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APPROXIMATION BY POLYNOMIALS WITH INTEGRAL COEFFICIENTS

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APPROXIMATION BY POLYNOMIALS WITH INTEGRAL COEFFICIENTS

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To the memory of SEABURY COOK

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PREFACE

Results in the approximation of functions by polynomials with coefficients which are integers have been appearing since that of Pál in 1914. The body of results has grown to an extent which seems to justify the present book. The intention here is to make these results as accessible as possible.

Aside from the intrinsic interest to the pure mathematician, there is the likelihood of important applications to other areas of mathematics; for example, in the simulation of transcendental functions on computers. In most computers, fixed point arithmetic is faster than floating point arithmetic and it may be possible to take advantage of this fact in the evaluation of integral polynomials to create more efficient simulations. Another promising area for applications of this research is in the design of digital filters. A central step in the design procedure is the approximation of a desired system function by a polynomial or rational function. Since only finitely many binary digits of accuracy actually can be realized for the coefficients of these functions in any real filter the problem amounts (to within a scale factor) to approximation by polynomials or rational functions with integral coefficients. For more details one may consult this author's listing in the Bibliography. It would be gratifying to the author if this book stimulates research in this direction.

Most of the results here have already appeared in the literature. However, for the expert, we mention the following exceptions: Corollaries 7.17, 7.20, Propositions 7.16, 9.8, and Theorems 9.7, 9.9, 9.10, 9.11, A.4, A.5.

It is a pleasure to acknowledge the help of many people in the writing of this book. It was my advisor, Edwin Hewitt, who initially brought the problem to my attention. G. G. Lorentz suggested the book itself. In learning the subject, especially as it relates to number theory, I am indebted to a number of valuable conversations with David Cantor. I would also like to express my gratitude for the support of the institutions listed at the end of the Bibliography and to the Air Force Office of Scientific Research for partial support from grants numbered AFOSR 71-2030 and AFOSR 78-3599. Finally, I thank Mrs. Joyce Kepler for her excellent services as typist.

Riverside
February, 1976

La vie est breve:
Un peu d'espoir,
Un peu de rêve
Et puis–bonsoir!   Leon Montenaeken

APPENDIX

APPROXIMATION AT ALGEBRAIC INTEGERS

The purpose of this appendix is to establish some fundamental theorems on approximation by integral polynomials on finite sets of algebraic integers. The first three derive from a conversation with David Cantor. These are essential in the more general situation. Throughout, the symbol z will be used only to represent an element of $\mathbf{C}$ and n-tuples of complex numbers will be written out in full. Also, A will be any discrete subring of $\mathbf{C}$ of rank 2 and L the unique imaginary quadratic field such that $A \subset I_L$ (Proposition 1.10).

The first theorem tells us that we can approximate by integral polynomials on any incomplete set of conjugate algebraic integers over the imaginary quadratic field containing A. In the case of a complete set of conjugate algebraic integers, the situation is just the opposite; we can approximate only what we can interpolate. This is a corollary of Proposition 3.7.

THEOREM A.1. *Let $\alpha_1, \ldots, \alpha_n$ be a complete set of conjugate algebraic integers over L, ε any positive number, and $z_2, \ldots, z_n$ any complex numbers. Then there is a polynomial $q \in A[z]$ such that*

$$|q(\alpha_j) - z_j| < \varepsilon, \qquad 2 \leqslant j \leqslant n.$$

PROOF. From Proposition 1.11 we know that there is a positive rational integer m such that $mI_L \subset A$. If the theorem were true for I_L in place of A we could find $q_0 \in I_L[z]$ satisfying

$$|q_0(\alpha_j) - z_j/m| < \varepsilon/m, \qquad 2 \leqslant j \leqslant m$$

from which the conclusion of the general theorem follows if we take $q = mq_0$. Thus we can assume from the outset that $A = I_L$.

