Partially Ordered Abelian Groups with Interpolation

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American Mathematical Society
PARTIALLY ORDERED
ABELIAN GROUPS
WITH INTERPOLATION
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Preface

Ordered algebraic structures—particularly ordered fields, ordered groups, and ordered vector spaces—have a well-established tradition in mathematics, partly due to their intrinsic interest, and partly due to their applications in other areas. A new branch of this subject, motivated by $K$-theoretic applications (as sketched in the Prologue and the Epilogue), has grown out during the past decade. This branch, which is mainly concerned with partially ordered abelian groups satisfying the Riesz interpolation property ("interpolation groups", for the sake of abbreviation), is the subject of the present book. The purpose of the book is to provide a solid foundation in the theory of interpolation groups, for the use of researchers or students pursuing the applications. In particular, detailed developments are presented for that part of the subject on which a majority of the current applications rest. That a few topics related to newly developing applications are not included is due partly to the lack of a good perspective on these evolving developments, and partly to the considerable additional space that would be required to treat them coherently.

Although interpolation groups are defined as purely algebraic structures, their development has been strongly influenced by functional analysis. The interpolation property itself was introduced in F. Riesz’s fundamental 1940 paper, as a key to his investigations of partially ordered real vector spaces, and the study of ordered vector spaces with the interpolation property has been continued by many functional analysts since that time. A number of the techniques developed in this area have been adapted to interpolation groups, after being stripped of scalars. On the other hand, parts of functional analysis dealing with compact convex sets have been applied to interpolation groups via certain dual objects. Any partially ordered abelian group possessing an “order-unit” (i.e., a positive element whose positive multiples provide upper bounds for all elements of the group) has a natural dual object called its “state space”, consisting of all normalized positive real-valued homomorphisms on the group. This state space is a compact convex subset of the space of all real-valued functions on the group, and in the case of an interpolation group, the state space is a very special type of compact convex set known as a “Choquet simplex”. A major line of devel-
development for interpolation groups has concerned the interrelationships among an interpolation group, its state space, and the partially ordered real Banach space of all affine continuous real-valued functions on the state space.

This cross-cultural development has left interpolation groups somewhat estranged from both algebraists, who may feel intimidated by compact convex sets, and functional analysts, who may feel handicapped by the lack of scalars. The intention of this book is to make the subject of interpolation groups accessible to readers from each culture, by providing sufficient details for both the algebraic and analytic aspects of the subject. Aside from a few basic concepts from the general theories of abelian groups and partially ordered sets, the algebraic theory of interpolation groups is developed directly from the definitions. A bit of the language of category theory is used in places, and the reader may need to remind himself of a few basic concepts such as products and coproducts, direct and inverse limits, and pullbacks and pushouts. On the other hand, the analytic theory of interpolation groups depends on selected nontrivial portions of the theory of compact convex sets. This is developed, as needed, from an elementary level, assuming only that the reader is (or can become) familiar with some very basic functional analysis, such as linear topological spaces, the Hahn-Banach Theorem, and the Riesz Representation Theorem. Readers with stronger backgrounds can easily skim over any familiar parts of this development.

The first four chapters of the book are purely algebraic, beginning with introductions to partially ordered abelian groups in general and to interpolation groups in particular. Of necessity, the highest density of definitions occurs in these introductory chapters, and some readers may therefore wish to skim the first two chapters, referring back to them later as needed. Chapter 3 is concerned with “dimension groups”, which may be described as upward-directed interpolation groups in which positive integers can be cancelled from inequalities. (The importance of the class of dimension groups lies in the fact that in most applications, all the interpolation groups which appear are dimension groups.) A major milestone of the theory is located here, namely, a structural characterization of dimension groups as direct limits of finite products of copies of the integers. Chapter 4, another introductory chapter, is concerned with states (normalized positive real-valued homomorphisms on a partially ordered abelian group with an order-unit) as individual functions. Existence theorems for states, formulas for the ranges of values of states on group elements, and conditions for the uniqueness of states are developed here.

In Chapter 5, a purely functional-analytic interlude occurs, designed to introduce the more algebraic reader to some basic aspects of compact convex sets. In addition to presenting a number of basic concepts, the Krein-Mil'nan Theorem (the fundamental existence theorem for extreme points of compact convex sets) is derived. The following two chapters introduce the connections through which functional analysis is brought to bear on interpolation groups. Chapter 6 is concerned with the state space (the collection of all states) on a partially ordered abelian group with order-unit, and with the structure of the state space as a
compact convex subset of the linear topological space of all real-valued functions on the group, while Chapter 7 is concerned with the natural representation of the group (via evaluations) as affine continuous real-valued functions on the state space. Also included here is another section of functional analysis, on spaces of affine continuous real-valued functions on compact convex sets.

A long-range goal is to describe any interpolation group with order-unit as completely as possible in terms of affine continuous functions on its state space. The cases in which this goal is most readily accessible are developed in Chapters 8 and 9, namely, the cases of interpolation groups with a sufficiently large supply of direct sum decompositions, and Dedekind \( \sigma \)-complete lattice-ordered abelian groups. Here is a second milestone: a complete description of any Dedekind \( \sigma \)-complete lattice-ordered abelian group with order-unit in terms of continuous real-valued functions on a basically disconnected compact Hausdorff space (the collection of extremal states).

The remaining portions of functional analysis needed in the book, namely, some of the general properties of Choquet simplices and their affine continuous function spaces, are developed in Chapters 10 and 11. In particular, Choquet simplices are characterized as those compact convex sets on which the space of affine continuous real-valued functions is an interpolation group. The following chapter concerns completions of interpolation groups with respect to pseudometrics derived from states, a process by which certain questions and calculations can be reduced to the case of a Dedekind complete lattice-ordered abelian group.

In the remainder of the book, some of the fruits of the combined algebraic and analytic developments appear. For example, a characterization of the closure of the image of the natural affine continuous function representation of an interpolation group with order-unit is obtained in Chapter 13. This result provides a strong approximation to the desired description of the interpolation group in terms of affine continuous functions on its state space and may be regarded as another milestone of the theory, in spite of its rather technical nature. One effective application of this result is the development, in Chapter 14, of a complete description and classification of all "simple" dimension groups (i.e., dimension groups with no nontrivial ideals) in terms of Choquet simplices.

Chapter 15 is concerned with the class of those interpolation groups (with order-unit) which are complete with respect to the natural norm derived from their states. For such a group, the approximation theorems of Chapter 13 yield an exact description of the group as isomorphic to a well-described group of affine continuous functions. In addition, there is a strong duality between the algebraic structure of the group and the geometric structure of its state space. For instance, there are natural bijections between the extremal states and the maximal ideals of the group, and between the closed faces of the state space and the norm-closed ideals of the group. In Chapter 16, which may be viewed as a corollary of the preceding chapter, the results from the norm-complete case find application to groups which either satisfy the countably infinite version of the interpolation property or are monotone \( \sigma \)-complete.
The final chapter is a signpost indicating one of the directions in which open problems may be found. It is an introduction to the problem of classifying dimension groups arising as extensions of one given dimension group by another and also serves to present some techniques for constructing new dimension groups as extensions of old ones. A list of some open problems concerning interpolation groups is given following the Epilogue.

To use this book for an introductory course on interpolation groups, we suggest developing the basic concepts and methods as far as the classification of simple dimension groups (Chapter 14) and the structure theory of archimedean norm-complete dimension groups (Chapter 15). In such a course, the following sections may be omitted without loss of continuity: Chapter 1—Pullbacks, pushouts, and coproducts; Chapter 2—Extensions; Products, pullbacks, and pushouts; Chapter 3—Products, pullbacks, and pushouts; Chapter 4—Additional uniqueness criteria; Chapter 5—Categorical concepts; Faces of probability measures; Chapter 6—Products and limits; Change of order-unit; Chapter 9—Additional examples; Chapter 10—Complementary faces; Categorical properties; Chapter 11—Inverse limits; Complementary faces; Chapter 14—Finite-dimensional state spaces; Finite rank; Chapter 15—Norm-completions. In a few instances, an item from one of these sections is referred to in an example or a discussion in another section, but this should cause the reader little trouble.

A more drastic shortcut, for the reader who wishes to see the general structure theory of interpolation groups before delving into the supporting technical details, would be to skip over most of Chapters 8–13, referring back to the statements of the results in these chapters as needed. In such a case, the reader may find it helpful to read through the definition of a simplex and the section on Choquet simplices in Chapter 10, as well as the statements of Theorem 11.4 and Corollary 13.6, before embarking on Chapter 14.

We have resisted the temptation to coin any abbreviation for the term "partially ordered abelian group". (In particular, the ugliness of the acronym "poag" makes it easy to reject.) In discussions and in statements of theorems, we have pedantically included the phrase "as an ordered group" wherever it seemed useful to avoid potential confusion. This usage is intended to make hypotheses and conclusions more readily apparent to the browsing reader, and any reader can more easily disregard redundant usage than worry over omissions.

It is a pleasure to acknowledge the influences which led to my interest in and research on interpolation groups: George Elliott's work on the classification of ultramatricial algebras and approximately finite-dimensional $C^*$-algebras and David Handelman's work on the structure of $K_0$-continuous regular rings and finite Rickart $C^*$-algebras, via the partially ordered Grothendieck group $K_0$. Their lead in developing other applications for partially ordered $K_0$ has helped to maintain the momentum of the research and to attract other mathematicians to the area. In regard to the manuscript, I would like to thank Robert Phelps for reading an early draft of the functional analysis sections and offering comments which helped improve the exposition.
Prologue: Partially Ordered Grothendieck Groups

Motivation for the study of partially ordered abelian groups has come from many different parts of mathematics, for mathematical systems with compatible order and additive (or linear) structures are quite common. This is particularly evident in functional analysis, where spaces of various kinds of real-valued functions provide impetus for investigating partially ordered real vector spaces. In the past decade, the observation that a Grothendieck group (such as $K_0$ of a ring or algebra) often possesses a natural partially ordered abelian group structure has led to new directions of investigation, whose goals have been to develop structure theories for certain types of partially ordered abelian groups to the point where effective application to various Grothendieck groups is possible. These recent developments in the area of partially ordered abelian groups are the subject of this book. In order to give the reader some hint of the motivations for these developments, we present here a sketch of the construction of Grothendieck groups as abelian groups equipped with pre-orderings that are often partial orderings, together with brief sketches of several situations in which the theory of partially ordered abelian groups (particularly those known as interpolation groups) can be applied, via the Grothendieck groups $K_0$, to the study of certain rings and $C^*$-algebras. This discussion is admittedly cursory, in the interest of avoiding technicalities, and for reasons of space. For the same reasons, we do not discuss the recent applications of interpolation groups to topological Markov chains [88, 22, 65, 68] or to positive polynomials and compact group actions [70, 71, 73, 74]. We shall continue the discussion in the Epilogue, where further applications of the theory developed in the book will be exhibited.

