals of the form

$$\int e^{ik\phi(z)} a(z)\, dz.$$

The method of stationary phase, for evaluating such integrals for k large, is discussed in detail following Hörmander's idea of making use of Morse's lemma. It is then applied to $(\star)$ to explain the phenomenon of shift in phase, when light rays pass through caustics. In the Appendix to this Chapter, we present a proof of Morse's lemma and its extensions to the existence of polynomial normal forms for functions satisfying the Milnor condition. The technique of proof, due to Moser and Palais, will recur several times in various contexts where normal forms of different kinds of geometric objects are required. This chapter is rather short and is intended mainly to motivate the type of asymptotic developments we consider in Chapters II and VII.

Chapter II. Differential Operators and Asymptotic Solutions

We discuss the Luneburg-Lax-Ludwig technique for constructing asymptotic solutions for differential equations $P(x, D, \tau)$ involving a large parameter τ. This technique involves solving the characteristic or eikonal equation and then inductively solving a series of transport equations. We show how these can be solved using methods of symplectic geometry and in doing so also show how the characteristic and transport equation, viewed symplecticly, make sense in the vicinity of caustics even though the asymptotic expansions blow up at caustics. This enables us, in §6, to analytically "continue" our asymptotic expansions through caustics making the appropriate phase adjustments. In §7, we apply these results to the time independent Schrödinger equation and derive the Bohr-Sommerfeld quantization conditions of Keller and Maslov. We also derive an asymptotic formula for the fundamental solution of the time dependent Schrödinger equation and show that it can be interpreted, as Feynmann does, as the evaluation of a Feynmann integral by means of stationary phase[1].

Chapter III. Geometrical Optics

We indicate how the techniques of the preceding chapter can be applied to optics. The point of view we emphasize here is that geometrical optics is just the study of the bicharacteristics of Maxwell's equation. In particular we develop the elementary laws of refraction and reflection from this point of view, and discuss focusing and magnification. We spend more time than the reader may think is warranted discussing Gaussian optics. However, this section is partly intended to motivate our study of the linear symplectic group a little later on. We also want to emphasize that the abstract mathematical considerations of Chapters II and IV are quite closely connected to concrete physical applications. However the treatment is not meant to be complete. There are two excellent and highly readable texts on the subject: Born and Wolf, "Principles of Physical Optics" and Luneburg, "Mathematical Theory of Optics", and the reader is referred to them for a detailed treatment. We finally discuss Maxwell's equations themselves.

The first three chapters can be viewed as principally motivational. The main formal development starts with Chapter IV.

[1] The techniques we describe in this chapter have been further developed by the mathematicians of the NYU school to obtain deep results in physical optics, acoustics and scattering theory. We regret very much that the limitations of this book prevent us from touching on their remarkable work.

Chapter IV. Symplectic Geometry

We have already made sporadic use of symplectic methods. Our intention here is to provide a much more systematic treatment than in the earlier chapters. We begin by giving Weinstein's proof of Darboux's theorem (and the Kostant-Weinstein generalization which says that near a Lagrangian submanifold Λ a given symplectic manifold looks locally like $T^*\Lambda$.) In §2 we discuss linear symplectic geometry, the Langrangian Grassmannian and its universal covering space, and define the Maslov index following the ideas of Leray and Souriau. In §3 we discuss Hörmander's cross ratio construction of the Maslov class. This material is based on §3.3 of Hörmander's Acta paper on Fourier integral operators, with some simplifications suggested to us by Kostant. In §§4 and 5 we develop the main facts we will need in Chapters VI and VII about Lagrangian manifolds. In preparation for Chapters VI and VII we have put considerable stress on their "functorial" properties. In §6 we discuss periodic orbits of Hamiltonian systems and make an application to the Kepler problem (following Moser). In §7 we discuss symplectic manifolds on which a group G acts transitively. Finally, in §8 we discuss relations between symplectic geometry and the calculus of variations.

