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Noncommutative Harmonic Analysis

Michael E. Taylor

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Providence, Rhode Island

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Introduction

This book began as lecture notes for a one semester course at Stony Brook, on noncommutative harmonic analysis. We explore some basic roles of Lie groups in linear analysis, concentrating on generalizations of the Fourier transform and the study of naturally occurring partial differential equations.

The Fourier transform is a cornerstone of analysis. It is defined by

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = (2\pi)^{-n/2} \int_{R^n} f(x) e^{-ix\cdot\xi}\, dx, \tag{I.1}$$

where, given $f \in L^1$, one has $\hat{f} \in L^\infty$. If f and all its derivatives are rapidly decreasing (one says $f \in \mathcal{S}(R^n)$), so is $\hat{f}$. If you define

$$\mathcal{F}^* f(\xi) = \tilde{f}(\xi) = (2\pi)^{-n/2} \int_{R^n} f(x) e^{ix\cdot\xi}\, dx, \tag{I.2}$$

then one has the Fourier inversion formula

$$\mathcal{F}^*\mathcal{F} = \mathcal{F}\mathcal{F}^* = I \quad \text{on } \mathcal{S}, \tag{I.3}$$

and hence $\mathcal{F}$ extends uniquely to a unitary operator on $L^2(R^n)$; this is the Plancherel theorem. These facts are proved in Appendix A, at the end of this monograph.

The Fourier transform is very useful in deriving solutions to the classical partial differential equations with constant coefficients, because of the intertwining property

$$\mathcal{F}(\partial/\partial x_j) = i\xi_j \mathcal{F}, \tag{I.4}$$

which implies

$$\mathcal{F}P(D) = P(\xi)\mathcal{F}, \tag{I.5}$$

where $D_j = (1/i)\partial/\partial x_j$ and $P(\xi)$ is any polynomial. For example, the fundamental solution to the heat equation $P(t,x)$, defined by

$$\begin{aligned}(\partial/\partial t)P(t,x) &= \Delta P(t,x),\\ P(0,x) &= \delta(x),\end{aligned} \tag{I.6}$$

is transformed to an ODE with parameters for the partial Fourier transform $\hat{P}(t,\xi)$:

$$(d/dt)\hat{P}(t,\xi) = -|\xi|^2\hat{P}(t,\xi), \qquad \hat{P}(0,\xi) = (2\pi)^{-n/2}, \tag{I.7}$$

which implies

$$\hat{P}(t,\xi) = (2\pi)^{-n/2}e^{-t|\xi|^2}, \tag{I.8}$$

and computing the inverse Fourier transform of this Gaussian (see Appendix A) gives

$$P(t,x) = (2\pi)^{-n}\int e^{-t|\xi|^2+ix\cdot\xi}\,d\xi = (4\pi t)^{-n/2}e^{-|x|^2/4t}. \tag{I.9}$$

It is suggestive, and quite handy, to use the operator notation

$$P(t,x) = e^{t\Delta}\delta(x), \tag{I.10}$$

where, for a decent function f, $f(\Delta)$ is defined to be $\mathcal{F}^{-1}f(-|\xi|^2)\mathcal{F}$.

From here we can apply a nice analytical device, known as the subordination identity, to derive the Poisson integral formula in the upper half space. The subordination identity is

$$e^{-yA} = \frac{1}{2}y\pi^{-1/2}\int_0^\infty e^{-y^2/4t}e^{-tA^2}t^{-3/2}\,dt, \quad \text{if } A > 0. \tag{I.11}$$

We will give a proof of (I.11) at the end of this introduction. By the spectral theorem (or the Fourier integral representation for $A = \sqrt{-\Delta}$), this identity also holds for a positive selfadjoint operator A. We can apply it to $A = \sqrt{-\Delta}$, substituting (I.9) for $e^{t\Delta}\delta(x)$, to get

$$\begin{aligned} e^{-y(-\Delta)^{1/2}}\delta(x) &= \frac{1}{2}y\pi^{-1/2}\int_0^\infty e^{-y^2/4t}e^{t\Delta}\delta(x)t^{-3/2}\,dt \\ &= (4\pi)^{-(n+1)/2}y\int_0^\infty e^{-y^2/4t}e^{-|x|^2/4t}t^{-(n+3)/2}\,dt \\ &= c_n y(y^2+|x|^2)^{-(n+1)/2}, \end{aligned} \tag{I.12}$$

where the last integral is easily evaluated using the substitution $s = 1/t$. If $n \geq 2$, then $(-\Delta)^{-1/2}e^{-y(-\Delta)^{1/2}}$ maps the space of compactly supported distributions $\mathcal{E}'(R^n)$ to the space of tempered distributions $\mathcal{S}'(R^n)$ and is obtained by integrating (I.12) with respect to y. We obtain

$$(-\Delta)^{-1/2}e^{-y(-\Delta)^{1/2}}\delta(x) = c_n'(y^2+|x|^2)^{-(n-1)/2}. \tag{I.13}$$

We can analytically continue these formulas to complex y such that $\operatorname{Re} y > 0$, and then pass to the limit of purely imaginary y, say $y = it$, to obtain a formula for the solution of the wave equation on $R \times R^n$:

$$(\partial^2/\partial t^2 - \Delta)u = 0, \qquad u(0,x) = f(x), \qquad u_t(0,x) = g(x), \tag{I.14}$$

which is

$$u(t,x) = \cos t(-\Delta)^{1/2} f + (-\Delta)^{-1/2} \sin t(-\Delta)^{1/2} g. \tag{I.15}$$

In fact we have

$$(-\Delta)^{-1/2} e^{it(-\Delta)^{1/2}} \delta(x) = c_n' \lim_{\varepsilon \downarrow 0} (|x|^2 - (t - i\varepsilon)^2)^{-(n-1)/2}. \tag{I.16}$$

Taking real and imaginary parts, we can read off the finite propagation speed, and the fact that the strict Huygens principle holds when $n \geq 3$ is odd. For example, if $n = 3$, some elementary distribution theory applied to (I.16) yields

$$(-\Delta)^{-1/2} \sin t(-\Delta)^{1/2} \delta(x) = (4\pi t)^{-1} \delta(|x| - t). \tag{I.17}$$

In the last paragraphs we have sketched one line of application of Fourier analysis, which is harmonic analysis on Euclidean space, to constant coefficient PDE. One reason such analysis works so neatly is that such operators commute with translations, i.e., with the regular representation of R^n on $L^2(R^n)$ defined by

$$R(y)u(x) = u(x+y), \qquad x, y \in R^n. \tag{I.18}$$

The Fourier transform provides a spectral representation of $R(y)$, i.e., intertwines it with

$$\mathcal{F}^{-1} R(y) \mathcal{F} u(\xi) = e^{iy\cdot\xi} u(\xi), \tag{I.19}$$

and thus any operator commuting with $R(y)$ is intertwined with a multiplication operator by $\mathcal{F}$. This restatement of (I.5) is obvious since (I.4) is just a differentiated form of (I.19). From this point of view we can look upon the functions of differential operators such as (I.12) or (I.16) as being obtained by synthesis of very simple operators (scalars) on the irreducible representation spaces into which the representation (I.18) decomposes.

Many partial differential equations considered classically, particularly boundary problems for domains with simple shapes, exhibit noncommutative groups of symmetries, and noncommutative harmonic analysis arises as a tool in the investigation of these equations. The connection between solving equations on domains bounded by spheres and harmonic analysis on orthogonal groups is one basic case. Sometimes symmetries are manifested not simply as groups of motions on the underlying spatial domain, but in more subtle fashions. For example, for the harmonic oscillator Hamiltonian $-\Delta + |x|^2$, on R^n, we have the rotation group $\mathrm{SO}(n)$ acting as a group of symmetries in an obvious manner, but actually $\mathrm{SU}(n)$ acts as a group of symmetries, by restricting the "metaplectic representation," introduced in Chapter 1 and discussed further in Chapter 11. As another such example, we mention the action of the "$ax + b$ group" on L^2 functions on a cone, which will be explored in Chapter 7.

Harmonic analysis, commutative and noncommutative, plays an important role in contemporary investigations of linear PDE. For example, pseudodifferential operators are certain variable coefficient versions of Fourier multipliers (convolution operators). They can be used to impose an approximate translation symmetry on elliptic operators, such as Laplace operators on general Riemannian manifolds, and are particularly effective in treating various analytical aspects of elliptic equations and elliptic boundary problems. Fourier integral operators allow one to impose a translational symmetry on an operator with simple characteristics (and real principal symbol). However, when a Fourier integral operator is applied to an operator with multiple characteristics, in cases where the characteristic set can be put in some sort of normal form via a symplectic transformation, one is often led to a problem in noncommutative harmonic analysis. These notes will not be concerned much with such aspects of harmonic analysis. Presentations in the commutative case can be found in [**121**, **234**, **239**], and an introduction to "noncommutative microlocal analysis" in [**235**]. Although we do not consider the general theory here, one thread to be followed through these notes is the use of noncommutative harmonic analysis to treat certain important model cases of operators with double characteristics.

The basic method of noncommutative harmonic analysis, generalizing the use of the Fourier transform, is to synthesize operators on a space on which a Lie group has a unitary representation from operators on irreducible representation spaces. Thus one is led to determine what the irreducible unitary representations of a given Lie group are, and how to decompose a given representation into irreducibles. This general study is far from complete, though a great deal of progress has been made on important classes of Lie groups.

These notes begin with an introductory Chapter 0, dealing with some basic concepts of Lie group representation theory. We lay down some of the foundations for the study of strongly continuous representations of a Lie group G, and representations induced on various convolution algebras, including the algebra of compactly supported distributions on G, and the universal enveloping algebra of G, which includes its Lie algebra. Members of the Lie algebra, or more general elements of the universal enveloping algebra, are typically represented by unbounded operators, a fact which requires some technical care. Spaces of smooth vectors for a representation, introduced by Gårding originally to complete some arguments of Bargmann, and also analytic vectors, provide useful tools to handle these technical problems. Chapter 0 also includes some descriptions of various types of Lie groups one runs into, such as nilpotent, solvable, semisimple, etc.

In Chapter 1, we begin our study of analysis on specific Lie groups with a look at the Heisenberg group. This nilpotent Lie group has a representation theory that is at once simple and rich in structure. Its irreducible unitary representations were classified by Stone and von Neumann many years ago, when the group was studied because of its occurrence in basic quantum theory. Much terminology associated with the group, in addition to its name (actually, many physicists call it the Weyl group), celebrates this. For example, certain of the group's irreducible unitary representations are called Schrödinger representations, and other unitarily equivalent forms are called Bargmann-Fok representations. We obtain an analogue of the Fourier inversion formula easily in this case, and postpone proving that all irreducible unitary representations are given until Chapter 5. A study of the Heisenberg group naturally introduces other important Lie groups, in particular the symplectic group, acting as a group of automorphisms of the Heisenberg group, and the unitary group, arising as a maximal compact subgroup of the symplectic group. It might seem unorthodox to treat the Heisenberg group before the unitary group or other compact Lie groups, since most of its irreducible unitary representations are infinite-dimensional. However, it is this group for which it is easiest to develop harmonic analysis in closest analogy to the development on Euclidean space R^n. This may be traced to the fact that the Heisenberg group really has the simplest representation theory, compared to compact groups like $\mathrm{SU}(n)$ or $\mathrm{SO}(n)$, for which combinatorial considerations arise even to list all their irreducible representations. That the classification is particularly simple for $\mathrm{SU}(2)$ and $\mathrm{SO}(3)$ still does not make harmonic analysis on these groups as easy as on the Heisenberg group, partly for the basic reason that integrals are easier to work with than infinite series, and, not unrelated, partly because of the richer set of automorphisms of the Heisenberg group, particularly groups of dilations.

Chapter 2 studies the unitary groups $\mathrm{U}(n)$, and Chapter 3 pursues some general results about compact Lie groups. The complete description of the irreducible representations of $\mathrm{U}(n)$ and $\mathrm{SU}(n)$, for general n, spills over into Chapter 3, where it is produced as a consequence of the Theorem of the Highest Weight. In fact, we give a complete proof of this theorem only for $\mathrm{SU}(n)$. Chapter 3 contains a number of general results about compact Lie groups, such as the Peter-Weyl theorem and the Borel-Weil theorem. We have also emphasized the study of compact group actions on eigenspaces of Laplace operators in Chapter 3. The interplay between the orthogonal group and harmonic analysis on spheres is discussed in Chapter 14. Here we have tried to include short, simple derivations for a number of classical identities from the theory of spherical harmonics.

In Chapter 5 we discuss a major general tool for constructing representations of Lie groups, the method of induced representations. We discuss Mackey's results on systems of imprimitivity and applications to some important classes of Lie groups, such as the groups of isometries of Euclidean space and some other groups that are semi-direct products. This chapter also gives a proof of the Stone-von Neumann theorem classifying the irreducible unitary representations of the Heisenberg group. Chapter 6 pursues the study of nilpotent Lie groups. The theory of the Heisenberg group plays a crucial role here; the representation theory in the general case is reduced by an inductive argument to that of the Heisenberg group.

Chapter 7 gives a brief study of harmonic analysis on cones. The basic analytical tool for this study is the Hankel transform

$$H_\nu f(\lambda) = \int_0^\infty f(r) J_\nu(\lambda r) r \, dr. \tag{I.20}$$

The map H_ν is unitary from $L^2(R^+, r\,dr)$ onto $L^2(R^+, \lambda\,d\lambda)$, and the Hankel inversion formula says

$$H_\nu(H_\nu f)(r) = f(r). \tag{I.21}$$

We produce the Hankel transform as an operator intertwining two irreducible unitary representations of the "$ax + b$ group," a two-dimensional solvable Lie group whose irreducible unitary representations were classified in Chapter 5. We remark that special cases of the Hankel transform arise in taking Fourier transforms of radial functions on R^n. In fact, if $f(x) = f(r)$ on R^n, a change of variable yields

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int_0^\infty f(r)\psi_n(r|\xi|)r^{n-1}\,dr \tag{I.22}$$

where

$$\begin{aligned}\psi_n(r) &= \int_{S^{n-1}} e^{i(x\cdot\omega)r}\,d\omega \\ &= A_{n-1}\int_{-1}^{1} e^{irs}(1-s^2)^{(n-3)/2}\,ds \\ &= (2\pi)^{n/2}r^{1-n/2}J_{n/2-1}(r)\end{aligned} \tag{I.23}$$

in view of the integral formula

$$J_\nu(z) = [\Gamma(\tfrac{1}{2})\Gamma(\nu+\tfrac{1}{2})]^{-1}(z/2)^\nu \int_{-1}^{1}(1-t^2)^{\nu-1/2}e^{izt}\,dt \tag{I.24}$$

for the Bessel function. The unitarity of H_ν and the Hankel inversion formula, for $\nu = \frac{1}{2}n - 1$, follow from the Fourier inversion formula and Plancherel theorem. These results are not accidentally special cases of the result (I.21), but conceptually so, since Euclidean space R^n is the cone over S^{n-1}, and both sets of formulas arise from the spectral representation of the Laplace operator.

In Chapter 8 we study the group $\mathrm{SL}(2,R)$. This group covers the Lorentz group $\mathrm{SO}_e(2,1)$, and hence is important for relativistic physics. We derive the classification of the irreducible unitary representations of $\mathrm{SL}(2,R)$ achieved in the famous paper of V. Bargmann [**12**], and study a number of topics in harmonic analysis on this group, and also on the two dimensional hyperbolic space. Chapter 9 studies $\mathrm{SL}(2,\mathbf{C})$, which covers the Lorentz group $\mathrm{SO}_e(3,1)$, and describes its repesentations, given by Gelfand et al. [**71**]. More general Lorentz groups $\mathrm{SO}_e(n,1)$ are also introduced. Our study of their representations is somewhat less complete, though we do describe the principal series, which provides "almost all" the irreducible unitary representations for n odd. Chapter 10 considers the actions of $\mathrm{SO}_e(n,1)$ as groups of conformal transformations on balls and spheres. We show that the existence of a certain (nonunitary) representation of $\mathrm{SO}_e(n,1)$ on the space of harmonic functions on the unit ball in R^n implies Poisson's integral formula for the solution to the Dirichlet problem in the ball.

In Chapter 11 we return to the symplectic group $\mathrm{Sp}(n,R)$, introduced in Chapter 1, and make a detailed study of the metaplectic representation. Our study uses the Cartan decomposition of the symplectic group, a special case of a decomposition that plays an important role in the general study of semisimple Lie groups. The Cartan decomposition is useful in studying the metaplectic representation, because the action of the double cover of $\mathrm{U}(n) \subset \mathrm{Sp}(n, R)$ takes a particularly simple form in the Bargmann-Fok representation.