A Grothendieck group is an invariant attached to a collection of objects, such as the vector bundles on a given topological space, the finitely generated projective modules over a given ring, or the projection operators associated with a given $C^*$-algebra. The construction only requires a collection of objects equipped with a notion of isomorphism and some means of combining any two objects from the collection into a third object. While it is traditional to develop Grothendieck

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Versions of some of the discussions in the Prologue and the Epilogue appear in [54, 55].
groups using short exact sequences as the means of combining objects, in many important applications this reduces to direct sums (or direct products). Since the construction process is simpler using direct sums, we shall restrict our discussion to that case.

Thus for our basic data we take a set $\mathcal{P}$ of objects in some category, such that $\mathcal{P}$ has a zero object and every finite set of objects in $\mathcal{P}$ has a direct sum (coproduct) in $\mathcal{P}$. The easiest algebraic system to build using this data is an abelian semigroup, whose elements are the isomorphism classes of objects in $\mathcal{P}$ and whose operation is addition induced from the direct sum operation in $\mathcal{P}$. However, the semigroup obtained in this manner need not have cancellation and so cannot always be embedded in an abelian group. This problem can be circumvented either by reducing the semigroup modulo a suitable congruence relation or, more conveniently, by using an equivalence relation on $\mathcal{P}$ slightly coarser than isomorphism, as follows.

Objects $A, B \in \mathcal{P}$ are said to be stably isomorphic (in $\mathcal{P}$) if and only if there is an object $C \in \mathcal{P}$ such that $A \oplus C \cong B \oplus C$. Stable isomorphism is an equivalence relation on $\mathcal{P}$, and we write $[A]$ for the stable isomorphism class of an object $A \in \mathcal{P}$. Let $\text{Grot}(\mathcal{P})^+$ denote the collection of all stable isomorphism classes in $\mathcal{P}$. The direct sum operation in $\mathcal{P}$ induces an addition operation in $\text{Grot}(\mathcal{P})^+$, where $[A] + [B] = [A \oplus B]$ for all $A, B \in \mathcal{P}$, and using this operation, $\text{Grot}(\mathcal{P})^+$ becomes an abelian semigroup with cancellation. By formally adjoining additive inverses to $\text{Grot}(\mathcal{P})^+$, we obtain an abelian group $\text{Grot}(\mathcal{P})$, the Grothendieck group of $\mathcal{P}$. (Reminder: If short exact sequences are available in $\mathcal{P}$, the Grothendieck group constructed from $\mathcal{P}$ using short exact sequences may well be different from the group constructed here.) All elements of $\text{Grot}(\mathcal{P})$ have the form $[A] - [B]$ for $A, B \in \mathcal{P}$, and elements $[A] - [B]$ and $[C] - [D]$ in $\text{Grot}(\mathcal{P})$ are equal if and only if $A \oplus D$ is stably isomorphic to $B \oplus C$.

For example, if $\mathcal{P}$ is the collection of all real (complex) vector bundles on some compact Hausdorff space $X$, then $\text{Grot}(\mathcal{P})$ is the real (complex) $K$-group $K^0(X)$. For another example, let $R$ be a ring with 1, and let $\mathcal{P}$ be the collection of all finitely generated projective right $R$-modules (i.e., all direct summands of free right $R$-modules of finite rank). In this case, $\text{Grot}(\mathcal{P})$ is the algebraic $K$-group $K_0(R)$. Alternatively, $K_0(R)$ may be constructed by taking $\mathcal{P}$ to be the category of all rectangular matrices over $R$. Then the objects in $\mathcal{P}$ are all idempotent square matrices over $R$, and the direct sum of idempotent matrices $e$ and $f$ is the block matrix $\begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}$. In case $R$ is a $C^*$-algebra, the set of idempotent matrices over $R$ may be reduced to the set of self-adjoint idempotent matrices (see [52, Chapter 19]).

In order not to lose track of the original semigroup $\text{Grot}(\mathcal{P})^+$ inside $\text{Grot}(\mathcal{P})$, we make it the “positive cone” of an order relation. Namely, for $x, y \in \text{Grot}(\mathcal{P})$, we define $x \leq y$ if and only if $y - x$ lies in $\text{Grot}(\mathcal{P})^+$. This relation is a pre-order (i.e., a reflexive, transitive relation) which is invariant under translation (that is, $x \leq y$ implies $x + z \leq y + z$). The combined structure $(\text{Grot}(\mathcal{P}), +, \leq)$ is called
a pre-ordered abelian group. In case the relation ≤ is a partial order (i.e., an
antisymmetric pre-order), Grot(ℛ) is a partially ordered abelian group.

Various relatively mild assumptions on ℛ will force Grot(ℛ) to be partially
ordered. A common assumption is that all the objects in ℛ are “directly finite”,
i.e., objects A, B ∈ ℛ can satisfy A ⊕ B ≅ A only if B is a zero object. For
example, if ℛ is the collection of all finitely generated projective right modules
over a unital ring R, the objects in ℛ are directly finite if and only if all square
matrices over R that satisfy xy = 1 also satisfy yx = 1. In particular, this holds
if R is commutative, or if R is a directed union of finite-dimensional algebras,
or if R is noetherian on either side. It also holds if R is a unit-regular
ring, meaning that for any x ∈ R there exists a unit (invertible element) u ∈ R such
that xux = x, for in that case the objects in ℛ may be cancelled from direct
sums [63, Theorem 2; 49, Theorem 4.5].

In the case that ℛ is the collection of all finitely generated projective right
modules over a unital ring R, the module R plays a special role in ℛ, for every
object in ℛ is isomorphic to a direct summand of a finite direct sum of copies of
R. As a consequence, [R] plays a special role in K₀(R): given any x ∈ K₀(R)
there exists a positive integer n such that x ≤ n[R]. By virtue of this property,
[R] is called an order-unit of K₀(R).

The construction of K₀(R) for a ring R without 1 is not based on some class
of non-unital projective modules but instead is obtained from the “unification”
of R, namely the ring R¹ based on the abelian group ℤ × R, with multiplication
given by the rule (m, r)(n, s) = (mn, ms + nr + rs). The original ring R may
be identified with the set {0} × R, which is an ideal of R¹, and then R¹/R ≅ ℤ.
Then K₀(R) is defined to be the subgroup of K₀(R¹) consisting of those
elements [A] − [B] in K₀(R¹) for which A/AR and B/BR are (stably) isomorphic
free abelian groups, and K₀(R) is equipped with the pre-ordered abelian group
structure inherited from K₀(R¹). In general, K₀(R) need not have an order-unit.
To take the place of an order-unit, we may use the subset D(R) of K₀(R)⁺
consisting of those elements x ∈ K₀(R) for which 0 ≤ x ≤ [R¹].

Grothendieck groups having been constructed, two basic meta-questions arise:
What sort of information about the data ℛ is stored in Grot(ℛ), and how may
this information be retrieved? For instance, since Grot(ℛ) is an invariant of ℛ,
there can be situations in which it can be proved that data ℛ and ℛ' are not
equivalent by showing that Grot(ℛ) and Grot(ℛ') are not isomorphic. Also, by
its construction Grot(ℛ) reflects the arrangement of direct sum decompositions
in ℛ, and we may ask how much of the direct sum decomposition structure of ℛ,
and what other structural information about ℛ, may be recovered from
Grot(ℛ). By way of illustration, we shall discuss a number of situations in
which these questions have been successfully answered. Due to the author’s bias,
these examples are K₀’s of certain rings and C*-algebras, but the patterns of
these examples are to be expected in Grothendieck groups of other mathematical
systems.
- Irrational Rotation Algebras. Let \( T \) be the unit circle in the plane, and let \( C(T) \) denote the algebra of all continuous complex-valued functions on \( T \). We may view the functions in \( C(T) \) as bounded linear operators on the Hilbert space \( L^2(T) \) (where a function from \( C(T) \) acts on functions from \( L^2(T) \) by multiplication). Given a positive real number \( \alpha \), let \( \rho_\alpha : T \to T \) be counterclockwise rotation through the angle \( 2\pi \alpha \). The rule \( \rho_\alpha^* (f) = f \rho_\alpha \) then defines a bounded linear operator \( \rho_\alpha^* \) on \( L^2(T) \), and we let \( A_\alpha \) denote the closed self-adjoint subalgebra of bounded linear operators on \( L^2(T) \) generated by \( C(T) \) and \( \rho_\alpha^* \). This algebra is known as the "transformation group \( C^* \)-algebra of the rotation \( \rho_\alpha \)".

It is fairly easy to distinguish among the algebras \( A_\alpha \) for rational \( \alpha \), and it is also easy to distinguish the rational cases from the irrational cases. However, distinguishing among the irrational cases is a subtler problem, which was only solved when Pimsner, Voiculescu, and Rieffel calculated \( K_0 \) of these algebras. Namely, for any irrational number \( \alpha \in (0,1) \), there is an ordered group isomorphism from \( K_0(A_\alpha) \) onto the subgroup \( \mathbb{Z} + \alpha \mathbb{Z} \) of \( \mathbb{R} \), under which the canonical order-unit \( \{A_\alpha\} \) in \( K_0(A_\alpha) \) corresponds to the real number 1 [103, Corollary 2.6; 104, Corollary 1; 108, Theorem 1].