Chapter V. Geometric Quantization

In this chapter we discuss the current state of knowledge concerning the geometry of quantization as introduced (independently) by Kostant and Souriau. The main function of this chapter, as far as the rest of the book is concerned, is the introduction of metalinear structures, half-forms, and metaplectic structures. These notions will be of crucial importance for us in Chapters VI and VII where we develop the symbol calculus in the metalinear category. The first two sections deal with the concept, due to Kostant and Souriau, of prequantization. This has the effect of selecting, among symplectic manifolds, those satisfying certain integrality conditions. (In the case of relativistic particles this has the effect of constraining the "spin" to take on half integral values. In the case of the hydrogen atom this constrains the negative energy levels to their appropriate discrete values.) In §3 we introduce the notion of polarization, a concept introduced independently by Kostant and Souriau in the real case, while Auslander and Kostant introduced the notion of a complex polarization. The rest of the chapter represents joint research by Blattner, Kostant, and Sternberg, dealing with half-forms and a pairing between sections associated to different polarizations, together with some physical examples due to Simms. The concept of a metaplectic structure is introduced together with the pairing, mentioned above which can be viewed as a generalization of the Fourier transform. The metaplectic representation of Segal-Shale-Weil is constructed by geometric methods using the pairing, and is applied to the construction of "symplectic spinors". In a sense, this chapter is the most incomplete in the book. It is clear that the quantization scheme proposed here is not broad enough to include all interesting cases. A preliminary computation, outlined at the end of the chapter, indicates that the requirement that two polarizations be unitarily related—an essential ingredient in the scheme—is only slightly less restrictive than they be "Heisenberg related", i.e. that the pairing be geometrically equivalent to the classical Fourier transform. Thus more sophisticated procedures must be developed, presumably along the lines of closer and closer approximations to the Feynmann path integral method. Even within the current framework many basic questions remain unanswered: What are the appropriate conditions guaranteeing the convergence of the integrals involved in the pairing? What is the correct formulation of the pairing in the case of complex polarizations, and its relation with such objects as the Bergmann kernel? Is there a discrete analogue of the notion of polarization, with an appropriate pairing so as to tie in with the theory of theta functions? Must one look at generalized sections, associated polarizations, or higher cohomologies in order to construct the quantized group representations? To what extent are the recent examples, of Vergne and Rothschild-Wolf, where the group representation is not independent of the polarization, a consequence of geometrical pathology of the polarization in question? What is the relation between polarizations and pairings as introduced here geometrically to formally similar objects arising in the p-adic theory, especially in Weil's fundamental papers? In short, it is quite clear that the subject matter of this chapter is only in its preliminary stages of development, and it is to be hoped that the present chapter will be completely out of date within a few years.

Chapter VI. Geometric Aspects of Distributions

This is perhaps the central chapter of the book from the mathematical point of view. In it we develop the theory of generalized functions (or sections of a vector bundle) in terms of their behavior under smooth maps. A *density* on a manifold X is an object which locally looks like a function, but transforms, under change of coordinates, in such a way that integration makes sense. If μ is a density and v is a smooth function of compact support on a manifold X, we define $\mu(v)$ by $\int_X v\mu$, so that densities define linear functionals on $C_0^\infty(X)$. By a generalized density we then mean any continuous linear functional on $C_0^\infty(X)$.

Let X and Y be manifolds and $f\colon X \to Y$ a proper mapping. If μ is a generalized density on X the "push-forward" $f_* \mu$ is defined by the formula

$$f_* \mu(v) = \mu(f^* v), \qquad v \in C_0^\infty(Y).$$

It turns out that in certain instances one can define the "pull-back" $f^* \mu$ of a generalized function on Y. (For example this is always the case if f is a fiber mapping.) The purpose of this chapter is to develop systematically the theory of distributions in such a way as to permit maximum interplay between these two functors.

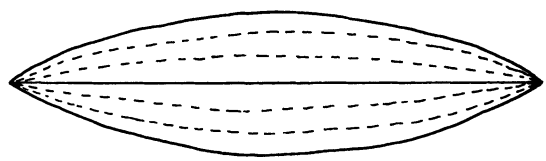
In §1 we indicate the basic properties of these two functors, and use them to define a special class of distributions called δ-distributions and derive a fixed point formula. In §2 we give a few applications of the ideas discussed in §1; in particular we prove a few theorems about characters of induced representations (in the theory of group representations) and sketch a proof of the Atiyah-Bott fixed point formula. In §3 we define the wave front set of a distribution. It turns out that from our functorial point of view it is convenient to do this using the Radon transform rather than the Fourier transform as Hörmander does. (This is because the Radon transform can be defined as a composition of a "push-forward" and a "pull-back".) Therefore, we have included in this section a discussion of the basic properties of the Radon transform. The idea of using the Radon transform was pointed out to us by Dave Schaeffer in a lecture in which he expounded some fundamental papers by Ludwig who had developed the theory of singularities from this point of view. Ludwig's papers have served as a guide to us in constructing much of the theory of this chapter. In addition we discuss a few other Radon-like transforms which are defined by push-pull operations and show that the only "elliptic" examples have properties very similar to the classical Radon transforms on rank one homogeneous spaces.

In §4 we discuss a larger class of distributions than those of §1. This class is obtained by starting with the δ-function on the real line and applying the operations of pull-back, push-forward, differentiation and multiplication by smooth functions. What one gets is approximately the class of distributions discussed by Hörmander in his basic paper on Fourier integral operators (though his way of defining these distributions is quite different from ours.) In §5 we develop a symbol calculus for these distributions, again making heavy use of the two functorialities. Our symbols differ a little from those of Hörmander in that instead of using the Maslov line bundle, as Hörmander does, to handle phase adjustments we use the metaplectic structure discussed in Chapter V. In particular, our symbols are "half-forms" instead of half-densities. We get a slightly less general theory than that of Hörmander. (The manifolds we consider have to satisfy $w_1(X)^2 = 0$, $w_1(X)$ being the first Stiefel-Whitney class.) However, we feel this disadvantage is outweighed by the advantage that the symbols are less complicated. We indicate how our theory needs to be modified to cover the more general situation. In §6 we discuss the calculus of Fourier integral operators and derive some applications to partial differential equations.