Chapter 12 is devoted to spinors. The spin groups are double covers of $\mathrm{SO}(n)$, and more generally of $\mathrm{SO}_e(p, q)$. We define Dirac operators on manifolds with spin structures, but do not really pursue harmonic analysis for these objects, though there is a great deal that has been done. Some of this is hinted at in the final chapter of these notes, and material given here on spinors, it is hoped, will make this intelligible.

Chapter 13 is a very brief introduction to the general theory of noncompact semisimple Lie groups, illustrated to some degree by the material of Chapters 8–11. We describe some general results on principal series and discrete series, with no proofs but references to some basic sources. Much more extensive introductions to this vast topic can be found in the last two chapters of Wallach [**253**], and in the two volume treatise of Warner [**256**]. This subject is under intensive development at present, some of which is described in the more recent reports [**137**, **140**]. Also see the new book [**136**].

The course from which these notes grew was given to students who had been through one semester with the book of Varadarajan [**246**]. The present monograph is addressed to people with a basic understanding of Fourier analysis (commutative harmonic analysis) and functional analysis, such as covered in one of the basic texts, e.g., Reed and Simon [**201**], Riesz-Nagy [**202**], or Yosida [**267**]. A concise statement of the ideal preparation would be "Chapters 6, 9, and 11 of Yosida," together with the first few chapters of a standard Lie group text, such as [**246**] or [**100**]. We have included four appendixes, one on the Fourier transform and tempered distributions, one on the spectral theorem, emphasizing the use of tempered distributions in the proof of the spectral theorem, one on the Radon transform on Euclidean space, proving some results needed in Chapter 6, and one on analytic vectors, discussing some technical analytical points often encountered in passing from a representation of a Lie algebra to a representation of a Lie group.

These notes focus on ways specific Lie groups arise in analysis. The emphasis is on multiplicity, rather than unity. We do treat some general approaches to certain classes of Lie groups, especially the classes of compact and of nilpotent Lie groups, and to a lesser extent the class of semisimple Lie groups, but our emphasis is on the particular.

One unifying theme is the relationship between harmonic analysis and linear PDE, the use of results from each of these fields to advance the study of the other. In addition to certain operators with double characteristics, mentioned earlier, we also study the "classical" PDE, the Laplace, heat, and wave equations, in various contexts. For example, we show how exact solutions to the wave equation store up a great deal of information about harmonic analysis on spheres and hyperbolic space. For the past twenty years or so it has been an active line of inquiry to see how the spectral analysis of the Laplacian on a general manifold (especially in the compact case) stores up geometric information, and how such analysis can be achieved by parametrices for the heat equation and for the wave equation.

To give another illustration of this point of view, we mention a PDE approach to the proof of the subordination identity (I.11). We begin with the observation that a direct proof of (I.12), for $n = 1$, will imply the identity (I.11), on the operator level, with A replaced by $(-d^2/dx^2)^{1/2}$, which in turn implies (I.11) for all positive real A, by the spectral theorem for $-d^2/dx^2$. Thus it suffices to obtain a proof of the Poisson integral formula for $n = 1$. But

$$
\begin{aligned}
\text{(I.25)} \qquad e^{-y(-d^2/dx^2)^{1/2}}\delta(x) &= (2\pi)^{-1}\int_{-\infty}^{\infty} e^{-y|\xi|+ix\xi}\,d\xi \\
&= (2\pi)^{-1}\int_{0}^{\infty} (e^{-y\xi+ix\xi}+e^{-y\xi-ix\xi})\,d\xi \\
&= (2\pi)^{-1}[(y-ix)^{-1}+(y+ix)^{-1}] \\
&= \pi^{-1}y/(y^2+x^2).
\end{aligned}
$$

This proves the subordination identity (I.11). A more classical method of proof is to take the Mellin transform with respect to y of both sides of (I.11), deducing that (I.11) is equivalent to the duplication formula

$$\pi^{1/2}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+\tfrac{1}{2})$$

for the gamma function. The subordination identity will make another appearance, in Chapter 1, §7, in a study of various PDEs on the Heisenberg group.

The reader will find that, once the material of the introductory Chapter 0 is understood subsequent chapters can be read in almost any order, and by and large depend on each other only loosely. For example, if one wanted to read about the theory of nilpotent Lie groups, one would need only §§1 and 2 of Chapter 1 and §§1 and 2 of Chapter 5, before tackling Chapter 6. In particular, later sections of chapters with numerous sections, dealing with special problems, can be bypassed without affecting one's understanding of subsequent chapters.

APPENDIX A

The Fourier Transform and Tempered Distributions

As mentioned in the introduction, the Fourier transform is defined by

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int_{R^n} f(x) e^{-ix\cdot\xi}\, dx. \tag{A.1}$$

We also denote $\hat{f}(\xi)$ by $\mathcal{F}f(\xi)$. If $f \in L^1(R^n)$, it is clear that $\hat{f} \in L^\infty(R^n)$. We want to introduce some other spaces of functions and generalized functions, discuss the behavior of the Fourier transform theorem, and prove the Fourier inversion formula. For a more complete treatment, see Chapter VI of Yosida [**267**], or Chapter I of Taylor [**234**].

One important space is $\mathcal{S}(R^n)$, the Schwartz space of rapidly decreasing functions. One says $f \in \mathcal{S}(R^n)$ when f is C^∞ and

$$x^\beta D^\alpha f(x) \in L^\infty(R^n) \tag{A.2}$$

for all multi-indices α and β. Here, if $\alpha = (\alpha_1, \dots, \alpha_n)$, we set

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}; \qquad D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n} \tag{A.3}$$

where $D_j = (1/i)\partial/\partial x_j$. It is easy to see that if $f \in \mathcal{S}(R^n)$, so if $\hat{f}$, and

$$\xi^\alpha D^\beta \hat{f}(\xi) = \mathcal{F}(D^\alpha x^\beta f)(\xi). \tag{A.4}$$

Thus

$$\mathcal{F}\colon \mathcal{S}(R^n) \to \mathcal{S}(R^n). \tag{A.5}$$

We denote the space of continuous linear functionals on $\mathcal{S}(R^n)$ by $\mathcal{S}'(R^n)$. This is called the Schwartz space of tempered distributions.

If we set

$$\mathcal{F}^* f(\xi) = (2\pi)^{-n/2} \int f(x) e^{ix\cdot\xi}\, dx \tag{A.6}$$

we see that $\mathcal{F}^*\colon \mathcal{S}(R^n) \to \mathcal{S}(R^n)$ and, with respect to the natural inner product $(u, v) = \int u(x)\overline{v(x)}\, dx$, we have

$$(\mathcal{F}u, v) = (u, \mathcal{F}^* v), \qquad u, v \in \mathcal{S}(R^n). \tag{A.7}$$

Using this identity, we can extend $\mathcal{F}$ and $\mathcal{F}^*$ to $\mathcal{S}'(R^n)$ by duality:

$$\mathcal{F}, \mathcal{F}^* : \mathcal{S}'(R^n) \to \mathcal{S}'(R^n). \tag{A.8}$$

The following is the Fourier inversion formula.

PROPOSITION A.1. *If* $u \in \mathcal{S}$, *then*

$$\mathcal{F}\mathcal{F}^* u = \mathcal{F}^*\mathcal{F}u = u. \tag{A.9}$$

Hence

$$\mathcal{F}\mathcal{F}^* = \mathcal{F}^*\mathcal{F} = I \quad \text{on } \mathcal{S}'(R^n). \tag{A.10}$$

We sketch a proof of this. It suffices to show $\mathcal{F}^*\mathcal{F}u = u$ for $u \in \mathcal{S}$. Now

$$\begin{aligned} \mathcal{F}^*\mathcal{F}u(x) &= (2\pi)^{-n} \iint u(y) e^{i(x-y)\cdot\xi}\, dy\, d\xi \\ &= \lim_{\varepsilon\downarrow 0} (2\pi)^{-n} \iint u(y) e^{i(x-y)\cdot\xi - \varepsilon|\xi|^2}\, dy\, d\xi \\ &= \lim_{\varepsilon\downarrow 0} \int u(y) p(\varepsilon, x-y)\, dy \end{aligned} \tag{A.11}$$

where

$$p(\varepsilon, x) = (2\pi)^{-n} \int e^{ix\cdot\xi - \varepsilon|\xi|^2}\, d\xi. \tag{A.12}$$

Note that

$$p(\varepsilon, x) = \varepsilon^{-n/2} q(x/\sqrt{\varepsilon}) \tag{A.13}$$

where $q(x) = p(1, x)$. In a moment we will show that

$$p(\varepsilon, x) = (4\pi\varepsilon)^{-n/2} e^{-|x|^2/4\varepsilon}. \tag{A.14}$$

The derivation of this identity will also show that

$$\int_{R^n} q(x)\, dx = 1. \tag{A.15}$$

From this it is a simple exercise to deduce that

$$\lim_{\varepsilon\downarrow 0} \int u(y) p(\varepsilon, x-y)\, dy = u(x) \tag{A.16}$$

for any $u \in \mathcal{S}(R^n)$, and the Fourier inversion formula is proved.

To prove (A.14), we observe that $p(\varepsilon, x)$, defined by (A.12), is an entire analytic function of $x \in \mathbf{C}^n$; it is convenient to verify that

$$p(\varepsilon, ix) = (4\pi\varepsilon)^{-n/2} e^{|x|^2/4\varepsilon}, \qquad x \in R^n, \tag{A.17}$$

from which (A.14) follows by analytic continuation. Now

$$\begin{aligned} p(\varepsilon, ix) &= (2\pi)^{-n} \int e^{-x\cdot\xi - \varepsilon|\xi|^2}\, d\xi \\ &= (2\pi)^{-n} e^{|x|^2/4\varepsilon} \int e^{-|x/2\sqrt{\varepsilon} + \sqrt{\varepsilon}\xi|^2}\, d\xi \\ &= (2\pi)^{-n} \varepsilon^{-n/2} e^{|x|^2/4\varepsilon} \int_{R^n} e^{-|\xi|^2}\, d\xi. \end{aligned} \tag{A.18}$$

So to prove (A.17), it remains to show that

$$\int_{R^n} e^{-|\xi|^2}\,d\xi = \pi^{n/2}. \tag{A.19}$$

Now if $A = \int_{-\infty}^{\infty} e^{-\xi^2}\,d\xi$, then the left side of (A.19) is equal to A^n. But then for $n = 2$ we can use polar coordinates:

$$A^2 = \int_{R^2} e^{-|\xi|^2}\,d\xi = \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\,dr\,d\theta = \pi.$$

This completes the proof of the identity (A.17), hence of (A.14).

In light of (A.7) and the Fourier inversion formula, we see that, for $u, v \in \mathcal{S}$,

$$(\mathcal{F}u, \mathcal{F}v) = (u, v) = (\mathcal{F}^*u, \mathcal{F}^*v).$$

Thus $\mathcal{F}$ and $\mathcal{F}^*$ extend uniquely from $\mathcal{S}(R^n)$ to $L^2(R^n)$, and are inverse to each other. Thus we have the Plancherel theorem,

PROPOSITION A.2. *The Fourier transform*

$$\mathcal{F}\colon L^2(R^n) \to L^2(R^n) \tag{A.20}$$

is unitary, with inverse $\mathcal{F}^*$.

As mentioned in the introduction, the calculation of the Gaussian integral (A.12) gives the explicit formula for the fundamental solution to the heat equation:

$$e^{t\Delta}\delta(x) = (4\pi t)^{-n/2} e^{-|x|^2/4t}, \qquad t > 0,\ x \in R^n. \tag{A.21}$$

It was also shown in the introduction how to find the Poisson kernel

$$e^{-y\sqrt{-\Delta}}\delta(x) = c_n(y^2 + |x|^2)^{-(n+1)/2}$$

and also the fundamental solution to the wave equation $(\partial^2/\partial t^2 - \Delta)u = 0$, on $R \times R^n$. See (I.12)–(I.17) of the introduction. Here we give some complementary results on solutions to some basic linear PDE on R^n. Namely, we analyze the resolvent of the Laplace operator:

$$(\lambda^2 - \Delta)^{-1}\delta(x) = (2\pi)^{-n}\int_{R^n} (\lambda^2 + |\xi|^2)^{-1} e^{ix\cdot\xi}\,d\xi. \tag{A.22}$$

This Fourier integral is not absolutely convergent, if $n \geq 2$, but can be interpreted in the sense of tempered distributions. It is convenient to use the identity

$$(\lambda^2 - \Delta)^{-1}\delta(x) = \int_0^{\infty} e^{t(\Delta - \lambda^2)}\delta(x)\,dt, \tag{A.23}$$

when $\lambda > 0$, and plug in (A.21), to get

$$(\lambda^2 - \Delta)^{-1}\delta(x) = (4\pi)^{-n/2}\int_0^{\infty} e^{-|x|^2/4t - t\lambda^2} t^{-n/2}\,dt. \tag{A.24}$$

Note that this integral was evaluated in the introduction for $n = 1$ and 3. The identity (I.11) (complemented by the identity (7.89) of Chapter 1) gives, for

$$K(r,\lambda,\nu) = \int_0^\infty e^{-r^2/4t} e^{-t\lambda^2} t^{-1-\nu}\, dt, \tag{A.25}$$

the formulas

$$\begin{aligned} K(r,\lambda,-\tfrac{1}{2}) &= \pi^{-1/2}\lambda^{-1}e^{-\lambda r}, \\ K(r,\lambda,\tfrac{1}{2}) &= 2\pi^{-1/2}r^{-1}e^{-\lambda r}. \end{aligned} \tag{A.26}$$

It is clear that for any *odd* integer n, we can evaluate the integral in (A.24) by differentiating repeatedly (A.26) with respect to r; so on R^{2k+1},

(A.27)

$$\begin{aligned} (\lambda^2 - \Delta)^{-1}\delta(x) &= (4\pi)^{-k-1/2}K(r,\lambda,-\tfrac{1}{2}+k) \\ &= (4\pi)^{-k-1/2}(-2r^{-1}\partial/\partial r)^k(\pi^{-1/2}\lambda^{-1}e^{-\lambda r}) \qquad (r = |x|) \end{aligned}$$

in view of the general identity

$$(\partial/\partial r)K(r,\lambda,\nu) = -\tfrac{1}{2}rK(r,\lambda,\nu+1). \tag{A.28}$$

On R^{2k}, one has

$$(\lambda^2 - \Delta)^{-1}\delta(x) = (4\pi)^{-k}K(r,\lambda,k-1). \tag{A.29}$$

This is not an elementary function. In fact, use of polar coordinates for the Laplace operator as in Chapter 4, §1, shows (A.29) satisfies a modified Bessel equation, as a function of r. We can also see this from (A.25), if we note that, in addition to (A.28), we have

$$(\partial/\partial\lambda)K(r,\lambda,\nu) = -2\lambda K(r,\lambda,\nu-1). \tag{A.30}$$

Also, a change of variable in the integral (A.25) gives

$$K(r,\lambda,\nu) = \lambda^{2\nu}K(\lambda r,1,\nu), \tag{A.31}$$

and if we combine (A.31) and (A.30) we get

$$(\partial/\partial r)K(r,\lambda,\nu) + (2\nu/\lambda)K(r,\lambda,\nu) = -2\lambda K(r,\lambda,\nu-1). \tag{A.32}$$

If we apply $\partial/\partial r$ to both sides of (A.32) and use (A.28), we get

$$(\partial^2/\partial r^2)K + 2\nu\lambda^{-1}(\partial/\partial r)K = \lambda r K(r,\lambda,\nu), \tag{A.33}$$

wich can be converted to the modified Bessel equation. One has the identity

$$\frac{1}{2}(\lambda r/2)^\nu \int_0^\infty e^{-r^2/4t}e^{-t\lambda^2}t^{-1-\nu}\, dt = \lambda^{2\nu}K_\nu(\lambda r); \tag{A.34}$$

see Lebedev [**148**], page 119. Hence, on R^n,

$$(\lambda^2 - \Delta)^{-1}\delta(x) = 2(4\pi)^{-n/2}(2\lambda/r)^{n/2-1}K_{n/2-1}(\lambda r). \tag{A.35}$$

The identity (A.34) gives this for $\lambda > 0$. Then we get it for $\lambda^2 \in \mathbf{C} \setminus R^-$, by analytic continuation.

One could also tackle the resolvent by looking at the Fourier transform on the right side of (A.22). If we use (I.21) to express this Fourier transform as a Hankel transform, the formula (A.35) is seen to be equivalent to the special case when $\nu = \frac{1}{2}n - 1$ of the classical identity

$$\int_0^\infty (\lambda^2 + s^2)^{-1} J_\nu(sr) s^{\nu+1}\, ds = \lambda^\nu K_\nu(\lambda r).$$

See Lebedev [**154**], page 133, for a proof of this identity (in the case of absolute convergence) and further generalizations.

