Mathematical Surveys and Monographs

Number 19

Contributions to the Theory of Transcendental Numbers

Gregory V. Chudnovsky

American Mathematical Society

http://dx.doi.org/10.1090/surv/019

Contributions to the

THEORY OF TRANSCENDENTAL NUMBERS

MATHEMATICAL SURVEYS AND MONOGRAPHS · *Number 19*

Contributions to the

THEORY OF TRANSCENDENTAL NUMBERS

BY

GREGORY V. CHUDNOVSKY

AMERICAN MATHEMATICAL SOCIETY
PROVIDENCE, RHODE ISLAND

Portions of this volume were translated
from the Russian by G. A. Kandall

1980 *Mathematics Subject Classification*. Primary 10F35, 10F37.

Library of Congress Cataloging in Publication Data

Chudnovsky, G. (Gregory), 1952–

Contributions to the theory of transcendental numbers.

(Mathematical surveys and monographs; no. 19)

Bibliography: p.

Includes index.

1. Numbers, Transcendental. I. Title. II. Series.

QA247.5.C48 1984 512′.73 83-15728

ISBN 0-8218-1500-8

TABLE OF CONTENTS

To My Parents

PREFACE

Transcendental number theory, which began its development in the time of Euler, is particularly attracted to classical constants of analysis, especially to those connected with the exponential function. The irrationality and transcendence proofs of π and e, achieved by the successive efforts of Euler, Lambert, Hermite and Lindemann mark the spectacular beginning of transcendental number theory. Hilbert's seventh problem (or, as Gel′fond [**1**] put it, the Euler-Hilbert problem) of the transcendence of α^{β} for algebraic $\alpha \neq 0, 1$ and algebraic irrational β, was solved partially by Gel′fond and Kuzmin (1929) and completely by Gel′fond and Schneider (1934). These results mark a mature state of transcendental number theory, with the Gel′fond-Schneider method as its main instrument for studying the algebraic independence of values of functions satisfying algebraic differential (and functional) equations. In modern form, transcendental number theory and related studies of diophantine approximations exploit a combination of various auxiliary algebraic techniques with analytic approximation and interpolation methods traced back to the works of Hermite, Siegel, Gel′fond and Schneider and with important multidimensional generalizations introduced by Baker. We refer the reader to the excellent book of Baker [**2**] for an overall picture of the field. Progress in this field over the last decade has been connected with the introduction of new functional and algebraic techniques from different fields of mathematics, including mathematical physics. This interaction with the outside world has not yet fully developed and applications of transcendental number theory are still scarce; though one amazing example of the absence of particle production for a particular σ-model that is conditional on the transcendence result of Gel′fond [**1**] was presented by Lüscher [**3**].

This volume consists of a collection of papers devoted primarily to transcendental number theory and diophantine approximations written by the author since 1970. Most of the materials included in this volume are English translations of Russian manuscripts of the author, written in 1970 (Chapter 6), 1974 (Chapters 1 and 2 and parts of Chapter 5), 1974–1977 (Chapters 3, 9, Appendix and parts

0076-5376/84 \$1.00 + \$.25 per page

of Chapter 7). These papers and other papers included in this volume were available to specialists in manuscript form, but this is the first time that they have been collected together and published.

Though the earlier papers have been preserved in the form in which they were prepared initially, the volume is organized in such a way as to reflect recent progress and to allow readers to follow recent developments in the field. As a guide to the volume we have included an expanded and updated text of the author's address to the 1978 International Mathematical Congress as an Introductory paper. We also made an effort to interrelate different directions of the research, and some papers in the volume were updated to include cross-references and to unify the exposition, though for the sake of completeness some repetitions still remain.

The Introductory paper surveys the progress in transcendental number theory with particular emphasis on transcendence, algebraic independence and measures of the algebraic independence of numbers connected with exponential and elliptic functions. Special sections of the Introduction are devoted to period relations for Abelian varieties of CM-type, the author's results on the Schanuel conjecture, measures of algebraic independence of periods and quasi-periods of elliptic curves, etc. Proofs of the appropriate results are presented in different papers (chapters) of this volume.

Chapter 1 deals with the proofs of the author's 1974 results on the Schanuel conjecture, which show, essentially, that the number of algebraically independent elements in the set S for complex numbers of the form $\{e^{\alpha_i\beta_j}\}$, $\{\alpha_i, e^{\alpha_i\beta_j}\}$, $\{\alpha_i, \beta_j, e^{\alpha_i\beta_j}\}$ $(1 \leqslant i \leqslant N,\ 1 \leqslant j \leqslant M)$ grows logarithmically with $\min(N, M)$. The author has improved his results since then (now we know that in a set S of the form above there are not less than $|S|/(M+N)$ algebraically independent elements). However, the methods of Chapter 1 are still novel and introduce a variety of useful auxiliary algebraic techniques to deal with specializations of algebraic varieties over fields of finite degrees of transcendence, especially in the cases, when the degrees and sizes of the coefficients of equations defining algebraic varieties grow independently. The techniques of Chapter 1 also serve as a prelude to a different elementary technique used to prove more powerful results (see, especially, Chapters 2, 4, 7 and 8).