In §7 we describe how our distributions behave under composition with differential operators and show that on the symbol level $\sigma(P\mu)$ is given by applying the transport equation to $\sigma(\mu)$ just as we did in Chapter II in a somewhat more elementary setting. As an application we describe the progressing and the regressing fundamental solutions of the wave equation

$$\frac{d^2\mu}{dt^2} - \Delta\mu = 0$$

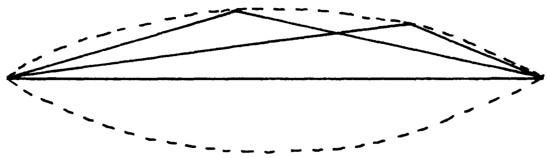
for the Laplace-Beltrami operator Δ on a compact manifold. In §8 we use the results of §7 to obtain some theorems in spectral theory. In particular we obtain Hörmander's asymptotic formula for the spectral function of a positive self-adjoint elliptic operator on a compact manifold, and a result of Chazarain-Duistermaat-Guillemin on the singularities of the Fourier transform of the spectral function. This result asserts, roughly speaking, that the knowledge of the eigenvalues of the Laplacian gives the lengths of the closed geodesics. To use the phraseology of Mark Kac, one can "hear" the lengths of the closed geodesics.[2] (This result is connected with a physical observation of Hermholtz in his study of stringed instruments. Helmholtz was puzzled by the following problem: Why is it that a bowed string, which is a "forced vibration", should yield (approximately) the same note as a plucked string, which is a "free vibration". The frequency of a forced vibration should be the same as the frequency of the forcing term (the bow). There must be some mechanism whereby the string triggers the bow to execute a forcing action at exactly the free frequency of the string.) Helmholtz examined the motion of strings using his 'vibrating microscope' (today we would use a stroboscope) and discovered the following result: The plucked string vibrates between the extreme positions as indicated:



where the central line represents the rest position of the string. On the other hand, the instantaneous positions of the bowed string are of the form

[2] The earliest reference that we were able to find in the scientific literature to this inverse problem is in a report to the British Association by Sir Arthur Schuster in 1882 on spectroscopy. Until that time, the primary function of the analysis of the spectra of atoms and molecules was to identify the chemicals in question. (Indeed the most striking results were those of Kirchoff in determining the chemistry of the sun's atmosphere by analysing the absorbtion lines of the solar spectrum.) In fact, the study of spectra was known as "spectrum analysis". In this report, Schuster suggested that the primary function of the study of spectra in the future would be to analyze the structure of atoms and molecules, and coined the name "spectroscopy" for this new science. He writes:

> "but we must not too soon expect the discovery of any grand and very general law, for the constitution of what we call a molecule is no doubt a very complicated one, and the difficulty of the problem is so great that were it not for the primary importance of the result which we may finally hope to obtain, all but the most sanguine might well be discouraged to engage in an inquiry which, even after many years of work, may turn out to have been fruitless. We know a great deal more about the forces which produce the vibrations of sound than about those which produce the vibrations of light. To find out the different tunes sent out by a vibrating system is a problem which may or may not be solvable in certain special cases, but it would baffle the most skillful mathematician to solve the inverse problem and to find out the shape of a bell by means of the sounds which it is capable of sending out. And this is the problem which ultimately spectroscopy hopes to solve in the case of light. In the meantime we must welcome with delight even the smallest step in the desired direction."



where the kink in the string moves around the circuit with the velocity of sound along the string, and it is this motion of the kink which triggers the bow to adhere and detach from the string. Thus, in a bowed string, we hear the period of the kink, i.e. the length of the "closed geodesic". In an appendix to this Chapter we describe Gelfand's celebrated results on the Plancherel formula for the complex semi-simple Lie groups in terms of the method of stationary phase.

Chapter VII. Compound Asymptotics

We go back to the type of problems discussed in Chapter II. We have more machinery at our disposal now, so we can formulate the results of Chapter II more systematically.We define objects on manifolds which we call *asymptotics*. They are functions (densities, half-densities, etc.) which depend on a large parameter τ. Any two such objects are identified if they have the same asymptotic growth as $\tau \to \infty$. Following Leray we define the Fourier transform of an asymptotic on $\mathbf{R}^n$, and use it to analyze the singularities of the asymptotic. More generally given an asymptotic $[\gamma]$ on a manifold X we define a subset $F[\gamma]$ of T^*X which we call its "frequency set" (in analogy with the wave front set of Hörmander) and which gives us rather precise information about where high frequency oscillations of $[\gamma]$ are located. Asymptotics have some obvious functorial properties which we discuss in §3.

