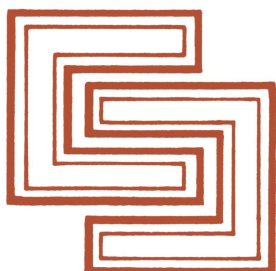


SYMPLECTIC GROUPS

By O. T. O'MEARA



MATHEMATICAL SURVEYS

NUMBER 16

Published by the American Mathematical Society

SYMPLECTIC GROUPS

MATHEMATICAL SURVEYS · *Number 16*

SYMPLECTIC GROUPS

BY

O. T. O'MEARA

1978

AMERICAN MATHEMATICAL SOCIETY
PROVIDENCE, RHODE ISLAND

TO JEAN

– my wild Irish rose

Library of Congress Cataloging in Publication Data

O'Meara, O. Timothy, 1928—

Symplectic groups.

(Mathematical surveys; no. 16)

Based on lectures given at the University of Notre Dame, 1974–1975.

Bibliography: p.

Includes index.

1. Symplectic groups. 2. Isomorphisms (Mathematics)

I. Title. II. Series: American Mathematical Society.

Mathematical surveys; no. 16.

QA171.046

512'.2

78-19101

ISBN 0-8218-1516-4

AMS (MOS) subject classifications (1970). Primary 15–02, 15A33, 15A36,
20–02, 20B25, 20D45, 20F55, 20H05, 20H20, 20H25; Secondary 10C30,
13–02, 20F35, 20G15, 50D25, 50D30.

Copyright © 1978 by the American Mathematical Society

Reprinted with corrections, 1982

Printed in the United States of America

All rights reserved except those granted to the United States Government.
Otherwise, this book, or parts thereof, may not be reproduced in any form
without permission of the publishers.

CONTENTS

Preface.....	ix
Prerequisites and Notation.....	xi
Chapter 1. Introduction	
1.1. Alternating Spaces.....	1
1.2. Projective Transformations.....	9
1.3. Residues.....	12
1.4. Transvections.....	15
1.5. Matrices.....	17
1.6. Projective Transvections.....	18
1.7. Some Theorems about SL_n	19
1.8. Comments.....	20
Chapter 2. Generation Theorems	
2.1. Generation by Transvections in Sp_n	23
2.2. Elementary Generation of Sp_n	31
2.3. Comments.....	33
Chapter 3. Structure Theorems	
3.1. Orders of Symplectic Groups.....	35
3.2. Centers.....	37
3.3. Commutator Subgroups.....	38
3.4. Simplicity Theorems.....	41
3.5. Comments.....	43
Chapter 4. Symplectic Collinear Transformations	
4.1. Collinear Transformations.....	45
4.2. Symplectic Collinear Transformations.....	49
4.3. Hyperbolic Transformations.....	53
Chapter 5. The Isomorphisms of Symplectic Groups	
5.1. Groups with Enough Projective Transvections.....	57
5.2. Preservation of Projective Transvections.....	58
5.3. The Isomorphism Theorems in General.....	67
5.4. 4-Dimensional Groups in Characteristic 2.....	70
5.5. Bounded Modules over Integral Domains.....	92
5.6. The Isomorphism Theorems over Integral Domains.....	98
5.7. Comments.....	106
Chapter 6. The Nonisomorphisms between Linear and Symplectic Groups	
6.1. The Nonisomorphisms.....	111
Index.....	121

PREFACE

My goal in these lectures is the isomorphism theory of symplectic groups over integral domains as illustrated by the theorem

$$\mathrm{PSp}_n(o) \cong \mathrm{PSp}_{n_1}(o_1) \Leftrightarrow n = n_1 \text{ and } o \cong o_1$$

for dimensions ≥ 4 . This is a sequel to my *Lectures on Linear Groups* where there was a similar objective with the linear groups in mind. Once again I will start from scratch assuming only basic facts from a first course in algebra plus a minimal number of references to the *Linear Lectures*. The simplicity of $\mathrm{PSp}_n(F)$ will be proved. My approach to the isomorphism theory will be more geometric and more general than the *CDC* approach that has been in use for the last ten years and that I used in the *Linear Lectures*. This geometric approach will be instrumental in extending the theory from subgroups of PSp_n ($n \geq 6$), where it is known, to subgroups of $\mathrm{P}\Gamma\mathrm{Sp}_n$ ($n \geq 4$), where it is new⁽¹⁾. There will be an extensive investigation and several new results⁽¹⁾ on the exceptional behavior of subgroups of $\mathrm{P}\Gamma\mathrm{Sp}_4$ in characteristic 2.