APPENDIX B

The Spectral Theorem

Our purpose here is to give a brief discussion of the spectral theorem. We consider several forms. The best known form is

THE SPECTRAL THEOREM. *If A is a selfadjoint operator on a Hilbert space H, then there is a strongly countably additive projection-valued measure dE such that*

$$Au = \int_{-\infty}^{\infty} \lambda \, dE_\lambda u, \qquad u \in \mathcal{D}(A). \tag{B.1}$$

This result is a consequence of the

STRONG SPECTRAL THEOREM. *If A is a selfadjoint operator on a separable Hilbert space H, there is a σ-compact space Ω, a Borel measure μ on Ω, a unitary map*

$$W: L^2(\Omega, d\mu) \to H, \tag{B.2}$$

and a real-valued measurable function $a(x)$ on Ω, such that

$$W^{-1}AWf(x) = a(x)f(x), \qquad Wf \in \mathcal{D}(A). \tag{B.3}$$

In order to establish this result, we will work with the unitary group

$$U(t) = e^{itA}. \tag{B.4}$$

For a given $\xi \in H$, let H_ξ denote the closed linear span of $U(t)\xi$ for $t \in R$. If $H_\xi = H$, we say ξ is a *cyclic vector.* If H_ξ is not all of H, note that $H_\xi^\perp$ is invariant under $U(t)$. Using this, it is not difficult to decompose H into a (possibly countably infinite) direct sum of spaces H_{ξ_j}. It will suffice to establish the Strong Spectral Theorem on each factor, so we will establish the following result.

PROPOSITION B.1. *If $H = H_\xi$ (so there is a cyclic vector ξ), then we can take $\Omega = R$, and there exists a positive Borel measure μ on R and a unitary map $W: L^2(R, d\mu) \to H$ so that*

$$W^{-1}AWf(x) = xf(x), \qquad Wf \in \mathcal{D}(A), \tag{B.5}$$

or equivalently,

$$W^{-1}U(t)Wf = e^{itx}f(x), \qquad f \in L^2(R, d\mu). \tag{B.6}$$

The measure μ on R will be the Fourier transform

$$\mu = \hat{\varsigma}(\tau) \tag{B.7}$$

where

$$\varsigma(t) = (2\pi)^{-1}(e^{itA}\xi, \xi). \tag{B.8}$$

A priori, it is not at all clear that (B.7) defines a measure, though, since clearly $\varsigma \in L^\infty(R)$, we see that μ is a *tempered distribution*. We will show that μ is indeed a positive measure during the course of our argument. We will in the meantime use notation anticipating that μ is a measure:

$$(e^{itA}\xi, \xi) = \int_{-\infty}^{\infty} e^{it\tau}\, d\mu(\tau), \tag{B.9}$$

keeping in mind that the Fourier transform in (B.9) is to be interpreted in the sense of tempered distributions. As for the map W, we first define

$$W\colon S(R) \to H, \tag{B.10}$$

where $S(R)$ is the Schwartz space of rapidly decreasing functions, by

$$W(f) = f(A)\xi, \tag{B.11}$$

where we define the operator $f(A)$ by the formula

$$f(A) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} \hat{f}(t)e^{itA}\, dt. \tag{B.12}$$

The reason for this notation will become apparent shortly; see (B.21). Note that, using (B.9), we have

$$\begin{aligned}(f(A)\xi, g(A)\xi) &= (2\pi)^{-1}\left(\int \hat{f}(s)e^{isA}\xi\, ds, \int \hat{g}(t)e^{itA}\xi\, dt\right)\\ &= (2\pi)^{-1}\iint \hat{f}(s)\overline{\hat{g}(t)}(e^{i(s-t)A}\xi, \xi)\, ds\, dt\\ &= (2\pi)^{-1}\iiint \hat{f}(s)\overline{\hat{g}(t)}e^{i(s-t)\tau}\, d\mu(\tau)\, ds\, dt\\ &= \int f(\tau)\overline{g(\tau)}\, d\mu(\tau).\end{aligned} \tag{B.13}$$

For $f, g \in S(R)$, the last "integral" is to be interpreted as $\langle f\overline{g}, \mu\rangle$.

Now, if $g = f$, the left side of (B.13) is $\|f(A)\xi\|^2$, which is ≥ 0. Hence,

$$\langle |f|^2, \mu\rangle \geq 0 \quad \text{for all } f \in S(R). \tag{B.14}$$

This is enough to imply that the tempered distribution μ is actually a positive measure! Knowing this, we can now interpret (B.13) as implying that W has a unique continuous extension

$$W\colon L^2(R, d\mu) \to H, \tag{B.15}$$

and W is an isometry. Furthermore, since ξ is assumed to be cyclic, it is easy to see the range of W must be dense in H, so W must be unitary.

Now from (B.12) it follows that, if $f \in S(R)$,

$$e^{isA} f(A) = \varphi_s(A) \tag{B.16}$$

where

$$\varphi_s(\tau) = e^{is\tau} f(\tau). \tag{B.17}$$

Hence

$$W^{-1} e^{isA} f(A)\xi = W^{-1}\varphi_s(A)\xi = e^{is\tau} f(\tau). \tag{B.18}$$

Since $\{f(A)\xi : f \in S\}$ is dense in H, we obtain

$$W^{-1} e^{isA} = e^{is\tau} W^{-1}, \tag{B.19}$$

which proves (B.6), and hence Proposition B.1.

Note that since $W^{-1} e^{itA} W = e^{it\tau}$, the formula (B.12) implies

$$W^{-1} f(A) W = f(\tau), \tag{B.20}$$

which justifies the notation $f(A)$ in (B.12). In the language of (B.1), we have

$$f(A)u = \int f(\lambda)\, dE_\lambda u. \tag{B.21}$$

If a selfadjoint operator A has the representation (B.5), one says A has *simple spectrum.* It follows from Proposition B.1 that a selfadjoint operator has simple spectrum if and only if it has a cyclic vector.

We remark that a more typical method of showing there is a positive measure μ such that (B.9) holds is to invoke Bochner's theorem, characterizing the Fourier transform of positive measures as positive definite functions. It is well known that, using the theory of tempered distributions, one can give a simple proof of Bochner's theorem. Our proof of the spectral theorem has avoided this detour entirely, working directly with tempered distributions.

In general, the spectral theorem is considered to be nonconstructive in nature. However, especially if A has simple spectrum, or spectrum of low multiplicity, the method of proof of the spectral theorem can sometimes be implemented to give an explicit spectral representation of A. Doing this involves knowing explicitly the unitary group $U(t) = e^{itA}$, and particularly knowing explicitly the function $\varsigma(t) = (e^{itA}\xi, \xi)$ for some cyclic vector ξ, and then finding explicitly the Fourier transform of this function, which is known generally to be some positive measure. We can illustrate this program in the case

$$A = D = i d/dx, \tag{B.22}$$

so

$$e^{itD} f(x) = f(x - t). \tag{B.23}$$

If we pick $\xi \in \mathcal{S}(R)$ such that $\hat{\xi}(\tau)$ never vanishes, then

$$\varsigma(t) = (e^{itD}\xi, \xi) = \xi * \check{\xi}(t) \tag{B.24}$$

where $\check{\xi}(t) = \overline{\xi}(-t)$, so we have $\hat{\varsigma}(\tau) = |\hat{\xi}(\tau)|^2$, i.e.,

$$d\mu = |\hat{\xi}(\tau)|^2 \, d\tau \quad \text{on } R. \tag{B.25}$$

In this case the unitary map

$$W \colon L^2(R, |\hat{\xi}(\tau)|^2 \, d\tau) \to L^2(R, dx) \tag{B.26}$$

is given by

$$Wf = f(D)\xi = \hat{f} * \xi, \tag{B.27}$$

and we have $W^{-1}e^{itD}Wf(\tau) = e^{it\tau}f(\tau)$. If we take ξ to be an approximate delta function, then in the limit W becomes the Fourier transform.

Of course, as has been shown several times in this monograph, an explicit knowledge of the unitary group e^{itA} is a very convenient tool for studying the spectral resolution of A. In fact, the spectral measure is given by

$$dE_\lambda = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{itA - i\lambda t} \, dt, \tag{B.28}$$

where the right side is interpreted a priori as an operator valued tempered distribution. Compare the spectral analysis of the Laplace operator on spheres and hyperbolic space, deduced from a knowledge of the fundamental solution of the wave equation, in Chapters 4 and 8.

One can generalize the notion of a cyclic vector to the case of a k-tuple of commuting selfadjoint operators $(A_1, \ldots, A_k)$. More precisely, suppose the unitary groups $U_j(t) = e^{itA_j}$ all commute. We say $\xi \in H$ is cyclic if the closed linear span of $U_1(t_1) \cdots U_k(t_k)\xi$ is all of H. In that case, the proof of Proposition B.1 generalizes to show that there is a positive measure μ on R^k and a unitary map $W \colon L^2(R^k, d\mu) \to H$ such that $W^{-1}A_jWf(x) = x_jf(x)$, $Wf \in \mathcal{D}(A_j)$. We would say $(A_1, \ldots, A_k)$ has simple spectrum. If a single operator $A = A_1$ does not have simple spectrum, it would elucidate its spectral behavior to find a k-tuple of commuting operators such that $(A_1, \ldots, A_k)$ has simple spectrum. More generally, given a selfadjoint operator B, without simple spectrum, we might try to find $(A_1, \ldots, A_k)$, commuting and with simple spectrum, such that $W^{-1}BWf(x) = \beta(x)f(x)$, $Wf \in \mathcal{D}(B)$. For example, for the Laplace operator Δ on R^k, one takes $(A_1, \ldots, A_k) = (D_1, \ldots, D_k)$, $D_j = i\partial/\partial x_j$. Of course, given such B, one would like such $A_1, \ldots, A_k$ to arise in some "natural" fashion. In many cases, such as $B = \Delta$ on the sphere S^n, $n \geq 2$, it is natural to write B as a function of some selfadjoint operators which commute with B, but not with each other, and which generate a noncommutative group of transformations acting irreducibly on each eigenspace of B. Thus noncommutative harmonic analysis arises naturally in the search for a "simple spectrum."

As an application of the spectral theorem, we prove the following result, a version of Schur's lemma, characterizing irreducible group actions, which is used from time to time in this monograph (see Chapter 0, Theorem 1.6).

PROPOSITION B.2. *A group G of unitary operators on a Hilbert space H is irreducible if and only if, for any bounded linear operator A on H,*

$$UA = AU \quad \text{for all } U \in G \Rightarrow A = \lambda I. \tag{B.29}$$

PROOF. First, if $H_0 \subset H$ is a closed invariant subspace and P is the orthogonal projection on H_0, then (B.29) implies H_0 is 0 or H, so clearly (B.29) implies irreducibility. Conversely, suppose G is irreducible, and let $A \in \mathcal{L}(H)$ be such that

$$UA = AU \quad \text{for all } U \in G. \tag{B.30}$$

The unitarity of U implies the same identity also holds for A^*, so considering $A + A^*$ and $A - A^*$, we can assume without loss of generality that A is selfadjoint in (B.30). Now (B.30) continues to hold with A replaced by any polynomial in A, and then, by the spectral theorem, we can deduce that, if $A = \int \lambda \, dE_\lambda$, then, for any Borel set $\mathcal{B}$, $E(\mathcal{B})U = UE(\mathcal{B})$ for all $U \in G$. But then the irreducibility implies $E(\mathcal{B}) = 0$ or I, since $E(\mathcal{B})$ is a projection on an invariant subspace. Since this holds for any Borel set $\mathcal{B}$, we have $A = \lambda I$ for some λ. This completes the proof.

We say a few words about the general justification of the assertion that for any selfadjoint operator A, the operator iA generates a unitary group (B.4), which is part of Stone's theorem. First, if A is bounded, the operator e^{itA} can be defined by a convergent power series expansion, and power series manipulation gives $e^{isA}e^{itA} = e^{i(s+t)A}$ and $(e^{itA})^* = e^{-itA}$. For unbounded A, there are direct demonstrations that iA generates a unitary group, but it is perhaps simpler to use von Neumann's trick, and consider the unitary operator $V = (A+i)(A-i)^{-1}$. Then $B_1 = (V + V^*)/2$, $B_2 = (V - V^*)/2i$ form a pair of commuting bounded selfadjoint operators, to which the spectral theorem proved above applies. Then the spectral theorem for A is a corollary of that for (B_1, B_2). Of course, granted the spectral theorem for A, the fact that iA generates a unitary group is a simple consequence.

We conclude this appendix by mentioning one important connection between the unitary group e^{itA} and the notion of selfadjointness. A symmetric operator A_0 with domain $\mathcal{D}$ is said to be essentially selfadjoint if there exists a unique selfadjoint operator A such that $\mathcal{D} \subset \mathcal{D}(A)$, the domain of A, and $A_0 u = Au$ for $u \in \mathcal{D}$. For symmetric operators, this extends Proposition 2.2 of Chapter 0.

PROPOSITION B.3. *Let A_0 be an operator on a Hilbert space H, with domain $\mathcal{D}$. Assume $\mathcal{D}$ is dense in H. Let $U(t)$ be a unitary group, with infinitesimal generator iA, so A is selfadjoint, and*

$$U(t) = e^{itA}. \tag{B.31}$$

Suppose $\mathcal{D} \subset \mathcal{D}(A)$ and $A_0 u = Au$ for $u \in \mathcal{D}$, or equivalently,

$$\lim_{t \to 0} t^{-1}(U(t)u - u) = A_0 u \quad \text{for all } u \in \mathcal{D}. \tag{B.32}$$

Also suppose $\mathcal{D}$ is invariant under $U(t)$:

$$U(t)\mathcal{D} \subset \mathcal{D}. \tag{B.33}$$

Then A_0 is essentially selfadjoint; A is its unique selfadjoint extension. Suppose furthermore that

$$A_0 \colon \mathcal{D} \to \mathcal{D}. \tag{B.34}$$

Then A_0^k, with domain $\mathcal{D}$, is essentially selfadjoint, for each positive integer k.

PROOF. We use the following well known criterion for essential selfadjointness (see, e.g., [**202**]). A_0 is essentially selfadjoint if and only if the range of $i \pm A_0$ is dense in H. So suppose $v \in H$ and

$$((i \pm A_0)u, v) = 0 \quad \text{for all } u \in \mathcal{D}. \tag{B.35}$$

Using (B.33) together with the fact that $A_0 = A$ on $\mathcal{D}$, we have

$$((i \pm A_0)u, U(t)v) = 0 \quad \text{for all } t \in R,\ u \in \mathcal{D}. \tag{B.36}$$

Consequently $\int \rho(t)U(t)v\,dt$ is orthogonal to the range of $i \pm A_0$, for any $\rho \in L^1(R)$. Choosing $\rho(t) \in C_0^\infty(R)$ an approximate identity, we approximate v by elements of $\mathcal{D}(A)$, indeed of $\mathcal{D}(A^k)$ for all k. Thus we can suppose in (B.35) that $v \in \mathcal{D}(A)$. Hence, taking adjoints, we have

$$(u, (i \pm A)v) = 0 \quad \text{for all } u \in \mathcal{D}. \tag{B.37}$$

Since $(i \pm A)v \in H$, and $\mathcal{D}$ is dense in H, it follows that (B.37) holds for all $u \in H$, hence for $u = v$. But then the imaginary part is $\|v\|^2$, so (B.35) implies $v = 0$. This proves the first part of the proposition. Granted (B.34), the same proof works with A_0 replaced by A_0^k (but $U(t)$ unaltered), so the proposition is proved.

An example of an application of Proposition B.3 is Chernoff's proof [**38**] that on a complete Riemannian manifold M, with $\mathcal{D} = C_0^\infty(M)$, all powers of the Laplace operator are essentially selfadjoint on $\mathcal{D}$. This is because, with

$$B_0 = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}, \qquad \mathcal{D}(B_0) = C_0^\infty \oplus C_0^\infty, \tag{B.38}$$

the group

$$U(t) = e^{tB_0} \tag{B.39}$$

is the fundamental solution to the wave equation

$$\partial^2 U/\partial t^2 - \Delta U = 0, \tag{B.40}$$

and its unitarity is equivalent to the standard energy conservation law for the wave equation, while *finite propagation speed* for solutions to the wave equation implies $\mathcal{D}(B_0)$ is preserved by $U(t)$. Note that

$$B_0^2 = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix}, \tag{B.41}$$

so essential selfadjointness of Δ and its powers follows.

APPENDIX C

The Radon Transform on Euclidean Space

In this appendix we give a brief discussion of the Radon transform on Euclidean space R^n, defined as follows. If H is any $(n-1)$-dimensional linear subspace of R^n, $w \in R^n/H$, set

$$\mathcal{R}f(w, H) = \int_H f(y + w)\, dy, \tag{C.1}$$

where H gets induced Lebesgue measure. This Radon transform is useful in commutative harmonic analysis, and also played a role in Chapter 6, in deriving the Plancherel formula on a nilpotent Lie group. Other variants of the Radon transform include transforms on spheres (see Chapter 4) and also on hyperbolic space and more general symmetric spaces (see Helgason [**102**, **103**]). Our goal will be to obtain a Plancherel formula for (C.1), since that was used in Chapter 6. This will be a simple consequence of the Plancherel formula for the Fourier transform.

