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Introduction to Various Aspects of Degree Theory in Banach Spaces

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Preface

As its title indicates this book is not meant to be an encyclopedic presentation of the present state of degree theory. Likewise, the list of references is not meant to be a complete bibliography. The choice of subjects treated is determined partly by the fact that the book is written from the point of view of an analyst, partly by the desire to throw light on the theory from various angles, and partly the choice is subjective.

A detailed description of the subjects treated is contained in §10 of the following introduction. At this point we mention, together with relevant references, some subject matter belonging to degree theory which is not treated in this book.

(a) The Leray-Schauder degree theory in Banach spaces may be extended to linear convex topological spaces as already noticed by Leray in [**35**]. A complete self-contained treatment may be found in Nagumo's 1951 paper [**43**].

(b) Already, in his 1912 paper [**9**], Brouwer established degree theory for certain finite-dimensional spaces which are not linear. For later developments in this direction and for bibliographies, see, e.g., the books by Alexandroff-Hopf [**2**], by Hurewicz-Wallmann [**29**], and by Milnor [**39**]. In their 1970 paper [**20**] Elworthy and Tromba established a degree theory in certain infinite-dimensional manifolds. See also the systematic exposition by Borisovich-Zvyagin-Sapronov [**6**].

(c) There are various degree theories for mappings which are not of the type treated by Leray and Schauder: a theory by Browder and Nussbaum on "intertwined" maps [**11**] and a theory of "A-proper" maps by Browder and Petryshyn [**12**] in which the degree is not integer-valued but a subset of the integers. See also [**10**]. Moreover, the method used in the Fenske paper [**21**] mentioned in §8 of the following introduction applies also to mappings which are "α-contractions." Various mathematicians considered degree theory for multiple-valued maps. A survey of these generalizations and further bibliographical data may be found in the book [**37**] by Lloyd.

(d) Some important theorems based on degree theory (e.g., those contained in §§2–3 of Chapter 5 and parts of Chapters 8 and 9 of this book) and some generalizations thereof can be derived by the use of cohomology theory in infinite-dimensional spaces. See the paper by Geba and Granas [**22**], in particular Chapter IX, and the book by Granas [**25**], in particular Chapter 11.

(e) The axiomatic treatment of degree theory by Amann and Weiss [**3**].

(f) The "coincidence degree theory" of Mawhin [**38**].

(g) For supplementary reading and further references we mention books by the following authors: Krasnoselskiĭ [**31**], 1956, J. Cronin [**14**], 1964, J. T. Schwartz [**51**], 1965, K. Deimling [**15**], 1974, E. Zeidler [**54**], 1976, F. Browder [**10**], 1976, G. Eisenack–C. Fenske [**19**], 1978, N. G. Lloyd [**37**], 1978. We also mention the paper [**52**], 1980 by H. W. Siegbert which contains an interesting exposition of the history of finite-dimensional degree theory from Gauss via Kronecker to Brouwer.

(h) Ever since it was established by Leray and Schauder [**36**], degree theory in Banach spaces has been an important tool for the treatment of boundary value problems (including periodicity problems) for ordinary and partial differential equations, for integral equations, and for eigenvalue and bifurcation problems. These applications to analysis are not treated in the present book, but the reader will find a number of them discussed in many of the books quoted in part (g) of this preface. See, e.g., Chapters III–VI of [**31**], Chapters II and IV of [**14**], and Chapter 9 of [**37**]. The Cronin book [**14**] discusses, in particular, papers up to 1962 using the "Cesari method." For the role of degree theory in the further development of this method, now usually referred to as the "Alternative Method," the reader may consult the bibliography on pp. 232–234 in *Nonlinear Phenomena in Mathematical Science*, ed. V. Lakshmikantham, Academic Press, 1982.

We finish this preface with a word on the organization of this book. In addition to the introduction the book consists of nine chapters. Each chapter (except Chapter 1) is divided into sections and each section into subsections. (The notation "1.2" in Chapter 3 means subsection 1.2 in that chapter; "(1.2)" refers to a formula in that chapter. However §1.2 refers to §2 in Chapter 1.) Each chapter is followed by notes containing some of the proofs and historical remarks. Finally, there are two Appendixes A and B.

My thanks go to my colleagues and friends Professors Lamberto Cesari, Charles Dolph, and R. Kannan for many encouraging conversations.

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APPENDIX A

The Linear Homotopy Theorem

§1. Motivation for the theorem and the method of proof

1.1. Let E be a Banach space, and let

$$l(x) = x - L(x) \tag{1.1}$$

be a nonsingular L.-S. map, $E \to E$. We want to define an index $j(l)$ which indicates how often E is covered by $l(E)$. Since l is a nonsingular L.-S. map, it maps, according to §§1.11 and 1.12, E onto E in a one-to-one fashion. Therefore we should define $j(l)$ as a number of absolute value 1. In order to decide for which l to define $j(l)$ as $+1$ and for which as -1, we consider the finite-dimensional case.

Let $E = E^n$ be an oriented space of finite dimension n (see subsections 1.21 and 1.22 in Chapter 6 for the concept of orientation). It is then natural to define $j(l)$ as $+1$ if l preserves the orientation (as does, e.g., the identity map I) and as -1 otherwise. By the sections just quoted, this is equivalent to defining

$$j(l) = \begin{cases} +1 & \text{if } \det l > 0, \\ -1 & \text{if } \det l < 0, \end{cases} \tag{1.2}$$

where $\det l$ denotes the determinant of l. (Note that $\det l \neq 0$ since l is supposed to be nonsingular.) Now definition (1.2) in the form stated cannot be used in general Banach spaces since it depends on determinants; however, it can be reformulated in a way to become meaningful for Banach spaces. This is possible on account of the following facts (a) and (b) which are well known in linear algebra, being a consequence of the existence of the Jordan normal form for $n \times n$ matrices (see, e.g., [**26**, §58 and §77]): (a) if L has real eigenvalues greater than 1, say $\lambda_1 > \lambda_2 > \cdots > \lambda_r > 1$, and if for $\rho = 1, \dots, r$, $\mu_\rho = \mu(\lambda_\rho)$ is the generalized multiplicity of λ_ρ, i.e., the dimension of the subspace E^ρ of E^n which consists of those $x \in E^n$ for which

$$(\lambda_\rho I - L)^m x = \theta$$

for some positive integer m, then

$$\operatorname{sign} \det l = (-1)^{\sum_{\rho=1}^{r} \mu_\rho}; \tag{1.3}$$

(b) if L has no real eigenvalues greater than 1, then

$$\operatorname{sign}\det l = +1. \tag{1.4}$$

It follows from (1.3) and (1.4) that definition (1.2) is equivalent to

$$j(l) = \begin{cases} (-1)^{\sum_{\rho=1}^{r} \mu_\rho} & \text{in case (a)}, \\ +1 & \text{in case (b)}. \end{cases} \tag{1.5}$$

As is well known and as will be seen in the following sections, the right member of (1.5) makes sense for an arbitrary Banach space E if l is a nonsingular linear L.-S. map $E \to E$. We note that Definition 1.6 in Chapter 2 appears as a theorem in the original Leray-Schauder theory (see [**36**, II. 11]).

In §2 of this appendix we recall background material from the spectral theory of linear operators M, referring for proofs mainly to the presentation given in [**18**, Chapter VII]. Since in general the spectrum of M contains complex numbers, we deal here with a complex Banach space Z. In order to apply the theory to the given real Banach space E, we have to "complexify" E, i.e., embed it into a complex Banach space Z. This is done in §3. §4 discusses the index $j(l)$ in E as defined by (1.5) and also its definition for the complexification Z of E. Finally, in §5, the linear homotopy theorem (cf. subsection 1.3 in Chapter 2) is proved.

§2. Background material from the spectral theory in a complex Banach space Z

2.1. DEFINITION. Let Z be a complex Banach space, let M be a linear bounded operator on Z, and let I denote the identity map on Z. Then the set of complex numbers λ for which the inverse $R_\lambda(M)$ of $\lambda I - M$ exists as a bounded operator $Z \to Z$ is called the resolvent set of M and will be denoted by $\rho(M)$. The complement $\sigma(M)$ of $\rho(M)$ in the λ-plane is called the spectrum of M. $\sigma(M)$ is closed and bounded. A subset $\sigma_0(M)$ of $\sigma(M)$ is called a spectral set of M if it is open and closed with respect to $\sigma(M)$ [**18**, pp. 566, 567, 572].

2.2. For fixed M the domain of $R_\lambda(M)$ as a function of λ is the set $\rho(M)$, and its range is a subset of the family of bounded linear operators on Z. This family is a Banach space with the linear operations defined in the obvious way and with the norm defined [**18**, pp. 475, 477] by

$$\|M\| = \sup_{\|z\| \le 1} \|M(z)\|. \tag{2.1}$$

This Banach space will be denoted by Z_1, and for M fixed we consider $R_\lambda(M)$ as a map $\rho(M) \to Z_1$. The topology of Z_1 induced by the norm (2.1) is called the uniform operator topology.

2.3. An example for a spectral subset of the spectrum $\sigma(M)$ is $\sigma(M)$ itself, and so is every isolated point of $\sigma(M)$.

2.4. LEMMA. *$\rho(M)$ is open, and $\sigma(M)$ is a nonempty closed bounded set. $R_\lambda(M)$ as a function of λ is analytic at each $\lambda \in \rho(M)$, i.e., at such λ the* *derivative of $R_\lambda(M)$ with respect to λ exists. Obviously*

$$\sum_{n=0}^{\infty} \|M\|^n |\lambda|^{-(n+1)}$$

converges if

$$|\lambda| > \|M\|. \tag{2.2}$$

Moreover for such λ

$$R_\lambda(M) = \sum_{n=0}^{\infty} \frac{M^n}{\lambda^{n+1}}. \tag{2.3}$$

For a proof of this lemma see [**18**, pp. 566, 567].

2.5. LEMMA. *Let Δ be a closed not necessarily bounded subset of $\rho(M)$. Then $R_\lambda(M)$ is bounded on Δ.*

PROOF. On the bounded closed set $\{\lambda \mid |\lambda| \leq \|M\| + 1\}$ the boundedness of $R_\lambda(M)$ follows from its analyticity. But on the set $\{\lambda \mid |\lambda| > \|M\| + 1\}$ it follows from (2.3).