Thus if \( A_\alpha \cong A_\beta \) for some irrational numbers \( \alpha, \beta \in (0,1) \), there must be an ordered group isomorphism \( f \) of \( \mathbb{Z} + \alpha \mathbb{Z} \) onto \( \mathbb{Z} + \beta \mathbb{Z} \) such that \( f(1) = 1 \). After approximating elements of \( \mathbb{Z} + \alpha \mathbb{Z} \) and \( \mathbb{Z} + \beta \mathbb{Z} \) by rational numbers, clearing denominators, and using the relation \( f(1) = 1 \), it is easily seen that \( \mathbb{Z} + \alpha \mathbb{Z} = \mathbb{Z} + \beta \mathbb{Z} \). From this a quick computation leads to the conclusion that either \( \beta = \alpha \) or \( \beta = 1 - \alpha \), whence either \( \rho_\beta = \rho_\alpha \) or \( \rho_\beta = \rho_\alpha^{-1} \).

- Ultramatricial Algebras. Fix a field \( F \). A matricial \( F \)-algebra is any \( F \)-algebra that is isomorphic to a finite direct product of full matrix algebras over \( F \). (In case \( F \) is algebraically closed, the matricial \( F \)-algebras are exactly the finite-dimensional semisimple \( F \)-algebras.) An ultramatricial \( F \)-algebra is an \( F \)-algebra that is a union of a countable ascending sequence of matricial subalgebras (equivalently, any \( F \)-algebra that is isomorphic to a direct limit of a countable sequence of matricial \( F \)-algebras and \( F \)-algebra homomorphisms). It is easily checked that matricial \( F \)-algebras are unit-regular. Hence, any unital ultramatricial \( F \)-algebra \( R \) is unit-regular, and so \( K_0(R) \) is a partially ordered abelian group. Using \( F \)-algebra unifications, it follows that \( K_0 \) of any ultramatricial \( F \)-algebra is partially ordered.

These partially ordered abelian groups may be used to classify ultramatricial \( F \)-algebras, following a method of Elliott. If \( R \) and \( S \) are unital ultramatricial \( F \)-algebras, then \( R \cong S \) if and only if there exists an ordered group isomorphism of \( K_0(R) \) onto \( K_0(S) \) mapping \([R]\) to \([S]\) [35, Theorem 4.3; 49, Theorem 15.26]. If \( R \) and \( S \) are non-unital ultramatricial \( F \)-algebras, then \( R \cong S \) if and only if there exists an ordered group isomorphism of \( K_0(R) \) onto \( K_0(S) \) mapping \( D(R) \) onto \( D(S) \) [35, Theorem 4.3].

The partially ordered abelian groups which can appear as \( K_0 \) of ultramatricial
\(F\)-algebras are just those which are isomorphic (as ordered groups) to direct limits of countable sequences of finite products of copies of \(\mathbb{Z}\) \([35, \text{Theorems 5.1, 5.5}].\) However, it is usually impossible to check directly whether a given partially ordered abelian group is isomorphic to such a direct limit. Some obvious properties of these direct limits are that they are countable, they are directed (upward and downward), and they are unperforated (any \(x\) satisfying \(nx \geq 0\) for some \(n \in \mathbb{N}\) also satisfies \(x \geq 0\)). A more fundamental property, also easily checked, is the Riesz interpolation property: given any \(x_1, x_2, y_1, y_2\) such that \(x_i \leq y_j\) for all \(i, j\), there exists \(z\) such that \(x_i \leq z \leq y_j\) for all \(i, j\). The direct limits of sequences of finite products of copies of \(\mathbb{Z}\) were characterized by Effros, Handelman, and Shen as exactly those countable partially ordered abelian groups which are directed and unperforated and which satisfy the Riesz interpolation property \([30, \text{Theorem 2.2}; 52, \text{Corollary 21.8}].\)

Partially ordered abelian groups with the latter three properties are now called dimension groups. Thus, given a partially ordered abelian group \(G\) and an order-unit \(u \in G\), there exists an ordered group isomorphism of \(G\) onto \(K_0\) of some unital ultramatricial \(F\)-algebra \(R\), with \(u\) corresponding to \([R]\), if and only if \(G\) is a countable dimension group. For the non-unital case, replace the order-unit \(u\) by an upward-directed subset \(D \subseteq G^+\) such that every element of \(G^+\) is a sum of elements from \(D\) and such that any element of \(G^+\) which lies below an element of \(D\) must lie in \(D\). Then there exists an ordered group isomorphism of \(G\) onto \(K_0\) of some ultramatricial \(F\)-algebra \(R\), with \(D\) mapping onto \(D(\mathbb{R})\), if and only if \(G\) is a countable dimension group. (These results are obtained by combining Elliott’s results \([35, \text{Theorems 5.1, 5.5}]\) with those of Effros, Handelman, and Shen \([30, \text{Theorem 2.2}].\))

For example, the subgroup \(\{a/2^n \mid a \in \mathbb{Z} \text{ and } n \in \mathbb{N}\}\) of \(\mathbb{Q}\) appears as \(K_0\) of a direct limit of matrix algebras

\[
M_2(F) \to M_4(F) \to M_8(F) \to \cdots
\]

using block diagonal maps \(x \mapsto \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \). The lexicographic product of \(\mathbb{Z}\) with itself appears as \(K_0\) of a direct limit

\[
F \times M_2(F) \to F \times M_3(F) \to F \times M_4(F) \to \cdots
\]

using maps \((x, y) \mapsto (x, \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix})\). The subgroup \(\mathbb{Z} + \mathbb{Z}\sqrt{2}\) of \(\mathbb{R}\) appears as \(K_0\) of a direct limit

\[
F \times M_2(F) \to M_3(F) \times M_4(F) \to M_7(F) \times M_{10}(F) \to \cdots
\]

using maps

\[
(x, y) \mapsto \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & y \end{pmatrix}
\]

An algorithm for obtaining \(\mathbb{Z} + a\mathbb{Z}\) (where \(a\) is a positive irrational number) as \(K_0\) of an ultramatricial algebra, by using the continued fraction expansion of \(a\), was developed by Effros and Shen \([32, \text{Theorem 3.2}].\)
The ability to realize any countable dimension group as $K_0$ of an ultramatri-
cial algebra is an aid to constructing examples, since dimension groups are easier
to construct than ultramatriacial algebras. For instance, we may use this method
to construct algebras with various sorts of ideal-theoretic structure, for this corre-
sponds directly to ideal-theoretic structure in $K_0$. An ideal in a partially ordered
abelian group $G$ is any directed subgroup $H$ of $G$ such that whenever $x_1, x_2 \in H$
and $y \in G$ with $x_1 \leq y \leq x_2$, then $y \in H$. For any unital ultramatriacial algebra
$R$ (or, more generally, for any unit-regular ring), the lattice of two-sided ideals of
$R$ is isomorphic to the lattice of ideals of $K_0(R)$ [49, Corollary 15.21]. Given any
countable ordinal $\alpha$, it is easy to construct a countable dimension group, with an
order-unit, whose lattice of ideals is isomorphic to the interval $[1, \alpha + 1]$ (see [50,
Proposition 1.3]). Consequently, there exists a unital ultramatriacial $F$-algebra
whose lattice of two-sided ideals is isomorphic to $[1, \alpha + 1]$. Similarly, there ex-
ists a unital ultramatriacial $F$-algebra whose lattice of ideals is anti-isomorphic
to $[1, \alpha]$. (These ideal-theoretic examples are the basic ingredients in two corre-
sponding module-theoretic examples constructed by the author [50, Corollaries
1.6, 1.7].)

Any family of ultramatriacial algebras defined by properties which are reflected
in $K_0$ may be classified by a corresponding family of countable dimension groups.
For example, since an ultramatriacial algebra $R$ is simple if and only if $K_0(R)$ is
simple (i.e., the only ideals in $K_0(R)$ are $K_0(R)$ and $\{0\}$), the family of simple
unital ultramatriacial $F$-algebras is classified by the family of countable simple di-
msion groups with order-unit. The task of constructing all (countable) simple
dimension groups was initiated by Effros, Handelman, and Shen [30, Lemmas
3.1, 3.2, Theorem 3.5] and completed by the author and Handelman [57, Theo-
rem 4.11].

- **Approximately Finite-Dimensional $C^*$-Algebras.** A complex $C^*$-
algebra is approximately finite-dimensional (abbreviated AF) if it is isomorphic
(as a $C^*$-algebra) to a direct limit of a countable sequence of finite-dimensional
$C^*$-algebras. Since all finite-dimensional $C^*$-algebras are semisimple, any AF
$C^*$-algebra $A$ contains a dense ultranatricial complex $*$-subalgebra $R$. More-
over, the inclusion map $R \rightarrow A$ induces an ordered group isomorphism of $K_0(R)$
onto $K_0(A)$ (see [52, Corollary 19.10] for the unital case). Hence, $K_0$ of any AF
$C^*$-algebra is a countable dimension group. Bratteli proved that AF $C^*$-algebras
are determined up to isomorphism by their dense ultramatriacial $*$-subalgebras
[15, Theorem 2.7; 52, Theorem 20.7]. Consequently, AF $C^*$-algebras are classi-
fied in terms of countable dimension groups, via $K_0$, in exactly the same manner
as ultramatriacial algebras [35, Theorems 4.3, 5.1, 5.5; 30, Theorem 2.2; 52,
Theorems 20.7, 21.10].

- **Pseudo-Rank Functions.** A ring $R$ is (von Neumann) regular provided
that for each $x \in R$ there exists $y \in R$ such that $yx = x$. (This ensures a
large supply of idempotents, for whenever $xyz = x$, the elements $xy$ and $yz$ are idempotent.) A (normalized) pseudo-rank function on a regular ring $R$ with 1 is any map $N$ from $R$ to the unit interval $[0,1]$ such that (a) $N(1) = 1$; (b) $N(xy) \leq N(x)$ and $N(xy) \leq N(y)$ for all $x, y \in R$; (c) $N(e + f) = N(e) + N(f)$ for all orthogonal idempotents $e, f \in R$. For example, if $R$ is the ring of all $n \times n$ matrices over a field $F$, then normalized matrix rank defines a pseudo-rank function $N$ on $R$ (that is, $N(x) = \text{rank}(x)/n$ for all matrices $x \in R$). For another example, if $R$ is a ring of subsets of some set $X$, and $X \in R$, then a pseudo-rank function on $R$ is just a nonnegative finitely additive measure $\mu$ on $R$ such that $\mu(X) = 1$.