In fact, if ω is a unit vector orthogonal to H, note that

$$\hat{f}(s\omega) = (2\pi)^{-n/2} \int e^{-isw} \mathcal{R}f(w, H)\, dw, \tag{C.2}$$

provided R^n/H is made isomorphic to R via $\omega + H \mapsto 1$. Now the Plancherel formula $\|f\|_{L^2} = \|\hat{f}\|_{L^2}$ on R^n gives

$$\|f\|_{L^2}^2 = \frac{1}{2} \int_{-\infty}^{\infty} \int_{S^{n-1}} |\hat{f}(s\omega)|^2 |s^{n-1}|\, d\omega\, ds. \tag{C.3}$$

Meanwhile, the Plancherel formula for the Fourier transform on R gives, in view of (C.2),

$$\int_{-\infty}^{\infty} |\hat{f}(s\omega)|^2\, ds = (2\pi)^{-(n-1)/2} \int |\mathcal{R}f(w, H)|^2\, dw. \tag{C.4}$$

In order to combine (C.3) and (C.4), note that

$$\int_{-\infty}^{\infty} |\hat{g}(s)|^2 |s^\alpha|\, ds = \int_{-\infty}^{\infty} |\widehat{D^{\alpha/2} g}(s)|^2\, ds, \tag{C.5}$$

so we get

$$\|f\|_{L^2}^2 = \frac{1}{2}(2\pi)^{-(n-1)/2} \int_{\mathcal{G}} \int_{-\infty}^{\infty} |D_w^{(n-1)/2} \mathcal{R}f(w, H)|^2 \, dw \, d\,\mathrm{vol}(H), \tag{C.6}$$

as our Plancherel formula for the Radon transform. Here $d\,\mathrm{vol}(H)$ is the invariant volume element on the Grassmannian manifold of hyperplanes in R^n, normalized so that $\mathrm{vol}(\mathcal{G}) = \mathrm{vol}(S^{n-1})$.

We can polarize (C.6) and get, for $f, g \in L^2(R^n)$,

$$\begin{aligned}(f, g)_{L^2} &= c_n \int_{\mathcal{G}} \int_{\mathbf{R}} |D_w^{(n-1)/2}| \mathcal{R}f(w, H) \overline{|D_w^{(n-1)/2}| \mathcal{R}g(w, H)} \, dw \, d\,\mathrm{vol}(H) \\ &= c_n \int_{\mathcal{G}} \int_{\mathbf{R}} |D_w^{(n-1)}| \mathcal{R}f(w, H) \overline{\mathcal{R}g(w, H)} \, dw \, d\,\mathrm{vol}(H).\end{aligned} \tag{C.7}$$

Note that

$$n \,\text{odd} \Rightarrow |D_w^{(n-1)}| = \pm D_w^{n-1}, \tag{C.8}$$

but if n is even, $|D_w^{n-1}|$ is not a local operator. Taking g to be highly concentrated near a point $p \in R^n$, we see that

$$\mathcal{R}\delta_p(w, H) = \delta(\omega_H \cdot p - w) \tag{C.9}$$

where ω_H is a unit normal to H, determining an isomorphism $R^n/H \approx R$. Thus we get the inversion formula

$$f(p) = c_n \int_{\mathcal{G}} |D_w^{n-1}| \mathcal{R}f(\omega_H \cdot p, H) \, d\,\mathrm{vol}(H). \tag{C.10}$$

The Radon transform provides a nice tool for proving the Huygens principle for constant coefficient hyperbolic systems when n is odd; this arises from the fact that (C.8) is a local operator in that case. It also is a tool for scattering theory. See Lax and Phillips **[150]** for more about these uses of the Radon transform.

APPENDIX D

Analytic Vectors, and Exponentiation of Lie Algebra Representations

We discuss briefly the role of analytic vectors in passing from representations of a Lie algebra to representations of a Lie group.

Let $X_1, \dots, X_k$ be a set of (unbounded) skew adjoint opertors on a Hilbert space H. We say $u \in H$ is an analytic vector for $(X_1, \dots, X_k)$ if $u \in \mathcal{D}(X^\alpha)$ for all α, where $X^\alpha = X_{\alpha(1)} X_{\alpha(2)} \cdots X_{\alpha(N)}$, and, for some $C < \infty$, $K < \infty$,

$$\|X^\alpha u\| \le C(K|\alpha|)^{|\alpha|}. \tag{D.1}$$

We will be interested in the case when ω is a representation of a Lie algebra $\mathfrak{g}$ by skew adjoint operators, $X_j = \omega(L_j)$, $L_1, \dots, L_k$ forming a basis of $\mathfrak{g}$. We denote by $\mathfrak{A}_K$ the set of analytic vectors satisfying (D.1), for some $C < \infty$. We make the following hypothesis:

(D.2) For some K, $\mathfrak{A}_K$ is dense in H.

Note that, if $u \in \mathfrak{A}_K$, then, for $X = \sum a_j X_j$, $\sum |a_j| \le 1$,

$$e^{tX} u = \sum_{j=0}^{\infty} (t^j / j!) X^j u \tag{D.3}$$

is a power series converging, in the norm topology on H, for $|t| < (eK)^{-1}$, even complex t. If X_j are skew adjoint, then e^{tX} is unitary for t real.

We suppose $X_j = \omega(L_j)$ where ω is a representation of the Lie algebra $\mathfrak{g}$ by skew adjoint operators such that

$$\omega([L, M])u = \omega(L)\omega(M)u - \omega(M)\omega(L)u \tag{D.4}$$

for $u \in \mathfrak{A}_K$. Note that $\mathfrak{A}_K$ is invariant under $\omega(L)$, $L \in \mathfrak{g}$. We want to obtain a "local representation" ω_b of a neighborhood U of e in G, a Lie group with Lie algebra, by unitary operators on H. Indeed, if $L = \sum a_j L_j$, $\sum |a_j| < R = (Ke)^{-1}$, set $X = \omega(L)$ and

$$\omega_b(\exp L)u = e^{\omega(L)} u = \sum_{j=0}^{\infty} (1/j!) X^j u, \qquad u \in \mathfrak{A}_K. \tag{D.5}$$

We suppose L belongs to a small enough neighborhood U_0 of $0 \in \mathfrak{g}$ that $\exp : \mathfrak{g} \to G$ maps U_0 diffeomorphically onto a neighborhood U of $e \in G$, and that $L = \sum a_j L_j$, $\sum |a_j| < R$ on U_0.

In such a case, (D.5) defines a function on U taking values in the set of unitary operators on H. We need to show it is a local representation. If $V \subset U$ is a sufficiently small neighborhood of e (in particular, we need $V \cdot V \subset U$), we claim that, under the definition (D.5) of ω_b,

$$\omega_b(g_1 g_2)u = \omega_b(g_1)\omega_b(g_2)u, \qquad g_j \in V, u \in H. \tag{D.6}$$

It suffices to establish this identity for $u \in \mathfrak{A}_K$. This is essentially a consequence of the Campbell-Hausdorff formula, which says

$$\exp X \exp Y = \exp \Phi(X, Y) \tag{D.7}$$

where $\Phi(X, Y)$ is given by a convergent power series for X and Y small. The power series is of a "universal" sort:

$$\Phi(X, Y) = X + Y + \tfrac{1}{2}[X, Y] + \cdots. \tag{D.8}$$

See, e.g., Varadarajan [**246**]. The fact that (D.6) holds when $g_j = \exp X_j$, X_j small, $u \in \mathfrak{A}_K$, is just manipulation of power series, using the fact that the form of (D.8) is preserved on applying the representation ω of $\mathfrak{g}$ to both sides.

Thus we have produced a "local representation" of a neighborhood U of $e \in G$. We want to extend this to a global representation, provided G is simply connected. Any $g \in G$ can be written in the form

$$g = g_1 \cdots g_M, \qquad g_j \in V \ (V \subset U \text{ as above}), \tag{D.9}$$

so we would like to set

$$\omega(g)u = \omega_b(g_1) \cdots \omega_b(g_M)u. \tag{D.10}$$

We need to show that the representation (D.10) is independent of the representation (D.9). What is the same, if

$$e = g_1 \cdots g_N, \qquad g_j \in V, \tag{D.11}$$

we need to show that

$$\omega_b(g_1) \cdots \omega_b(g_N)u = u, \qquad \text{all } u \in H. \tag{D.12}$$

In (D.11) we could express each g_j as a product of terms in a very much smaller neighborhood $V^{\#}$ of e, so in proving (D.12) we can assume that each g_j belongs to as small a neighborhood of e as desired (of course, N gets quite large). The idea is to connect the points $p_0 = e$, $p_1 = g_1, \dots, p_j = g_1 \cdots g_j, \dots, p_N = e$ by a closed analytic loop. If G is simply connected, we can analytically contract the loop, leaving it fixed at e. Thus the points p_j trace out analytic curves $p_j(t)$, with $p_j(0) = p_j$, $p_j(1) = e$. So we define $g_j(t)$ by $p_j(t) = p_{j-1}(t)g_j(t)$. We can suppose that each $g_j(t)$ belongs to V for $0 \le t \le 1$. In fact, we can suppose $\operatorname{dist}(g_j(t), e) \le C/N$. Note that $g_1(t) \cdots g_N(t) = e$ for all t. Consider

$$E(t)u = \omega_b(g_1(t)) \cdots \omega_b(g_N(t))u. \tag{D.13}$$

Note that, if t is close to 1, all partial products $g_1(t)\cdots g_J(t)$ belong to V, so, by (D.6), for some $\varepsilon > 0$, we have

$$E(t)u = u \quad \text{for } 1-\varepsilon \le t \le 1. \tag{D.14}$$

Now if $u \in \mathfrak{A}_K$ we claim (D.13) is analytic in t. This gives $E(t)u = u$ for all $t \in [0,1]$, if $u \in \mathfrak{A}_K$, and hence for all u, since $\mathfrak{A}_K$ is dense in H. Thus ω is well defined by (D.10), on all of G, if G is simply connected.

To see that (D.13) is analytic in t for $u \in \mathfrak{A}_K$, note that if $\mathcal{B}_W$ denotes the set of $u \in H$ such that the function $\omega_b(g)u$ (which has been defined for $g \in U$) is analytic for $g \in W$, a neighborhood of $e \in G$, then of course $\mathfrak{A}_K \subset \mathcal{B}_U \subset \mathcal{B}_W$ if $W \subset U$, and if $g \in V$, then

$$\omega_b(g)u \in \mathcal{B}_{\mathrm{Ad}\, gW} \quad \text{if } u \in \mathcal{B}_W, \tag{D.15}$$

as follows from

$$\omega_b(g')\omega_b(g)u = \omega_b(g)\omega_b(g^{-1}g'g)u. \tag{D.16}$$

Now if $\operatorname{dist}(g_j(t), e) \le C/N$, then $\prod_J^N \mathrm{Ad}\, g_j(t)$ has a limited effect, so we can suppose that $u \in \mathcal{B}_W$ and $\omega_b(g_J(t))\cdots\omega_b(g_N(t))u \in \mathcal{B}_W$ for each J. Then the analyticity of (D.13) in t for $u \in \mathfrak{A}_K$ is clear.

Now that $\omega(g)$ has been defined, as a unitary operator, for each $g \in G$, we can show it is a representation of G, i.e.,

$$\omega(g_1g_2)u = \omega(g_1)\omega(g_2)u, \quad \text{if } g_j \in G. \tag{D.17}$$

Indeed, we know (D.17) holds for $g_j \in V$. But if $u \in \mathfrak{A}_K$ then both sides are analytic in (g_1, g_2), so (D.17) holds for $u \in \mathfrak{A}_K$. Since $\mathfrak{A}_K$ is assumed dense in H, (D.17) holds in general.

Such conclusions as were reached here were applied in Chapters 1 and 11 to the metaplectic representation, where the representation ω of $\mathrm{sp}(n, R)$ was defined by

$$\omega(Q) = iQ(X, D), \tag{D.18}$$

with $Q(x, \xi)$ a second order homogeneous polynomial. We need to know that hypothesis (D.2) is satisfied in this case. Indeed, we have, for some K,

$$\mathfrak{A}_K \supset \{p(x)e^{-|x|^2/2} : p(x) \text{ polynomial in } x\}, \tag{D.19}$$

the linear span of the Hermite functions, discussed in §6 of Chapter 1. We leave (D.19) as an exercise with the hint that it follows from (D.49). Clearly the right side of (D.19) is dense in $L^2(R^n)$.

This method of exponentiating a representation of a Lie algebra $\mathfrak{g}$ to the Lie group G depends on prescribing a dense space $\mathfrak{A}_K$ of analytic vectors. Such a method is never doomed to failure; in fact, we have the following important result, due to Harish-Chandra [**92**], Nelson [**183**], and Gårding [**69**].

THEOREM D.1. *If ω is a strongly continuous unitary representation of G on H, then for some $K < \infty$,*

$$\mathfrak{A}_K \text{ is dense in } H. \tag{D.20}$$

We will indicate a proof of this, along the lines of the proof given by Nelson [**183**] and by Gårding [**69**]. We include some details, particularly since this result is often stated in the weaker form

$$\bigcup_K \mathfrak{A}_K \text{ is dense in } H. \tag{D.21}$$

A different approach to the stronger result (D.20) is given in [**84**]. The proof involves approximating $u \in H$ by

$$\rho(t)u = \int_G h(t,x)\omega(x)u\,dx, \tag{D.22}$$

where $h(t,x)$ is the heat kernel

$$h(t,x) = e^{t\Delta}\delta(x). \tag{D.23}$$

Δ is the Laplace operator on G, which we assume endowed with a left invariant metric. It is easy to show that, for any neighborhood U of $e \in G$, $\int_U h(t,x)\,dx \to 1$ as $t \to 0$, while $\int_{G\setminus U} h(t,x)\,dx \to 0$; also $h(t,x) \geq 0$. Thus $\rho(t)u \to u$ in H as $t \to 0$. It remains to show that there exists $K < \infty$ such that, for $0 < t \leq 1$, $\rho(t)u \in \mathfrak{A}_K$ for all $u \in H$. Estimates on $h(t,x)$ were proved in [**69**]. The argument we give here is a special case of an argument from Cheeger, Gromov, and Taylor [**36**]. In this approach, one gets estimates on the Schwartz kernel of $f(\sqrt{-\Delta})$ for a general class of *even* functions $f(\lambda)$, including

$$f_t(\lambda) = e^{-t\lambda^2}, \tag{D.24}$$

by writing

$$\begin{aligned} f(\sqrt{-\Delta}) &= (2\pi)^{-1/2}\int_{-\infty}^{\infty} \hat{f}(s)e^{is\sqrt{-\Delta}}\,ds \\ &= (2\pi)^{-1/2}\int_{-\infty}^{\infty} \hat{f}(s)\cos s\sqrt{-\Delta}\,ds, \end{aligned} \tag{D.25}$$

and exploiting *finite propagation speed* for solutions to the wave equation

$$(\partial^2/\partial s^2 - \Delta)u = 0,$$

which implies

$$\cos s\sqrt{-\Delta}\delta_p(x) = 0 \quad \text{if } \operatorname{dist}(p,x) > |s|. \tag{D.26}$$

We get

(D.27)

$$f(\sqrt{-\Delta})\delta_p(x) = (2/\pi)^{1/2}\int_r^{\infty} \hat{f}(s)\cos s\sqrt{-\Delta}\delta_p(x)\,ds, \quad \text{if } \operatorname{dist}(p,x) > r.$$

Now, suppose $f(\lambda)$ satisfies the following estimates:

$$|\hat{f}^{(k)}(s)| \le C_1(k/A)^k \varphi(s), \tag{D.28}$$

for some $\varphi \in L^1$. We say $f \in \mathcal{F}^{\varphi,A}$. If we set

$$\psi(r) = \int_r^\infty \varphi(s)\,ds, \tag{D.29}$$

then

$$\int_r^\infty |f^{(k)}(s)|\,ds \le C_1(k/A)^k \psi(r). \tag{D.30}$$

Since we have the elementary L^2 operator norm bound $\|\cos s\sqrt{-\Delta}\| \le 1$, we easily deduce the following. Let $\operatorname{dist}(p,q) = r + 2a$. Then

$$\|\Delta^k f(\sqrt{-\Delta})\Delta^l u\|_{L^2(B_a(x))} \le C((2k+2l)/A)^{2k+2l}\psi(r)\|u\|_{L^2(B_a(p))} \tag{D.31}$$

provided $u \in L^2$ has support in the ball $B_a(p)$ of radius a, centered at p. Thus, for such u, and for $|y| \le A/2$, the power series

$$\begin{aligned} v(x,y) &= \sum_{k=0}^\infty ((2k)!)^{-1} y^{2k}(-\Delta)^k f(\sqrt{-\Delta})u \\ &= (\cosh y\sqrt{-\Delta}) f(\sqrt{-\Delta})u(x) \quad \text{(formally)} \end{aligned} \tag{D.32}$$

converges to an element of $L^2(B_a(x) \times [-A/2, A/2])$. Moreover,

$$(\Delta_G + \partial^2/\partial y^2)v = 0, \tag{D.33}$$

and, by (D.31),

$$\|v\|_{L^2(B_a(x))\times[-A/2,A/2]} \le C\psi(r)\|u\|_{L^2(B_a(p))}. \tag{D.34}$$