2.6. DEFINITION. $\mathcal{F}(M)$ denotes the family of all complex-valued functions f which are defined and analytic in some open neighborhood U of $\sigma(M)$. (U may depend on f.)

2.7. Let σ_0 be a spectral set of M, and let V_0 be an open set in the complex λ-plane containing σ_0 but no point of $\sigma - \sigma_0$ where $\sigma = \sigma(M)$. Let $f \in \mathcal{F}(M)$ and let U_f be the domain of definition of f. Let Γ_0 and Γ_1 be two rectifiable Jordan curves lying in $V_0 \cap U_f$ oriented in the counterclockwise sense and such that σ_0 lies in the open bounded sets whose boundaries are Γ_0 and Γ_1. Then

$$\int_{\Gamma_0} f(\mu) R_\mu(M)\, d\mu = \int_{\Gamma_1} f(\mu) R_\mu(M)\, d\mu. \tag{2.4}$$

We note that the integrands in (2.4) are operator-valued. For the definition of such integrals and for the proof of the "Cauchy theorem" (2.4), we refer the reader to [**18**, pp. 224–227].

2.8. DEFINITION. For $f \in \mathcal{F}(M)$ we define $f(M)$ by setting

$$f(M) = \frac{1}{2\pi i} \int_\Gamma f(\mu) R_\mu(M)\, d\mu, \tag{2.5}$$

where Γ is a rectifiable Jordan curve of the following properties: (i) $\sigma(M)$ lies in the bounded open domain whose boundary is Γ; (ii) Γ is contained in the domain U_f of definition for f; (iii) Γ is oriented in the counterclockwise sense. The uniqueness of Definition 2.5 follows from Lemma 2.7.

2.9. LEMMA. *Let Γ be a rectifiable Jordan curve satisfying conditions* (i), (ii) *and* (iii) *of subsection* 2.8. *Then*

$$M = \frac{1}{2\pi i} \int_\Gamma \mu R_\mu(M)\, d\mu, \tag{2.6}$$

$$I = \frac{1}{2\pi i} \int_\Gamma R_\mu(M)\, d\mu. \tag{2.7}$$

If, moreover, f and g are elements of $\mathcal{F}(M)$ with domains U_f and U_g resp. and if, in addition to the above requirements, $\Gamma \in U_f \cap U_g$, then

$$f(M) \cdot g(M) = \frac{1}{2\pi i} \int_\Gamma f(\mu) g(\mu) R_\mu(M)\, d\mu. \tag{2.8}$$

For the proof we refer the reader to the proof of Theorem 10, p. 468 in [**18**], which contains the assertions of the present lemma.

2.10. DEFINITION. A bounded linear map $P\colon Z \to Z$ is called a projection if

$$P^2 = P. \tag{2.9}$$

2.11. LEMMA. *Let σ_0 be a spectral set of M. Let Γ_0 be as in Lemma 2.7. Then the operator P_{σ_0} defined by*

$$P_{\sigma_0} = \frac{1}{2\pi i} \int_{\Gamma_0} R_\mu(M)\, d\mu \tag{2.10}$$

is a projection.

PROOF. Let $\sigma_0^* = \sigma - \sigma_0$. By the definition of spectral sets, there exist bounded open sets V_0, V_0^*, U_0, U_0^* such that $\sigma_0 \subset V_0 \subset \overline{V}_0 \subset U_0$, $\sigma_0^* \subset V_0^* \subset \overline{V}_0^* \subset U_0^*$ with $\overline{U}_0$ and $\overline{U}_0^*$ disjoint and such that $\Gamma_0 = \partial V_0$ and $\Gamma^* = \partial V^*$ are rectifiable Jordan curves. Let $\Gamma = \Gamma_0 \cup \Gamma^*$.

Then (2.8) holds if we set

$$f(\mu) = g(\mu) = \begin{cases} 1, & \mu \in U_0, \\ 0, & \mu \in U_0^*. \end{cases} \tag{2.11}$$

But with this choice of f and g we see from (2.10) that the right member of (2.8) equals P_{σ_0} since $f(\mu) = g(\mu) = 0$ for $\mu \in \Gamma_0^*$, and taking account also of (2.5), we see that $f(M) = g(M) = P_{\sigma_0}$. Thus the left member of (2.8) equals $P_{\sigma_0}^2$.

2.12. LEMMA. *Let σ_0 and σ_1 be two disjoint spectral sets for M. Let Γ_0 and P_{σ_0} be defined as in subsection 2.11, and let Γ_1 and P_{σ_1} be defined correspondingly with respect to σ_1. Then*

$$P_{\sigma_0} \cdot P_{\sigma_1} = P_{\sigma_1} \cdot P_{\sigma_0} = \theta_1 \tag{2.12}$$

where $\theta_1 x = \theta$ for every $x \in E$.

PROOF. Since σ_0, σ_1, and $\sigma_2 = \sigma - (\sigma_1 \cup \sigma_2)$ are closed bounded sets, there exist for $i = 0, 1, 2$ bounded open sets V_i, U_i such that $\sigma_i \subset V_i \subset \overline{V}_i \subset U_i$ with the sets $\overline{U}_1, \overline{U}_2, \overline{U}_3$ being pairwise disjoint and such that $\Gamma_i = \partial V_i$ is a rectifiable Jordan curve which we assume to be oriented in the counterclockwise sense. If we set

$$f(\mu) = \begin{cases} 1 & \text{for } \mu \in U_0, \\ 0 & \text{for } \mu \in U_1 \cup U_2, \end{cases} \qquad g(\mu) = \begin{cases} 1 & \text{for } \mu \in U_1, \\ 0 & \text{for } \mu \in U_0 \cup U_2, \end{cases}$$

then with $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$

$$P_{\sigma_0} = \frac{1}{2\pi i} \int_\Gamma f(\mu) R_\mu(M)\, d\mu = f(M),$$

$$P_{\sigma_1} = \frac{1}{2\pi i} \int_\Gamma g(\mu) R_\mu(M)\, d\mu = g(M),$$

and application of (2.8) yields the assertion (2.12) since $f(\mu) \cdot g(\mu) = 0$ for all $\mu \in \Gamma$.

2.13. LEMMA. *Let σ_0 be a spectral set of M, let P_{σ_0} be the projection* (2.10), *and let Z_0 be the range of P_{σ_0}, i.e., $Z_0 = P_{\sigma_0} Z$. Then $MZ_0 \subset Z_0$ and σ_0 is the spectrum of the restriction of M to Z_0.*

For a proof see [**18**, pp. 574, 575].

2.14. LEMMA. *The restriction M_{σ_0} of M to $Z_0 = P_{\sigma_0} Z$ is given by*

$$M_{\sigma_0} = \int_{\Gamma_0} \mu R_\mu(M)\, d\mu \tag{2.13}$$

where Γ_0 is as in subsection 2.7.

PROOF. Let Γ be as in Lemma 2.9, and let V_0 and V be the bounded open sets whose boundaries are Γ_0 and Γ resp. Then M is given by (2.6), i.e., by (2.5) with $f(\mu) = \mu$. Now $z = P_{\sigma_0}(z)$ if and only if $z \in Z_0 = P_{\sigma_0} Z$. Therefore $M(z) = MP_{\sigma_0}(z)$ for $z \in Z_0$. But P_{σ_0} is given by (2.10) or what is the same

$$P_{\sigma_0} = \frac{1}{2\pi i} \int_\Gamma g(\mu) R_\mu(M)\, d\mu, \tag{2.14}$$

with $g(\mu)$ as in (2.11). The assertion (2.13) follows now from (2.5) and (2.14) on application of (2.8).

2.15. So far we assumed of the linear operator M only boundedness or what is the same continuity. In the remainder of this section we require complete continuity, i.e., we make the additional assumption that for any bounded set $\beta \subset E$ the closure of $M(\beta)$ is compact. The definitions and assertions stated below are classical.

Proofs may be found, e.g., in [**18**, p. 577ff.].

2.16. The spectrum $\sigma(M)$ of a completely continuous linear operator M on E is at most countable. If $\sigma(M)$ contains infinitely many points, then 0 is the only accumulation point of $\sigma(M)$. It follows that each $\lambda_0 \in \sigma(M)$ which is different from 0 is a spectral set σ_0 of M. For the projection P_{σ_0} defined by (2.10) we write:

$$P_{\lambda_0} = \frac{1}{2\pi i} \int_{\Gamma_0} R_\mu(M)\, d\mu. \tag{2.15}$$

For Γ_0 we may and will take a circle with center λ_0 and radius smaller than the distance of λ_0 from the rest of $\sigma(M)$. Γ_0 is supposed to be oriented in the counterclockwise sense. We recall that the points of $\sigma(M)$ are also referred to as eigenvalues of M.

2.17. Let $\lambda_0 \in \sigma(M)$ and $\lambda_0 \neq 0$. Then

$$P_{\lambda_0} Z = \{z \in Z \mid (\lambda_0 I - M)^n = 0\} \tag{2.16}$$

for some positive integer n. The dimension $\nu(\lambda_0)$ of the linear subspace $P_{\lambda_0} Z$ of Z is finite and is called the generalized multiplicity of λ_0 as an eigenvalue of M.

2.18. Let $\lambda_0 \in \sigma(M)$ and $\lambda_0 \neq 0$. Then the equation

$$\lambda_0 z - M(z) = \theta \tag{2.17}$$

has nontrivial solutions (i.e., solutions $z \neq 0$) called eigenelements of M to the eigenvalue λ_0. These eigenelements obviously form a linear subspace of E. The dimension $r(\lambda_0)$ of this subspace satisfies the inequality $1 \leq r(\lambda_0) \leq \nu(\lambda_0)$ and is called the multiplicity of λ_0 as an eigenvalue of M.

2.19. LEMMA. *Let M be a completely continuous linear map $Z \to Z$ and suppose that the linear map $m = I - M$ mapping Z onto Z is one-to-one. Then the inverse $n = m^{-1}$ exists and is of the form $m(\xi) = \xi - N(\xi)$, where N is completely continuous on Z.*

PROOF. By assumption m is a linear, continuous, and therefore bounded, one-to-one map of Z onto Z. By a well-known theorem (see, e.g., [**18**, p. 57]) these properties imply that the linear map $n = m^{-1}$ is bounded. Now if $\xi = z - M(z)$, then $n(\xi) = z = \xi + M(z) = \xi + M(n(\xi))$. Since n is bounded, i.e., continuous, and M is completely continuous, the map $N(\xi) = -M(n(\xi))$ is completely continuous.