Pseudo-rank functions on $R$ correspond to certain real-valued functions on $K_0(R)$, normalized with respect to the canonical order-unit $[R]$. Namely, a state on $(K_0(R), [R])$ is any group homomorphism $s : K_0(R) \to \mathbb{R}$ such that $s(K_0(R)^+) \subseteq \mathbb{R}^+$ and $s([R]) = 1$. There is a canonical bijection between the set of such states and the set of pseudo-rank functions on $R$ [56, Proposition 2.4; 49, Proposition 17.12]. Thus questions of existence and/or uniqueness for pseudo-rank functions may be translated into questions of existence and/or uniqueness for states.

For example, the author and Handelman showed that any nonzero partially ordered abelian group with an order-unit has at least one state [56, Corollary 3.3; 49, Corollary 18.2]. Since $K_0$ of any nonzero unit-regular ring is nonzero and partially ordered, it follows that any nonzero unit-regular ring has at least one pseudo-rank function [56, Corollary 3.5; 49, Corollary 18.5]. The author and Handelman also developed various criteria for a nonzero partially ordered abelian group $G$ with an order unit $u$ to have a unique state. For instance, this happens if and only if there exist integers $s > t > 0$ such that given any $x, y \in G^+$ with $x + y = u$, there is some $n \in \mathbb{N}$ for which either $ntx \leq nsy$ or $nty \leq nsx$. As a consequence, a nonzero unit-regular ring $R$ possesses a unique pseudo-rank function if and only if there exist integers $s > t > 0$ such that given any orthogonal idempotents $e, f \in R$ with $e + f = 1$, there is some $n \in \mathbb{N}$ for which either the direct sum of $nt$ copies of $eR$ embeds in the direct sum of $ns$ copies of $fR$ or the direct sum of $nt$ copies of $fR$ embeds in the direct sum of $ns$ copies of $eR$ [56, Theorem 4.6; 49, Theorem 18.16].

- **Hiatus.** Other uses of Grothendieck groups occur in cases where a structure theory for a class of partially ordered abelian groups is used, via $K_0$, to derive a corresponding structure theory for a class of rings or $C^*$-algebras. As these applications are more easily discussed with the relevant ordered group theory at hand, we postpone them until the Epilogue. In these applications, the Grothendieck groups that appear satisfy the Riesz interpolation property and in fact are usually dimension groups. Therefore our main interest is in developing structure theory for dimension groups, although in many instances the development relies only on the interpolation property.
Notational Conventions

The main purpose of this section is to provide a quick review of those basic concepts from the general theory of partially ordered sets that will be needed in the book. As the few notations and definitions from abelian group theory that we use are quite standard, we do not review them.

A partial order on a set $X$ is any reflexive, antisymmetric, transitive relation on $X$. In most cases, partial orders are denoted $\leq$. When this notation is used, associated relations $\geq, <, >$ are defined on $X$ so that $x \geq y$ if and only if $y \leq x$, while $x < y$ if and only if $x \leq y$ but $x \neq y$, and $x > y$ if and only if $y < x$. A partially ordered set (or poset) is a set $X$ equipped with a specified partial order. The dual ordering on a partially ordered set $(X, \leq)$ is the partial order $\geq$. To avoid confusion, the dual ordering is often denoted $\leq^*$. The dual of $(X, \leq)$ is the partially ordered set $(X, \leq^*)$.

Let $(X, \leq)$ be a partially ordered set, and let $Y$ be an arbitrary set. There is a natural partial order $\leq$ on the set $X^Y$ of all functions from $Y$ to $X$, where $f \leq g$ if and only if $f(y) \leq g(y)$ for all $y \in Y$. This partial order is called the pointwise ordering on $X^Y$. With respect to the pointwise ordering, $f < g$ means only that $f \leq g$ and $f(y) < g(y)$ for at least one $y \in Y$. For uniformly strict inequalities in $X^Y$, we use the notation $\ll$. Thus $f \ll g$ if and only if $f(y) < g(y)$ for all $y \in Y$. The corresponding partial order $\leq$ on $X^Y$ is called the strict ordering. (That is, $f \leq g$ if and only if either $f \ll g$ or $f = g$.)

Intervals in a partially ordered set $(X, \leq)$ are defined in the same manner as intervals on the real line, and with the same notation. Thus for elements $a, b \in X$, the closed interval $[a, b]$ is the set of those $x \in X$ for which $a \leq x \leq b$, and the open interval $(a, b)$ is the set of those $x \in X$ for which $a < x < b$. (Of course, if $a \leq b$, then $[a, b]$ and $(a, b)$ are empty.)

A partially ordered set $X$ is totally ordered if every pair of elements $x, y \in X$ are comparable, i.e., either $x \leq y$ or $y \leq x$. In this case, the partial order $\leq$ on $X$ is called a total order. The set $X$ is upward (downward) directed provided every finite subset of $X$ has an upper (lower) bound in $X$. If every finite subset of $X$ has a supremum (least upper bound) and an infimum (greatest lower bound) in $X$, then $X$ is a lattice. A lattice $X$ is complete provided every subset of $X$ (including
the empty subset) has a supremum and an infimum in \( X \). (The notations \( \wedge \) and \( \vee \) are used for infima and suprema.) A \textit{boolean algebra} is a lattice \( X \) which is distributive (that is, 
\[
x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \quad \text{and} \quad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)
\]
for all \( x, y, z \in X \) and also complemented (that is, \( X \) contains a smallest element 0 and a largest element 1, and for each element \( x \in X \) there is an element \( y \in X \) such that \( x \wedge y = 0 \) and \( x \vee y = 1 \)).

Let \( X \) and \( Y \) be partially ordered sets. A map \( f: X \to Y \) is said to be \textit{order-preserving} (or \textit{isotone}) if \( f \) preserves all the order relations in \( X \), that is, whenever \( x, y \in X \) with \( x \leq y \), then \( f(x) \leq f(y) \). Dually, \( f \) is \textit{order-reversing} (or \textit{antitone}) provided that whenever \( x, y \in X \) with \( x \leq y \), then \( f(x) \geq f(y) \). An \textit{order-isomorphism} of \( X \) onto \( Y \) is a bijection \( f: X \to Y \) such that \( f \) and \( f^{-1} \) are both order-preserving. Such a map preserves whatever infima and suprema may exist in \( X \). Namely, if an element \( a \in X \) is the infimum (supremum) of a subset \( A \subseteq X \), then \( f(a) \) is the infimum (supremum) of \( f(A) \) in \( Y \). An \textit{order anti-isomorphism} of \( X \) onto \( Y \) is a bijection \( f: X \to Y \) such that \( f \) and \( f^{-1} \) are both order-reversing. (Then \( f \) provides an order-isomorphism of \( X \) onto the dual of \( Y \).) Such a map reverses infima and suprema; namely, if an element \( a \in X \) is the infimum (supremum) of a subset \( A \subseteq X \), then \( f(a) \) is the supremum (infimum) of \( f(A) \) in \( Y \). An \textit{order-automorphism} of \( X \) is any order-isomorphism of \( X \) onto itself, while an \textit{order anti-automorphism} of \( X \) is any order anti-isomorphism of \( X \) onto itself.

Finally, let \( X \) and \( Y \) be lattices. A \textit{lattice homomorphism} from \( X \) to \( Y \) is any map \( f: X \to Y \) that preserves finite infima and suprema, that is,
\[
f(x_1 \wedge \cdots \wedge x_n) = f(x_1) \wedge \cdots \wedge f(x_n),
\]
\[
f(x_1 \vee \cdots \vee x_n) = f(x_1) \vee \cdots \vee f(x_n)
\]
for all \( x_1, \ldots, x_n \in X \). All lattice homomorphisms are order-preserving. A \textit{lattice isomorphism} of \( X \) onto \( Y \) is any bijection \( f: X \to Y \) such that \( f \) and \( f^{-1} \) are both lattice homomorphisms. A map \( f: X \to Y \) is a lattice isomorphism if and only if it is an order-isomorphism.

We follow the prevalent American usage (as distinct from, e.g., Bourbaki) in letting \( \mathbb{N} \) denote the set \( \{1, 2, 3, \ldots\} \) of positive integers. The symbol \( \mathbb{Z}^+ \) is used for the set \( \{0, 1, 2, \ldots\} \) of nonnegative integers, in order to be consistent with the notation \( G^+ \) for the subset \( \{x \in G \mid x \geq 0\} \) of a partially ordered abelian group \( G \).
Epilogue: Further $K$-Theoretic Applications.

We resume the discussion begun in the Prologue, giving brief sketches of other situations in which the theory of interpolation groups can be applied, via $K_0$, to the study of certain rings and $C^*$-algebras.

- **Pseudo-Rank Function Spaces and Trace Spaces.** The collection $P(R)$ of all pseudo-rank functions on a regular ring $R$ (with 1), like the collection of all states on a partially ordered abelian group with order-unit, is naturally a compact convex set. Namely, $P(R)$ is a compact convex subset of the product space $R^R$ [49, Proposition 16.17], and in fact $P(R)$ is a Choquet simplex [49, Theorem 17.5]. Moreover, the canonical bijection between $S(K_0(R), [R])$ and $P(R)$ is an affine homeomorphism [49, Proposition 17.12]. Hence, to realize a given Choquet simplex $K$ as $P(R)$ for some regular ring $R$, it suffices to realize $K$ as $S(K_0(R), [R])$. In the metrizable case, this can always be done, because of our ability to realize any countable dimension group as $K_0$ of an ultramatricial algebra. Namely, if $K$ is metrizable, then $K$ is affinely homeomorphic to $S(G, u)$ for some countable simple dimension group $(G, u)$ with order-unit (Theorem 14.12). Since there exists a simple unital ultramatricial algebra $R$ with $(K_0(R), [R]) \cong (G, u)$, we get $P(R)$ affinely homeomorphic to $K$. This proves the author's result that every metrizable Choquet simplex is affinely homeomorphic to $P$ of a simple unit-regular ring [48, Theorem 5.1; 49, Theorem 17.23]. Some nonmetrizable Choquet simplices may be realized via the observation that $K_0$ of a tensor product of ultramatricial algebras is isomorphic to a corresponding tensor product of the $K_0$'s of the factors [59, Proposition 3.4]. As a consequence, the author and Handelman proved that given any nonempty direct product $X$ of compact metric spaces, there exists a tensor product $R$ of simple ultramatricial algebras such that $P(R)$ is affinely homeomorphic to $M_1^+(X)$ [59, Theorem 5.1].