Since G is a homogeneous Riemannian manifold, we can apply regularity estimates for solutions to the elliptic equation (D.33), uniformly, to get

$$\left[\int_{B_a(p)} |f(\sqrt{-\Delta})\delta_{p'}(x)|^2\,dp'\right]^{1/2} \le C\psi(r). \tag{D.35}$$

(Most of the effort in **[36]** was devoted to the study of nonhomogeneous manifolds, with semibounded Ricci tensor, where this last piece of reasoning must be replaced by very much more elaborate arguments.) Similarly we obtain

$$\|\Delta_{p'}^l f(\sqrt{-\Delta})\delta_{p'}(x)\|_{L^2(B_a(p))} \le C(2l/A)^{2l}\psi(r), \tag{D.36}$$

so for each x we can set

$$w(x,p',y) = \sum_{l=0}^\infty ((2l)!)^{-1} y^{2l}\Delta_{p'}^l f(\sqrt{-\Delta})\delta_{p'}(x) \tag{D.37}$$

and get a harmonic function of (p',y) on $B_a(p) \times [-A/2, A/2]$ whose L^2 norm satisfies

$$\|w(x,\cdot,\cdot)\|_{L^2(B_a(p)\times[-A/2,A/2])} \le C\psi(r), \tag{D.38}$$

and which is equal to $f(\sqrt{-\Delta})\delta_{p'}(x)$ at $y=0$. Since

$$(\Delta_{p'} + \partial^2/\partial y^2)w(x,p',y) = 0, \tag{D.39}$$

we can again apply the elliptic regularity estimates, uniformly, as before, and deduce

$$|w(x,p,y)| \le C\psi(r). \tag{D.40}$$

Furthermore, since in exponential coordinates (D.39) has analytic coefficients, we can appeal to analytic regularity of solutions to an elliptic equation (see, e.g., [**121**, **179**]), and in light of the homogeneity of G we deduce, for $x' \in B_a(x)$,

$$|D^\alpha_{x'} f(\sqrt{-\Delta})\delta_p(x')| \le C(C|\alpha|)^{|\alpha|}\psi(r). \tag{D.41}$$

Here, $D^\alpha_{x'}$ is computed in exponential coordinates centered at $x \in G$; $\operatorname{dist}(p,x) = r + 2a$.

In order to finish the proof of Theorem D.1, we need to check the functions $f_t(\lambda) = e^{-t\lambda^2}$, to see to what classes $\mathcal{F}^{\varphi,A}$ they belong. We have

LEMMA D.2. *For all $t>0$, and for all $A>0$, we have*

$$f_t(\lambda) = e^{-t\lambda^2} \in \mathcal{F}^{\varphi,A}, \tag{D.42}$$

with

$$\varphi(s) = e^{-(1-\varepsilon)s^2/4t}. \tag{D.43}$$

Of course, $\varphi(s)$ depends on t; our notation suppresses this dependence. The important fact is that A is independent of t. This is why we have (D.20) rather than merely the weaker result (D.21). It remains only to establish this lemma. Note that

$$\hat{f}_t(s) = t^{-1/2}\hat{f}_1(t^{-1/2}s), \tag{D.44}$$

so

$$\hat{f}_t^{(k)}(s) = t^{-(k+1)/2}\hat{f}_1^{(k)}(t^{-1/2}s). \tag{D.45}$$

We will estimate $\hat{f}^{(k)}_{1/4}(s)$, since

$$\hat{f}_{1/4}(s) = \pi^{-1/2}e^{-s^2}. \tag{D.46}$$

Then

$$\hat{f}^{(k)}_{1/4}(s) = (-1)^k H_k(s)e^{-s^2}, \tag{D.47}$$

where $H_k(s)$ is the kth Hermite polynomial. In order to estimate (D.47), we will make use of the following generating function identity,

$$\sum_{n=0}^{\infty} H_n(s)^2\eta^n/2^n n! = (1-\eta^2)^{-1/2}\exp(2s^2\eta/(1+\eta)), \qquad |\eta|<1, \tag{D.48}$$

which is a special case of the identity (7.16) proved in Chapter 1. Pick η positive but fairly small. We obtain that, for any given $\varepsilon > 0$,

$$|H_n(s)|^2/2^n n! \le Ce^{\varepsilon s^2}|\eta|^{-n}. \tag{D.49}$$

It follows that

$$|\hat{f}^{(k)}_{1/4}(s)| \le C[(2/|\eta|)^k k!]^{1/2} e^{-(1-\varepsilon)s^2}. \tag{D.50}$$

Consequently,

$$\begin{aligned} |\hat{f}^{(k)}_t(s)| &\le C[(2/|\eta|)^k k!]^{1/2} t^{-(k+1)/2} e^{-(1-\varepsilon)s^2/4t} \\ &= C[(2/|\eta| t)^k k!]^{1/2} t^{-1/2} e^{-(1-\varepsilon)s^2/4t}. \end{aligned} \tag{D.51}$$

The square root of the term in square brackets justifies the assertion that (D.42) holds for all $A > 0$, and completes the proof of Lemma D.2.

In view of the easily established estimate that the volume of the set of points in G within a distance r of e is bounded by $C_1 e^{C_2 r}$ as $r \to \infty$, the proof of Theorem D.1 now follows.

We include here a brief discussion of the connection between analytic vectors and essential selfadjointness. Let A_0 be a symmetric operator with domain $\mathcal{D}$ in a Hilbert space H. Suppose $\mathcal{D}$ is dense in H. A theorem of Nelson [**183**] states that if $\mathcal{D}$ consists of analytic vectors for A_0, then A_0 is essentially selfadjoint on $\mathcal{D}$. We will give a short proof of the following weaker result, which is nevertheless of use.

PROPOSITION D.3. *Let $\mathfrak{A}_K$ be as defined above with $X_1 = iA_0$. Suppose that, for some K, $\mathcal{D} \subset \mathfrak{A}_K$. Then A_0 is essentially selfadjoint.*

PROOF. Starting from the power series (D.3) for $X = iA_0$, $u \in \mathcal{D}$, we get a vector-valued holomorphic function $U(t)u = e^{itA_0}u$ for $|t| < (eK)^{-1}$, and the argument sketched in the first part of this appendix extends $U(t)$ to $t \in R$ as a unitary group. Thus $U(t) = e^{itA}$ where A is a selfadjoint extension of A_0. If A_1 is any other selfadjoint extension, then it generates a unitary group $U_1(t) = e^{itA_1}$. But $U_1(t)u = U(t)u$ for $u \in \mathcal{D}$, and if $\mathcal{D}$ is dense this implies $U_1(t) = U(t)$, so the selfadjoint extension of A_0 is unique.

We will discuss one final method for exponentiating a Lie algebra representation. This method does not involve analytic vectors, and in fact is closely related to Proposition B.3.

Let $x_1, \dots, x_l$ be a basis for $\mathfrak{g}$ as a linear space; suppose $x_1, \dots, x_{l_0}$ actually *generate* $\mathfrak{g}$ as a Lie algebra. Let $\mathcal{D}$ be a dense linear subspace of H and suppose $\rho\colon \mathfrak{g} \to \operatorname{End}(\mathcal{D})$ is a representation of $\mathfrak{g}$. Thus we suppose

$$\rho(x_j) = iX_j^0\colon \mathcal{D} \to \mathcal{D}, \tag{D.52}$$

and $\rho([x_j, x_k]) = [iX_j^0, iX_k^0]$. Suppose each X_j^0 is a symmetric (possibly unbounded) operator on $\mathcal{D}$.

Now suppose that, for $1 \le j \le l_0$, each X_j^0 has a selfadjoint extension X_j, and

$$e^{itX_j}: \mathcal{D} \to \mathcal{D}, \qquad 1 \le j \le l_0. \tag{D.53}$$

We will not a priori make such an assumption on $\rho(x)$ for other elements $x \in \mathfrak{g}$. Note that, by Proposition B.3, (D.53) implies that X_j^0 is essentially selfadjoint on $\mathcal{D}$, for $1 \le j \le l_0$. The hypothesis (D.52) implies $\mathcal{D}$ is a space of smooth vectors for each such X_j.

We will make some further hypotheses on $\mathcal{D}$. We suppose $\mathcal{D}$ has the structure of a Hausdorff locally convex topological vector space such that

$$\mathcal{D} \hookrightarrow H \text{ is continuous.} \tag{D.54}$$

Furthermore, we suppose

$$\text{the maps (D.52) and (D.53) are continuous on } \mathcal{D}, \tag{D.55}$$

and

(D.56)

for $v \in \mathcal{D}$, $V_j(t) = e^{itX_j}v$ is continuous in t with values in $\mathcal{D}, 1 \le j \le l_0$.

From the formula

$$t^{-1}(e^{itX_j}v - v) = it^{-1}\int_0^t e^{isX_j}X_j^0 v\, ds, \qquad v \in \mathcal{D}, \tag{D.57}$$

it follows that, if $\mathcal{D}$ is sequentially complete,

$$v \in \mathcal{D} \Rightarrow t^{-1}(e^{itX_j}v - v) \to iX_j^0 v \quad \text{in } \mathcal{D}, \qquad 1 \le j \le l_0. \tag{D.58}$$

The result we aim to prove is

THEOREM D.4. *Let G be the simply connected Lie group with Lie algebra $\mathfrak{g}$, and let $\rho\colon \mathfrak{g} \to \operatorname{End}(\mathcal{D})$ be a representation of $\mathfrak{g}$ by skew-symmetric operators on a dense subspace $\mathcal{D} \subset H$. Under hypotheses* (D.52)–(D.56), *there is a stongly continuous unitary representation π of G on H such that*

$$x \in \mathfrak{g},\ v \in \mathcal{D} \Rightarrow t^{-1}(\pi(\exp tx)v - v) \to \rho(x)v \quad \textit{as } t \to 0. \tag{D.59}$$

This result is related to a theorem of R. Moore (Bull. Amer. Math. Soc. **71** (1965), 903–908), but has the advantage that (D.53) is assumed only for a set of generators of the Lie algebra, not for all elements of the Lie aglebra. As simple examples show, the condition (D.53) is often much easier to check in interesting cases, than such a stronger condition.

As a tool in proving Theorem D.4, we will need

LEMMA D.5. *For $t \in R$, $1 \le j \le l_0$, $1 \le k \le l$, we have*

$$ie^{itX_j}X_k^0 = \rho(\operatorname{Ad}(\exp tx_j)x_k)e^{itX_j} \quad \textit{on } \mathcal{D}. \tag{D.60}$$

PROOF. Fix $v \in \mathcal{D}$, and let

$$Y_{jk}^{\#}(t) = ie^{itX_j}X_k^0 v, \tag{D.61}$$

and

$$Z_{jk}^{\#}(t) = \rho(\mathrm{Ad}(\exp tx_j)x_k)e^{itX_j}v. \tag{D.62}$$

Then

$$Y_{jk}^{\#}(0) = Z_{jk}^{\#}(0) = X_k^0 v. \tag{D.63}$$

Also

$$(d/dt)Y_{jk}^{\#}(t) = iX_j^0 Y_{jk}^{\#}(t). \tag{D.64}$$

Write

$$\rho(\mathrm{Ad}(\exp tx_j)x_k) = i\sum_{\nu} \varsigma_{jk}^{\nu}(t)X_\nu^0 \quad \text{on } \mathcal{D}, \tag{D.65}$$

where $\varsigma_{jk}^{\nu}(t)$ are real-valued C^∞ functions of t. Thus

$$Z_{jk}^{\#}(t) = i\sum_{\nu} \varsigma_{jk}^{\nu}(t)X_\nu^0 e^{itX_j}v. \tag{D.66}$$

In view of (D.57), the continuity of X_ν^0 on $\mathcal{D}$, and the fact that $\varsigma_{jk}^{\nu}(t)$ are scalars, it follows that $Z_{jk}^{\#}(t)$ is differentiable in t, and the derivatives can be computed using the product rule. Since clearly $(d/dt)\sum_\nu \varsigma_{jk}^{\nu}(t)x_\nu = [x_j, \mathrm{Ad}(\exp tx_j)x_k]$, we obtain

$$\begin{aligned}(d/dt)Z_{jk}^{\#}(t) &= i[X_j^0, \rho(\mathrm{Ad}(\exp tx_j)x_k)]e^{itX_j}v \\ &\quad + i\rho(\mathrm{Ad}(\exp tx_j)x_k)X_j^0 e^{itX_j}v \\ &= iX_j^0 Z_{jk}^{\#}(t).\end{aligned} \tag{D.67}$$

Now let

$$W(t) = Y_{jk}^{\#}(t) - Z_{jk}^{\#}(t), \tag{D.68}$$

so $W(0) = 0$ and

$$dW/dt = iX_j^0 W(t). \tag{D.69}$$

It follows that $\|W(t)\|^2$ is differentiable, and

$$\begin{aligned}(d/dt)\|W(t)\|^2 &= (W'(t), W(t)) + (W(t), W'(t)) \\ &= i(X_j^0 W(t), W(t)) - i(W(t), X_j^0 W(t)) \\ &= 0.\end{aligned} \tag{D.70}$$

Hence $W(t) = 0$ for all t, so (D.61) and (D.62) coincide for all $v \in \mathcal{D}$. This proves the lemma.

To proceed further, we introduce certain auxiliary norms. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_N)$, $1 \le \alpha_j \le l$, set $|\alpha| = N$, let

$$X^\alpha = X_{\alpha_1}^0 \cdots X_{\alpha_N}^0 \quad \text{on } \mathcal{D}, \tag{D.71}$$

and, for $v \in \mathcal{D}$, let

$$p_m(v) = \left(\sum_{|\alpha| \le m} \|X^\alpha v\|^2 \right)^{1/2}. \tag{D.72}$$

Let $\overline{\mathcal{D}}_m$ denote the completion of $\mathcal{D}$ with respect to this norm, so $\mathcal{D} \hookrightarrow \overline{\mathcal{D}}_m$ and $\overline{\mathcal{D}}_m \hookrightarrow H$ are continuous injections. It is immediate that $p_{m-1}(X_j^0 v) \le p_m(v)$ for all $v \in \mathcal{D}$, and hence X_j^0 has a unique continuous extension

$$X_j' \colon \overline{\mathcal{D}}_m \to \overline{\mathcal{D}}_{m-1} \tag{D.73}$$

for each positive integer m.

LEMMA D.6. *We have*

$$e^{itX_j} \colon \overline{\mathcal{D}}_m \to \overline{\mathcal{D}}_m, \quad \textit{continuous}, \ 1 \le j \le l_0, \tag{D.74}$$

with e^{itX_j} actually a strongly continuous semigroup on each $\overline{\mathcal{D}}_m$.

PROOF. In view of Lemma D.5, we see that

$$v \in \mathcal{D} \Rightarrow X_{\alpha_1}^0 \cdots X_{\alpha_N}^0 e^{itX_j} v = e^{itX_j} Y_{\alpha_1} \cdots Y_{\alpha_N} v$$

where $Y_\nu = (1/i)\rho(\mathrm{Ad}(\exp -tx_j)x_\nu)$ is a linear combination of $X_1^0, \ldots, X_l^0$. The present lemma is immediate.

The following lemma will justify our operations in the proof of Theorem D.4. Let

$$E_j(t, v) = e^{itX_j} v, \qquad 1 \le j \le l_0. \tag{D.75}$$

LEMMA D.7. *For each $m \ge 1$,*

$$E_j \colon R \times \overline{\mathcal{D}}_m \to \overline{\mathcal{D}}_{m-1} \quad \textit{is a } C^1 \textit{ map.} \tag{D.76}$$

PROOF. The differentiability of E_j follows from the formula (D.57), and the continuity of the derivative of E_j follows from Lemma D.6.