§3. The complexification Z of a real Banach space E

3.1. The construction of Z from E is analogous to the construction of complex numbers as ordered pairs of real numbers: the elements z of Z are ordered pairs (x, y) of points of the real Banach space E. If $z_i = (x_i, y_i)$, $i = 1, 2$, are two elements of Z, then their sum is defined by $z_1 + z_2 = (x_1 + x_2, y_1 + y_2)$, and if $c = \alpha + i\beta$, α, β real, is a complex number, then cz_1 is defined by

$$cz_1 = (\alpha x_1 - \beta y_1, \alpha y_1 + \beta x_1). \tag{3.1}$$

Finally if as usual $\|\cdot\|$ denotes the norm in E, we set for the point $z = (x, y) \in Z$

$$\|z\|_1 = \sup_{\phi} \|x \cos\phi + y \sin\phi\|. \tag{3.2}$$

3.2. In the case that E is the real line, then Z is the complex plane. The reader may verify that then $\|z\|_1$ is the absolute value of the complex number z.

3.3. The number $\|z\|_1$ defined by (3.2) is a norm, i.e., for points z, z_1, z_2 in Z and complex number c

$$\|z\|_1 \geq 0 \tag{3.3}$$

with the equality holding if and only if $z = \theta$, the zero element of Z, and

$$\|z_1 + z_2\| \leq \|z_1\| + \|z_2\|, \tag{3.4}$$

$$\|cz\|_1 = |c| \cdot \|z\|. \tag{3.5}$$

3.4. With the norm $\|z\|_1$ the linear space Z is complete and thus a Banach space.

The verification of the assertions contained in subsections 3.3 and 3.4 is of course based on the corresponding properties of the Banach space E of which Z is the complexification. We leave this verification to the reader.

3.5. It is easily verified from our definitions that $\|(x, \theta_1)\|_1 = \|x\|$. Thus the linear one-to-one map E into Z given by

$$x \to (x, \theta) \tag{3.6}$$

is norm preserving. Identifying $x \in E$ with $(x, \theta) \in Z$, we consider E as a subset of Z. This identification also allows us to write $\|z\|$ instead of $\|z\|_1$ without ambiguity.

3.6. From the multiplication rule (2.1) we see that $(\theta, y) = i(y, \theta)$ for any $y \in E$ where as usual $i = (0, 1)$, the imaginary unit of the complex plane. Therefore $z = (x, y) = (x, \theta) + (\theta, y) = (x, \theta) + i(y, \theta)$, and by our above identification we see that every $z \in Z$ may be written uniquely in the form

$$z = x + iy, \qquad x \in E,\ y \in E. \tag{3.7}$$

3.7. For later use we note the inequality

$$\tfrac{1}{2}(\|x\| + \|y\|) \leq \|z\| \leq \|x\| + \|y\| \quad \text{for } z = x + iy. \tag{3.8}$$

The right part of this inequality is a restatement of the triangle inequality (3.4). To prove the left part we note that $\|x\| = \|x \cos 0 + y \sin 0\| \leq \sup_\phi \|x \cos \phi + y \sin \phi\| = \|z\|$ by (3.2). In the corresponding way we see that $\|y\| \leq \|z\|$. Thus $\|x\| + \|y\| \leq 2\|z\|$.

3.8. DEFINITION. Let L be a linear bounded operator $E \to E$. Then the map $M\colon Z \to Z$ defined for $z = x + iy$ by

$$M(z) = L(x) + iL(y) \tag{3.9}$$

is called the complex extension of L.

3.9. LEMMA. (i) *M is linear*; (ii) *M is bounded*; (iii) *M is completely continuous if and only if L is completely continuous*; (iv) *if L is continuous, then $m = I - M$ is nonsingular if and only if $l = I - L$ is not singular (I denotes the identity map in the respective space)*; (v) *$m(x) = \theta$ has a solution $z \neq \theta$ if and only if $l(x) = \theta$ has a solution $x \neq \theta$*; (vi) *m is not singular (i.e., $m(Z) = Z$) if and only if m is one-to-one, i.e., if $z = \theta$ is the only root of $m(z) = \theta$.*

PROOF. The proof of (i) consists of an obvious verification.

Proof of (ii). Let μ be a bound for L. Then by (3.9), (3.4), and (3.8)

$$\|M(z)\| \leq \|L(x)\| + \|L(y)\| \leq \mu(\|x\| + \|y\|) \leq 2\mu\|z\|.$$

Proof of (iii). Suppose L is completely continuous. Then, by (ii), M is continuous and it remains to show that if $z_n = x_n + iy_n$ where x_n and y_n are bounded sequences in E, then the $M(z_{n_i})$ converge for some subsequence z_{n_i} of the z_n. Now by (3.8) the boundedness of the z_n implies the boundedness of the x_n and y_n. It therefore follows from the complete continuity of L that for some subsequence n_i of the integers the $L(x_{n_i})$ and $L(y_{n_i})$ converge. But then for $z_{n_i} = x_{n_i} + iy_{n_i}$ the sequence $M(z_{n_i}) = L(x_{n_i}) + iL(y_{n_i})$ converges. The converse is obvious since L is a restriction of M to E.

Proof of (iv): If $z = x + iy$, $\varsigma = \xi + i\eta$, with x, y, ξ, η elements of E, then the equation $m(z) = \varsigma$ is equivalent to the couple of equations $l(x) = \xi$, $l(y) = \eta$. This obviously implies that l maps E onto E if and only if m maps Z onto Z. We obtain a proof of assertion (v) if in the proof of (iv) we set $\varsigma = \xi = \eta = \theta$. Finally assertion (vi) follows from assertions (iv) and (v) in conjunction with part (iv) of Lemma 12 in Chapter 1, the latter being valid also for complex Banach spaces.

3.10. DEFINITION. With E considered as subset of Z (cf. subsection 3.5), the elements of E are called the real elements of Z. If $z = x + iy$ with x, y real, the element $\bar{z} = x - iy$ is called conjugate to z and the map $z \to \bar{z}$ is called a conjugation. A subspace Z_1 of Z is called invariant under conjugation if $z \in Z_1$ implies that $\bar{z} \in Z_1$.

3.11. LEMMA. *Let M be a bounded linear map $Z \to Z$. Then M is the complex extension of a linear bounded map $L\colon E \to E$ if and only if for all $z \in Z$*

$$M(\bar{z}) = \overline{M(z)}. \tag{3.10}$$

PROOF. The necessity is obvious from Definition 3.8. Then let M be a linear bounded map $Z \to Z$ satisfying (3.10). Now for each $x \in E$, $M(x)$ is an element of Z. Therefore

$$M(x) = L_1(x) + iL_2(x), \tag{3.11}$$

where $L_1(x)$ and $L_2(x)$ are elements of E.

It will be sufficient to show that $L_2(x) = \theta$, for then M will be the complex extension of L_1. Now by (3.11) for $z = x + iy$

$$M(z) = M(x) + iM(y) = L_1(x) - L_2(y) + i(L_2(x) + L_1(y)). \tag{3.12}$$

This holds for all $z \in Z$. Therefore

$$M(\bar{z}) = L_1(x) + L_2(y) + i(L_2(x) - L_1(y)). \tag{3.13}$$

On the other hand, we see from (3.12) that

$$\overline{M(z)} = L_1(x) - L_2(y) - i(L_2(x) + L_1(y)). \tag{3.14}$$

By assumption (3.10) the left members of (3.13) and (3.14) are equal. Comparing the imaginary parts of the latter equalities, we see that $L_2(x) = -L_2(x)$, i.e., $L_2(x) = \theta$ as we wanted to prove.

3.12. LEMMA. *Let Z_1 be a linear subspace of Z and let E_1 be the set of real elements in Z_1. It is asserted:*

(i) *if Z_1 is invariant under conjugation, then for each $z = x + iy \in Z_1$ the elements x and y belong to E_1;*

(ii) *if, besides satisfying the assumption of* (i), *Z_1 is finite dimensional, then there exists a base for Z_1 which consists of elements of E_1;*

(iii) *a finite-dimensional subspace Z_1 of Z is invariant under conjugation if and only if there exists a projection $P\colon Z$ onto Z_1, such that*

$$P(\bar{z}) = \overline{P(z)} \quad \textit{for all } z \in Z. \tag{3.15}$$

PROOF. (i) If $z = x+iy \in Z_1(x, y \text{ real})$, then by assumption $\bar{z} = x - iy \in Z_1$. Consequently $x = (z + \bar{z})/2$ and $y = (z - \bar{z})/2i$ lie in $Z_1 \cap E = E_1$.

Proof of (ii). E_1 is obviously a linear subspace of the real space E.

Moreover, if $b_1, b_2, \ldots, b_r$ are linearly independent elements of this subspace, they are also linearly independent as elements of Z_1. For if

$$\sum_{j=1}^{r} c_j b_j = \theta,$$

with $c_j = \alpha_j + i\beta_j$, a_j, β_j real, then

$$\sum_{j=1}^{r} \alpha_j b_j + i \sum_{j=1}^{r} \beta_j b_j = \theta,$$

and thus

$$\sum_{j=1}^{r} \alpha_j b_j = \sum_{j=1}^{r} \beta_j b_j = \theta.$$

Therefore $\alpha_j = \beta_j = 0$, $j = 1, 2, \ldots, r$, on account of the independence of the b_j as elements of E_1. Thus $c_j = \alpha_j + i\beta_j = 0$. This proves the independence of the b_j as elements of Z_1. It follows that the subspace E_1 of E is finite dimensional since Z_1 is finite dimensional. Then let $b_1, \ldots, b_n$ be a base for E_1. Since the b_j are independent also as elements of Z_1, it remains to show that they span Z_1. Now let $z = x + iy \in Z_1$. Then there exist real numbers α_j, β_j such that

$$x = \sum_{j=1}^{n} \alpha_j b_j, \qquad y = \sum_{j=1}^{n} \beta_j b_j, \tag{3.16}$$

and thus

$$z = x + iy = \sum_{j=1}^{n} (\alpha_j + i\beta_j) b_j.$$

Proof of (iii). The coefficients α_j in the first sum in (3.16) are obviously linear in x for x in the finite-dimensional subspace E_1 of E. They are therefore also continuous (see, e.g., [**18**, p. 245]). Consequently, by the Hahn-Banach theorem, these bounded linear functions $\alpha_j = \alpha_j(x)$ can be extended to bounded linear functions $A_j(x)$ defined for all $x \in E$. We now define for $j = 1, 2, \ldots, n$ continuous linear functionals C_j on Z by setting for $z = x + iy$

$$C_j(z) = A_j(x) + iA_j(y). \tag{3.17}$$

Then

$$C_j(\overline{z}) = \overline{C_j(z)} \tag{3.18}$$

since the $A_j(x)$ are real-valued. We now set

$$P(z) = \sum_{j=1}^{n} C_j(z) b_j, \tag{3.19}$$

where the b_j form a real base for Z_1. We then see from (3.18) that (3.15) is satisfied. We now show that P is a projection, i.e., that (2.9) is satisfied. Since the b_j are real, we see from (3.19) and (3.17) that

$$P^2(z) = P(P(z)) = \sum_{j=1}^{n} b_j C_j \left(\sum_{l=1}^{n} C_l(z) b_l \right) = \sum_{j=1}^{n} b_j \sum_{l=1}^{n} C_l(z) C_j(b_l). \tag{3.20}$$