By Proposition 1.7, I_L is a discrete subring of $\mathbf{C}$ of rank 2. Then $(I_L)^n$ is a discrete subgroup of $\mathbf{C}^n$ with rank $2n$. We identify $\mathbf{C}^n$ with $\mathbf{R}^{2n}$ by the map $(z_1, \ldots, z_n) \to (\operatorname{Re} z_1, \operatorname{Im} z_1, \ldots, \operatorname{Re} z_n, \operatorname{Im} z_n)$. Thus, by Cassels [**59**, Theorem VI, p. 78], $(I_L)^n$ is a lattice in $\mathbf{R}^{2n}$. Consider the complex linear transformation T of $\mathbf{C}^n$ represented by the matrix

$$\begin{bmatrix} 1 & \alpha_1 & \cdots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \cdots & \alpha_2^{n-1} \\ & & \cdots & \\ 1 & \alpha_n & \cdots & \alpha_n^{n-1} \end{bmatrix} \tag{1}$$

with respect to the canonical basis of $\mathbf{C}^n$ as a complex linear space. It is clear from definitions that T is also a real linear transformation of the $2n$-dimensional real linear space underlying $\mathbf{C}^n$. We denote this real linear transformation by the different symbol T_0, since, in general, we will have $\det T \neq \det T_0$. Set

$$c_1 = 2^{n-1/2}(\det T_0)\delta\big((I_L)^n\big)$$

and $c_2 = c_3 = \cdots = c_{2n} = 2^{-1/2}$, where $\delta((I_L)^n)$ denotes the determinant of the lattice $(I_L)^n$ (Cassels [59, p. 10]). Then by Cassels [59, Theorem III, p. 73] there exists $(y_1, \ldots, y_n)$ in $(I_L)^n$ such that $(y_1, \ldots, y_n) \neq (0, \ldots, 0)$ and

$$|\mathrm{Re}(w_j)| < 2^{-1/2}, \quad |\mathrm{Im}(w_j)| < 2^{-1/2}, \qquad 2 \leqslant j \leqslant n,$$

where $(w_1, \ldots, w_n) = T(y_1, \ldots, y_n)$. Consequently,

$$|w_j|^2 = |\mathrm{Re}(w_j)|^2 + |\mathrm{Im}(w_j)|^2 < 1, \qquad 2 \leqslant j \leqslant n.$$

If we define $\tilde{q}(z) = y_1 + y_2 z + \cdots + y_n z^{n-1}$ then we have, by the representation (1) for T,

$$|\tilde{q}(\alpha_j)| = |w_j| < 1, \qquad 2 \leqslant j \leqslant n. \tag{2}$$

Since L has characteristic zero, the minimal polynomial over L of the set of conjugates $\alpha_1, \ldots, \alpha_n$ is separable, i.e., the α's are distinct. Since the determinant of (1) is $\Pi_{1 \leqslant i < j \leqslant n}(\alpha_j - \alpha_i)$, T is a nonsingular transformation. Thus, since $(y_1, \ldots, y_n) \neq (0, \ldots, 0)$, we have $w_j = \tilde{q}(\alpha_j) \neq 0$ for some j. Since the α's are conjugate to α_j, this implies that

$$\tilde{q}(\alpha_j) \neq 0, \qquad 1 \leqslant j \leqslant n.$$

Now suppose that p is any polynomial with complex coefficients and degree $< n - 1$. If $[p]$ denotes a polynomial obtained by replacing each coefficient of p by a nearest integer (i.e., element of I_L) we have

$$|([p] - p)(\alpha_j)| \leqslant \sum_{m=0}^{n-2} \delta|\alpha_j^m|, \qquad 2 \leqslant j \leqslant n, \tag{3}$$

where δ is defined in Proposition 1.2. By (2) there is a positive integer k such that

$$\left(\sum_{m=0}^{n-2} \delta|\alpha_j^m|\right)|\tilde{q}(\alpha_j)|^k < \varepsilon, \qquad 2 \leqslant j \leqslant n. \tag{4}$$