In §4 we develop a symbol calculus for asymptotics. Here these are two parallel theories—one associated to "exact" Lagrangian manifolds (for which $(1/2\pi)\alpha$ is exact) and one associated to "integral" Lagrangian manifolds for which $(1/2\pi)\alpha \in H^1(\Lambda, Z)$, i. e. $1/(2\pi)\alpha$ has integral cycles. The first of these is associated to asymptotics depending on a continuous parameter and the second to asymptotics associated to a discrete parameter. (These subtleties did not enter in Chapter VI because there we dealt with homogeneous Lagrangian manifold and α vanishes when restricted to a homogeneous Lagrangian manifold.) We then discuss the subprincipal symbol and the transport equation for asymptotics and indicate how integral asymptotics should be used, in conjuction with half forms, to obtain quantization conditions. In particular, our point of view is quite different from that espoused in Chapter II.

Even in dimension zero the asymptotic property of an arbitrary asymptotic can be quite complicated. The only general result known is Bernstein's theorem which describes asymptotic properties of integrals of the form

$$\int a(z)e^{i\tau\alpha(z)}\,dz$$

for functions α with isolated singularities. This is discussed in §5.

In §§6-9 we discuss asymptotic properties of certain kinds of generic asymptotics, such as those occurring in optics in the vicinity of a simple caustic. The simple caustic case is discussed in §6, and used to obtain results of Ludwig on uniform asymptotic expansions. To obtain analogous results for more complicated types of caustics we need canonical form theorems for the associated phase functions. These canonical forms are discussed in §§7 and 8. Finally in §9 we indicate some generalizations of the results of §6. The results of §§6-9 are joint research of Guillemin and Schaeffer, the results being obtained in 1972. Since then, two articles have appeared covering similar grounds, one by Duistermaat and one by Arnol'd. We feel that the point of view developed here, and the results obtained are sufficiently different from the above-mentioned articles to warrent publication in their original form.

For the reader who prefers to deal directly with the mathematical theory, unencumbered by historical allusions or physical applications we recommend that he begin with Chapter VI, and refer back to those sections of Chapters IV and V as they become necessary.

As we indicated in the above chapter by chapter analysis, much of the present book represents joint work with others. Many of the results of Chapter IV were obtained jointly with Kostant; Chapter V represents joint work of Blattner, Kostant, and Sternberg. The second half of Chapter VII is joint work of Guillemin and Schaeffer and many comments by Schaeffer were extremely helpful in the development of Chapter VII. As many of the results are presented here for the first time, we are grateful to Bob Blattner, Bert Kostant and Dave Schaeffer for allowing the material to be published as part of this book.

The book itself is based on an intensive joint Harvard-MIT course in the fall of 1973, together with seminars over the past few years. We wish to thank John Guckenheimer and Marty Golubitsky for their contribution to the course and seminar, and to Molly Scheffé for taking careful notes on which a lot of the current manuscript is based. Above all, we wish to thank Mary McQuillin for her assistance in seeing this book through its many editorial stages.

Both authors were supported in part by the National Science Foundation and the second author by the John Simon Guggenheim Memorial Foundation to whom we wish to express our thanks.

Notation

In general, we follow the notation used in the text *Advanced Calculus* by Loomis and Sternberg, Addison-Wesley Publishing Co., Reading, Mass. 1968, with some slight changes. The letters X, Y, Z, W, M will be used for differentiable manifolds; usually M will denote a general differentiable manifold, and X a manifold carrying some additional structures. Points on these manifolds will be denoted by lower case letters such as x, y, z, etc. Local coordinates will be written as $(x^1, \ldots, x^n)$. If M is a manifold and T^*M is its cotangent bundle, then the local coordinates on T^*M associated with the local coordinates $(x^1, \ldots, x^n)$ on M will be $(x^1, \ldots, x^n, \xi_1, \ldots, \xi_n)$ and, sometimes $(q^1, \ldots, q^n, p_1, \ldots, p_n)$. The fundamental one form on the cotangent bundle will be written as α, so that, in terms of these local coordinates, $\alpha = \xi_1\, dx^1 + \cdots + \xi_n\, dx^n$ or $\alpha = p_1\, dq^1 + \cdots + p_n\, dq^n$. We will use bold face greek letters, usually $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, or $\boldsymbol{\zeta}$ to denote either tangent vectors or vector fields. The tangent space to a manifold M at a point x will be denoted by TM_x, and so a typical element of this tangent space will be $\boldsymbol{\xi} \in TM_x$. If f: $M \to N$ is a smooth map between differentiable manifolds, its differential at the point x will be denoted by df_x, so that df_x: $TM_x \to TN_{f(x)}$. The "pull back" of a differential form θ on N by f will be denoted by $f^*\theta$. Thus, for example, if θ is a *linear* differential form on N, and $\theta_y \in T^*N_y$ is its value at $y \in N$, then for $\boldsymbol{\xi} \in TM_x$ we have