Since $b_l \in E_1$, $C_j(b_l) = A_j(b_l) = \alpha_j(b_l)$. But $\alpha_j(b_l) = 0$ if $j \neq l$, and 1 for $j = l$ as is seen from (3.16) with $x = b_l$. Therefore

$$C_j(b_l) = \begin{cases} 1 & \text{if } j = l, \\ 0 & \text{if } j \neq l, \end{cases}$$

and we see from (3.20) and (3.19) that

$$P^2(z) = \sum_{j=1}^{n} b_j C_j(z) = P(z).$$

This finishes the proof that the invariance of Z_1 under conjugation implies the existence of a projection P satisfying (3.15).

Conversely if such projection P exists, then for $z \in Z_1$ we see that $z = P(z)$, and therefore by (3.15) that $\overline{z} = \overline{P(z)} = P(\overline{z})$ which shows that $\overline{z} \in Z_1$.

§4. On the index j of linear nonsingular L.-S. maps on complex and real Banach spaces

4.1. Definition. Let Z be a complex Banach space. Let

$$m(z) = z - M(z) \tag{4.1}$$

be a nonsingular L.-S. map $Z \to Z$. Then with $\mathcal{R}(\lambda)$ denoting the real part of the complex number λ, the index $j(m)$ of m is defined as follows: if M has no eigenvalues λ with $\mathcal{R}(\lambda) > 1$ we set $j(m) = 1$. If M does have eigenalues λ with $\mathcal{R}(\lambda) \geq 1$, we denote them by $\lambda_1, \lambda_2, \ldots, \lambda_s$ and define

$$j(m) = (-1)^{\sum_{i=1}^{s} \nu_i}, \tag{4.2}$$

where $\nu_i = \nu(\lambda_i)$ is the generalized multiplicity of λ_i as an eigenvalue of M (see subsection 2.17 for the definition of "generalized multiplicity," and see §1 for a motivation for the definition of $j(m)$). Note that by subsection 2.16 the number of eigenvalues λ of M with $\mathcal{R}(\lambda) \geq 1$ is finite and by subsection 2.17 the generalized multiplicity by $\nu(\lambda)$ of an eigenvalue λ of M is finite.

4.2. DEFINITION. Let E be a real Banach space, and let

$$l(x) = x - L(x) \tag{4.3}$$

be a nonsingular L.-S. map $E \to E$. An eigenvalue λ_0 of L is then a real number such that for some point $x \in E$ different from θ

$$L(x) = \lambda_0 x. \tag{4.4}$$

If Z is the complexification of E and M the complex extension of L, it is clear from §3 that λ_0 is an eigenvalue of L if and only if λ_0 is a real eigenvalue of M. We define the generalized multiplicity of an eigenvalue $\lambda_0 \neq 0$ of L as the dimension $\mu(\lambda_0)$ of the space

$$E_{\lambda_0} = \{x \in E \mid (\lambda_0 I - L)^n x = \theta\} \tag{4.5}$$

for some integer $n \geq 1$. $\mu(\lambda_0)$ is finite, for it is clear that $\mu(\lambda_0) \leq \nu(\lambda_0)$, the generalized multiplicity of λ_0 as an eigenvalue of M (see subsection 2.17; actually $\mu(\lambda_0) = \nu(\lambda_0)$ as we will see in subsection 4.11).

4.3. DEFINITION. Let l and L be as in subsection 4.2. We define the index $j(l)$ to be 1 if L has no eigenvalues > 1. If L has eigenvalues > 1, we denote them by $\lambda_1 > \lambda_2 > \cdots > \lambda_s$ and define the index $j(l)$ by

$$j(l) = (-1)^{\sum_{\rho=1}^{s} \mu_\rho}, \tag{4.6}$$

where $\mu_\rho = \mu(\lambda_\rho)$, $\rho = 1, 2, \ldots, s$. Note that this definition agrees with (1.5), which was shown to agree in the finite-dimensional case with (1.2). Note also that $\lambda = 1$ is not an eigenvalue of L since l is not singular.

4.4. DEFINITION. A linear map l of the real Banach space E into itself is said to be type I^- if it can be constructed as follows: let E^1 be a one-dimensional subspace of E and let E^2 be a complementary subspace such that every $x \in E$ can be written in the form (cf. Lemma 4 of Chapter 1)

$$x = x_1 + x_2, \qquad x_1 \in E^1, \; x_2 \in E^2. \tag{4.7}$$

Then l maps $x = x_1 + x_2$ into $-x_1 + x_2$ (cf. §8 of the introduction). Maps of type I^- in a complex Banach space Z are defined correspondingly.

Note that $l(x) = x - 2x_1$, i.e., $l(x)$ is of the form (4.3) with $L(x) = 2x_1$. Since x_1 lies in the one-dimensional space E^1, $L(x)$ is completely continuous. Thus l is an L.-S. map. Since obviously l maps E onto E, the map l is not singular. Thus $j(l)$ is defined for a map l of type I^-.

4.5. LEMMA.

$$j(l) = \begin{cases} +1 & \textit{if } l = I, \textit{ the identity map}, \\ -1 & \textit{if } l \textit{ is of type } I^-. \end{cases} \tag{4.8}$$

PROOF. If $l = I$ then the $L(x)$ in (4.3) equals θ for all x. Therefore (4.4) for $x \neq \theta$ is satisfied only with $\lambda_0 = 0$. Thus L has no eigenvalues ≥ 1 which by Definition 4.3 proves the first part of assertion (4.8). Suppose now l is of the I^- type. Then with the notation used in (4.7), $l(x) = -x_1 + x_2 = x - 2x_1$, i.e., $L(x) = 2x_1$ by (4.3) and the eigenvalue equation (4.4) reads $2x_1 = \lambda_0 x = \lambda_0(x_1 + x_2)$. This implies $x_2 = \theta$ and $\lambda_0 = 2$. Thus 2 is the only eigenvalue of L, and

$$j(l) = (-1)^{\mu(2)}. \tag{4.9}$$

Here $\mu(2)$ is the dimension of the space given by (4.5) with $\lambda_0 = 2$ and $L(x) = 2x_1$. Now for $i = 1, 2$ let P_i be the projection $x \to x_i$ (cf. (4.7)). Then $\lambda_0 I - L = 2(P_1 + P_2) - 2P_1 = 2P_2$. Thus, by (4.5), E_2 consists of those x for which $P_2^n x = 0$ for some integer $n \geq 1$. But for these integers $P_2^n = P_2$ since P_2 is a projection, and we see that $E_2 = \{x \in E \mid P_2(x) = \theta\} = E^1$. Thus $\mu(2) = \dim E^1 = 1$. This together with (4.9) proves the second part of the assertion (4.8).

Since L.-S. homotopy (see subsection 1.2 of Chapter 2) is transitive, it follows from Lemma 4.5 that the linear homotopy theorem (stated in subsection 1.3 of Chapter 2) is equivalent to the following one.

4.6. THEOREM. *Let l_0 and l_1 be two linear nonsingular L.-S. maps $E \to E$. Then l_0 and l_1 are linearly L.-S. homotopic (see Definition 1.2 in Chapter 2) if and only if*

$$j(l_0) = j(l_1). \tag{4.10}$$

Now for questions of continuity it is preferable to deal with the complexification Z of E and the complex extension M of the operator L given by (4.3). For if M changes continuously, a real eigenvalue of M may become complex. But in this case the eigenvalue of L as an operator on E "disappears" since the concept of a complex eigenvalue makes no sense in the real space E. We therefore state the following theorem whose proof will be given in §5.

4.7. THEOREM. *Let l_0 and l_1 be as in Theorem 4.6. Let Z be the complexification of E, and for $i = 0, 1$ let M_i be the complex extension of $L_i = I - l_i$ on E, and let $m_i(z) = z - M_i(z)$. It is asserted that the equality*

$$j(m_0) = j(m_1) \tag{4.11}$$

(see Definition 4.1) holds if and only if there exists an L.-S. linear homotopy $m(x,t) = z - M(z,t)$, $0 \leq t \leq 1$, between $m_0(z) = m(z,0)$ and $m_1(z) = m(z,1)$ which satisfies for $z \in Z$ and $t \in [0,1]$

$$M(\bar{z}, t) = \overline{M(z,t)}. \tag{4.12}$$

The remainder of §4 is devoted to showing

4.8. THEOREM. *Theorems 4.6 and 4.7 are equivalent.*

The proof of this theorem requires some lemmas.

4.9. LEMMA. *Let E and Z be as in Theorems 4.7 and 4.8. Let L be a continuous linear map $E \to E$, and let M be its complex extension. Then* (i) *the spectrum $\sigma(M)$ of M is symmetric with respect to the real axis of the complex λ-plane, and* (ii) *for $\lambda \in \rho(M)$*

$$R_{\overline{\lambda}}(M)(z) = \overline{R_\lambda(M)(\overline{z})} \tag{4.13}$$

(*see Definition* 2.1).

PROOF. (i) $\sigma(M)$ and the resolvent set $\rho(M)$ are disjoint subsets of the λ-plane whose union is the whole λ-plane. Therefore assertion (i) is equivalent to the assertion that $\rho(M)$ is symmetric to the real λ-axis. Let $\lambda_0 \in \rho(M)$.