Parallel procedures can be used to realize metrizable Choquet simplices as trace spaces of unital $C^*$-algebras. Given a unital $C^*$-algebra $A$, the set $A_{sa}$ of self-adjoint elements of $A$ becomes a partially ordered real vector space with positive cone

$$\{ x \in A_{sa} \mid \text{spectrum}(x) \subseteq \mathbb{R}^+ \}$$
and order-unit 1 [52, Proposition 6.1]. A state on \( A \) is any linear functional \( A \to C \) which restricts to a state on \( (A_{sa}, 1) \). Since all states on \( (A_{sa}, 1) \) extend uniquely to states on \( A \), the collection of all states on \( A \) may be identified with the state space of \( (A_{sa}, 1) \). A tracial state on \( A \) is any state \( t \) such that \( t(xx^*) = t(x^*x) \) for all \( x \in A \), and a normalized finite trace on \( A \) is the restriction to \( A_{sa}^+ \) of any tracial state. The trace space of \( A \) is the collection \( T(A) \) of all normalized finite traces on \( A \). We identify \( T(A) \) with the collection of all tracial states on \( A \), which is a compact convex subset of \( S(A_{sa}, 1) \). In fact, \( T(A) \) is a Choquet simplex [110, Theorem 3.1.18].

For a matrix algebra \( M_n(C) \), the trace space \( T(M_n(C)) \) is a singleton, as is the state space of \( (K_0(M_n(C)), [M_n(C)]) \). Using the observation that the functors \( T(—) \) and \( S(K_0(—), [—]) \) convert finite products and finite coproducts and convert direct limits to inverse limits, it follows that the trace space of any unital AF C*-algebra \( A \) is affinely homeomorphic to \( S(K_0(A), [A]) \) [13, Corollary 3.2]. Given any metrizable Choquet simplex \( K \), there is a countable simple dimension group \( (G, u) \) with order-unit such that \( S(G, u) \) is affinely homeomorphic to \( K \) (Theorem 14.12). Hence, by choosing a simple unital AF C*-algebra \( A \) for which \( (K_0(A), [A]) \approx (G, u) \), we obtain Blackadar’s result that any metrizable Choquet simplex \( K \) is affinely homeomorphic to \( T(A) \) for some simple unital AF C*-algebra \( A \) [13, Theorem 3.9].

- **Metrically Complete Regular Rings.** A norm-like function \( N^* \) may be defined on any regular ring \( R \) (with 1) by setting \( N^*(x) \) equal to the supremum of the values \( N(x) \) for \( N \in P(R) \) (or zero, if \( R \) has no pseudo-rank functions). It is easily checked that the rule \( d(x, y) = N^*(x - y) \) then defines a pseudo-metric \( d \) on \( R \) [53, Lemma 1.2]. In case \( d \) is a metric and \( R \) is complete with respect to \( d \), we say that \( R \) is \( N^* \)-complete. For instance, if there exists a positive integer \( n \) such that all nilpotent elements \( x \in R \) satisfy \( x^n = 0 \), then \( N^*(y) \geq 1/n \) for all nonzero elements \( y \in R \), and so \( R \) is \( N^* \)-complete [53, Theorem 1.3]. This occurs, for instance, if \( R \) can be embedded in a direct product of \( n \times n \) matrix rings over division rings.

Because of the canonical bijection between \( P(R) \) and \( S(K_0(R), [R]) \), it follows that \( ||xR|| = N^*(x) \) for all \( x \in R \), and so \( N^* \)-completeness of \( R \) is related to norm-completeness of \( K_0(R) \). Specifically, the author proved that if \( R \) is \( N^* \)-complete, then \( (K_0(R), [R]) \) is an archimedean norm-complete interpolation group [53, Theorem 2.11]. Since archimedean groups are unperforated, \( K_0(R) \) is of course a dimension group in this case. Thus the structure theory for archimedean norm-complete dimension groups developed in Chapter 15 applies to \( K_0 \) of any \( N^* \)-complete regular ring \( R \).

For example, the lattice anti-isomorphism between the lattice of closed faces of \( S(K_0(R), [R]) \) and the lattice of norm-closed ideals of \( K_0(R) \) obtained from Theorem 15.24 yields a lattice anti-isomorphism between the lattice of closed faces of \( P(R) \) and the lattice of \( N^* \)-closed two-sided ideals of \( R \). Under this
anti-isomorphism, the singleton closed faces of $P(R)$ (i.e., the faces $\{N\}$ where $N$ is an extreme point of $P(R)$) correspond to the maximal two-sided ideals of $R$ (all of which are $N^*$-closed [53, Corollary 1.14]). Because of the Krein-Mil'man Theorem, the only closed face of $P(R)$ which contains $\partial_s P(R)$ is $P(R)$ itself. Hence, applying the lattice anti-isomorphism, the intersection of the maximal two-sided ideals of $R$ must be zero [53, Corollary 4.6]. Alternatively, this can be obtained by using Corollary 15.29 to see that the intersection of the maximal ideals of $K_0(R)$ is zero.

The affine continuous function representation for archimedean norm-complete dimension groups (Theorem 15.7) provides an affine continuous function representation for $K_0(R)$. Under the canonical affine homeomorphism between $S(K_0(R), [R])$ and $P(R)$, a discrete extremal state $s$ on $(K_0(R), [R])$ corresponds to an extremal pseudo-rank function $P$ with a discrete range of values. Specifically, if $s(K_0(R)) = (1/m)\mathbb{Z}$ for some $m \in \mathbb{N}$, then $P(R) = \{0, \frac{1}{m}, \frac{2}{m}, \ldots, 1\}$, and this occurs if and only if $R/\ker(P)$ is isomorphic to an $m \times m$ matrix ring over a division ring. Set $A_P = (1/m)\mathbb{Z}$ in this case, and for all other extremal pseudo-rank functions $P$ set $A_P = R$. Then there is a natural isomorphism of $(K_0(R), [R])$ onto $(A, 1)$, where

$$A = \{q \in \text{Aff}(P(R)) \mid q(P) \in A_P \text{ for all } P \in \partial_s P(R)\}$$

[53, Theorem 4.11].

To give an easy application of this affine continuous function representation of $K_0(R)$, assume, for some fixed positive integer $t$, that all simple artinian factor rings of $R$ (if there are any) are $t \times t$ matrix rings (over some other rings, not necessarily division rings). Then if $P \in \partial_s P(R)$ and $R/\ker(P)$ is isomorphic to an $m \times m$ matrix ring over a division ring, we must have $t|m$, whence $1/t \in A_P$. As a result, the constant function $1/t$ belongs to the group $A$ given above. From the isomorphism of $(K_0(R), [R])$ onto $(A, 1)$, it follows that $[R] = t[C]$ for some $[C] \in K_0(R)^+$. Since $R$ is unit-regular [53, Theorem 2.3], the module $R$ is isomorphic to a direct sum of $t$ copies of $C$, whence the ring $R$ is isomorphic to a $t \times t$ matrix ring (over the endomorphism ring of $C$) [53, Corollary 4.14].

- **Aleph-Nought-Continuous Regular Rings.** In any regular ring $R$, the collection $L(R_R)$ of principal right ideals forms a lattice, with finite intersections for finite infima and finite sums for finite suprema [49, Theorems 1.1, 2.3]. The ring $R$ is said to be $\aleph_0$-continuous if the lattice $L(R_R)$ is $\aleph_0$-continuous in the sense that (a) every countable subset of $L(R_R)$ has an infimum and a supremum in $L(R_R)$; (b) whenever $A \in L(R_R)$ and $B_1 \subseteq B_2 \subseteq \cdots$ in $L(R_R)$, then $A \wedge (\bigvee B_i) = \bigvee (A \wedge B_i)$; (c) whenever $A \in L(R_R)$ and $B_1 \supseteq B_2 \supseteq \cdots$ in $L(R_R)$, then $A \vee (\bigwedge B_i) = \bigwedge (A \vee B_i)$. (Since $L(R_R)$ is anti-isomorphic to the lattice of principal left ideals of $R$ [49, Theorem 2.5], this definition is left-right symmetric.) Equivalently, $R$ is $\aleph_0$-continuous if and only if given any countably generated right (left) ideal $I$ of $R$, there exists a principal right (left) ideal $J \supseteq I$.
such that every nonzero right (left) ideal contained in $J$ has nonzero intersection with $I$ [49, Corollary 14.4].

Handelman proved that every $\mathcal{N}_0$-continuous regular ring is unit-regular [64, Theorem 3.2; 49, Theorem 14.24], and the author proved that every $\mathcal{N}_0$-continuous regular ring is $N^*$-complete [53, Theorem 1.8]. Hence, the structure theories for archimedean norm-complete dimension groups and $N^*$-complete regular rings yield a structure theory for $\mathcal{N}_0$-continuous regular rings. However, the structure theory for $\mathcal{N}_0$-continuous regular rings was first derived from the structure theory for monotone $\sigma$-complete interpolation groups, as follows.

Handelman, Higgs, and Lawrence proved that $K_0$ of any $\mathcal{N}_0$-continuous regular ring $R$ is a (directed) monotone $\sigma$-complete interpolation group [72, Proposition 2.1] and that such groups are archimedean [72, Theorem 1.3]. Since $K_0(R)$ is archimedean, they obtained $\ker(P(R)) = 0$, from which it follows that the intersection of the maximal two-sided ideals of $R$ is zero [72, Theorem 2.3]. If $M$ is any maximal two-sided ideal of $R$, then the existence of a state on $(K_0(R/M), [R/M])$ implies the existence of a pseudo-rank function $P$ on $R/M$, and $\ker(P) = 0$ because $R/M$ is a simple ring. As a consequence, $R/M$ contains no uncountable direct sums of nonzero principal right or left ideals, and using this countability condition, Handelman proved that $R/M$ is a right and left self-injective ring [64, Corollary 3.2]. Thus, since the intersection of the maximal two-sided ideals of $R$ is zero, $R$ is a subdirect product of simple right and left self-injective rings.