$$\langle \boldsymbol{\xi}, (f^*\theta)_x \rangle = \langle df_x \boldsymbol{\xi}, \theta_{f(x)} \rangle,$$

where $\langle\ ,\ \rangle$ gives the pairing between vectors and covectors. We will use the symbol $D_{\boldsymbol{\xi}}$ to denote the Lie derivative with respect to the vector field $\boldsymbol{\xi}$. Thus, if Ω is a k-form, $D_{\boldsymbol{\xi}}\Omega$ is its Lie derivative with respect to $\boldsymbol{\xi}$; if η is another vector field, $D_{\boldsymbol{\xi}}\eta = [\boldsymbol{\xi}, \eta]$ is the Lie bracket; if u is a function, we frequently write $\boldsymbol{\xi}u$ for $D_{\boldsymbol{\xi}}u$.

In an oriented Riemannian or psuedoriemmanian manifold of dimension n, we have the "star operator", denoted by $\star$, mapping k forms into $n - k$ forms. Thus, for example, if S is an oriented surface in Euclidean three space, $\mathbf{R}^3$ and u is a smooth function on $\mathbf{R}^3$, we write $\iint_S \star\, du$ for the expression $\iint_S (\partial u/\partial n)\, dS$ that occurs in many of the older analysis texts.

TABLE OF CONTENTS

Appendix II. Various Functorial Constructions

In this appendix we collect a few of the functorial facts that are used throughout the book. We first discuss the notion of morphism of vector bundles, a concept introduced by Atiyah and Bott in their celebrated paper on the fixed point theorem. We then discuss the notion of fiber product, and show how it relates to various constructions that we describe in various chapters.

§1. The category of smooth vector bundles.

In this section we briefly establish notation concerning the calculus in the category of smooth vector bundles. An object in this category consists of a real (resp., complex) C^∞ vector bundle E over the C^∞ manifold X:

$$\begin{array}{c} E \\ \downarrow \\ X \end{array}$$

To describe the morphisms in this category we recall the following: Let Y be a differentiable manifold and f: $Y \to X$ a smooth map. Then $f^\# E$ is a vector bundle over Y, whose fibers are given by $(f^\# E)_y = E_{f(y)}$. If F is a vector bundle over Y then $\mathrm{Hom}(f^\# E, F)$ is also a vector bundle over Y whose fiber at y is $\mathrm{Hom}((f^\# E)_y, F_y) = \mathrm{Hom}(E_{f(y)}, F_y)$. It makes sense to talk of a smooth section of this vector bundle. Following Atiyah-Bott [2] of Chapter VI, we define a morphism

$$\begin{array}{ccc} F & & E \\ \downarrow & \xrightarrow[\mathfrak{f}=(f,r)]{} & \downarrow \\ Y & & X \end{array}$$

to consist of the pair $\mathbf{f} = (f, r)$ where $f\colon Y \to X$ is a smooth map and r is a smooth section of $\mathrm{Hom}(f^{\#}E, F)$. If $\mathbf{g} = (g, t)$,

$$\mathbf{g}\colon \quad \begin{array}{ccc} G & & F \\ \downarrow & \xrightarrow[\mathbf{g}=(g,t)]{} & \downarrow \\ Z & & Y \end{array}$$

is a morphism from G to F then $\mathbf{f} \circ \mathbf{g}$ is defined by $\mathbf{f} \circ \mathbf{g} = (f \circ g, w)$ where w is the section of

$$\mathrm{Hom}(G, (f \circ g)^{\#} E) = \mathrm{Hom}(G, g^{\#}(f^{\#} E))$$

given by

$$w(z) = r(g(z)) \circ t(z)$$

where

$$t(z) \in \mathrm{Hom}(G_z, F_{g(z)}) \quad \text{and} \quad r(g(z)) \in \mathrm{Hom}(F_{g(z)}, E_{f\circ g(z)})$$

so that

$$w(z) \in \mathrm{Hom}(G_z, E_{(f\circ g)(z)}) = \mathrm{Hom}(G, (f \circ g)^{\#} E)_z.$$

It is easy to see that the various axioms for a category are satisfied. For any vector bundle $E \to X$ we let $\Gamma(E)$ denote the space of C^∞ sections of E and $\Gamma_0(E)$ the space of C^∞ sections with compact support. If $\mathbf{f}\colon F \to E$ is a morphism then it induces a linear map $\mathbf{f}^*\colon \Gamma(E) \to \Gamma(F)$ by

$$(\mathbf{f}^* s)(y) = r(y)s(f(y)) \in F_y$$

for any section s of E. It is clear that

$$\mathbf{f}^*(\varphi s) = (f^*\varphi)\mathbf{f}^* s$$

for any C^∞ function φ on X where $f^*\varphi(y) = \varphi(f(y))$. It is also clear that the map sending f to f^* is a contravariant functor to the category whose objects are $\Gamma(E)$'s.