We have to prove

$$\overline{\lambda}_0 \in \rho(M). \tag{4.14}$$

Now by the definition of $R_\lambda(M)$

$$(\lambda_0 I - M)R_{\lambda_0}(M)(z) = z \tag{4.15}$$

for all $z \in Z$. But by (3.10) for $u \in Z$

$$\overline{(\lambda_0 I - M)(u)} = (\overline{\lambda}_0 I - M)(\overline{u}).$$

Applying this relation with $u = R_{\lambda_0}(M)(z)$ we see from (4.15) that

$$(\overline{\lambda}_0 I - M)\overline{R_{\lambda_0}(M)(z)} = \overline{z}.$$

Replacing z here by $\overline{z}$ we see that

$$(\overline{\lambda}_0 I - M)\overline{R_{\lambda_0}(M)(\overline{z})} = z$$

for all $z \in Z$. This shows that

$$S(z) = \overline{R_{\lambda_0}(M)(\overline{z})} \tag{4.16}$$

is a right inverse of $(\overline{\lambda}_0 I - M)$. But interchanging the factors in (4.15) we conclude that $S(z)$ is also a left inverse of $\overline{\lambda}_0 I - M$. Thus S is an inverse of $\overline{\lambda}_0 I - M$. Since S is bounded, the assertion (4.14) follows by definition of $\rho(M)$.

Proof of (ii). By definition, $R_{\overline{\lambda}_0}(M)$ is the inverse of $\overline{\lambda}_0 I - M$. Therefore by (4.16)

$$R_{\overline{\lambda}_0}(M)(z) = S(z) = \overline{R_{\lambda_0}(M)(\overline{z})}.$$

This proves assertion (ii).

4.10. LEMMA. *Let E, Z, L, M be as in Lemma 4.9. Let λ_0 be an isolated point of the spectrum $\sigma(M)$ of M such that by Lemma 4.9 $\overline{\lambda}_0$ is also an isolated point of $\sigma(M)$. Let c_0 be a counterclockwise-oriented circle with center λ_0 and radius r_0 where r_0 is such that the closure of the circular disk $B(\lambda_0, r_0)$ contains no point of $\sigma(M)$ other than λ_0. Let c_0' be the counterclockwise-oriented circle with center $\overline{\lambda}_0$ and radius r_0. Let P_{λ_0} and $P_{\overline{\lambda}_0}$ be the linear operators defined by*

$$P_{\lambda_0} = \frac{1}{2\pi i}\int_{c_0} R_\mu(M)\,d\mu, \qquad P_{\overline{\lambda}_0} = \frac{1}{2\pi i}\int_{c_0'} R_\mu(M)\,d\mu. \tag{4.17}$$

Then the ranges $Z_{\lambda_0} = P_{\lambda_0} Z$ and $Z_{\overline{\lambda}_0} = P_{\overline{\lambda}_0} Z$ are conjugate to each other, i.e., if z varies over Z_{λ_0}, then $\overline{z}$ varies over $Z_{\overline{\lambda}_0}$.

PROOF. λ_0 and $\overline{\lambda}_0$ are spectral sets (see subsections 2.1 and 2.3). Therefore, by subsection 2.11, P_{λ_0} and $P_{\overline{\lambda}_0}$ are projections. We will show first our lemma follows from the relation

$$P_{\overline{\lambda}_0}(\overline{z}) = \overline{P_{\lambda_0}(z)} \quad \text{for all } z \in Z. \tag{4.18}$$

Indeed, since P_{λ_0} is a projection, $z \in Z_{\lambda_0} = P_{\lambda_0} Z$ if and only if $z = P_{\lambda_0}(z)$ or $\overline{z} = \overline{P_{\lambda_0}(z)}$, or, by (4.18), $\overline{z} = P_{\overline{\lambda}_0}(\overline{z})$. But since $P_{\overline{\lambda}_0}$ is a projection, the last equality is a necesary and sufficient condition for $\overline{z}$ to be an element of $P_{\overline{\lambda}_0} Z = Z_{\overline{\lambda}_0}$.

Thus for the proof of our lemma it remains to verify (4.18). For this purpose we set $\mu = \overline{\lambda}_0 + r_0 e^{i\phi}$ and $\lambda = \overline{\mu} = \lambda_0 + r_0 e^{-i\phi}$ such that $d\mu = i r_0 e^{i\phi}\, d\phi$ and $d\lambda = -i r_0 e^{-i\phi}\, d\phi$. We then see from (4.17) and (4.13) that

$$\begin{aligned} P_{\overline{\lambda}_0}(\overline{z}) &= \frac{1}{2\pi}\int_0^{2\pi} R_\mu(M)(\overline{z}) r_0 e^{i\phi}\, d\phi = \frac{1}{2\pi}\int_0^{2\pi} \overline{R_\lambda(M)(z)} r_0 e^{i\phi}\, d\phi \\ &= \overline{\frac{1}{2\pi}\int_0^{2\pi} R_\lambda(M)(z) r_o e^{-i\phi}\, d\phi} = \overline{-\frac{1}{2\pi i}\int_0^{2\pi} R_\lambda(M)(z)(-i r_0 e^{-i\phi})\, d\phi} \\ &= \overline{-\frac{1}{2\pi i}\int_{-c_0} R_\lambda(M)(z)\, d\lambda} = \overline{\frac{1}{2\pi i}\int_{c_0} R_\lambda(M)(z)\, d\lambda} = \overline{P_{\lambda_0}(z)}. \end{aligned}$$

This proves (4.18), and thus our lemma.

4.11. COROLLARY TO LEMMA 4.10. (i) *The spaces Z_{λ_0} and $Z_{\overline{\lambda}_0}$ have the same dimension; thus $\nu(\lambda_0) = \nu(\overline{\lambda}_0)$ (cf. subsection 2.17).*

(ii) *If λ_0 is real, then Z_{λ_0} is invariant under conjugation.*

(iii) *If λ_0 is real, then the generalized multiplicity $\nu(\lambda_0)$ of λ_0 as an eigenvalue of M equals the generalized multiplicity $\mu(\lambda_0)$ of λ_0 as an eigenvalue of L (cf. subsections 2.17 and 4.2).*

PROOF. Assertions (i) and (ii) are obvious consequences of Lemma 4.10. As to assertion (iii) we noted already in subsection 4.2 that $\mu(\lambda_0) \le \nu(\lambda_0)$. It remains to prove that

$$\nu(\lambda_0) \le \mu(\lambda_0). \tag{4.19}$$

Now it follows from assertion (ii) of our corollary and Lemma 3.12 that $Z_{\lambda_0} = P_{\lambda_0} Z$ has a base $b_1, b_2, \ldots, b_r$ consisting of elements of E. Then

$$r = \nu(\lambda_0),$$

since r and $\nu(\lambda_0)$ both equal the dimension of Z_{λ_0} (cf. subsection 2.17). Since each $b_j \in Z_{\lambda_0}$, we have (also by subsection 2.17)

$$(\lambda_0 I - M)^n b_j = \theta, \qquad j = 1, 2, \ldots, \nu(\lambda_0) \tag{4.20}$$

for some n. But by definition (3.9) of the extension M of L, we see that $M(b_j) = L(b_j)$ since $b_j \in E$. Thus by (4.20) $(\lambda_0 I - L)^n b_j = 0$, $j = 1, 2, \ldots, \nu(\lambda_0)$. The asserted inequality (4.19) now follows from the definition of $\mu(\lambda_0)$ given in 4.2 since λ_0 is real.

4.12. LEMMA. *We use the notation of Lemma* 4.10. *We assume now that the linear map* $l(x) = x - L(x)$ *is a nonsingular L.-S. map* $E \to E$. *Let* $m(z)$ *be its complex extension. Then*

$$j(m) = j(l). \tag{4.21}$$

PROOF. By Lemma 3.9 the nonsingularity of l implies that $m(z) = z - M(z)$ is a nonsingular L.-S. map $Z \to Z$. Thus both members of (4.21) are defined. To prove their equality we note first: if λ_0 is a complex eigenvalue of M, then so is $\overline{\lambda}_0$ (see Lemma 4.9), and $\nu(\lambda_0) = \nu(\overline{\lambda}_0)$ by part (i) of Corollary 4.11. Thus $\nu(\lambda_0) + \nu(\overline{\lambda}_0)$ is an even number. Therefore in Definition 4.1 (see in particular (4.2)) we may omit those $\nu_i = \nu(\lambda_i)$ for which λ_i is complex. But for a real eigenvalue λ_i of M we know from part (i) of the corollary 4.11 that $\nu(\lambda_i) = \mu(\lambda_i)$, and the definition of $j(l)$ in subsection 4.3 (see in particular (4.6)) shows that $j(m) = j(l)$.

4.13. We are now ready to prove Theorem 4.8. In the present section we show that Theorem 4.7 implies Theorem 4.6.

(a) Suppose (4.10) is satisfied. We have to show that l_0 and l_1 are linearly L.-S. homotopic. Let m_0 and m_1 be the complex extensions of l_0 and l_1 resp. Then, by Lemma 4.12, the equality (4.10) implies (4.11). Therefore, by Theorem 4.7 there exists a homotopy $m(z,t) = z - M(z,t)$ of the properties asserted in that theorem. But by Lemma 3.11 the relation (4.12) implies that for each $t \in [0,1]$ the map $M(z,t)$ is the complex extension of a bounded linear map $L(x,t): E \to E$ which by Lemma 3.9 is completely continuous and such that $l(x,t) = x - L(x,t)$ is nonsingular. This $l(x,t)$ gives the desired homotopy between l_0 and l_1.

(b) Assume now that l_0 and l_1 are linearly L.-S. homotopic. We have to prove (4.10). By assumption there exists a linear L.-S. homotopy $l(x,t) = x - L(x,t)$ connecting l_0 with l_1. For each $t \in [0,1]$ let $m(x,t) = z - M(z,t)$ be the complex extension of $l(x,t)$. Then $m(z,t)$ is an L.-S. homotopy connecting $m_0(z) = m(z,0)$ with $m_1 = m(z,1)$ which by Lemma 3.11 satisfies (4.12). The assertion (4.10) follows now from Theorem 4.7 in conjunction with Lemma 4.12.