The structure theory for monotone $\sigma$-complete interpolation groups (Chapter 16) was developed by the author, Handelman, and Lawrence [60] and applied to $K_0(R)$. For example, the affine continuous function representation for such groups (Corollary 16.15) led to a complete representation of $K_0(R)$ in terms of affine continuous functions on $P(R)$ [60, Theorem II.15.1]. As a consequence, if all simple factor rings of $R$ are $t \times t$ matrix rings (for some fixed positive integer $t$), then $R$ is a $t \times t$ matrix ring [60, Theorem II.15.3].

Also, the characterizations of various types of quotient groups of monotone $\sigma$-complete interpolation groups (Theorems 16.27, 16.29, 16.30) yield characterizations of various types of factor rings of $R$. For instance, from the characterization of Dedekind $\sigma$-complete quotient groups (Theorem 16.30) it follows that a factor ring $R/J$ is $\mathcal{N}_0$-continuous with “general comparability” if and only if $J$ equals the kernel of some compact basically disconnected subset of $\partial_e P(R)$ [60, Theorem II.14.13]. (A regular ring $S$ satisfies general comparability if and only if for every $x, y \in S$ there exists a central idempotent $e \in S$ such that $exS$ embeds in $eyS$ while $(1-e)yS$ embeds in $(1-e)xS$. In case $S$ is unit-regular, this holds if and only if $(K_0(S), [S])$ satisfies general comparability [60, Proposition II.14.2].)

- **Finite Rickart $C^*$-Algebras.** A Rickart $C^*$-algebra is a $C^*$-algebra $A$ within which the right annihilator of any element $x$ (that is, the right ideal
\{a \in A \mid xa = 0\}\) equals the principal right ideal generated by some projection \(p\) (that is, \(p = p^* = p^2\)). This is a generalization of the concept of an \(AW^*\)-algebra, which is a \(C^*\)-algebra in which the right annihilator of any subset is a principal right ideal generated by a projection. In particular, all von Neumann algebras (\(W^*\)-algebras) are Rickart \(C^*\)-algebras. A finite \(C^*\)-algebra is a unital \(C^*\)-algebra \(A\) such that all elements \(x \in A\) satisfying \(xx^* = 1\) also satisfy \(x^*x = 1\).

The \(K\)-theory of a finite Rickart \(C^*\)-algebra \(A\) can be investigated with the aid of an auxiliary \(\mathbb{K}_0\)-continuous regular ring \(R\) which is also \(*\)-regular, i.e., there is an involution \(*\) on \(R\) such that every principal right ideal of \(R\) is generated by a projection. Handelman proved that \(A\) is a \(*\)-subring of an \(\mathbb{K}_0\)-continuous \(*\)-regular ring \(R\) such that the only projections in \(R\) are those in \(A\) \([64, \text{Theorem } 2.1]\). The ring \(R\) is essentially unique (up to a \(*\)-ring isomorphism which is the identity on \(A\)), and is called the regular ring of \(A\). Handelman also proved that the inclusion map \(A \to R\) induces an isomorphism of \((K_0(A), [A])\) onto \((K_0(R), [R])\). (A proof for the case that \(A\) has no one-dimensional representations is given in \([54, \text{Theorem } 5.2]\).)

In particular, \(K_0(A)\) is a monotone \(\sigma\)-complete interpolation group, and the structure theory for such groups yields a corresponding structure theory for \(A\), in exactly the same manner as for \(\mathbb{K}_0\)-continuous regular rings. (However, this structure theory was first derived from the structure theory for \(\mathbb{K}_0\)-continuous regular rings, via the regular ring of \(A\).) For example, \(A\) is a subdirect product of simple \(AW^*\)-algebras, and so \(A\) can be embedded in a finite \(AW^*\)-algebra \([72, \text{Theorem } 3.1]\). For another example, if the dimension of every finite-dimensional irreducible representation of \(A\) is divisible by a fixed positive integer \(t\), then \(A\) is a \(t \times t\) matrix ring over some other finite Rickart \(C^*\)-algebra \([60, \text{Theorem III.16.8}]\).

- **Extensions of AF \(C^*\)-Algebras.** The classification of AF \(C^*\)-algebras in terms of countable dimension groups can also be used to classify various types of AF \(C^*\)-algebras. For example, the simple unital AF \(C^*\)-algebras are classified by the countable simple dimension groups with order-unit, and these are determined by Proposition 14.3 and Theorem 14.16. A natural next step is to study extensions of one simple AF \(C^*\)-algebra by another, as follows.

Let \(A\) and \(C\) be AF \(C^*\)-algebras, with \(C\) unital. A unital \((C^*\)-algebra) extension of \(A\) by \(C\) is a diagram

\[
0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0
\]

where \(B\) is a unital \(C^*\)-algebra, \(\phi\) is a \(C^*\)-algebra isomorphism of \(A\) onto a closed two-sided ideal of \(B\), and \(\psi\) is a \(C^*\)-algebra map that induces an isomorphism of \(B/\phi(A)\) onto \(C\). Elliott and Brown proved that \(B\) is necessarily AF \([36, \text{Corollary } 3.3; \text{16, Theorem; } 29, \text{Theorem } 9.9]\), so that \(B\) can be classified in terms of \(K_0(B)\). To classify the unital extensions of \(A\) by \(C\), apply \(K_0\) to the
entire extensions. Thus $K_0$ of an extension as given above yields a diagram

$$0 \rightarrow (K_0(A), D(A)) \xrightarrow{K_0(\phi)} (K_0(B), [B]) \xrightarrow{K_0(\psi)} (K_0(C), [C]) \rightarrow 0$$

which is a dimension group extension of $(K_0(A), D(A))$ by $(K_0(C), [C])$ [67, Lemma I.5; 58, p. 84]. Conversely, any dimension group extension of $(K_0(A), D(A))$ by $(K_0(C), [C])$ is equivalent to $K_0$ of a unital extension of $A$ by $C$ [67, Proposition I.7; 58, p. 84].

Several definitions of equivalence for unital extensions of $A$ by $C$ are possible. For classification via $K_0$, define unital extensions

$$0 \rightarrow A \xrightarrow{\phi_1} B_1 \xrightarrow{\psi_1} C \rightarrow 0$$

and

$$0 \rightarrow A \xrightarrow{\phi_2} B_2 \xrightarrow{\psi_2} C \rightarrow 0$$

to be equivalent if and only if there exists a commutative diagram

$$0 \rightarrow A \xrightarrow{\phi_1} B_1 \xrightarrow{\psi_1} C \rightarrow 0$$

$$\begin{array}{ccc}
\alpha \downarrow & & \beta \downarrow \\
0 \rightarrow A \xrightarrow{\phi_2} B_2 \xrightarrow{\psi_2} C \rightarrow 0
\end{array}$$

in which $\beta$ is a $C^*$-algebra isomorphism of $B_1$ onto $B_2$ while $\alpha$ and $\gamma$ are "approximately inner" $C^*$-algebra automorphisms of $A$ and $C$ (that is, $\alpha$ and $\gamma$ are norm-limits of sequences of inner automorphisms). The given unital extensions of $A$ by $C$ are equivalent in this sense if and only if the corresponding dimension group extensions of $(K_0(A), D(A))$ by $(K_0(C), [C])$ are equivalent [67, Lemma I.5; 17, Theorem 1; 58, Proposition 9.1].

Therefore the problem of classifying all unital extensions of $A$ by $C$ can be translated into the problem of classifying all dimension group extensions of $(K_0(A), D(A))$ by $(K_0(C), [C])$.

For example, the author and Handelman proved some general existence theorems for unital extensions of $A$ by $C$ in this manner. If $A$ has no nonzero unital homomorphic images, [58, Proposition 7.5] shows that the generating interval $D(A)$ satisfies the hypotheses of Theorem 17.19, and so there exist dimension group extensions of $(K_0(A), D(A))$ by $(K_0(C), [C])$. Consequently, there exist unital extensions of $A$ by $C$ in this case [58, Theorem 9.9]. In particular, this holds if $A$ is simple and unitless.

In case $A$ and $C$ are simple and $A$ is unitless, a unital extension of $A$ by $C$ has only three closed ideals (i.e., there are only three closed ideals in the algebra $B$ in the middle of the extension), namely, 0, the whole algebra, and the image of $A$. More generally, a unital extension of $A$ by $C$ is stenotic provided every ideal of the extension either contains or is contained in the image of $A$. Stenotic dimension group extensions are defined in the same way, and a unital extension of $A$ by $C$ is stenotic if and only if $K_0$ of the extension is a stenotic extension of dimension groups.

The author and Handelman classified stenotic dimension group extensions in several cases [58, Section VII] and translated the results into classifications of stenotic unital $C^*$-algebra extensions in the corresponding cases [58, Section...
EPILOGUE: FURTHER K-THEORETIC APPLICATIONS.

IX]. For example, let \( H \) be a nonzero dimension group, let \( D \) be a generating interval in \( H^+ \) satisfying the condition of Theorem 17.19, and let \( m \) be a positive integer. If \( H \) has an order-unit, the equivalence classes of stenotic dimension group extensions of \( (H, D) \) by \((Z, m)\) can be arranged in bijection with the group \( H/mH \) [58, Theorem 7.14]. Correspondingly, let \( A \) be a nonzero AF \( C^* \)-algebra with no nonzero unital homomorphich images, and assume that \( A \) contains a projection \( p \) such that \( ApA \) is dense in \( A \). Then for any positive integer \( m \), the equivalence classes of stenotic unital extensions of \( A \) by \( M_m(C) \) can be arranged in bijection with the group \( K_0(A)/mK_0(A) \) [58, Theorem 9.10].