Let p be the Lagrange interpolating polynomial such that $p(\alpha_j) = z_j/\tilde{q}^k(\alpha_j)$, $2 \leqslant j \leqslant n$. Then $\deg p < n - 1$ so (3) and (4) give

$$\begin{aligned}|[p](\alpha_j)\tilde{q}^k(\alpha_j) - z_j| &= |[p](\alpha_j)\tilde{q}^k(\alpha_j) - p(\alpha_j)\tilde{q}^k(\alpha_j)| \\ &\leqslant |([p] - p)(\alpha_j)|\,|\tilde{q}(\alpha_j)|^k \\ &< \varepsilon, \qquad 2 \leqslant j \leqslant n.\end{aligned}$$

The conclusion of the theorem holds if we set $q = [p]\tilde{q}^k$. □

Utilizing some ideas from algebraic number theory, it is possible to give a shorter proof of Theorem A.1 as follows. We assume as before that $A = I_L$. Let p be the minimal polynomial of $\alpha_1, \ldots, \alpha_n$. Adjoin a root θ of p to L to get a field $F = L(\theta)$. Then p is the minimal polynomial of θ and p factors into linear factors over the completion $\hat{L} = \mathbf{C}$ of L with respect to the usual Archimedean valuation $|\cdot|$ on L. Let $|\cdot|_1, \ldots, |\cdot|_n$ be the extensions to F of $|\cdot|$ corresponding to the linear factors $x - \alpha_1, \ldots, x - \alpha_n$, respectively (Bachman [**64**, p. 133]). Then any element of $L(\theta)$ can be expressed in the form $p'(\theta)$ with $p' \in L[x]$ and $|p'(\theta)|_i = |p'(\alpha_i)|$.

There exists $b \in I_L$ such that $\theta_1 = b\theta$ is integral over I_L. Since $1, \theta, \ldots, \theta^{n-1}$ is a base for $L(\theta)$ over L, we see that $1, \theta_1, \ldots, \theta_1^{n-1}$ is also a base and consists of elements integral over I_L. Thus by Lang [**70**, Proposition 6, proof, p. 6] there exists $c \in I_L$, $c \neq 0$, such that $I_F \subset c^{-1}(I_L + I_L\theta_1 + \cdots + I_L\theta_1^{n-1})$; hence

$$I_F \subset c^{-1}(I_L + I_L\theta + \cdots + I_L\theta^{n-1}). \tag{$*$}$$

Since L is dense in $\mathbf{C}$ we can assume that $z_2, \ldots, z_n \in L$. By the very strong approximation theorem (O'Meara [**63**, p. 77]) there exists $a \in F$ such that

$$|a - c^{-1}z_i|_i < |c|_i^{-1}\varepsilon, \qquad 2 \leqslant i \leqslant n,$$

and $|a|_\mathfrak{p} \leqslant 1$, $\mathfrak{p}$ non-Archimedean. Since $|a|_\mathfrak{p} \leqslant 1$ for all non-Archimedean $\mathfrak{p}$, $a \in I_F$. Thus, by ($*$) there is a polynomial $q \in I_L[x]$ such that $c^{-1}q(\theta) = a$. Then

$$|c^{-1}q(\theta) - c^{-1}z_i|_i < |c|_i^{-1}\varepsilon, \qquad 2 \leqslant i \leqslant n;$$

hence $|q(\theta) - z_i|_i < \varepsilon$, $2 \leqslant i \leqslant n$, and $|q(\alpha_i) - z_i| < \varepsilon$, $2 \leqslant i \leqslant n$.

Our next result generalizes this theorem to the case of an arbitrary finite number of incomplete sets of conjugate algebraic integers.