4.14. In this section we show that Theorem 4.6 implies Theorem 4.7.

(a) For $i = 0,1$ let $m_i(z) = z - M_i(z)$ be a linear nonsingular L.-S. map $Z \to Z$ which is the complex extension of the linear nonsingular L.-S. map $l_i(x) = x - L_i(x): E \to E$, and suppose that (4.11) holds. Now by Lemma 4.12 the relation (4.11) implies (4.10). Consequently by Theorem 4.6 there exists a linear L.-S. homotopy $l(x,t) = x - L(x,t)$ connecting $l_0 = l(x,0)$ with $l_1 = l(x,1)$. Then the complex extension $m(z,t) = z - M(z,t)$ of $l(x,t)$ is a linear L.-S. homotopy which connects m_0 with m_1 and which, by Lemma 3.11, satisfies (4.12).

(b) Again, for $i = 0,1$, let $m_i(z) = z - M_i(z)$ be a nonsingular linear L.-S. map $Z \to Z$ which is the extension of a linear nonsingular L.-S. map $l_i(x) = x - L_i(x): E \to E$. Suppose now there exists a linear L.-S. homotopy which connects m_0 with m_1 and satisfies (4.12). From the latter relation and from Lemma 3.11 we conclude that there exists a linear L.-S. homotopy which connects l_0 and l_1 and that therefore by Theorem 4.6, $j(l_0) = j(l_1)$. The asserted relation (4.11) follows now from Lemma 4.12.

§5. Proof of the linear homotopy theorem

5.1. The linear homotopy theorem (see subsection 1.3 of Chapter 2) is, as already stated in the last paragraph of subsection 4.5, equivalent to Theorem 4.6, which in turn, by Theorem 4.8, is equivalent to Theorem 4.7. It is thus sufficient to prove Theorem 4.7. We start by proving that if a homotopy $m_t(z) = m(x, t)$ of the properties stated in that theorem exists, then (4.11) is true. To do this we will show that $j(m_t)$ is constant for $t \in [0, 1]$. Since $j(m_t)$ is integer-valued it will be sufficient to prove the "continuity" of that number or, more precisely, to prove the following theorem.

5.2. THEOREM. *Let Z be the complexification of the real Banach space E, and let $m_0(z) = z - M_0(z)$ be the complex extension of the nonsingular linear L.-S. map $l_0(x) = x - L_0(x)$ mapping E into E. Then there exists a positive number ε_0 of the following property: if $m(z) = z - M(z)$ is the complex extension of a linear nonsingular L.-S. map $l\colon E \to E$ and if*

$$\|m - m_0\| = \|M - M_0\| < \varepsilon < \varepsilon_0, \tag{5.1}$$

then

$$j(m) = j(m_0). \tag{5.2}$$

For the proof we need the following well-known elementary lemma.

5.3. LEMMA. *Let Z be an arbitrary complex Banach space, and let n_0 be a bounded linear map $Z \to Z$ which has a bounded everywhere-defined inverse n_0^{-1}. Then every linear bounded map $n\colon Z \to Z$ satisfying*

$$\|n - n_0\| < \|n_0^{-1}\|^{-1} \tag{5.3}$$

has a bounded everywhere-defined inverse. (As the following proof shows, the lemma is also valid in a real Banach space E.)

PROOF. Suppose first that $n_0 = I$. Then the assumption (5.3) reads

$$\|n - I\| < 1, \tag{5.4}$$

from which it easily follows that

$$n_1 = \sum_{i=0}^{\infty} (I - n)^i \tag{5.5}$$

converges (with $(I - n)^0 = I$). Moreover, writing $n = I - (I - n)$, one sees that $nn_1 = n_1 n = I$, and thus $n_1 = n^{-1}$.

In the general case we see from (5.3) that

$$\|I - n_0^{-1} n\| = \|n_0^{-1}(n_0 - n)\| \leq \|n_0^{-1}\| \, \|n_0 - n\| < 1. \tag{5.6}$$

Thus (5.4) is satisfied if n is replaced by $n_0^{-1}n$. Thus $n_2 = (n_0^{-1}n)^{-1}$ exists and $(n_2 n_0^{-1})n = n_2(n_0^{-1}n) = I$. This shows that $n_2 n_0^{-1}$ is a left inverse of n. Similarly, we see from the inequality $\|I - nn_0^{-1}\| < 1$, which is proved the same way as (5.6), that n has a left inverse. But the existence of a left and a right inverse proves the existence of a unique inverse.

5.4. PROOF OF THEOREM 5.2. We distinguish two cases: (A) M_0 has no eigenvalues λ with real part $\mathcal{R}(\lambda) \geq 1$; (B) M_0 does have such eigenvalues.

Proof in case (A). We define the subset $\Delta = \Delta(M_0)$ of the λ-plane by $\Delta = \{\lambda \mid \mathcal{R}(\lambda) \geq 1\}$. Then Δ is a closed subset of $\rho(M_0)$. Therefore by Lemma 2.5 there exists a positive number μ_0 such that

$$\|R_\lambda(M_0)\| < \mu_0/2 \quad \text{for } \lambda \in \Delta. \tag{5.7}$$

We now claim that if M is a linear completely continuous map $Z \to Z$ satisfying

$$\|M - M_0\| < \tfrac{1}{2}(\mu_0/2)^{-1}, \tag{5.8}$$

then

$$\Delta \subset \rho(M). \tag{5.9}$$

Indeed if $n_0(z) = \lambda z - M_0(z)$ and $n(z) = \lambda z - M(z)$ with $\lambda \in \Delta$, then by (5.8), by (5.7), and by the definition of R_λ

$$\|n - n_0\| = \|M - M_0\| < (\mu_0/2)^{-1} < \|R_\lambda(M_0)\|^{-1} = \|n_0^{-1}\|^{-1}. \tag{5.10}$$

But by Lemma 5.3 this inequality implies that $n^{-1} = R_\lambda(M)$ exists, as a bounded operator, i.e., that $\lambda \in \rho(M)$ as asserted.

It is now easy to see that (5.8) implies the asserted equality (5.2). Indeed $j(m_0) = 1$ by Definition 4.1. But by definition of Δ we see from (5.9) that no eigenvalue λ of M has a real part ≥ 1. Thus, again by Definition 4.1, $j(m) = 1$.

Proof in case (B). $\sigma(M_0)$ is closed and bounded (see subsection 2.1). Moreover by Lemma 3.9 M_0 is completely continuous and therefore (see subsection 2.16) zero is the only accumulation point of $\sigma(M)$. It follows that $\sigma(M_0)$ contains only a finite number of points λ with $R(\lambda) \geq 1$. Moreover, if we denote them by $\lambda_1^0, \lambda_2^0, \ldots, \lambda_s^0$, then there exists a positive number r of the following properties: if for $i = 1, 2, \ldots, s$, d_i is the open circular disk with center λ_i^0 and radius r, then (a) λ_i^0 is the only eigenvalue of M_0 in the closed disk $\bar{d}_i$; (b) the $\bar{d}_i$ are disjoint; (c) for those i for which $\mathcal{R}(\lambda_i^0) > 1$ the disk $\bar{d}_i$ is contained in the open half plane $\mathcal{R}(\lambda) > 1$; (d) for those i for which λ_i^0 is not real the disk $\bar{d}_i$ does not intersect the real axis; (e) for those $\lambda \in \sigma(M_0)$ for which $\mathcal{R}(\lambda) < 1$ we have $\mathcal{R}(\lambda) < 1 - r$; (f) $0 < r < 1$. In our present case (B) we define

$$\Delta = \Delta(M_0) = \{\lambda \mid \mathcal{R}(\lambda) \geq 1 - r\} - \bigcup_{j=1}^{s} d_j. \tag{5.11}$$

Then Δ is a closed subset of $\rho(M_0)$. As in case (A) we conclude from this the existence of a positive μ_0 such that (5.7) holds and for completely continuous linear M satisfying (5.8) the inclusion (5.9) holds.

To prove that (5.1) with small enough ε_0 implies the assertion (5.2), we choose our notation in such a way that $\lambda_1^0, \lambda_2^0, \ldots, \lambda_r^0$ are real and $\lambda_{r+1}^0, \lambda_{r+2}^0, \ldots, \lambda_s^0$ are complex, $0 \leq r \leq s$ ($r = 0$ means that there are no real numbers among the λ_i^0, and $r = s$ that all of them are real; in any case $s \geq 1$ by assumption). Since M_0 is the complex extension of a completely continuous operator $L_0: E \to E$, we see from Lemma 4.9 and the corollary to Lemma 4.11 that if λ is one of the complex eigenvalues $\lambda_{r+1}^0, \ldots, \lambda_0^s$ ofM_0, then so is $\overline{\lambda}$ and $\nu^0(\lambda) = \nu^0(\overline{\lambda})$, where $\nu^0(\lambda)$ denotes the generalized multiplicity of λ as an eigenvalue of M_0. Consequently by definition (4.2) of $j(m)$

$$j(m_0) = (-1)^{\sum_{\rho=1}^{r} \nu(\lambda_\rho^0)}, \tag{5.12}$$

where it is understood that the sum stands for zero if $r = 0$. Suppose now that M is the complex extension of a linear completely continuous operator $L: E \to E$ and that M satisfies (5.8). Then (5.9) is true and therefore by definition (5.11) of Δ

$$\sigma(M) \cap \{\lambda \mid \mathcal{R}(\lambda) \geq 1 - r\} \subset \bigcup_{j=1}^{s} d_j = \bigcup_{j=1}^{r} d_j + \bigcup_{j=r+1}^{s} d_j.$$

Now by property (d) of the radius r of the disks d_j, these disks do not intersect the real axis for $j = r + 1, \ldots, s$. Thus the eigenvalues of M contained in these disks are complex, and the argument used to establish (5.12) shows that

$$j(m) = (-1)^{\sum \nu(\lambda)},$$

where the sum is extended over those eigenvalues λ of M which lie in one of the d_j for $j = 1, 2, \ldots, r$ and where $\nu(\lambda)$ denotes the generalized multiplicity of λ as an eigenvalue of M. Thus if $\lambda_1^j, \lambda_2^j, \ldots, \lambda_{r_j}^j$ are those eigenvalues of M which lie in d_j for $j = 1, 2, \ldots, r$, then

$$j(m) = (-1)^{\sum_{j=1}^{r} \sum_{i=1}^{r_j} \nu(\lambda_i^j)}. \tag{5.13}$$

Comparison of (5.12) with (5.13) shows that our assertion (5.2) will be proved once it is shown that

$$\nu(\lambda_j^0) = \sum_{i=1}^{r_j} \nu(\lambda_i^j), \qquad j = 1, 2, \ldots, r. \tag{5.14}$$

We recall that here the multiplicity ν at the left refers to M_0 while the ν's at the right refer to M.