Similarly, we can consider the subcategory of proper morphisms, consisting of those $\mathbf{f}$ for which $f\colon Y \to X$ is a proper map. Then we get a contravariant functor $E \to \Gamma_0(E)$, $\mathbf{f} \to \mathbf{f}^*$.

We denote by $F \boxtimes E$ the vector bundle over $Y \times X$ whose fiber at (y, x) is $F_y \otimes E_x$. If F and E are fibers over the same space X then $F \otimes E$ denotes the vector bundle over X whose fiber at x is $F_x \otimes E_x$. The map $\mathbf{\Delta} = (\Delta, \mathrm{id})$,

$$\begin{array}{ccc} F \otimes E & & F \boxtimes E \\ \downarrow & \to & \downarrow \\ X & & X \times X, \end{array}$$

where $\Delta(x) = (x,x)$ is clearly a morphism, is called a diagonal map. More generally, if G is any vector bundle over X then a morphism of the form

$$\begin{array}{ccc} G & & F \boxtimes E \\ \downarrow & \xrightarrow[(\Delta,r)]{} & \downarrow \\ X & & X \times X \end{array}$$

(so that $r(x)\colon F_x \otimes E_x \to G_x$) will be called a diagonal morphism.

If s_1 and s_2 are sections of F and E then $s_1 \boxtimes s_2$ is the section of $F \boxtimes E$ given by $s_1 \boxtimes s_2(y,x) = s_1(y) \otimes s_2(x)$ and $s_1 \otimes s_2 = \mathbf{\Delta}^*(s_1 \boxtimes s_2)$ is given by $s_1 \otimes s_2(x) = s_1(x) \otimes s_2(x)$.

If $\mathbf{f}_1 : F_1 \to E_1$ and $\mathbf{f}_2 : F_2 \to E_2$ are morphisms then we get a well-defined morphism $\mathbf{f}_1 \boxtimes \mathbf{f}_2 : F_1 \boxtimes F_2 \to E_1 \boxtimes E_2$; and, if F_1, F_2 are vector bundles over the same space Y while E_1, E_2 are vector bundles over the same space X then $\mathbf{f}_1 \otimes \mathbf{f}_2 : F_1 \otimes F_2 \to E_1 \otimes E_2$ with

$$\mathbf{\Delta} \circ (\mathbf{f}_1 \otimes \mathbf{f}_2) = (\mathbf{f}_1 \boxtimes \mathbf{f}_2) \circ \mathbf{\Delta}.$$

In particular,

$$\mathbf{f}_1^* s_1 \otimes \mathbf{f}_2^* s_2 = (\mathbf{f}_1 \otimes \mathbf{f}_2)^* (s_1 \otimes s_2).$$

We will not dwell any further on these considerations of a purely functorial nature, leaving the reader to fill in the details.

Any morphism $\mathbf{f} = (f,r)$ has a canonical factorization as

$$\mathbf{f} = \boldsymbol{\pi} \circ \boldsymbol{\iota}.$$

Here

$$\boldsymbol{\iota} = (\iota, \mathrm{id}) \colon \quad \begin{array}{ccc} F & & p^{\#} F \\ \downarrow & \to & \downarrow \\ Y & & Y \times X \end{array}$$

where $p\colon Y \times X \to X$ is projection onto the first factor giving $(p^{\#} F)_{(y,x)} = F_y$ and where $\iota(y) = (y, f(y))$ so that $(\iota^{\#} p^{\#} F)_y = F_y$ and so id makes sense. Also $\boldsymbol{\pi} = (\pi, w)$ where π is projection onto the second factor and $w(y,x)\colon E_x \to F_y$ is given by $w(y,x) = r(y)$.

If $\mathbf{f}\colon F \to E$ is a morphism and S is a subset of F then we define the subset $\mathbf{f}_* S$ of E by saying that $e \in E_x$ belongs to $\mathbf{f}_* S$ if and only if $r_y(e) \in S$ for some $y \in f^{-1}(x)$, where $f = (f,r)$. Thus

$$\mathbf{f}_* S = \bigcup_{y \in Y} r_y^{-1}(S \cap F_y).$$

It is easy to check that if g: $G \to F$ is a morphism and R is a subset of G then $f_*(g_* R) = (f \circ g)_* R$.