We are now ready to prove Theorem D.4. We begin by defining a function π from G to the set $U(H)$ of unitary operators on H. Suppose

$$g = \exp(t_1 x_{\alpha_1}) \cdots \exp(t_N x_{\alpha_N}) \in G \qquad (1 \le \alpha_\nu \le l_0). \tag{D.77}$$

We want to set

$$\pi(g) = e^{it_1 X_{\alpha_1}} \cdots e^{it_N X_{\alpha_N}}. \tag{D.78}$$

Our main task is to show that (D.78) is independent of the representation (D.77), since each $g \in G$ has one, and indeed, many representations of this form. To do this, it is equivalent to show that, if

$$e = \exp(t_1 x_{\alpha_1}) \cdots \exp(t_N x_{\alpha_N}), \tag{D.79}$$

then

$$e^{it_1 X_{\alpha_1}} \cdots e^{it_N X_{\alpha_N}} = I. \tag{D.80}$$

Most of the work in doing this is accomplished by

LEMMA D.8. *Let $t_\nu(s)$ be C^∞ functions of s and suppose*

$$g = \exp(t_1(s)x_{\alpha_1}) \cdots \exp(t_N(s)x_{\alpha_N}) \tag{D.81}$$

be independent of s. Then

$$U(s) = e^{it_1(s)X_{\alpha_1}} \dots e^{it_N(s)X_{\alpha_N}} \tag{D.82}$$

is independent of s.

PROOF. Lemma D.7 implies that, if we set

$$\mathcal{E}(t_1, \dots, t_N, v) = e^{it_1 X_{\alpha_1}} \dots e^{it_N X_{\alpha_N}} v, \tag{D.83}$$

then

$$\mathcal{E} : R^N \times \overline{\mathcal{D}}_{m+N} \to \overline{\mathcal{D}}_m \quad \text{is a } C^1 \text{ map,} \tag{D.84}$$

and we can use the chain rule to evaluate $(d/ds)\mathcal{E}(t_1(s), \dots, t_N(s), v)$. We obtain, for $v \in \mathcal{D}$,

(D.85)

$$\begin{aligned}(d/ds)U(s)v = {} & i(dt_1/ds)X^0_{\alpha_1} e^{it_1(s)X_{\alpha_1}} \dots e^{it_N(s)X_{\alpha_N}} v \\ & + e^{it_1(s)X_{\alpha_1}}(i(dt_2/ds)X^0_{\alpha_2})e^{it_2(s)X_{\alpha_2}} \dots e^{it_N(s)X_{\alpha_N}} v \\ & + \cdots \\ & + e^{it_1(s)X_{\alpha_1}} \dots e^{it_{N-1}(s)X_{\alpha_{N-1}}}(i(dt_N/ds)X^0_{\alpha_N})e^{it_N(s)X_{\alpha_N}} v,\end{aligned}$$

and using Lemma (D.5), we push all the terms $i(dt_\nu/ds)X^0_\nu$ to the far left, and obtain

$$(d/ds)U(s)v = A(s)U(s)v, \qquad v \in \mathcal{D}, \tag{D.86}$$

where

$$\begin{aligned}A(s) = {} & (dt_1/ds)\rho(x_{\alpha_1}) + (dt_2/ds)\rho(\mathrm{Ad}(\exp t_1 x_{\alpha_1})x_{\alpha_2}) \\ & + \cdots + (dt_N/ds)\rho(\mathrm{Ad}(\exp(t_1 x_{\alpha_1}) \cdots \exp(t_{N-1}x_{\alpha_{N-1}}))x_{\alpha_N}).\end{aligned} \tag{D.87}$$

Now if we replace ρ by a faithful finite-dimensional representation ρ_0 of $\mathfrak{g}$, which certainly exponentiates to a locally faithful representation π_0 of G, we see that

$$0 = (d/ds)\pi_0(g) = A_0(s)\pi_0(g), \tag{D.88}$$

where $A_0(s)$ is as in (D.87) with ρ replaced by ρ_0. Since $A_0(s) = 0$ and ρ_0 is faithful, it follows that the element of $\mathfrak{g}$ to which ρ is applied in (D.87) must vanish; hence $A(s) = 0$. This proves Lemma D.8.

We resume the task of showing that (D.79) implies (D.80). Define a path γ from the interval $[0, t_1 + \cdots + t_N]$ to G by

$$\gamma(s) = \exp(t_1 x_{\alpha_1}) \cdots \exp(t_\nu x_{\alpha_\nu}) \exp((s - t_1 - \cdots - t_\nu)x_{\alpha_{\nu+1}}) \tag{D.89}$$

if $t_1 + \cdots + t_\nu \le s \le t_1 + \cdots + t_{\nu+1}$. Then (D.79) implies this path is closed. By the device of writing $\exp t_\nu x_{\alpha_\nu} = \exp(t_\nu x_{\alpha_\nu}/K) \cdots \exp(t_\nu x_{\alpha_\nu}/K)$ (K factors), we can suppose the points $\gamma(t_\nu)$, $1 \le \nu \le N$, are closely spaced. We can perturb γ

slightly to produce a *smooth* closed curve $\tilde{\gamma}(s)$, with $\tilde{\gamma}(t_\nu) = \gamma(t_\nu)$ for $1 \le \nu \le N$. Assuming G is simply connected, we can produce a smooth contraction $\tilde{\gamma}_\tau(s)$ to the point e; $\tilde{\gamma}_0(s) = \tilde{\gamma}(s)$, $\tilde{\gamma}_1(s) = e$, $\tilde{\gamma}_\tau(0) = \tilde{\gamma}_\tau(t_1 + \cdots + t_N) = e$. Consider the curves

$$\text{(D.90)} \qquad \begin{gathered} \sigma_1(\tau) = \tilde{\gamma}_\tau(t_1) \\ \sigma_\nu(\tau) = \tilde{\gamma}_\tau(t_1 + \cdots + t_{\nu-1})^{-1}\tilde{\gamma}_\tau(t_1 + \cdots + t_\nu), \qquad 2 \le \nu \le N. \end{gathered}$$

We have each $\sigma_\nu(\tau)$, $0 \le \tau \le 1$, taking values in a neighborhood of $e \in G$ which can be arranged to be as small as desired, and

$$\text{(D.91)} \qquad e = \sigma_1(\tau)\sigma_2(\tau)\cdots\sigma_N(\tau), \qquad \tau \in [0,1].$$

Recall that

$$\text{(D.92)} \qquad \sigma_\nu(0) = \exp(t_\nu x_{\alpha_\nu}), \qquad \sigma_\nu(1) = e.$$

Now we claim there exist smooth $s_{\mu\nu}(\tau)$ such that

$$\text{(D.93)} \qquad \sigma_\nu(\tau) = \exp(s_{1\nu}(\tau)x_{\beta_1})\cdots\exp(s_{M\nu}(\tau)x_{\beta_M}), \qquad 1 \le \beta_\mu \le l_0,$$

for $0 \le \tau \le 1$. This is an immediate consequence of the following simple lemma.

LEMMA D.9. *There exists a smooth map*

$$\text{(D.94)} \qquad \Phi\colon R^L \to G$$

of the form

(D.95)

$$\begin{aligned} \Phi(s_1, \ldots, s_L) = {} & \exp(b_{11}x_{\gamma_{11}})\cdots\exp(b_{1K_1}x_{\gamma_{1K_1}})\exp(s_1 x_{\beta_1})\exp(-b_{1K_1}x_{\gamma_{1K_1}}) \\ & \cdots\exp(-b_{11}x_{\gamma_{11}})\cdots \\ & \cdot\exp(b_{L1}x_{\gamma_{L1}})\cdots\exp(s_L x_{\beta_L})\cdots\exp(-b_{L1}x_{\gamma_{L1}}) \end{aligned}$$

where $b_{jk} \in R$, $1 \le \gamma_{jk} \le l_0$, $1 \le \beta_j \le l_0$, such that Φ has surjective differential at $(s_1, \ldots, s_L) = 0$.

PROOF. We need merely arrange the quantities $b_{jk}x_{\gamma_{jk}}$ and x_{β_j} such that $\mathrm{Ad}(\exp(b_{j1}x_{\gamma_{j1}})\cdots\exp(b_{jK_j}x_{\gamma_{jK_j}})x_{\beta_j}$, $1 \le j \le L$, span $\mathfrak{g}$ as a linear space.

Having established (D.93), if we plug this into (D.91) and apply Lemma D.8, we deduce that

$$U(\tau) = \prod_{\nu=1}^{N}(e^{is_{\nu 1}(\tau)X_{\beta_1}}\cdots e^{is_{\nu M}(\tau)X_{\beta_M}})$$

is independent of τ. Note that some of the $s_{\mu\nu}(\tau)$ are the constants $\pm b_{jk}$ and some are functions $s_j(\tau)$, $1 \le j \le L$, and we can arrange $s_j(1) = 0$, since $\sigma_\nu(1) = e$. The cancellations that accrue at $\tau = 1$ imply $U(1) = I$. Thus $U(0) = I$. We claim we can choose $\Phi = \Phi_\nu$ and $s_{\mu\nu}(\tau)$ so that $U(0)$ is equal to the left side of (D.80). Indeed, if all b_{1k} are taken to be 0 and $x_{\beta_1} = x_{\alpha_\nu}$ in (D.95), with the other $b_{jk}x_{\gamma_{jk}}$, x_{β_k} chosen as indicated in the proof of Lemma D.9, then we can obtain (D.93) with $s_{1\nu}(0)x_{\beta_1} = t_\nu x_{\alpha_\nu}$ and $s_{2\nu}(0) = \cdots = s_{M\nu}(0) = 0$, by constructing a right inverse Ψ_ν of Φ_ν, defined on a neighborhood of e in G, into R^L, such that $\Psi_\nu(e) = 0$ and Ψ_ν maps the one parameter group in G in the direction $x_{\alpha_\nu} = x_{\beta_1} \in \mathfrak{g} = T_eG$ to the line through the first coordinate vector in R^L. This can be arranged by the implicit function theorem. We can finally conclude that (D.79) implies (D.80).

We are now able to say that (D.77)–(D.78) produces a uniquely defined function $\pi: G \to U(H)$, and it is automatic from (D.78) that π is a group homomorphism. To check the strong continuity of π and verify (D.59), we will again use Lemma D.9, which guarantees the existence of a C^∞ map

$$\Psi: \mathcal{O} \to R^L, \tag{D.96}$$

$\mathcal{O}$ a neighborhood of e in G, such that $\Psi(e) = 0$ and $\Phi \circ \Psi = \text{id}$ on $\mathcal{O}$. By the fact that π is well defined, we can write

$$\begin{aligned}\pi(g) = e^{ib_{11}X_{\gamma_{11}}} \cdots e^{ib_{1K_1}X_{\gamma_{1K_1}}} e^{is_1(g)X_{\beta_1}} e^{-ib_{1K_1}X_{\gamma_{1K_1}}} \cdots e^{-ib_{11}X_{\gamma_{11}}} \\ \cdots e^{ib_{L1}X_{\gamma_{L1}}} \cdots e^{is_L(g)X_{\beta_L}} \cdots e^{-ib_{L1}X_{\gamma_{L1}}}\end{aligned} \tag{D.97}$$

where

$$\Psi(g) = (s_1(g), \dots, s_L(g)). \tag{D.98}$$

By (D.76) we deduce that

$$v \in \overline{\mathcal{D}}_{m+B} \quad \Rightarrow \quad g \mapsto \pi(g)v \text{ is a } C^1 \text{ map from } \mathcal{O} \text{ to } \overline{\mathcal{D}}_m, \tag{D.99}$$

where $B = L + 2(K_1 + \cdots + K_L)$, and we can calculate its derivative by the chain rule. Then the same sort of argument used to establish Lemma D.8 shows the limit in (D.59) exists for all $v \in \mathcal{D} \subset \overline{\mathcal{D}}_B$, and equals $\rho(x)v$. The proof of Theorem D.4 is complete. Let us remark that $\mathcal{D}$ is a space of smooth vectors for π and $\pi(g): \mathcal{D} \to \mathcal{D}$ for all $g \in G$.

Let us see how Theorem D.4 applies to the representation ω of $hp_2 \approx \text{sp}(n, R)$, given by

$$\omega(Q) = iQ(X, D), \tag{D.100}$$

as in Chapter 1, §4. We take

$$\mathcal{D} = \mathcal{S}(R^n). \tag{D.101}$$

As shown in Chapter 1, §4, $e^{itQ_j(X,D)}$ preserves $\mathcal{S}(R^n)$ for a *basis* Q_j of hp_2. It is even a little easier to see this for polynomials $Q(x, \xi)$ of the form x_jx_k and $\xi_j\xi_k$, which generate $hp_2 = \text{sp}(n, R)$. This provides an alternate method of showing that ω exponentiates to a representation of $\widetilde{\text{Sp}(n, R)}$. Compare Proposition 4.1 of Chapter 1.

From time to time, it has been suggested that it is enough for $\mathcal{D}$ to be a common dense domain of selfadjointness for $(1/i)\rho(x)$, preserved by $\rho(x)$, $x \in \mathfrak{g}$, in order to exponentiate ρ. We note that this is false, and refer to Nelson [**183**] for a counterexample; see also pages 296–296 of Warner [**256**].

References

1. R. Abraham and J. Marsden, *Foundations of mechanics*, Benjamin, Reading, Mass., 1978.

2. J. Adams, *Lectures on Lie groups*, Benjamin, Reading, Mass., 1969.

3. R. Anderson, *The Weyl functional calculus*, J. Funct. Anal. **4** (1969), 240–267.

4. V. Arnold, *Mathematical methods of classical mechanics*, Springer, New York, 1978.

5. M. Atiyah, *Characters of semi-simple Lie groups*, Lecture Notes, Univ. of Oxford, 1975.

6. M. Atiyah, R. Bott, and A. Shapiro, *Clifford modules*, Topology **3** (1964), 3–38.

7. M. Atiyah and W. Schmid, *A geometric construction of the discrete series for semi-simple Lie groups*, Invent. Math. **42** (1977), 1–62.

8. L. Auslander, *Lecture notes on nil-theta functions*, CBMS Regional Conf. Ser. in Math., No. 34, Amer. Math. Soc., Providence, R.I., 1977.

9. L. Auslander and B. Kostant, *Polarizations and unitary representations of solvable Lie groups*, Invent. Math. **14** (1971), 255–354.

10. L. Auslander and C. Moore, *Unitary representations of solvable Lie groups*, Mem. Amer. Math. Soc., No. 62, Amer. Math. Soc., Providence, R.I., 1966.

11. W. Baldoni Silva, *Unitary dual of* $\mathrm{Sp}(n,1)$, $n \geq 2$, Duke Math. J. **48** (1982), 549–583.

12. V. Bargmann, *Irreducible unitary representations of the Lorentz group*, Ann. of Math. **48** (1947), 568–640.

13. ——, *On a Hilbert space of analytic functions and an associated integral transform*, Comm. Pure Appl. Math. **14** (1961), 187–214.

14. H. Bateman, *Partial differential equations of mathematical physics*, Dover, New York, 1944.

15. R. Beals, P. Greiner, and N. Stanton, *The heat equation and the geometry of CR manifolds*, Bull. Amer. Math. Soc. (N.S.) **10** (1984), 275–276.

16. F. Berezin, *Laplace operators on semi-simple Lie groups*, Amer. Math. Soc. Transl. (2) **21** (1962), 239–339.

17. M. Berger, P. Gauduchon, and E. Mazet, *Le spectre d'une variété Riemanniene*, Lecture Notes in Math., vol. 194, Springer, 1971.

18. P. Bernat, *Sur les representations unitaries des groupes de Lie resolubles*, Ann. École Norm. Sup. **82** (1965), 37–99.

19. L. Bers, F. John, and M. Schechter, *Partial differential equations*, Interscience, New York, 1964.

20. L. Biedenharn and J. Louck, *Angular momentum in quantum physics*, The Racah-Wigner Algebra in Quantum Theory, Vols. 8 and 9, Encyclopedia of Math. and its Applications, Addison-Wesley, Reading, Mass., 1981.

21. R. Blattner, *Quantization and representation theory*, Proc. Sympos. Pure Math., Vol. 26, Amer. Math. Soc., Providence, R.I., 1972, pp. 147–166.

22. ——, *General background*, Harmonic Analysis and Representations of Semi-simple Lie Groups, D. Reidel, Boston, 1980.

23. A. Borel and N. Wallach, *Continuous cohomology, discrete subgroups, and representations of reductive groups*, Princeton Univ. Press, Princeton, N.J., 1980.

24. L. Boutet de Monvel, *Hypoelliptic operators with double characteristics and related pseudodifferential operators*, Comm. Pure Appl. Math. **27** (1974), 585–639.

25. L. Boutet de Monvel, A. Grigis, and B. Helffer, *Parametrixes d'operateurs pseudodifferentiels a caracteristiques multiples*, Astérisque **34–35** (1976), 93–121.

26. L. Boutet de Monvel and F. Treves, *On a class of pseudodifferential operators with double characteristics*, Comm. Pure Appl. Math. **27** (1974), 59–81.

27. F. Bruhat, *Sur les representations induites des groupes de Lie*, Bull. Soc. Math. France **84** (1956), 97–205.

28. J. Brezin, *Unitary representation theory for solvable Lie groups*, Mem. Amer. Math. Soc., No. 79, Amer. Math. Soc., Providence, R.I., 1968.