Now by subsection 2.17 and (2.15)

$$\nu(\lambda_j^0) = \dim P_j^0 Z, \qquad j = 1, 2, \ldots, r, \tag{5.15}$$

where

$$P_j^0 = \frac{1}{2\pi i} \int_{\partial d_j} R_\mu(M_0)\, d\mu \tag{5.16}$$

with the circle ∂d_j oriented in the counterclockwise sense. In the same way we see that

$$\nu(\lambda_i^j) = \dim P_i^j, \qquad i = 1, 2, \ldots, r_j, \tag{5.17}$$

where

$$P_i^j = \frac{1}{2\pi i} \int_{\partial d_i^j} R_\mu(M)\, d\mu \tag{5.18}$$

and where d_i^j is a circular disk with center λ_i^j and a radius so small that the closures $\overline{d}_i^j$ are disjoint and lie in d_j.

The assertion (5.14) now reads

$$\dim P_j^0 Z = \sum_{i=1}^{r_j} \dim P_i^j Z, \qquad j = 1, 2, \ldots, r. \tag{5.19}$$

If we set

$$P^j = \frac{1}{2\pi i} \int_{\partial d_j} R_\mu(M)\, d\mu, \qquad j = 1, 2, \ldots, r, \tag{5.20}$$

we see from the "Cauchy theorem" (2.4) that

$$P^j(Z) = \sum_{i=1}^{r_j} P_i^j Z \tag{5.21}$$

and from this equality together with Lemma 2.12 that $P^j Z$ is the direct sum of the spaces $P_1^j Z, P_2^j Z, \ldots, P_{r_j}^j Z$, and that therefore

$$\dim P^j Z = \sum_{i=1}^{r_j} \dim P_i^j Z. \tag{5.22}$$

Thus the assertion (5.19) may be written as

$$\dim P_j^0 Z = \dim P^j Z, \qquad j = 1, 2, \ldots, r. \tag{5.23}$$

We will prove that if μ_0 is as in (5.7), if $\varepsilon_1 = (\mu_0/2)^{-1}$ (cf. (5.8)), if

$$\varepsilon_2 = \mu_0^{-2} \min_{j=1,2,\ldots,r} \|P_j^0\|^{-1}, \tag{5.24}$$

and if

$$\varepsilon_0 = \min(\varepsilon_1/2, \varepsilon_2), \tag{5.25}$$

then (5.1) implies (5.23) (and thus (5.2)). For the proof we need two lemmas; the first is related to Lemma 5.3 while the second is due to J. Schwartz [**50**, p. 424].

5.5. LEMMA. *Let $n_0(z)$ be as in Lemma* 5.3, *and let n be a linear bounded map $Z \to Z$ satisfying*

$$\|n - n_0\| < (2\|n_0^{-1}\|)^{-1}. \tag{5.26}$$

Then n^{-1} exists and satisfies

$$\|n^{-1} - n_0^{-1}\| < 2\|n_0^{-1}\|^2 \|n - n_0\|. \tag{5.27}$$

5.6. LEMMA. *Let Q_0 and Q_1 be projections $Z \to Z$. Suppose that*

$$\|Q_1 - Q_0\| \leq \tfrac{1}{2}\|Q_0\|^{-1}. \tag{5.28}$$

Then the dimension δ_0 of $Q_0 Z$ equals the dimension δ_1 of $Q_1 Z$.

5.7. We postpone the proof of the two preceding lemmas and prove that (5.1) implies (5.23). Let $n_0(z) = \mu z - M_0(z)$ and $n(z) = \mu z - M(z)$. Then by (5.8) and (5.7)

$$\begin{aligned}\|n(z) - n_0(z)\| &= \|M(z) - M_0(z)\| < \tfrac{1}{2}(\mu_0/2)^{-1}\\ &< \tfrac{1}{2}\|R_\mu(M_0)\|^{-1} = \tfrac{1}{2}\|n_0^{-1}\|^{-1},\end{aligned}$$

i.e., (5.26) is satisfied. Therefore by Lemma 5.5 the inequality (5.27) holds:

$$\|R_\mu(M) - R_\mu(M_0)\| < 2\|R_\mu(M_0)\|^2\|M - M_0\|.$$

Thus by (5.1), (5.25) and (5.24)

$$\|R_\mu(M) - R_\mu(M_0)\| < 2\|R_\mu(M_0)\|^2\mu_0^{-2} \min_{j=1,2,\ldots,r} \|P_j^0\|^{-1},$$

and by (5.7)

$$\|R_\mu(M) - R_\mu(M_0)\| < \tfrac{1}{2} \min_{j=1,2,\ldots,r} \|P_j^0\|^{-1}. \tag{5.29}$$

But by (5.20) and (5.16)

$$\begin{aligned}\|P^j - P_j^0\| &= \frac{1}{2\pi}\left\|\int_{\partial d_j} R_\mu(M) - R_\mu(M_0)\, d\mu\right\|\\ &\le \frac{1}{2\pi}\int_0^{2\pi} \|R_\mu(M) - R_\mu(M_0)\|r|e^{i\phi}|\, d\phi.\end{aligned}$$

Therefore by (5.29) (recalling that $0 < r < 1$) we see that $\|P^j - P_j^0\| < \frac{1}{2}\|P_j^0\|^{-1}$. But by Lemma 5.6 this inequality implies the assertion (5.23).

It remains to prove Lemmas 5.5 and 5.6.

5.8. Proof of Lemma 5.5. That the inequality (5.26) implies the existence of n^{-1} follows from Lemma 5.3. Moreover it follows easily from the proof for that lemma (see in particular (5.5) and (5.6)) that

$$n^{-1}n_0 = (n_0^{-1}n)^{-1} = \sum_{j=0}^{\infty}(I - n_0^{-1}n)^j$$

or

$$n^{-1} = \sum_{j=0}^{\infty}(I - n_0^{-1}n)^j n_0^{-1}.$$

Thus

$$n^{-1} - n_0^{-1} = \sum_{j=1}^{\infty}(I - n_0^{-1}n)^j n_0^{-1} = (I - n_0^{-1}n)\sum_{k=0}^{\infty}(I - n_0^{-1}n)^k n_0^{-1}.$$

Therefore

$$\begin{aligned}\|n^{-1} - n_0^{-1}\| &\le \|n_0^{-1}\|\cdot\|I - n_0^{-1}n\|\cdot\sum_{k=0}^{\infty}\|I - n_0^{-1}n\|^k\\ &= \frac{\|n_0^{-1}\|\cdot\|I - n_0^{-1}n\|}{1 - \|I - n_0^{-1}n\|} \le \frac{\|n_0^{-1}\|^2\|n_0 - n\|}{1 - \|n_0^{-1}\|\cdot\|n_0 - n\|}.\end{aligned}$$

This inequality together with the assumption (5.26) proves the assertion (5.27).

5.9. PROOF OF LEMMA 5.6. We will establish the inequalities

$$\left.\begin{array}{l}\|Q_0 - Q_0Q_1\| \\ \|Q_1 - Q_1Q_0\|\end{array}\right\} < 1, \tag{5.30}$$

and then show that they imply our lemma. Since $Q_0^2 = Q_0$ we see that

$$\|Q_0 - Q_0Q_1\| = \|Q_0(Q_0 - Q_1)\| \leq \|Q_0\| \cdot \|Q_0 - Q_1\|.$$

By assumption (5.28) this inequality implies the first of the two inequalities (5.30). To prove the second one, we note that $\|Q_0\| \geq 1$ since Q_0 is the identity on its range Q_0Z. Using this and again the assumption (5.28), we see that

$$\begin{aligned}\|Q_1\| = \|Q_0 + Q_1 - Q_0\| &\leq \|Q_0\| + \|Q_1 - Q_0\| \\ &\leq \|Q_0\| + 1/(2\|Q_0\|) \leq \|Q_0\| + \|Q_0\|/2.\end{aligned}$$

Thus

$$\|Q_1\| < 2\|Q_0\|. \tag{5.31}$$

Since $Q_1^2 = Q_1$ we see that

$$\|Q_1 - Q_1Q_0\| = \|Q_1(Q_1 - Q_0)\| \leq \|Q_1\| \cdot \|Q_1 - Q_0\|.$$

But if follows from (5.31) and from (5.28) that here the right member is less than 1. This proves the second of the equalities (5.30).

To derive the lemma from (5.30), we set $Z_0 = Q_0Z$, $Z_1 = Q_1Z$, and denote by $(Q_0Q_1)_0$ the restriction of Q_0Q_1 to Z_0. Then $(Q_0Q_1)_0$ maps $Z_0 \to Z_0$, while Q_0 restricted to Z_0 is the identity on Z_0. Therefore the first of the inequalities (5.30) implies by Lemma 5.3 that $(Q_0Q_1)_0$ has an inverse on Z_0. Thus $Z_0 \subset \text{range of} (Q_0Q_1)_0 \subset \text{range of } Q_0Q_1 \subset \text{range of } Q_1$, and $Z_0 \subset Q_1Z = Z_1$. Therefore $\delta_0 = \dim Z_0 \leq \delta_1 = \dim Z_1$. In a similar way one derives from the second of the inequalities (5.30) that $\delta_1 \leq \delta_0$. Thus $\delta_1 = \delta_0$ as asserted.

5.10. We finished the proof that the equality (4.11) is necessary for the existence of the homotopy described in Theorem 4.7. We now turn to the sufficiency proof. We recall that $\lambda = 1$ is not an eigenvalue of the complex extension M_0 of L_0 since $m_0 = I - M_0$ is not singular. We consider first the special case that M_0 has no real eigenvalues > 1. In this case it follows from Lemma 4.12 and Definition 4.3 that $j(m_0) = j(l_0) = 1 = j(I)$. Therefore in the special case considered we have to prove

5.11. LEMMA. *Let M_0 satisfy the assumption of Theorem* 4.7. *In addition it is assumed that M_0 has no real eigenvalues $\lambda > 1$. Then there exists a linear L.-S. homotopy which connects m_0 with I and which satisfies* (4.12).