While the analytic methods of Chapter 1 are based on the Gel'fond-Schneider method, Chapter 2 applies Baker's analytic methods to the problem of algebraic independence. The results proved in Chapter 2 were first announced in 1974 [**4**]. Baker's method, developed in Chapter 4, is useful later (cf. Chapters 5 and 7). To suggest the complexity of the problems under consideration, we mention one result of Chapter 2, that there are at least two algebraically independent numbers among π, $\pi^{\sqrt{D}}$, $e^{i\pi\sqrt{D}}$. Still, we do not know whether π and e^{π} are algebraically independent, or even, whether $\pi/\ln \pi$ is rational.

Chapter 3 on "colored sequences" is based on two 1975 papers of the author which seem to have vanished in one of the East European magazines. Chapter 3 and Chapter 4, which immediately follow, deal with the criterion of the algebraic independence of two or more numbers. The starting point of all these studies is the apparent impossibility of extending the well-known Gel′fond [1] criterion of the transcendence of a single number θ to the case of the algebraic independence of numbers $\theta_1, \ldots, \theta_n$ for $n > 1$. Gel′fond's criterion (or its subsequent modifications) states that it is impossible for a transcendental number θ to have a "dense" sequence of algebraic approximations that are good: say, $|P(\theta)| < H(P)^{-Cd(P)}$ for infinitely many $P(x) \in \mathbf{Z}[x]$ and a sufficiently large $C > 0$. Though a straightforward generalization of this criterion to $n > 1$ numbers is impossible, various results that are important for applications are presented in Chapters 3 and 4. Chapter 3 deals mainly with the case, when the algebraic independence of several numbers depends on the measure of transcendence of each of them. Chapter 4 presents an attractive and simple criterion of the algebraic independence of two numbers in terms of a "dense" system of polynomials in two variables with rational integer coefficients, assuming small values at these numbers. This criterion and its n-dimensional generalization are the best possible, and constitute the necessary algebraic part in the proof of the algebraic independence of various numbers connected with exponential, elliptic and Abelian functions. See particularly the discussion of §6, Chapter 4 on analytic methods in the algebraic independence proofs.

Chapter 5 is devoted to diophantine approximations. It is in this chapter that we use Siegel's 1929 [5] method in the theory of diophantine approximations, which is based on the study of linear differential equations. In Siegel's method one recognizes important features of Hermite's 1873 construction [6] of approximations to exponential functions. Siegel's method was later further developed in the theory of E-functions. At the same time Siegel's (and Hermite's) work was generalized in the theory of Hermite-Padé approximations initiated by Mahler in 1935 (see his paper [7], published in 1968). It turns out that the combination of the author's conjugate auxiliary function method [8], explained in detail in Chapter 9, and Siegel's method gives a new best possible result on diophantine approximations of numbers generated by values of E-functions at rational points. Chapter 5 is devoted to an exposition of such results. First, in §1 of Chapter 5, we present a different approach to the Thue-Siegel theorem on (noneffective) rational approximations to algebraic numbers. This exposition follows the author's 1974 preprint [9] and uses the Gel′fond-Baker method. We also show how an alternative use of linear differential equations gives much better bounds on constants than those known from Baker's theory of linear forms in logarithms of algebraic numbers. Our results are applied to bound the number of solutions of Thue's diophantine equations. In §2 we describe the essence of Siegel's method [5] and in §§3, 4 we prove our main results on the best possible bounds of the measures of diophantine approximations of arbitrary numbers from the fields generated over $\mathbf{Q}$ by values of exponential and other E-functions, satisfying linear differential equations over $\mathbf{Q}(x)$, at rational points. The results proved here are based on the author's technique of G-invariance explained in Chapter 9, and we recommend the reader to get acquainted with Chapter 9,

§§0–4, before reading §§3, 4 of Chapter 5. The results of Chapter 5 were recently significantly extended by the author to a larger class of linear forms in values of E-functions and G-functions (including linear forms in algebraic numbers) in the direction of the effectivization of analogs of Roth's and Schmidt's theorems.

Chapter 6 is a translation of the author's paper of 1970 on diophantine sets, and represents an expanded version of his 1970 article **[13]**.