29. ——, *Harmonic analysis on nilmanifolds*, Trans. Amer. Math. Soc. **150** (1970), 611–618.

30. ——, *Harmonic analysis on compact solvmanifolds*, Lecture Notes in Math., Vol. 602, Springer, 1977.

31. R. Cahn, *The K-types of principal series representations*, manuscript.

32. R. Cahn and M. Taylor, *Asymptotic behavior of multiplicities of representations of compact groups*, Pacific J. Math. **83** (1979), 17–28.

33. E. Cartan, *The theory of spinors*, Dover, New York, 1981.

34. P. Cartier and J. Dixmier, *Vecteurs analytiques dans les representations des groupes de Lie*, Amer. J. Math. **80** (1958), 131–145.

35. J. Cheeger, *Spectral geometry of spaces with cone-like singularities*, Proc. Nat. Acad. Sci. U.S.A. **76** (1979), 2103–2106.

36. J. Cheeger, M. Gromov, and M. Taylor, *Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds*, J. Differential Geom. **17** (1982), 15–53.

37. J. Cheeger and M. Taylor, *On the diffraction of waves by conical singularities*, Comm. Pure Appl. Math. **35** (1982), 275–331, 487–529.

38. P. Chernoff, *Essential self-adjointness of powers of generators of hyperbolic equations*, J. Funct. Anal. **12** (1973), 401–414.

39. C. Chevalley, *Theory of Lie groups*, Vol. 1, Princeton Univ. Press, Princeton, N.J., 1946.

40. ——, *Invariants of finite groups generated by reflections*, Amer. J. Math. **77** (1955), 778–782.

41. R. Coifman and G. Weiss, *Analyze harmonique non-commutative sur certaine espaces homogenes*, Lecture Notes in Math., No. 242, Springer, 1971.

42. J. Dieudonné, *Special functions and linear representations of Lie groups*, CBMS Regional Conf. Ser. in Math., No. 42, Amer. Math. Soc., Providence, R.I., 1980.

43. J. Dixmier, *Sur les representations unitaires des groupes de Lie nilpotents.* I, Amer. J. Math. **81** (1959), 160–170; II, Bull. Soc. Math. France **85** (1957), 325–388; III, Canad. J. Math. **10** (1958), 321–348; IV, Canad. J. Math. **11** (1959), 321–344; V, Bull. Soc. Math. France **87** (1959), 65–79; VI, Canad. J. Math. **12** (1960), 324–352.

44. ——, *Representations integrable du groupe de De Sitter*, Bull. Soc. Math. France **89** (1961), 9–41.

45. ——, *Sur les representations unitaires des groupes de Lie resolubles*, Math. J. Okayama Univ. **11** (1962), 1–18.

46. ——, *Les C^* algebres et leurs representations*, Gauthier-Villars, Paris, 1964.

47. ——, *Les algebres d'operateurs dans l'espace Hilbertiens* (*algebres de von Neumann*), Gauthier-Villars, Paris, 1969.

48. V. Dobrev, G. Mack, V. Petkova, S. Petrova, and I. Todorov, *Harmonic analysis on the n-dimensional Lorentz group and its applications to conformal quantum field theory*, Lecture Notes in Physics, No. 63, Springer, New York, 1977.

49. J. J. Duistermaat, *Harmonic analysis on semi-simple Lie groups* (preprint).

50. N. Dunford and J. Schwartz, *Linear operators.* I, II, Interscience, New York, 1958, 1963.

51. H. Dym and H. McKean, *Fourier series and integrals*, Academic Press, New York, 1972.

52. A. Dynin, *An analogue of pseudodifferential operators on the Heisenberg group; symbolic calculus*, Soviet Math. Dokl. **17** (1976), 508–512.

53. ——, *Pseudodifferential operators on Heisenberg groups*, Sem. CIME, Bressanone, Italy, June, 1977.

54. L. Ehrenpreis, *Harmonic analysis on Lie groups*, manuscript.

55. ——, *The use of partial differential equations for the study of group representations*, Proc. Sympos. Pure Math., Vol. 26, Amer. Math. Soc., Providence, R.I., 1972, pp. 317–320.

56. L. Ehrenpreis and J. Mautner, *Some properties of the Fourier transform on semi-simple Lie groups.* I, Ann of Math. **61** (1955), 406–439; II, Trans. Amer. Math. Soc. **84** (1957), 1–55; III, Trans. Amer. Math. Soc. **90** (1959), 431–484.

57. M. Englefield, *Group theory and the coulomb problem*, Wiley-Interscience, New York, 1972.

58. A. Erdelyi, *Asymptotic expansions*, Dover, New York, 1965.

59. A. Erdelyi et al., *Higher transcendental functions*, Vol. I–III (Bateman Manuscript Project), McGraw-Hill, New York, 1953.

60. L. Falicov, *Group theory and its physical applications*, Chicago Lectures in Physics, Chicago Univ. Press, Chicago, Ill., 1966.

61. H. Fegan, *The fundamental solution of the heat equation on a compact Lie group*, J. Differential Geom. **18** (1983), 659–668.

62. G. Folland, *Subelliptic operators on the Heisenberg group*, Proc. Sympos. Pure Math., Vol. 26, Amer. Math. Soc., Providence, R.I., 1973, pp. 359–361.

63. ——, *Introduction to partial differential equations*, Princeton Univ. Press, Princeton, N.J., 1976.

64. G. Folland and E. Stein, *Estimates for the $\overline{\partial}_b$ complex and analysis on the Heisenberg group*, Comm. Pure Appl. Math. **27** (1974), 429–522.

65. H. Freudenthal and H. de Vries, *Linear Lie groups*, Academic Press, New York, 1969.

66. F. G. Friedlander, *Sound pulses*, Cambridge Univ. Press, 1958.

67. R. Gangolli, *Spectra of discrete subgroups*, Proc. Sympos. Pure Math., Vol. 26, Amer. Math. Soc., Providence, R.I., 1973.

68. L. Gårding, *Note on continuous representations of Lie groups*, Proc. Nat. Acad. Sci. U.S.A. **33** (1947), 331–332.

69. ——, *Vecteurs analytiques dans les representations des groupes de Lie*, Bull. Soc. Math. France **88** (1960), 73–93.

70. B. Gaveau, *Principe de moindre action, propagation de chaleur, et estimees sous elliptiques sur certains groups nilpotents*, Acta Math. **139** (1977), 95–153.

71. I. Gelfand, M. Graev, and N. Vilenkin, *Generalized functions*, Vol. 5, Academic Press, New York, 1966.

72. I. Gelfand, M. Graev, and I. Pyatetskii-Shapiro, *Representation theory and automorphic functions*, W. B. Saunders, Philadelphia, Penn., 1969.

73. I. Gelfand, R. Minlos, and Z. Shapiro, *Represenations of the rotation and Lorentz groups and their applications*, Pergamon Press, New York, 1963.

74. I. Gelfand and M. Naimark, *Unitare Darstellungen der Klassischen Gruppen*, Akademie-Verlag, Berlin, 1957.

75. D. Geller, *Fourier analysis on the Heisenberg group. I. Schwartz space*, J. Funct. Anal. **36** (1980), 205–254.

76. ____, *Local solvability and homogeneous distributions on the Heisenberg group*, Comm. Partial Differential Equations **5** (1980), 475–560.

77. H. Georgi, *Lie algebras in particle physics*, Benjamin, Reading, Mass., 1982.

78. S. Gindikin and F. Karpelevic, *Plancherel measure of Riemannian symmetric spaces of non-positive curvature*, Dokl. Akad. Nauk SSSR **145** (1962), 252–255.

79. R. Gillmore, *The classical limit of quantum nonspin systems*, J. Math. Phys. **20** (1979), 891–893.

80. R. Godement, *Sur la theorie des representations unitaires*, Ann. of Math. **53** (1951), 68–124.

81. ____, *A theory of spherical functions*. I, Trans. Amer. Math. Soc. **73** (1952), 496–556.

82. H. Goldstein, *Classical mechanics*, Addison-Wesley, New York, 1950.

83. R. Goodman, *Nilpotent Lie groups, structure and applications to analysis*, Lecture Notes in Math., No. 562, Springer, 1976.

84. ____, *Analytic and entire vectors for representations of Lie groups*, Trans. Amer. Math. Soc. **143** (1969), 55–76.

85. W. Greub, *Multilinear algebra*, Springer, New York, 1978.

86. A. Grossman, G. Loupias, and E. Stein, *An algebra of pseudodifferential operators and quantum mechanics in phase space*, Ann. Inst. Fourier (Grenoble) **18** (1969), 343–368.

87. V. Guillemin and S. Sternberg, *Geometric asymptotics*, Amer. Math. Soc., Providence, R.I., 1977.

88. ____, *The metaplectic representation, Weyl operators, and spectral theory*, J. Funct. Anal. **42** (1981), 128–225.

89. ____, *Symplectic techniques in physics*, Cambridge Univ. Press, 1984.

90. R. Gunning, *Lectures on Riemann surfaces*, Princeton Univ. Press, Princeton, N.J., 1967.

91. N. Xuan Hai, *The Fourier transform on nilpotent Lie groups*, Comm. Partial Differential Equations **8** (1983), 667–691.

92. Harish-Chandra, *Representations of semi-simple Lie groups*. I, Trans. Amer. Math. Soc. **75** (1953), 185–243; II, Trans. Amer. Math. Soc. **76** (1954), 26–65; III, Trans. Amer. Math. Soc. **76** (1954), 234–253; IV, Amer. J. Math. **77** (1955), 743–777; V, Amer. J. Math. **78** (1956), 1–41; VI, Amer. J. Math. **78** (1956), 564–628.

93. ____, *The Plancherel formula for the complex semi-simple Lie groups*, Trans. Amer. Math. Soc. **76** (1954), 485–528.

94. ____, *The Fourier transform on a semi-simple Lie algebra*. I, II, Amer. J. Math. **79** (1957), 87–120, 653–686.

95. ____, *Spherical functions on a semi-simple Lie group*. I, II, Amer. J. Math. **80** (1958), 241–310, 553–613.

96. ____, *Discrete series for semi-simple Lie groups*. I, II, Acta Math. **113** (1965), 241–318; **116** (1966), 1–111.

97. ____, *Harmonic analysis on semi-simple Lie groups*, Bull. Amer. Math. Soc. **76** (1970), 529–551.

98. ____, *Harmonic analysis on real reductive groups*. I, J. Funct. Anal. **19** (1975), 104–204; II, Invent. Math. **36** (1975), 1–55); III, Ann. of Math. **104** (1976), 117–201.

99. D. Hejhal, *The Selberg trace formula for* $\mathrm{PSL}(2, R)$, Lecture Notes in Math., Nos. 548, 1001, Springer, 1976, 1983.

100. S. Helgason, *Differential geometry and symmetric spaces*, Academic Press, New York, 1962.

101. ____, *Analysis on Lie groups and symmetric spaces*, CBMS Regional Conf. Ser. in Math., No. 14, Amer. Math. Soc., Providence, R.I., 1972.

102. ____, *Topics in harmonic analysis on homogeneous spaces*, Birkhauser, Boston, 1981.

103. ____, *The Radon transform*, Birkhauser, Boston, 1980.

104. ____, *A duality for symmetric spaces with applications to group representations*, Adv. in Math. **5** (1970), 1–154.

105. ____, *Invariant differential operators and eigenspace representations*, London Math. Soc. Lecture Notes, No. 34, Cambridge Univ. Press, 1979, pp. 236–286.

106. ____, *Wave equations on homogeneous spaces* (preprint).

107. ____, *Fundamental solutions of invariant differential operators on symmetric spaces*, Amer. J. Math. **86** (1964), 565–601.

108. B. Helffer and J. Nourrigat, *Caracterization des operateurs hypoelliptiques homogenes invariant a gauche sur un groupe de Lie nilpotent gradue*, Comm. Partial Differential Equations **4** (1979), 899–958.

109. L. Helms, *Introduction to potential theory*, Wiley-Interscience, New York, 1969.

110. R. Herman, *Lie groups for physicists*, Benjamin, Reading, Mass., 1966.

111. E. Hewitt and K. Ross, *Abstract harmonic analysis*, Springer, New York, I, 1963; II, 1970.

112. E. Hille and R. Phillips, *Functional analysis and semi-groups*, Colloq. Publ., Vol. 31, Amer. Math. Soc., Providence, R.I., 1957.

113. T. Hirai, *On irreducible representations of the Lorentz group of nth order*, Proc. Japan Acad. **38** (1962), 258–262.

114. ____, *The characters of irreducible representations of the Lorentz group of nth order*, Proc. Japan Acad. **41** (1965), 526–531.

115. ____, *The Plancherel formula for the Lorentz group of nth order*, Proc. Japan Acad. **42** (1965), 323–326.

116. ——, *Classification and the characters of irreducible representations of* SU$(p,1)$, Proc. Japan Acad. **42** (1966), 907–912.

117. ——, *The Plancherel formula for* SU(p,q), J. Math. Soc. Japan **22** (1970), 134–179.

118. E. Hobson, *The theory of spherical harmonics*, Chelsea, New York, 1965.

119. L. Hörmander, *The Weyl calculus of pseudodifferential operators*, Comm. Pure Appl. Math. **32** (1979), 359–443.

120. ——, *A class of pseudodifferential operators with double characteristics*, Math. Ann. **217** (1975), 165–188.

121. ——, *The analysis of linear partial differential operators.* I, II, Springer, New York, 1983; III, IV, 1985.

122. ——, *An introduction to complex analysis in several variables*, Van Nostrand, Princeton, N.J., 1966.

123. R. Howe, *On the role of the Heisenberg group in harmonic analysis*, Bull. Amer. Math. Soc. (N.S.) **3** (1980), 821–843.

124. ——, *Quantum mechanics and partial differential equations*, J. Funct. Anal. **38** (1980), 188–254.

125. ——, *Some results of Strichartz and of Rallis and Schiffman*, J. Funct. Anal. **32** (1979), 297–303.

126. ——, *Dual pairs and the classical equations of physics* (preprint).

127. A. Hulanicki, *The distribution of energy in the Brownian motion in the Gaussian field and analytic-hypoellipticity of certain subelliptic operators on the Heisenberg group*, Studia Math. **51** (1976), 165–173.

128. H. Jacobson and M. Vergne, *Wave and Dirac operators and representations of the conformal group*, J. Funct. Anal. **24** (1977), 52–106.

129. N. Jacobson, *Lie algebras*, Interscience, New York, 1965.

130. H. Jacquet and R. Langlands, *Automorphic forms on* GL_2, Lecture Notes in Math., No. 114, Springer, 1970.

131. F. John, *Plane waves and spherical means*, Interscience, New York, 1955.

132. F. Karpelevic, *The geometry of geodesics and the eigenfunctions of the Laplace-Beltrami operators on symmetric spaces*, Trans. Moscow Math. Soc. **14** (1965), 48–185.

133. M. Kashiwara, A. Kowata, K. Minemura, K. Okamoto, T. Oshima, and M. Tanaka, *Eigenfunctions of invariant differential operators on a symmetric space*, Ann. of Math. **107** (1978), 1–39.

134. A. Kirillov, *Elements of the theory of representations*, Springer, New York, 1976.

135. ——, *Unitary representations of nilpotent Lie groups* , Russian Math. Surveys **17** (1962), 53–104.

136. A. Knapp, *Representation theory of semisimple groups*, Princeton Univ. Press, Princeton, N.J., 1986.

137. A. Knapp and B. Speh, *Status of classification of unitary representations*, Lecture Notes in Math., No. 908, Springer, 1982, pp. 1–38.

138. ——, *Irreducible unitary representations of* SU(2, 2), J. Funct. Anal. **45** (1982), 41–73.

139. A. Knapp and E. Stein, *Intertwining operators for semi-simple Lie groups*, Ann. of Math. **93** (1971), 489–578.

140. A. Knapp and G. Zuckerman, *Classification theorems for representations of semi-simple Lie groups*, Lecture Notes in Math., No. 587, Springer, 1977, pp. 138–159.

141. J. Kohn and L. Nirenberg, *An algebra of pseudodifferential operators*, Comm. Pure Appl. Math. **18** (1965), 269–305.

142. B. Kostant, *Quantization and unitary representations*, Lecture Notes in Math., No. 170, Springer, 1970, pp. 87–208.

143. ——, *On the existence and irreducibility of certain series of representations*, Bull. Amer. Math. Soc. **75** (1969), 627–642.

144. T. Kotake and M. Narasimhan, *Regularity theorems for fractional powers of linear elliptic operators*, Bull. Soc. Math. France **90** (1962), 449–472.

145. H. Kraljevic, *Representations of the universal covering group of the group* SU$(n, 1)$, Glas. Mat. **8** (1973), 23–72.