THEOREM A.2. *Let*

$$\begin{matrix}\alpha_{11} & ,\ldots, & \alpha_{1r_1} \\ \alpha_{21} & ,\ldots, & \alpha_{2r_2} \\ & \cdots & \\ \alpha_{s1} & ,\ldots, & \alpha_{sr_s}\end{matrix}$$

be an array (not necessarily rectangular) with each row an incomplete set of conjugates integral over I_L and where the minimal polynomials $p_1, \ldots, p_s$ satisfied by the respective rows are distinct. If the array

$$\begin{matrix}z_{11} & ,\ldots, & z_{1r_1} \\ z_{21} & ,\ldots, & z_{2r_2} \\ & \cdots & \\ z_{s1} & ,\ldots, & z_{sr_s}\end{matrix}$$

consists of any complex numbers and $\varepsilon > 0$, *then there exists a* q *in* $A[z]$ *such that*

$$|q(\alpha_{ij}) - z_{ij}| < \varepsilon, \qquad 1 \leqslant i \leqslant s, 1 \leqslant j \leqslant r_i.$$

PROOF. Let $q_i' = \prod_{k \neq i} p_k$, $1 \leqslant i \leqslant s$. Then $q_i'(\alpha_{kl}) = 0$ if and only if $k \neq i$, by definition of the p's. For each i ($1 \leqslant i \leqslant s$) there exists by Theorem A.1 a q_i'' in $A[z]$ such that

$$\left| q_i''(\alpha_{ij}) - \frac{z_{ij}}{q_i'(\alpha_{ij})} \right| < \frac{\varepsilon}{|q_i'(\alpha_{ij})|}, \qquad 1 \leqslant j \leqslant r_i.$$

Thus $|(q_i'' q_i')(\alpha_{ij}) - z_{ij}| < \varepsilon$, $1 \leqslant j \leqslant r_i$. If we set $q = q_1'' q_1' + \cdots + q_s'' q_s'$ then $q(\alpha_{ij}) = (q_i'' q_i')(\alpha_{ij})$ since $q_k'(\alpha_{ij}) = 0$ if $k \neq i$. Thus

$$|q(\alpha_{ij}) - z_{ij}| = |(q_i'' q_i')(\alpha_{ij}) - z_{ij}| < \varepsilon$$

for all i, j. □

We note in passing that another way of looking at Theorem A.2 is the following.

COROLLARY A.3. *If* $\{\alpha_1, \ldots, \alpha_k\}$ *is any set of algebraic integers which does not contain a complete set of conjugates over* L, *then the set of* k*-tuples* $\{(q(\alpha_1), \ldots, q(\alpha_k)) \colon q \in A[z]\}$ *is dense in* $\mathbf{C}^k$.

Another way in which Theorem A.1 may be generalized is to require that the derivatives at the given points also approximate. We needed this extension of the theorem in Chapter 7.

THEOREM A.4. *Let* $z_1, \ldots, z_n$ *be a complete set of conjugate (over* L*) algebraic integers in* $\mathbf{C}$ *where* L *is any imaginary quadratic field. Let* f *and* h *be defined and holomorphic functions in an open set containing* $\{z_2, \ldots, z_n\}$ *and*

$$|f(z_j)/h(z_j)| < \infty$$

for $2 \leqslant j \leqslant n$. *If* m *is any nonnegative integer and* $\varepsilon > 0$, *then there exists an integral polynomial* q *in* $A[z]$ *such that*

$$|f^{(\nu)}(z_j) - (hq)^{(\nu)}(z_j)| < \varepsilon, \qquad 0 \leqslant \nu \leqslant m, \quad 2 \leqslant j \leqslant n.$$

PROOF. We can assume without loss of generality that $A = I_L$ as follows. By Proposition 1.11 there is a positive integer m_0 such that $m_0 I_L \subset A$. If we knew the result for $A = I_L$ then we could find a $q \in I_L[z]$ satisfying

$$|f^{(\nu)}(z_j)/m_0 - (hq)^{(\nu)}(z_j)| < \varepsilon/m_0, \qquad 0 \leqslant \nu \leqslant m, \quad 2 \leqslant j \leqslant n;$$

hence

$$|f^{(\nu)}(z_j) - (hm_0 q)^{(\nu)}(z_j)| < \varepsilon, \qquad 0 \leqslant \nu \leqslant m, \quad 2 \leqslant j \leqslant n,$$

and $m_0 q \in m_0 I_L[z] \subset A[z]$.