These notes are taken from lectures given at the University of Notre Dame during the school year 1974–1975. I would like to express my thanks to Alex Hahn, Kok-Wee Phan and Warren Wong for several stimulating discussions.

O. T. O'Meara

Notre Dame, Indiana
March 1976

⁽¹⁾The research aspects of these notes were supported by National Science Foundation grant MPS74-06446 A01.

PREREQUISITES AND NOTATION

We assume a knowledge of the basic facts about sets, groups, fields and vector spaces.

If X and Y are sets, then $\text{pow } X$ will denote the set of all subsets of X ; $X \subset Y$ will denote strict inclusion; $X - Y$ will denote the difference set; $X \rightarrow Y$ will denote a surjection, $X \hookrightarrow Y$ an injection, $X \xrightarrow{\sim} Y$ a bijection, and $X \dashrightarrow Y$ an arbitrary mapping. If $f: X \dashrightarrow Y$ is a mapping and Z is a subset of X , i.e., Z is an element or point in $\text{pow } X$, then fZ is the subset $\{fz | z \in Z\}$ of Y ; this provides a natural extension of $f: X \dashrightarrow Y$ to $f: \text{pow } X \dashrightarrow \text{pow } Y$, namely the one obtained by sending Z to fZ for all Z in $\text{pow } X$; if f is respectively injective, surjective, bijective, then so is its extension to the power sets.

If X is any additive group, in particular, if X is a field or a vector space, then X^\times will denote the set of nonzero elements of X ; if X is a field, then X^\times is to be regarded as a multiplicative group. Use \mathbb{F}_q for the finite field of q elements. By a line, plane, hyperplane, in a finite n -dimensional vector space we mean a subspace of dimension 1, 2, $n - 1$, respectively.

V will always denote an n -dimensional vector space over a (commutative) field F with $0 \leq n < \infty$. After the appropriate definitions have been made (in fact, starting with Chapter 2) it will be assumed that V is also a nonzero regular alternating space, i.e., that V is provided with a regular alternating form $q: V \times V \rightarrow F$ with $2 \leq n < \infty$. And V_1, F_1, n_1, q_1 will denote a second such situation.

These lectures on the symplectic group are a sequel to:

O. T. O'MEARA, *Lectures on linear groups*, CBMS Regional Conf. Ser. in Math., no. 22, Amer. Math. Soc., Providence, R.I., 1974, 87 pp.

which will be referred to as the *Linear Lectures*. In general we will try to keep things self-contained. Our general policy will be to redevelop concepts and restate propositions needed from the *Linear Lectures*, but not to rework proofs.