If $\mathbf{f}$: $F \to E$ is a morphism and W is a subset of E then we define the subset $\mathbf{f}^* W$ of F by setting

$$\mathbf{f}^* W = \bigcup_{y \in Y} r_y(W \cap E_{f(y)})$$

when $\mathbf{f} = (f, r)$. Again, it is easy to check that if $\mathbf{g}$: $G \to F$ is another morphism then $(\mathbf{f} \circ \mathbf{g})^* W = \mathbf{g}^* \mathbf{f}^* W$.

There is a natural functor, T^*, which goes from the category of differentiable manifolds and differentiable maps to the category of vector bundles. It assigns to each differentiable manifold X its cotangent bundle $T^*(X)$ and to each differentiable map f: $Y \to X$ the morphism $\mathbf{T}^*(f) = (f, df^t)$ where df_y: $T_y(Y) \to T_{f(y)}(X)$ is just the induced map of f on the tangent space (the Jacobian map) and df_y^t: $T^*_{f(y)} \to T^*_y(Y)$ is the transpose.

More generally, we can consider the functor which assigns to any vector bundle E over X the vector bundle $E \otimes T^*(X)$ and to any morphism $\mathbf{f}$: $F \to E$ the morphism $\mathbf{f} \otimes \mathbf{T}^*(f)$: $F \otimes T^*(Y) \to E \otimes T^*(X)$. If s is a smooth section of E which vanishes at the point x then $ds(x) \in E_x \otimes T_x^*(X)$ is well defined. (Indeed, compute the differential in terms of some trivialization. The fact that $s(x) = 0$ implies that the result is independent of the trivialization.) If $\mathbf{f}$: $F \to E$ and $y \in f^{-1}(x)$ then $\mathbf{f}^* s(x) = 0$ and $d(\mathbf{f}^* s)(y) = (r_y \otimes df_y^t) ds(x)$.

§2. The fiber product.

Let f: $A \to B$ and g: $C \to B$ be maps of sets. Then the *fiber product* $D = A \times_B C$ consists of those pairs (a, c) such that $f(a) = g(c)$. We shall usually denote this situation by a diagram such as:

$$\begin{tikzcd} D \arrow[r, dashed] \arrow[d, dashed] & C \arrow[d] \\ A \arrow[r] & B \end{tikzcd}$$

If we let $\Delta \subset B \times B$ denote the diagonal, $\Delta = \{(b, b)\}$, and consider (f, g): $A \times C \to B \times B$ where $(f, g)(a, c) = (f(a), g(c))$ then $D \subset A \times C$ is given by

$$D = (f, g)^{-1} \Delta.$$

This expression lends itself to a formulation in terms of differentialbe manifolds. If A, C, and B are manifolds with f and g smooth, then f and g are transversal (we write $f \pitchfork g$) if and only if (f, g) is transversal to Δ. In this case $(f, g)^{-1} \Delta$ will be a submanifold so that the fiber product D is again a manifold. Let us give various examples of this construction.

(i) *Pull-back of fiber bundles.* If $f\colon A \to B$ is a submersion then $f \pitchfork g$ for any g. In particular, if A is a fiber bundle over B then D is a fiber bundle over C, the pull-back bundle which we denote by $f^{\#}A$.

(ii) *Push-forward and pull-back under morphism.* Let $E \to X$ and $F \to Y$ be vector bundles and suppose that we are given a morphism from E to F as discussed in §1. This means that we are given a "graph" $\subset E \times F$ consisting of all points of the form $(x, f(x), r(x)u, u)$ where $u \in F_{f(x)}$. Let A denote this graph, which is a submanifold of $B = E \times F$. Let S be a subset of E. Then we may let $C = S \times F$, and form the fiber product D, which consists of all points of the form $(x, f(x), r(x)u, u)$ with $(x, r(x)u) \in S$. We may project D onto F, its image in F will be denoted by $f_* S$. If S is a submanifold and C is transversal to A then $f_* S$ is again a submanifold.

Similarly, suppose that S' is a subset of F and we let $C' = E \times S'$. Then the fiber product D' consists of those points $(x, f(x), r(x)u, u)$ with $(f(x), r(x)u) \in S'$. We can project onto E and obtain a set which we shall denote by $f^* S'$. If S' is a submanifold and C' is transversal to A then $f^* S'$ will be a submanifold of E.

As two examples of this construction, suppose first that $E = X$ and $F = Y$ are the zero bundles. Then $B = \operatorname{graph} f \subset X \times Y$ and $f_* S = f(S)$ and $f^* S' = f^{-1} S'$. Here $S \times Y$ is always transversal to graph f but the condition that $X \times S'$ be transversal to B is an honest restriction, indeed it is the condition that S' be transversal to f.