By Theorem A.1 there is a $\tilde{q} \in I_L[z]$ with $0 < |\tilde{q}(z_j)| < 1$, $2 \leqslant j \leqslant n$. Choose $\rho > 0$ so small that the disks of radius ρ and centers $\{z_2, \ldots, z_n\}$ are disjoint, lie within the domain of definition of f and h, and $0 < |q| < 1$ and $h \neq 0$ on $H = \bigcup_{j=2}^{n}(z_j + \rho D)$ where D is the closed unit disk. Let $\Delta = \max\{\nu! \rho^{-\nu}\}_{\nu=0}^{m}$.

Choose ε' so that $\Delta 2\varepsilon' < \varepsilon$ and then a positive integer λ such that

$$M\frac{\|h\|_H\|q\|_H^\lambda}{1-\|q\|_H} < \varepsilon'$$

where M is defined by $M = (\deg q)\max_{0\leqslant s<\deg q}\|x^s\|_H$. Then by a fundamental result due to Mergelyan (our Theorem 7.14) there is a polynomial p in $\mathbf{C}[z]$ such that

$$\|f/hq^\lambda - p\|_H < \varepsilon'/\|hq^\lambda\|_H;$$

hence

$$\|f - hq^\lambda p\|_H < \varepsilon'. \qquad (*)$$

Utilizing Lemma 4.4 we can write $hq^\lambda p = h\Sigma_{k\geqslant\lambda}h_k q^k$ where the h_k's are polynomials satisfying $\deg h_k < \deg q$. If we define q by $q = \Sigma_{k\geqslant\lambda}[h_k]q^k$ where $[h_k]$ stands for h_k with each coefficient replaced by a nearest element of I_L then $q \in I_L[z]$ and

$$\|hq^\lambda p - hq\|_H \leqslant \|h\|_H M\|q\|_H^\lambda/(1-\|q\|_H) < \varepsilon'.$$

From this and $(*)$ we have $\|f - hq\|_H < 2\varepsilon'$. From the Cauchy integral formula we have for $0 \leqslant \nu \leqslant m$ and $2 \leqslant j \leqslant n$

$$\begin{aligned}\left|f^{(\nu)}(z_j) - (hq)^{(\nu)}(z_i)\right| &= \left|\frac{\nu!}{2\pi i}\int_{C_j}\frac{f(\xi)-(hq)(\xi)}{(\xi - z_j)^{\nu+1}}\,d\xi\right| \\ &\leqslant \nu!\rho^{-\nu}\|f - hq\|_H < \Delta 2\varepsilon' < \varepsilon\end{aligned}$$

where C_j is the circle with center z_j and radius ρ. □

THEOREM A.5. *The preceding theorem* (A.4) *holds whenever* $z_2, \ldots, z_n$ *is a set of algebraic integers which contains no complete set of conjugate algebraic integers over* L.

PROOF. We can assume without loss of generality that $A = I_L$ by the same argument as before. Let $S_1, \ldots, S_k$ be the decomposition of $z_2, \ldots, z_n$ under the equivalence relation of conjugacy over L and $q_1, \ldots, q_k$ the corresponding minimal polynomials. For each l $(1 \leqslant l \leqslant k)$ apply Theorem A.4 to get $\tilde{q}_l$ in $A[z]$ satisfying

$$\left|f^{(\nu)}(z_j) - \left(h\Bigl(\prod_{j\neq l} q_j^m\Bigr)\tilde{q}_l\right)^{(\nu)}(z_j)\right| < \varepsilon, \qquad 0 \leqslant \nu \leqslant m, \quad z_j \in S_l.$$

It suffices to take $q = \Sigma_{l=1}^k(\tilde{q}_l\Pi_{j\neq l}q_j^m)$ since for z_j in S_l and $0 \leqslant \nu \leqslant m$

$$(hq)^{(\nu)}(z_j) = \left(h\tilde{q}_l\prod_{j\neq l}q_j^m\right)^{(\nu)}(z_j). \quad □$$

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