INDEX

- algebraic group, 20, 43
- alternating bilinear form, 1
- alternating matrix, 2
- alternating space, 1
- Artin, E., 21, 107
- bar symbol, 10
- big dilation, 20
- big group, 107
- Borel, A., 21, 107
- bounded module, 94
- Callan, D., 33
- Carter, R. W., 21
- center, 37
- Chevalley, C., 43
- Chevalley group, 20, 43
- classical group, 20
- coefficient, 94
- collinear group, 47
- collinear transformation, 46, 47
- commutator subgroup, 38
- component, 4
- degenerate space, 4
- degenerate transformation, 14
- Dickson, L. E., 21
- Dieudonné, J., 21, 33, 43, 106, 109
- discriminant, 2, 3
- elementary generation, 31
- elementary matrix, 18
- elementary transvection, 18
- enough hyperbolic transformations, 92
- enough projective hyperbolic transformations, 85
- enough projective transvections, 57
- enough transvections, 69
- exceptional automorphism, 90, 92
- field isomorphism, 45
- fixed space, 12, 18
- fractional ideal, 93
- free module, 94
- full of projective transvections, 111
- full of transvections, 119
- Fundamental Theorem of Projective Geometry, 47
- general geometric group, 11
- general linear group, 1, 11
- generation questions, 21, 33
- generation results, 20, 23, 25, 31
- geometric transformation, 9, 11
- group of collinear transformations, 47
- group of geometric transformations, 11
- group of linear transformations, 11
- group of projectivities, 11
- group of symplectic collinear transformations, 50
- group of symplectic similitudes, 50
- group of symplectic transformations, 12
- Hahn, A. J., ix, 108
- Hua, L.-K., 106, 107
- Humphreys, J. E., 107
- hyperbolic base, 7
- hyperbolic transformation, 24
- hyperplane, 10
- ideal, 93
- imprimitive, 42
- integral ideal, 93
- integral linear group, 96
- integral matrix, 97
- integral matrix group, 97
- integral symplectic group, 98
- invertible ideal, 93
- involution, 13
- isometry, 1
- isomorphism theorems, 69, 70, 91, 92, 99, 100, 104
- lattice, 95
- line, 10
- linear congruence group, 96
- linear group, 20
- matrix congruence group, 97
- matrix of a semilinear map, 46
- matrix of a space, 2
- McDonald, B. R., 108
- McQueen, L., 108
- Merzlyakov, Yu., 33, 43, 109
- module, in and on, 93
- multiplier, 49
- nonisomorphism theorems, 118, 120
- O'Meara, O. T., xi, 107, 108, 109
- orders of linear groups, 20

- orders of symplectic groups, 35
- orthogonal complement, 5
- orthogonal group, 20
- orthogonal splitting, 4
- permutation group, 42
- permute projectively, 49
- Phan, K.-W., ix
- plane, 10
- polarity, 12
- preservation of projective transvections, 57
- primitive, 42
- principal ideal, 93
- product of ideals, 93
- product of ideal with module, 93
- projective collinear group, 47
- projective collinear transformation, 47
- projective general linear group, 11
- projective geometric transformation, 10
- projective group of collinear transformations, 47
- projective group of linear transformations, 11
- projective group of symplectic collinear transformations, 51
- projective group of symplectic similitudes, 51
- projective group of symplectic transformations, 12
- projective hyperbolic transformation, 56
- projective integral linear group, 96
- projective integral matrix group, 97
- projective integral symplectic group, 98
- projective isomorphism of a base, 18
- projective linear congruence group, 96
- projective matrix congruence group, 97
- projective space, 10
- projective special linear group, 11
- projective symplectic collinear group, 51
- projective symplectic collinear transformation, 49
- projective symplectic congruence group, 98
- projective symplectic group, 12
- projective transvection, 18
- projectivity, 10, 11
- radiation, 11
- radical, 5
- radical splitting, 6
- regular space, 4
- regular transformation, 14
- Reiner, I., 107
- representation, 1
- representative, 47
- representative transvection, 18
- residual space, 12, 18
- residue, 12
- Rickart, C. E., 107
- Schreier, O., 106
- semilinear map, 45
- simplicity theorems, 20, 41, 42
- skewsymmetric matrix, 2
- Solazzi, R. E., 108
- special linear group, 11
- Spengler, U., 33
- splitting, 4
- Steinberg, R., 107
- structure theory, 43
- symplectic base, 7, 100
- symplectic collinear group, 50
- symplectic collinear transformation, 49, 50
- symplectic congruence group, 98
- symplectic group, 1
- symplectic similitude, 50
- Tits, J., 43, 108, 109
- totally degenerate space, 4
- totally degenerate transformation, 14
- transitive, 42
- transvection, 15
- underlying, 57
- unimodular matrix, 97
- unitary group, 20
- Van der Waerden, B. L., 21, 106
- Wan, Z.-X., 108
- Wang, Y.-X., 108
- Witt's Theorem, 9
- Wolff, H., 33
- Wong, W. J., ix, 108

