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By O. T. O’MEARA

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PREFACE

My goal in these lectures is the isomorphism theory of symplectic groups over integral domains as illustrated by the theorem

$$\text{PSp}_n(o) \cong \text{PSp}_{n_1}(o_1) \iff n = n_1 \text{ and } o \cong o_1$$

for dimensions $\geq 4$. This is a sequel to my Lectures on Linear Groups where there was a similar objective with the linear groups in mind. Once again I will start from scratch assuming only basic facts from a first course in algebra plus a minimal number of references to the Linear Lectures. The simplicity of $\text{PSp}_n(F)$ will be proved. My approach to the isomorphism theory will be more geometric and more general than the CDC approach that has been in use for the last ten years and that I used in the Linear Lectures. This geometric approach will be instrumental in extending the theory from subgroups of $\text{PSp}_n (n \geq 6)$, where it is known, to subgroups of $\text{PTSp}_n (n \geq 4)$, where it is new\(^1\). There will be an extensive investigation and several new results\(^1\) on the exceptional behavior of subgroups of $\text{PTSp}_4$ in characteristic 2.

These notes are taken from lectures given at the University of Notre Dame during the school year 1974-1975. I would like to express my thanks to Alex Hahn, Kok-Wee Phan and Warren Wong for several stimulating discussions.

O. T. O’Meara

Notre Dame, Indiana
March 1976

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PREREQUISITES AND NOTATION

We assume a knowledge of the basic facts about sets, groups, fields and vector spaces.

If $X$ and $Y$ are sets, then $\text{pow} \ X$ will denote the set of all subsets of $X$; $X \subset Y$ will denote strict inclusion; $X - Y$ will denote the difference set; $X \rightarrow Y$ will denote a surjection, $X \leftarrow Y$ an injection, $X \leftrightarrow Y$ a bijection, and $X \rightarrow Y$ an arbitrary mapping. If $f: X \rightarrow Y$ is a mapping and $Z$ is a subset of $X$, i.e., $Z$ is an element or point in $\text{pow} \ X$, then $fZ$ is the subset $\{fz | z \in Z\}$ of $Y$; this provides a natural extension of $f: X \rightarrow Y$ to $f: \text{pow} \ X \rightarrow \text{pow} \ Y$, namely the one obtained by sending $Z$ to $fZ$ for all $Z$ in $\text{pow} \ X$; if $f$ is respectively injective, surjective, bijective, then so is its extension to the power sets.

If $X$ is any additive group, in particular, if $X$ is a field or a vector space, then $\hat{X}$ will denote the set of nonzero elements of $X$; if $X$ is a field, then $\hat{X}$ is to be regarded as a multiplicative group. Use $F_q$ for the finite field of $q$ elements. By a line, plane, hyperplane, in a finite $n$-dimensional vector space we mean a subspace of dimension 1, 2, $n - 1$, respectively.

$V$ will always denote an $n$-dimensional vector space over a (commutative) field $F$ with $0 < n < \infty$. After the appropriate definitions have been made (in fact, starting with Chapter 2) it will be assumed that $V$ is also a nonzero regular alternating space, i.e., that $V$ is provided with a regular alternating form $q: V \times V \rightarrow F$ with $2 < n < \infty$. And $V_1, F_1, n_1, q_1$ will denote a second such situation.

These lectures on the symplectic group are a sequel to:


which will be referred to as the Linear Lectures. In general we will try to keep things self-contained. Our general policy will be to redevelop concepts and restate propositions needed from the Linear Lectures, but not to rework proofs.
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