Next let us take $E = T^*X$ and $F = T^*Y$, where $f\colon X \to Y$ induces the standard morphism on the cotangent bundles. In this case $B = \{(x, f(x), df^*\xi, \xi)\}$ is the *anti*-normal bundle to graph f—it becomes the normal bundle if we replace ξ by $-\xi$ and leave $df^*\xi$ alone in the above expression. Notice that B is a Lagrangian submanifold for $T^*X \times T^*Y$ if we use the symplectic structure $\Omega_y - \Omega_x$. Here f_* carries subsets of T^*X to subsets of T^*Y and f^* does the opposite. If the appropriate transversality conditions are satisfied, then f_* and f^* carry submanifolds into submanifolds. If Λ_X is a Lagrangian submanifold of T^*X and $\Lambda_X \times T^*Y$ intersects B transversally then it is proved in Chapter IV that $f_* \Lambda_X$ is Lagrangian. Similarly for f^*. In fact, the arguments given there extend to the case where the fiber product comes from a *clean intersection*: Suppose that

$$\begin{tikzcd} D \arrow[r, dashed] \arrow[d, dashed] & C \arrow[d] \\ A \arrow[r] & B \end{tikzcd}$$

is a fiber product diagram in which A, B, and C are manifolds and f and g are smooth maps, and it is *assumed* that the fiber product, D, is a submanifold of $A \times C$ and that the differentials induce a fiber product diagram at all points

$d \in D$. Then we say that we have a clean intersection, or that f and g intersect cleanly. Thus, if f and g intersect transversally then they intersect cleanly; but clean intersections need not be transversal (for instance $A = C$ and $f = g$ is a clean intersection).

Let X and Y be symplectic manifolds; let Γ be a Lagrangian submanifold of $X \times Y$; let Λ be a Lagrangian submanifold of X so that

$$\begin{tikzcd} Y & & Y \\ D \arrow[u, "\pi_Y"] \arrow[r, dashed] \arrow[d, dashed] & \Gamma \arrow[r] \arrow[d] & X \times Y \arrow[u] \\ \Lambda \arrow[r] & X & \end{tikzcd}$$

is a clean intersection fiber product. *Then* $\pi_Y \colon D \to Y$ *maps* D *onto a Lagrangian submanifold of* Y.

To prove this assertion for vector spaces: Let V and W be symplectic vector spaces, with C a Lagrangian subspace of $V \times W$ and L a Lagrangian subspace of V. Let F be defined by the fiber product diagram:

$$\begin{tikzcd} W & & W \\ F \arrow[u, "\pi_W"] \arrow[r, dashed] \arrow[d, dashed] & C \arrow[r] \arrow[d] & V \times W \arrow[u] \\ L \arrow[r] & V & \end{tikzcd}$$

Then $F \to C \xrightarrow{\pi_W} W$ *maps* F *onto a Lagrangian subspace of* W.

PROOF. Let $G = \pi_W F$, so that we have the exact sequence

$$0 \to \ker \rho \to F \xrightarrow{\rho} G \to 0$$

where ρ is the composite map $F \to C \to G$. Here $\ker \rho$ can be identified with the space of all $v \in L$ such that $\{v, 0\} \in C$. Let $(\ ,\ )_V$ and $(\ ,\ )_W$ be the symplectic forms of V and W, and let w_1, w_2 be elements of G. Then $w_i = \pi_W\{v_i, w_i\}$ with $\{v_i, w_i\} \in C$ and $v_i \in L$. Thus

$$(w_1, w_2)_W = (\{v_1, w_1\}, \{v_2, w_2\})_{V \times W} - (v_1, v_2)_V = 0$$

so G is isotropic. We must prove that $\dim G = \frac{1}{2} \dim W$. If the diagram were transversal, we would have $\dim F = \dim C + \dim L - \dim V = \frac{1}{2} \dim V$

$+ \frac{1}{2}(\dim V + \dim W) - \dim V = \frac{1}{2} \dim W$, $\ker \rho = \{0\}$. In general, let $I \subset V$ denote the image of $L \oplus C$ in V, so that I consists of all vectors $u = v_1 + v_2$ where $v_1 \in L$ and $(v_2, w_2) \in C$ for some w_2. If $\{v, 0\} \in \ker \rho$, then $(v, v_1)_V = 0$ since $v \in L$ and $(v, v_2) = 0$ since $\{v, 0\} \in C$. Thus v annihilates I. Conversely, if $(v, v_2) = 0$ for all v_2 with $\{v_2, w_2\} \in C$, then since C is maximally isotropic, $\{v, 0\} \in C$, and, similarly, if $(v, v_1) = 0$ for all $v_1 \in L$ then $v \in L$. Thus $\ker \rho$ consists of those $\{v, 0\}$ with $(v, I)_V = 0$, or $\dim \ker \rho = \dim V - \dim I$, so that

$$
\begin{aligned}
\dim G &= \dim F - \dim \ker \rho \\
&= \dim C + \dim L - \dim I - (\dim V - \dim I) \\
&= \tfrac{1}{2} \dim W,
\end{aligned}
$$

as was to be proved.

The corresponding theorem for manifolds now follows by applying the above argument pointwise in each tangent space.

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