Chapter 7 is devoted mainly to the transcendence and algebraic independence of numbers connected with elliptic and exponential functions. It contains a survey of the author's results prior to 1981 (in §1) and various methods of proofs of these results in §§2–5. Among these results are the algebraic independence of π/ω, η/ω for a period ω and a quasi-period $\eta = 2\zeta(\omega/2)$ of an elliptic curve defined over $\overline{\mathbf{Q}}$; the algebraic independence of π and ω, and of π/ω and $e^{i\pi\sqrt{D}}$ for a period ω of an elliptic curve with a complex multiplication by $\sqrt{D}$. In Chapter 7 we also present a discussion of properties of L-functions of elliptic curves and the Birch-Swinnerton-Dyer conjecture used to bound heights of integer points on elliptic curves (see §1, Introductory paper). We discuss proofs of the algebraic independence for values of elliptic (and exponential) functions in §§7–9. §§10 and 11 of Chapter 7 are devoted to proofs of the author's generalization of the Lindemann-Weierstrass theorem for arbitrary Abelian CM-varieties. We also present analogs of the Lindemann-Weierstrass theorem for arbitrary systems of functions satisfying differential equations and the laws of addition. These results were first announced in **[10]** and in the elliptic case were presented in **[11]**.

In Chapter 8 we give the bound on the measure of the algebraic independence of π/ω and η/ω, which is close to the best possible. Another important feature of this result is in its dependence of height of "H^{-C}" form. This result implies, e.g. an "H^{-C}" form of the measure of diophantine approximations for numbers such as $\Gamma(1/3)$, $\Gamma(1/4)$, etc. The proof of Theorem 1.1 in Chapter 8 is obviously extended to other cases of algebraically independent numbers connected with elliptic functions studied by the author (see §4 of the Introductory paper).

We leave beyond the scope of this book various results of the author on the measures of diophantine approximations of classical constants of analysis based on Padé-type approximation techniques. Some of these results are touched upon in **[12]**.

Chapter 9 presents a group of the author's results on a generalization of the Straus-Schneider theorem on the number of complex z such that a given meromorphic function f assumes, together with all its derivatives, rational integer values at z. Our exposition is based on unpublished manuscripts of 1977–1978 discussing various one and multidimensional generalizations of these theorems and intersects with **[8]**. Chapter 9 could serve as a good introduction to Chapter 5.

As an appendix we included a paper on the extremality of certain multidimensional manifolds prepared by A. I. Vinogradov and the author in 1976.*

* This paper was initially prepared for a Festschrift for the 85th birthday of I. M. Vinogradov.

This book was prepared through the generous effort of the American Mathematical Society and with the help of D. V. Chudnovsky. The author is deeply thankful to the Society and to its Editorial Staff for their patience during the preparation of this volume which lasted several long years. Unfortunately the author's state of health did not contribute positively towards the completion of this project. The author specially thanks Professor W. J. LeVeque for his kindness and encouragement, Dr. B. Silver and Mr. M. Carcieri for their care and help and Dr. G. Kandall for his translations. The author is indebted to Professor L. Bers for his help. This book is dedicated to the author's parents.

REFERENCES

1. A. O. Gel'fond, *Transcendental and algebraic numbers* (Moscow, 1952); Dover, New York, 1960.
2. A. Baker, *Transcendental number theory*, 2nd ed., Cambridge Univ. Press, Cambridge, 1979.
3. M. Lüscher, Nuclear Physics, **B135** (1978), 1–19.
4. G. V. Chudnovsky, Uspekhi Math. Nauk **31**, No. 4, (1976), 281–282 (Russian).
5. C. L. Siegel, Abh. Preuss. Akad. Wiss., No. 1 (1929).
6. Ch. Hermite, C. R. Acad. Sci. Paris **77** (1873), 18–24, 74–79, 226–233, 285–293.
7. K. Mahler, Composito Math. **19** (1968), 95–166.
8. G. V. Chudovsky, Ann. of Math. (2) **109** (1979), 353–377.
9. ______, Preprint IM-74-9, Institute of Mathematics, Kiev, USSR (Russian); in Topics of Number Theory, Colloquia Math. Societatis J. Bolyai, Vol. 13, North-Holland, Amsterdam, 1974, pp. 19–30.
10. ______, C. R. Acad. Sci. Paris Sér. A **288** (1979), 439–440.
11. ______, Invent. Math. **61** (1980), 267–280.
12. ______, *Bifurcation phenomena in mathematical physics and related topics*, D. Reidel Publishing Company, Boston, 1980, 448–510; Lecture Notes Phys., vol. 120, Springer, New York, 1980, pp. 103–150; Arithmetic and Geometry, v. I, Birkhäuser, Boston, Mass., 1983, 61–105.
13. ______, Uspekhi Math. Nauk **25** No. 4, (1970), 171, 185–186.

INDEX

ABCDEFGHIJ –CM–8987654