146. R. Kunze and E. Stein, *Uniformly bounded representations and harmonic analysis on the real unimodular group*. I, II, III, Amer. J. Math. **82** (1960), 1–62; **83** (1961), 723–786; **89** (1967), 385–442.

147. S. Lang, $\mathrm{SL}_2(R)$, Addison-Wesley, New York, 1975.

148. R. Langlands, *On the classification of irreducible representations of real algebraic groups*, IAM mimeographed lecture notes, 1973.

149. H. B. Lawson and M. L. Michelson, *The geometry of spinors*, Monograph, in preparation.

150. P. Lax and R. Phillips, *Scattering theory*, Academic Press, New York, 1967.

151. ——, *Scattering theory for automorphic functions*, Princeton Univ. Press, Princeton, N.J., 1976.

152. ——, *An example of Huygens' principle*, Comm. Pure Appl. Math. **31** (1978), 415–421.

153. ——, *The asymptotic distribution of lattice points in Euclidean and non-Euclidean spaces*, J. Funct. Anal. **46** (1982), 280–350.

154. N. Lebedev, *Special functions and their applications*, Dover, New York, 1972.

155. J. Leray, *Lagrangian analysis and quantum mechanics*, M.I.T. Press, Cambridge, Mass., 1972.

156. G. Lion and M. Vergne, *The Weil representation, Maslov index, and theta series*, Birkhauser, Boston, 1980.

157. R. Lipsman, *Group representations*, Lecture Notes in Math., No. 388, Springer, 1974.

158. L. Loomis, *Abstract harmonic analysis*, Van Nostrand, Princeton, N.J., 1953.

159. I. MacDonald, *Algebraic structure of Lie groups*, London Math. Soc. Lecture Notes, No. 34, Cambridge Univ. Press, 1979, pp. 91–150.

160. G. Mackey, *Unitary group representations in physics, probability, and number theory*, Benjamin, Reading, Mass., 1978.

161. ____, *The theory of unitary group representations*, Univ. of Chicago Press, Chicago, Ill., 1976.

162. ____, *Imprimitivity for representations of locally compact groups*, Proc. Nat. Acad. Sci. U.S.A. **35** (1949), 537–545.

163. ____, *Induced representations of locally compact groups.* I, II, Ann. of Math. **55** (1952), 101–139; **58** (1953), 193–221.

164. ____, *Unitary representations of group extensions*, Acta Math. **99** (1958), 265–311.

165. ____, *Infinite dimensional group representations*, Bull. Amer. Math. Soc. **69** (1963), 628–686.

166. H. McKean and I. Singer, *Curvature and the eigenvalues of the Laplacian*, J. Differential Geom. **1** (1967), 43–70.

167. J. Mather, *Differentiable invariants*, Topology **16** (1977), 145–155.

168. F. Mautner, *Unitary representations of locally compact groups*, Ann. of Math. **52** (1950), 528–556.

169. A. Melin, *Parametrix construction for right invariant differential operators on nilpotent groups*, Comm. Partial Differential Equations **6** (1981), 1363–1405.

170. R. Melrose and M. Taylor, *Near peak scattering and the corrected Kirchhoff approximation for convex obstacles*, Adv. in Math. **55** (1985), 242–315.

171. A. Menikoff and J. Sjöstrand, *On the eigenvalues of a class of hypoelliptic operators*, Math. Ann. **235** (1978), 55–85.

172. A. Messiah, *Quantum mechanics*, Wiley, New York, 1958.

173. K. Miller, *Hypoellipticity on the Heisenberg group*, J. Funct. Anal. **31** (1979), 306–320.

174. ____, *Parametrices for hypoelliptic operators on step two nilpotent Lie groups*, Comm. Partial Differential Equations **5** (1980), 1153–1184.

175. W. Miller, *Lie theory and special functions*, Academic Press, New York, 1968.

176. S. Minakshisundaram and A. Pleijel, *Eigenfunctions of the Laplace operator on Riemannian manifolds*, Canad. J. Math. **1** (1949), 242–256.

177. C. Moore, *Representations of solvable and nilpotent groups and harmonic analysis on nil and solvmanifolds*, Proc. Sympos. Pure Math., Vol. 26, Amer. Math. Soc., Providence, R.I., 1973.

178. C. Moore and J. Wolf, *Square integrable representations of nilpotent Lie groups*, Trans. Amer. Math. Soc. **185** (1973), 445–462.

179. C. Morrey, *Multiple integrals and the calculus of variations*, Springer, New York, 1966.

180. G. Mostow, *Strong rigidity of locally symmetric spaces*, Princeton Univ. Press, Princeton, N.J., 1973.

181. A. Nachman, *The wave equation on the Heisenberg group*, Comm. Partial Differential Equations **7** (1982), 675–714.

182. E. Nelson, *Operants, a functional calculus for non commuting operators*, Functional Analysis and Related Topics, Springer, New York, 1970, pp. 172–187.

183. ——, *Analytic vectors*, Ann. of Math. **70** (1959), 572–615.

184. ——, *Operator differential equations*, Mimeographed graduate lecture notes, Princeton, 1964.

185. E. Nelson and W. Stinespring, *Representations of elliptic operators in an enveloping algebra*, Amer. J. Math. **91** (1959), 547–560.

186. H. Nussensweig, *High frequency scattering by an impenetrable sphere*, Ann. Physics **34** (1965), 23–95.

187. F. Olver, *Introduction to asymptotics and special functions*, Academic Press, New York, 1974.

188. B. Orsted, *Induced representations and a new proof of the imprimitivity theorem*, J. Funct. Anal. **31** (1979), 355–359.

189. ——, *Conformally invariant differential equations and projective geometry*, J. Funct. Anal. **44** (1981), 1–23.

190. ——, *The conformal invariance of Huygens' principle*, J. Differential Geom. **16** (1981), 1–9.

191. U. Ottoson, *A classification of the unitary irreducible representations of* $\mathrm{SU}(N, 1)$, Comm. Math. Phys. **10** (1968), 114–131.

192. T. Parker, *Gauge theories on four dimensional Riemannian manifolds*, Comm. Math. Phys. **85** (1982), 563–602.

193. R. Parthasarathy, *Dirac operators and the discrete series*, Ann. of Math. **96** (1972), 1–30.

194. J. Peetre, *The Weyl transform and Laguerre polynomials*, Le Mathematische, Univ. de Catania Sem. 27, 301–323.

195. L. Pontrjagin, *Topological groups*, Princeton Univ. Press, Princeton, N.J., 1958.

196. W. Poor, *Differential geometric structures*, McGraw-Hill, New York, 1981.

197. L. Pukansky, *Lecons sur les representations des groupes*, Monographie de la Soc. Math. France, no. 2, Dunod, Paris, 1967.

198. ——, *On the characters and the Plancherel formula of nilpotent groups*, J. Funct. Anal. **1** (1967), 255–280.

199. L. Pukansky, *On the unitary representations of exponential groups*, J. Funct. Anal. **2** (1968), 73–113.

200. ——, *Representations of solvable Lie groups*, Ann. Sci. École Norm. Sup. **4** (1971), 464–608.

201. M. Reed and B. Simon, *Methods of mathematical physics*. I–IV, Academic Press, New York, I, 1975; II, 1975; III, 1978; IV, 1978.

202. F. Riesz and B. Sz. Nagy, *Functional analysis*, Ungar, New York, 1955.

203. W. Rossmann, *Analysis on real hyperbolic spaces*, J. Funct. Anal. **30** (1978), 448–477.

204. L. Rothschild and E. Stein, *Hypoelliptic differential operators and nilpotent groups*, Acta Math. **137** (1976), 247–320.

205. W. Rudin, *Function theory on the unit ball of* $\mathbf{C}^n$, Springer, New York, 1980.

206. W. Schmid, *On the characters of the discrete series*, Invent. Math. **30** (1975), 47–144.

207. ——, *Representations of semi-simple Lie groups*, London Math. Soc. Lecture Notes, No. 34, Cambridge Univ. Press, 1979, pp. 185–235.

208. R. Schrader and M. Taylor, *Small h asymptotics for quantum partition functions associated to particles in external Yang-Mills potentials*, Comm. Math. Phys. **92** (1984), 555–594.

209. A. Schwartz, *Smooth functions invariant under the action of a compact group*, Topology **14** (1975), 63–68.

210. L. Schwartz, *Theorie des distributions*. I, II, Hermann, Paris, 1957.

211. I. Segal, *Transformations for operators and symplectic automorphisms over a locally compact abelian group*, Math. Scand. **13** (1963), 31–43.

212. ——, *An extension of Plancherel's formula to separable unimodular groups*, Ann. of Math. **52** (1950), 272–292.

213. A. Selberg, *Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series*, J. Indian Math. Soc. **20** (1956), 47–87.

214. J. Serre, *Lie algebras and Lie groups*, Benjamin, New York, 1965.

215. ——, *Algebres de Lie semi-simples complexes*, Benjamin, New York, 1966.

216. D. Shale, *Linear symmetries of free Boson fields*, Trans. Amer. Math. Soc. **103** (1962), 149–167.

217. B. Simon, *The classical limit of quantum partition functions*, Comm. Math. Phys. **71** (1980), 247–276.

218. J. Sjöstrand, *Parametrices for pseudodifferential operators with multiple characteristics*, Ark. Mat. **12** (1974), 85–130.

219. J. Sniatycki, *Geometric quantization and quantum mechanics*, Springer, New York, 1980.

220. A. Sommerfeld, *Mathematische theorie der diffraktion*, Math. Ann. **47** (1896), 317–374.

221. J. Souriau, *Structure des systemes dynamique*, Dunod, Paris, 1970.

222. B. Speh, *The unitary dual of* $\mathrm{GL}(3, R)$ *and* $\mathrm{GL}(4, R)$, Math. Ann. **258** (1981), 113–133.

223. I. Stakgold, *Boundary value problems of mathematical physics*, Vol. 2, MacMillan, New York, 1968.

224. E. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, N.J., 1970.

225. ——, *Analysis in matrix spaces and some new representations of* SL(n, C), Ann. of Math. **86** (1967), 461–490.

226. E. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton Univ. Press, Princeton, N.J., 1971.

227. W. Stinespring, *A semi-simple matrix group is of type* I, Proc. Amer. Math. Soc. **9** (1958), 965–967.

228. R. Strichartz, *Harmonic analysis on hyperboloids*, J. Funct. Anal. **12** (1973), 341–383.

229. ——, *Fourier transforms and non-compact rotation groups*, Indiana Univ. Math. J. **24** (1974), 499–526.

230. ——, *Mean value properties of the Laplacian via spectral theory* (preprint).

231. M. Sugiura, *Unitary representations and harmonic analysis*, Wiley, New York, 1975.

232. C. Sutherland, *Cartan subalgebras, transverse measures, and non-type-*I *Plancherel formulae*, J. Funct. Anal. **60** (1985), 281–308.

233. D. Tartakoff, *Local analytic hypoellipticity of* $\Box_b$ *on nondegenerate Cauchy Riemann manifolds*, Proc. Nat. Acad. Sci. U.S.A. **75** (1978), 3027–3028.

234. M. Taylor, *Pseudodifferential operators*, Princeton Univ. Press, Princeton, N.J., 1981.

235. ——, *Noncommutative microlocal analysis*. I, Mem. Amer. Math. Soc., No. 313, Amer. Math. Soc., Providence, R.I., 1984.

236. ——, *Fourier integral operators and harmonic analysis on compact manifolds*, Proc. Sympos. Pure Math., Vol. 35, Part 2, Amer. Math. Soc., Providence, R.I., 1979, pp. 115–136.

237. ——, *Diffraction of waves by cones and polyhedra*, Proc. Conf. on Analytical and Numer. Approaches to Asymptotic Problems in Analysis, North Holland, Amsterdam, 1981, pp. 235–248.

238. A. Terras, *Fourier analysis on symmetric spaces and applications*. I, Springer-Verlag, New York, 1985.

239. F. Treves, *Introduction to pseudodifferential and Fourier integral operators*, Vols. 1 and 2, Plenum Press, New York, 1980.

240. ——, *Analytic hypoellipticity for a class of pseudodifferential operators*, Comm. Partial Differential Equations **3** (1978), 475–642.

241. M. Tsuchikawa, *On the representations of* SL$(3, C)$. I–III, Proc. Japan Acad. **43** (1967), 852–855; **44** (1968), 127–129; **44** (1968), 130–132.

242. A. Unterberger, *Oscillateur harmonique et operateurs pseudodifferentiels*, Ann. Inst. Fourier (Grenoble) **29** (1979), 201–221.

243. I. Vahutinskii, *Unitary representations of* GL$(3, R)$, Mat. Sb. **75** (1968), 303–320.

244. L. van Hove, *Sur certains representations unitaires d'un groupe infinie de transformations*, Acad. Roy. Belg. Cl. Sci. Mém. 26, no. 6, 1951.

245. B. L. van der Waerden, *Group theory and quantum mechanics*, Springer, New York, 1980.

246. V. Varadarajan, *Lie groups, Lie algebras, and their representations*, Prentice-Hall, New Jersey, 1974.

247. ——, *Infinitesimal theory of representations of semisimple Lie groups*, Harmonic Analysis and Representations of Semisimple Lie Groups, D. Reidel, Boston, 1980.

248. ——, *Harmonic analysis on real reductive groups*, Lecture Notes in Math., No. 576, Springer, 1976.

249. N. Vilenkin, *Special functions and the theory of group representations*, Trans. Math. Monographs, Vol. 22, Amer. Math. Soc., Providence, R.I., 1968.

250. D. Vogan, *Algebraic structure of irreducible representations of semisimple Lie groups*. I, Ann. of Math. **109** (1979), 1–60.

251. ——, *Representations of real reductive Lie groups*, Birkhauser, Boston, 1982.

252. A. Voros, *An algebra of pseudodifferential operators and asymptotics of quantum mechanics*, J. Funct. Anal. **29** (1978), 104–132.

253. N. Wallach, *Harmonic analysis on homogeneous spaces*, Marcel Dekker, New York, 1973.

254. ——, *Symplectic geometry and Fourier analysis*, Math. Sci. Press, Brookline, Mass., 1977.

255. ——, *Cyclic vectors and irreducibility for principal series representations*, Trans. Amer. Math. Soc. **158** (1971), 107–113.

256. G. Warner, *Harmonic analysis on semisimple Lie groups*, Springer, New York, 1972.

257. G. Watson, *A treatise on the theory of Bessel functions*, Cambridge Univ. Press, 1966.

258. A. Weil, *Sur certaines groupes d'operateurs unitaires*, Acta Math. **111** (1964), 143–211.

259. ——, *L'integration dans les groupes topologiques et ses applications*, Hermann, Paris, 1940.

260. A. Weinstein, *Lectures on symplectic manifolds*, CBMS Regional Conf. Ser. in Math., No. 29, Amer. Math. Soc., Providence, R.I., 1977.

261. H. Weyl, *The classical groups*, Princeton Univ. Press, Princeton, N.J., 1956.

262. ——, *Group theory and quantum mechanics*, Dover, New York, 1931.

263. E. Whittaker and G. Watson, *A course of modern analysis*, Cambridge Univ. Press, 1927.

264. E. Wigner, *On unitary representations of the inhomogeneous Lorentz group*, Ann. of Math. **40** (1939), 149–204.

265. J. Wolf, *Foundations of representation theory for semisimple Lie groups*, Harmonic Analysis and Representations of Semisimple Lie Groups, D. Reidel, Boston, 1980.

266. H. Yamabe, *On a deformation of Riemannian structures on compact manifolds*, Osaka J. Math. **12** (1960), 21–37.

267. K. Yosida, *Functional analysis*, Springer, New York, 1965.

268. D. Zelobenko, *Compact Lie groups and their representations*, Trans. Math. Monographs, Vol. 40, Amer. Math. Soc., Providence, R.I., 1973.

269. A. Zygmund, *Trigonometrical series*, Cambridge Univ. Press, 1959.

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This book explores some basic roles of Lie groups in linear analysis, with particular emphasis on the generalizations of the Fourier transform and the study of partial differential equations. It is a valuable resource for both graduate students and faculty, and requires only a background in Fourier analysis and basic functional analysis, plus the first few chapters of a standard text on Lie groups.

The basic method of noncommutative harmonic analysis, a generalization of Fourier analysis, is to synthesize operators on a space on which a Lie group acts from operators on an irreducible representation of the Lie group. Though the general study is far from complete, this book covers a great deal of the progress that has been made on important classes of Lie groups. Also, while the author does treat some general approaches, particularly for compact and nilpotent groups, the emphasis is on particular groups.

Unlike many other books on harmonic analysis, this book focuses on the relationship between harmonic analysis and partial differential equations. The author considers many classical PDEs, particularly boundary value problems for domains with simple shapes, that exhibit noncommutative groups of symmetries. The book also contains detailed work on the harmonic analysis of the Heisenberg group and harmonic analysis on cones.

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