Analogous patterns of course appear for classification of extensions of ultra-matricial algebras, as indicated in [51].
Open Problems

The most fundamental question relating to the material in this book is a general meta-question: find new situations in which partially ordered Grothendieck groups can be used to classify or provide structural information about some mathematical systems. Such situations will most likely demand new results in the theory of partially ordered abelian groups, particularly if the groups that appear are not interpolation groups. On a more mundane level, we present in this section a list of open problems more directly related to the theory of interpolation groups as developed in this book. Solutions to the more technical of these problems would help to smooth out the development of the theory, but might not be of sufficient general interest to justify publication. In any case, the author would certainly wish to hear of solutions to any of these problems.

- Perforation.
  1. Remove the “unperforated” hypothesis from as many theorems as possible. In particular, extend results from dimension groups to directed interpolation groups. For instance, does Theorem 13.5 hold for arbitrary interpolation groups? (A positive solution to problem 2 would provide a general technique for use with this problem.)

  2. Given a partially ordered abelian group \( G \), define relations \( \leq_1 \) and \( \leq_2 \) on \( G \) so that elements \( x, y \in G \) satisfy \( x \leq_1 y \) if and only if \( mx \leq my \) for some \( m \in \mathbb{N} \), while \( x \leq_2 y \) if and only if \( 2^nx \leq 2^ny \) for some \( n \in \mathbb{N} \). If \( G \) is an interpolation group, is either \((G, \leq_1)\) or \((G, \leq_2)\) an interpolation group? (In case \( \leq_1 \) or \( \leq_2 \) is not antisymmetric, factor an appropriate ideal out of \( G \), as in Proposition 1.1.)

- Dimension Groups.
  3. Is every dimension group isomorphic to a direct limit of a countable sequence of lattice-ordered abelian groups? (This holds for all countable dimension groups, because of Theorem 3.17. Moreover, every dimension group is isomorphic to a (possibly uncountable) direct limit of lattice-ordered abelian groups, by Theorem 3.21.)

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4. Is every (metrizable) Choquet simplex affinely homeomorphic to the state space of a (countable) archimedean simple dimension group with order-unit? (By Corollary 14.9 and Theorem 14.12, every (metrizable) Choquet simplex is affinely homeomorphic to the state space of a (countable) simple dimension group (not necessarily archimedean). Also, any Choquet simplex is affinely homeomorphic to the state space of an archimedean dimension group (not necessarily simple), by Corollary 11.5.)

5. There exists a metrizable Choquet simplex $K$, called the Poulsen simplex, such that $\partial_e K$ is dense in $K$ [105]. Moreover, $K$ is unique up to affine homeomorphism, and every metrizable Choquet simplex is affinely homeomorphic to a closed face of $K$ [92, Theorems 2.3, 2.5]. Find necessary and sufficient conditions on a countable dimension group $(G, u)$ with order-unit so that $S(G, u)$ is affinely homeomorphic to $K$. (By Theorem 14.12, there exists at least one countable dimension group with order-unit whose state space is affinely homeomorphic to $K$.) If possible, extend the conditions to uncountable dimension groups with order-unit and obtain nonmetrizable Choquet simplices with dense extreme boundaries.

6. Given a nonempty set $X$, we may construct a dimension group $(G, u)$ with order-unit and a map $f: X \to [0, u]$ with the following "semi-universal" mapping property: given any dimension group $(H, v)$ with order-unit and any map $g: X \to [0, v]$, there exists a normalized positive homomorphism (not necessarily unique) $h: (G, u) \to (H, v)$ such that $hf = g$. (Namely, choose an element $u \notin X$, let $G_0$ be a free abelian group with basis $X \cup \{u\}$, and make $G_0$ into a partially ordered abelian group with positive cone generated by the set $X \cup \{u\} \cup \{u - x \mid x \in X\}$. Then construct unperforated partially ordered abelian groups $G_0 \subseteq G_1 \subseteq \cdots$ such that whenever $x_1, x_2, y_1, y_2 \in G_n$ with $x_i \leq y_j$ for all $i, j$, there exists $z \in G_{n+1}$ such that $x_i \leq z \leq y_j$ for all $i, j$, and $G_{n+1}$ is generated by $G_n$ and these $z$'s. The union of the $G_n$ is the desired dimension group $G$, and the inclusion map $X \to [0, u]$ is the desired map $f$.) If $X$ is countably infinite, then $G$ is countable. Is $S(G, u)$ affinely homeomorphic to the Poulsen simplex? (It follows from the semi-universal mapping property that every countable dimension group with order-unit is isomorphic to a quotient of $G$. Consequently, because of Theorem 14.12, every metrizable Choquet simplex is affinely homeomorphic to a closed face of $S(G, u)$.) In the uncountable case, is $\partial_e S(G, u)$ always dense in $S(G, u)$?

- **Order-Unit Norms.**

7. Let $(G, u)$ be an interpolation group with order-unit, and let $H$ be an ideal of $G$. Is $\|x + H\| = \inf \{\|y\| : y \in x + H\}$ for all $x \in G$? (This holds for dimension groups by Proposition 7.15.)
8. Let \((G, u)\) be a partially ordered abelian group with order-unit, let \(\overline{G}\) be the norm-completion of \(G\), and let \(\psi: G \rightarrow \overline{G}\) be the natural map. Is \(\overline{G}\) partially ordered? Is \(\psi(u)\) an order-unit in \(\overline{G}\)? If \(G\) has interpolation, does \(\overline{G}\) have interpolation? Assuming the first two questions have positive answers, is \(\overline{G}\) norm-complete? In particular, does the order-unit metric on \(\overline{G}\) coincide with the metric obtained from the completion process? (All these questions have positive answers in case \(G\) is a dimension group, by Theorems 15.2 and 15.3.)

- **Norm-Completeness.**

9. Derive the structure theory for archimedean norm-complete dimension groups (e.g., Corollaries 15.15, 15.22, Theorems 15.20, 15.23) directly from the affine continuous function representation (Theorem 15.7).

10. Let \((G, u)\) be an archimedean norm-complete dimension group with order-unit. If \(\partial_e S(G, u)\) is compact and totally disconnected, does \((G, u)\) have general comparability? If \(\partial_e S(G, u)\) is compact and basically disconnected, is \(G\) Dedekind complete? If \(\partial_e S(G, u)\) is compact and extremally disconnected, is \(G\) Dedekind complete? (In case \(G\) satisfies countable interpolation, Theorems 16.29 and 16.30 provide positive answers for the first two questions.)

11. Let \((G, u)\) be an archimedean norm-complete dimension group with order-unit, and let \(H\) be an ideal of \(G\). If \(H^+ = \ker(X)^+\) for some \(X \subseteq S(G, u)\), is \(H\) norm-closed in \(G\)? (This holds if \(H = \ker(X)\), by Theorem 15.6, or if \(G\) has countable interpolation, by Theorem 16.12.)

12. Let \((G, u)\) be an archimedean norm-complete dimension group with order-unit, and let \(\theta\) be the continuous bijection of \(\partial_e S(G, u)\) onto MaxSpec\((G)\) given in Theorem 15.32. Does \(\theta\) map all closed subsets of \(\partial_e S(G, u)\) onto compact subsets of MaxSpec\((G)\) ?

13. Let \((G, u)\) be an archimedean norm-complete dimension group with order-unit, set \(S = S(G, u)\), and let \(\phi: G \rightarrow \text{Aff}(S)\) denote the natural map. Let \(p: S \rightarrow \{-\infty\} \cup \mathbb{R}\) be an upper semicontinuous convex function, and let \(q: S \rightarrow \mathbb{R} \cup \{\infty\}\) be a lower semicontinuous concave function. Assume that \(p \leq q\) and that \(p(s), q(s) \in s(G) \cup \{\pm \infty\}\) for all discrete \(s \in \partial_e S\). Does there exist \(x \in G\) such that \(p \leq \phi(x) \leq q\)? If \(q \geq 0\), does there exist \(x \in G^+\) such that \(p \leq \phi(x) \leq q\)? (Example 15.13 shows that this need not happen if in place of \(p(s), q(s) \in s(G) \cup \{\pm \infty\}\) it is assumed that \([p(s), q(s)] \cap s(G)\) is nonempty. On the other hand, in case \(G\) has countable interpolation and \(p, q\) are continuous real-valued functions, Theorem 16.18 provides positive answers for both questions.)

14. Let \(K\) be a Choquet simplex, and for \(x \in \partial_e K\) let \(A_x\) be either \(\mathbb{R}\) or \((1/m_x)\mathbb{Z}\) for some \(m_x \in \mathbb{N}\). Set
\[
A = \{f \in \text{Aff}(K) \mid f(x) \in A_x \text{ for all } x \in \partial_e K\}.
\]
Then \((A, 1)\) is an archimedean norm-complete partially ordered abelian group with order-unit, and by Theorem 15.7 every archimedean norm-complete dimension group with order-unit is isomorphic to some \((A, 1)\). Find necessary and sufficient conditions on the groups \(A_x\) for \(A\) to be a dimension group. (Example 15.12 shows that \(A\) does not always have interpolation.)

- **Countable Interpolation and Monotone \(\sigma\)-Completeness.**
  15. Is every directed abelian group with countable interpolation unperforated?

  16. Is every directed abelian group with countable interpolation isomorphic to a quotient group of a monotone \(\sigma\)-complete dimension group?

  17. Let \((G, u)\) be a partially ordered abelian group with order-unit, satisfying countable interpolation, and let \(F\) be a closed face of \(S(G, u)\). Find necessary and sufficient conditions on \(F\) for \(G/\ker(F)\) to be monotone \(\sigma\)-complete. (In case \(\partial_e F\) is compact, \(G/\ker(F)\) is monotone \(\sigma\)-complete if and only if \(\partial_e F\) is basically disconnected, by Theorems 16.27 and 16.30.)

  18. If \((G, u)\) is a monotone \(\sigma\)-complete interpolation group with order-unit, and \(X\) is a compact subset of \(\partial_e S(G, u)\), then \(X\) is an \(F\)-space by Theorem 16.22. Is \(X\) totally disconnected, or even basically disconnected? Equivalently, does every archimedean lattice-ordered quotient group of \(G\) satisfy general comparability, or even Dedekind \(\sigma\)-completeness? (In case \(X = \partial_e S(G, u)\), Theorems 16.27 and 16.30 provide positive answers for both questions. On the other hand, monotone \(\sigma\)-completeness may not be replaced by countable interpolation, since \(C(X, R)\) has countable interpolation whenever \(X\) is a compact Hausdorff \(F\)-space [112, Theorem 1.1], and there exist compact Hausdorff connected \(F\)-spaces [46, Example 2.8].)

  19. Let \((G, u)\) be a partially ordered abelian group with order-unit, satisfying countable interpolation, set \(S = S(G, u)\), and let \(\phi: G \to \text{Aff}(S)\) be the natural map. Let \(p: S \to \{\infty\} \cup \mathbb{R}\) be an upper semicontinuous convex function, and let \(q: S \to \mathbb{R} \cup \{\infty\}\) be a lower semicontinuous concave function. Assume that \(p \leq q\) and that \([p(s), q(s)] \cap s(G)\) is nonempty for all discrete \(s \in \partial_e S\). Does there exist \(x \in G\) such that \(p \leq \phi(x) \leq q\)? If \(q \geq 0\), does there exist \(x \in G^+\) such that \(p \leq \phi(x) \leq q\)? (Both questions have positive answers in case \(p\) and \(q\) are continuous and real-valued, by Theorem 16.18.)

  20. Characterize those Choquet simplices \(K\) for which \(\text{Aff}(K)\) has countable interpolation or is monotone \(\sigma\)-complete. For instance, if all compact subsets of \(\partial_e K\) are \(F\)-spaces, does \(\text{Aff}(K)\) have countable interpolation? (In case \(\partial_e K\) is compact, it follows from [112, Theorem 1.1] that \(\text{Aff}(K)\) has countable interpolation if and only if \(\partial_e K\) is an \(F\)-space. Moreover, \(\text{Aff}(K)\) is Dedekind \(\sigma\)-complete...
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if and only if \( \partial_e K \) is compact and basically disconnected, by Corollaries 12.19, 11.20, and 9.3.)

21. Let \( K \) be a Choquet simplex, and for \( x \in \partial_e K \) let \( A_x \) be either \( \mathbb{R} \) or \( (1/m_x)\mathbb{Z} \) for some \( m_x \in \mathbb{N} \). Set

\[
A = \{ f \in \text{Aff}(K) \mid f(x) \in A_x \text{ for all } x \in \partial_e K \}.
\]

Find necessary and sufficient conditions on the groups \( A_x \) for \( A \) to have countable interpolation, or for \( A \) to be monotone \( \sigma \)-complete. (Compare problem 14.)

- Extensions.

22. Let \( H \) be a dimension group, let \( D \) be a generating interval in \( H^+ \), and let \( (K,v) \) be a dimension group with order-unit. Find necessary and sufficient conditions for the existence of a dimension group extension of \( (H,D) \) by \( (K,v) \). (See Example 17.15 and Theorem 17.19.) In particular, does such an extension exist if for each \( a \in D \) and \( m \in \mathbb{N} \) there exists \( b \in D \) such that \( mb \in D \) and \( a \leq mb \)?

23. Let \( H \) and \( K \) be dimension groups. Find data with which to classify all dimension group extensions of \( H \) by \( K \). An extension

\[
0 \to H \buildrel {\tau} \over \to G \buildrel {\pi} \over \to K \to 0
\]

is essential provided every nonzero ideal of \( G \) has nonzero intersection with \( \tau(H) \), and it is stenotic provided every ideal of \( G \) either contains \( \tau(H) \) or is contained in \( \tau(H) \). Find data with which to classify all essential or stenotic dimension group extensions of \( H \) by \( K \). (See [58, Theorems 6.5, 6.6, 6.7].) Similarly, given a generating interval \( D \) in \( H^+ \) and an order-unit \( v \in K \), find data with which to classify all, all essential, or all stenotic dimension group extensions of \( (H,D) \) by \( (K,v) \). (See [58, Corollaries 7.10, 7.16, Theorems 7.11, 7.14].)

24. Let \( H \) be a dimension group, let \( D \) be a generating interval in \( H^+ \), and let \( (K,v) \) be a dimension group with order-unit. If there exists a stenotic dimension group extension of \( (H,D) \) by \( (K,v) \), then by [58, Lemma 7.6] for any \( a \in D \) there exist \( b \in D \) and \( n \in \mathbb{N} \) such that \((n+1)a \leq nb \). Conversely, if this condition holds, and if \( H \) and \( K \) are countable, does there exist a stenotic dimension group extension of \( (H,D) \) by \( (K,v) \)? (See Theorem 17.19 and [58, Corollary 7.12, Theorem 7.20]. If \( K \) is allowed to be uncountable, there need not exist any stenotic dimension group extensions of \( (H,D) \) by \( (K,v) \) [58, Example 7.17]. A positive answer in the countable case would imply that given any AF \( C^* \)-algebras \( A \) and \( C \) such that \( C \) is unital while \( A \) has no nonzero unital homomorphic images, there would exist a stenotic unital \( C^* \)-algebra extension of \( A \) by \( C \), because of [58, Proposition 9.5].)

25. Let \( H \) be a dimension group, let \( D \) be a generating interval in \( H^+ \), and let \( (K,v) \) be a dimension group with order-unit. For \( n \in \mathbb{N} \), let \( S_n \) be the family
of all equivalence classes of stenotic dimension group extensions of \((H,D)\) by \((K,nv)\), and set \(S = \bigcup S_n\). As shown in \([58, \text{Section VIII}]\), there are natural addition operations \(S_m \times S_n \to S_{m+n}\) for all \(m, n \in \mathbb{N}\), using which \(S\) becomes an abelian semigroup. In some cases, there exists an abelian group \(E\) such that \(S \cong E \times \mathbb{N}\) \([58, \text{Theorems 8.6, 8.7}]\). Is \(S\) always isomorphic to the direct product of an abelian group with \(\mathbb{N}\)? Does there exist a natural addition operation on \(S_1\) under which \(S_1\) becomes an abelian semigroup or an abelian group?

- **Tensor Products.**

26. Given partially ordered abelian groups \(G_1, \ldots, G_n\), their tensor product \(G_1 \otimes \cdots \otimes G_n\) can be made into a partially ordered abelian group with positive cone \((G_1 \otimes \cdots \otimes G_n)^+\) equal to the set of all sums of pure tensors of the form \(x_1 \otimes \cdots \otimes x_n\) where each \(x_i \in G_i^+\) \([59, \text{Proposition 2.1}]\). Is every such tensor product of interpolation groups an interpolation group? (By \([59, \text{Proposition 2.3}]\), every tensor product of dimension groups is a dimension group.)

27. What universal mapping properties are satisfied by tensor products in the category of partially ordered abelian groups? For instance, given partially ordered abelian groups \(G_1, G_2, G\), order-units \(u_i \in G_i\), and positive homomorphisms \(f_i: G_i \to G\), what conditions on \(f_1\) and \(f_2\) ensure that there exists a positive homomorphism \(f: G_1 \otimes G_2 \to G\) such that \(f(x \otimes u_2) = f_1(x)\) for all \(x \in G_1\) and \(f(u_1 \otimes y) = f_2(y)\) for all \(y \in G_2\)?

28. The tensor product of an infinite family \(\{(G_i, u_i) \mid i \in I\}\) of partially ordered abelian groups with order-unit is defined as the direct limit of the tensor products \((\bigotimes_{i \in A} G_i, \bigotimes_{i \in A} u_i)\) over nonempty finite subsets \(A \subseteq I\), using maps

\[
\left(\bigotimes_{i \in A} G_i, \bigotimes_{i \in A} u_i\right) \to \left(\bigotimes_{i \in B} G_i, \bigotimes_{i \in B} u_i\right)
\]

(for \(A \subseteq B\)) sending \(\bigotimes_{i \in A} x_i\) to \((\bigotimes_{i \in A} x_i) \otimes (\bigotimes_{i \in B \setminus A} u_i)\). Characterize those dimension groups with order-unit which are isomorphic to tensor products of countable dimension groups with order-unit or which are isomorphic to quotient groups of tensor products of countable dimension groups with order-unit. (All such dimension groups appear as \(K_0\) of tensor products of ultramatricial algebras, or as \(K_0\) of factor rings of tensor products of ultramatricial algebras, because of \([59, \text{Proposition 3.4}]\).)

29. Given an arbitrary (nonmetrizable) Choquet simplex \(K\), does there exist a family \(\{K_i \mid i \in I\}\) of metrizable Choquet simplices such that \(K\) is affinely homeomorphic to a closed face of the state space of \(\bigotimes_{i \in I} (\text{Aff}(K_i), 1)\)? (If so, \(K\) is affinely homeomorphic to \(\mathbb{P}(R)\) for some \(N^*\)-complete unit-regular ring \(R\), by the method of \([59, \text{Theorem 5.2}]\).)
• Miscellany.

30. Can every partially ordered abelian group be embedded in a simple interpolation group? (It can be shown that every partially ordered abelian group can be embedded in a (not necessarily simple) interpolation group.) Can every torsion-free (unperforated) partially ordered abelian group be embedded in a torsion-free (unperforated) simple interpolation group?

31. Let $G$ be an interpolation group, let $A$ be a finite group of ordered group automorphisms of $G$, and let $G^A$ be the fixed subgroup $\{x \in G \mid \alpha(x) = x \text{ for all } \alpha \in A\}$. Is $G^A$ an interpolation group? (This can be proved for a dimension group $G$ such that whenever $x < z$ in $G$, there exists $y \in G$ with $x \leq my \leq z$, where $m = \text{card}(A)$. It follows that the result holds for all simple dimension groups.)

32. In the category of convex sets, is every inverse limit of simplices a simplex? (This can be proved for inverse limits of simplices which are affinely isomorphic to bases for Dedekind complete lattice cones.)

33. Find a simpler derivation for Theorem 13.5, or for Corollary 13.6. In particular, can these results be proved without proving the Dedekind complete case first and using metric completions? (For one result that can be done without using metric completions, see the proof of the extremal state criterion for dimension groups discussed following Theorem 12.14.)
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