PROOF. For $t \in [0,1]$ we set $m(z,t) = z - (1-t)M_0(z)$. That $M(z,t) = (1-t)M_0(z)$ satisfies (4.12) is clear from Lemma 3.11. It remains to show that $m_t(z) = m(z,t)$ is nonsingular for all $t \in [0,1]$. For $t = 0$ and $t = 1$

the nonsingularity is obvious since $m(z,0) = m_0(z)$ and $m(z,1) = I$. Suppose then $m_{t_0}(z)$ to be singular for some t_0 in the open interval $(0,1)$. Then $z_0 - (1-t_0)M_0(z_0) = z_0$ for some $z_0 \neq \theta$, or $M(z_0) = \lambda_0 z_0$ with $\lambda_0 = (1-t_0)^{-1} > 1$. This contradicts our assumption that M_0 has no real eigenvalues > 1.

5.12. We suppose now that M_0 has real eigenvalues greater than 1. We denote them by $\lambda_1 > \lambda_2 > \cdots > \lambda_s$. Let $\sigma_0 = \sigma_0(M_0)$ be their union, and let $\sigma_1 = \sigma_1(M_0)$ be the complement of σ_0 in $\sigma(M_0)$:

$$\sigma(M_0) = \sigma_0 \cup \sigma_1. \tag{5.32}$$

Let

$$P_0 = P_{\sigma_0}, \qquad P_1 = P_{\sigma_1} \tag{5.33}$$

be the projections defined in Lemma 2.11 (cf. also subsection 2.12), and let

$$Z_0 = P_0 Z, \qquad Z_1 = P_1 Z. \tag{5.34}$$

Then as will be shown in subsection 5.13

$$Z = Z_0 \dotplus Z_1 \quad \text{(direct sum)} \tag{5.35}$$

with Z_0 being finite-dimensional. Corresponding to the decomposition (5.34) we set

$$M_0^0 = P_0 M_0, \qquad M_0^1 = P_1 M_0. \tag{5.36}$$

We will show that

$$M_0 = \begin{pmatrix} M_0^0 & 0 \\ 0 & M_0^1 \end{pmatrix} \tag{5.37}$$

(see subsection 5.15) and that σ_0 is the spectrum of M_0^0 and σ_1 the spectrum of M_0^1 (see subsection 5.16). Thus M_0^1 has no real eigenvalues > 1, and Lemma 5.11 may be applied to $m_0' = I_1 - M_0^1$, where I_1 denotes the identity map on Z_1. Discussion of the "finite-dimensional" part M_0^0 of M_0 will begin in subsection 5.20.

5.13. Proof of (5.35). Let V_0 and V_1 be open sets in the λ-plane containing σ_0 and σ_1 resp. with $\overline{V}_0$ and $\overline{V}_1$ disjoint. Let $\Gamma_0 = \partial V_0$ and $\Gamma_1 = \partial V_1$, where Γ_0 and Γ_1 are supposed to be rectifiable Jordan curves oriented in the counterclockwise sense. Then by subsections 2.11 and 2.12, by (5.33) and by Lemma 2.9

$$I = P_0 + P_1 = \frac{1}{2\pi i}\int_\Gamma R_\mu(M_0)\,d\mu, \qquad \Gamma_0 + \Gamma_1. \tag{5.38}$$

This shows that every $z \in Z$ can be represented by

$$z = z_0 + z_1, \qquad z_0 \in Z_0,\ z_1 \in Z_1. \tag{5.39}$$

The uniqueness of this representation follows from (2.12).

To establish (5.37) we prove the following lemma.

5.14. LEMMA.

$$M_0^0 = \frac{1}{2\pi i}\int_{\Gamma_0} \mu R_\mu(M_0)\,d\mu = M_0P_0, \tag{5.40}$$

$$M_0^1 = \frac{1}{2\pi i}\int_{\Gamma_1} \mu R_\mu(M_0)\,d\mu = M_0P_1, \tag{5.41}$$

where M_0^0 and M_0^1 are the operators defined in (5.36). *Moreover, for $z \in Z$,*

$$M_0(z) = M_0^0(z_0) + M_0^1(z), \tag{5.42}$$

$$M_0^0(z_1) = \theta, \qquad M_0^1(z_0) = \theta, \tag{5.43}$$

with z_0, z_1 as in (5.39).

PROOF. Let

$$f(\mu) = \begin{cases} 1 & \text{for } \mu \in \overline{V}_0, \\ 0 & \text{for } \mu \in \overline{V}_1, \end{cases}$$

where V_0 and V_1 are as in subsection 5.13. Then by (2.10) and (5.33)

$$P_0 = \frac{1}{2\pi i}\int_{\Gamma_0} R_\mu(M_0)\,d\mu = \frac{1}{2\pi i}\int_{\Gamma_0+\Gamma_1} f(\mu)R_\mu(M_0)\,d\mu,$$

and from (5.32) and Lemma 2.9 we see that

$$M_0 = \frac{1}{2\pi i}\int_{\Gamma_0+\Gamma_1} \mu R_\mu(M_0)\,d\mu.$$

Therefore from (2.8) with $g(\mu) = \mu$ on $\overline{V}_0 \cup \overline{V}_1$

$$P_0M_0 = M_0P_0 = \frac{1}{2\pi i}\int_{\Gamma_0\cup\Gamma_1} f(\mu)\mu R_\mu(M_0)\,d\mu = \frac{1}{2\pi i}\int_{\Gamma_0} \mu R_\mu(M_0)\,d\mu. \tag{5.44}$$

By definition (5.36) this equality proves (5.40). Assertion (5.41) is proved correspondingly.

To prove (5.42) we note that by (5.39) and (5.34)

$$\begin{aligned} M_0(z) &= M_0(z_0 + z_1) = M_0(z_0) + M_0(z_1) \\ &= M_0(P_0z) + M_0(P_1z) = M_0P_0(z_0) + M_0P_1(z_1). \end{aligned}$$

By (5.40) and (5.41) this proves (5.42). Finally by (5.36), (5.44), and (2.12)

$$M_0^0(z_1) = P_0M_0(z_1) = M_0P_0(z_1) = M_0P_0P_1(z) = 0.$$

This proves the first of the relations (5.43). The second one follows similarly.

5.15. PROOF OF (5.37). Since $P_0M_0 = M_0P_0$ by (5.44) and since $P_1M_0 = M_0P_1$ is established the same way,

$$M_0(z) = \begin{pmatrix} P_0M_0(z) \\ P_1M_0(z) \end{pmatrix} = \begin{pmatrix} M_0P_0(z) \\ M_0P_1(z) \end{pmatrix}.$$

Therefore by the direct decomposition (5.39)

$$M_0(z) = \begin{pmatrix} M_0P_0(z_0) & M_0P_0(z_1) \\ M_0P_1(z_0) & M_0P_1(z_1) \end{pmatrix}.$$

This relation proves (5.37) as we see from (5.40), (5.41), and (5.43).

5.16. ***Proof that σ_0 is the spectrum of M_0^0 and σ_1 is the spectrum of M_0^1.*** It follows from (5.42) and (5.43) that, restricted to Z_0, $M_0 = M_0^0$, and, restricted to Z_1, $M_0 = M_0^1$. Our assertion now follows from subsection 2.13.

This finishes the proof of the assertions contained in subsection 5.12. We next prove

5.17. LEMMA. (i) *Z_0 and Z_1 are invariant under conjugation (see Definition* 3.10); (ii) *Z_0 is finite dimensional.*

PROOF. (i) Let λ_j, $j = 1, 2, \ldots, s$, be as in subsection 5.12. Let d_j be a circular disk with center λ_j of radius so small that the closures $\bar{d}_j$ are disjoint and lie in V_0 (cf. subsection 5.13). Let the boundary ∂d_j of d_j be oriented in the counterclockwise sense, and let

$$Q_j = \frac{1}{2\pi i} \int_{\partial d_j} R_\mu(M_0)\, d\mu, \qquad j = 1, 2, \ldots, s. \tag{5.45}$$

Since the λ_j are the only eigenvalues of M_0 in V_0, it follows from the Cauchy theorem 2.7 and from 5.12 that

$$P_0 = \sum_{j=1}^{s} Q_j \tag{5.46}$$

and

$$Z_0 = P_0 Z = \sum_{j=1}^{s} Q_j Z. \tag{5.47}$$

Since each λ_j is a spectral set for M_0, each Q_j is a projection by Lemma 2.11, and by the Corollary 4.11 to Lemma 4.10 each of the spaces $Q_j Z$ is invariant under conjugation since λ_j is real. Thus we see from (5.47) that Z_0 is invariant under conjugation. Since Z also is invariant under conjugation, being the complexification of a real Banach space E, it now follows that Z_1 also is invariant under conjugation.

(ii) By Lemma 2.12 $Q_i Q_j = 0$ for $i \neq j$. It follows that the decomposition (5.47) is direct. Therefore

$$\dim Z_0 = \sum_{j=1}^{s} \dim Q_j Z_0.$$

But by 2.17 the dimension of $Q_j Z_0$ is the finite number $\nu(\lambda_j)$. Thus

$$\dim Z_0 = \sum_{j=1}^{s} \nu(\lambda_j). \tag{5.48}$$

5.18. Let I_0 and I_1 be the identity maps on Z_0 and Z_1 resp., and let, as always, I denote the identity map on Z. Then

$$m_0 = I - M_0 \begin{pmatrix} m_0^0 & 0 \\ 0 & m_0^1 \end{pmatrix}, \tag{5.49}$$

where

$$m_0^0 = I_0 - M_0^0, \qquad m_0^1 = I_1 - M_0^1. \tag{5.50}$$

This is an immediate consequence of (5.37).

5.19. LEMMA. *There exists an L.-S. homotopy $m_0^1(z_1, t) = z_1 - M_0^1(z_1, t)$, $z_1 \in Z_1$, $t \in [0,1]$, connecting m_0^1 with I_1 and satisfying*

$$M_0^1(\bar{z}, t) = \overline{M_0^1(z, t)}.$$

PROOF. The lemma will be proved once it is shown that M_0^1 satisfies the assumption made in Theorem 5.11 for M_0. That is, we have to verify: (i) Z_1 is the complexification of a real Banach space E_1; (ii)$m_0^1 = I_1 - M_0^1$ and I_1 are the complex extensions of linear nonsingular L.-S. maps on E_1; (iii) M_1^0 has no real eigenvalues > 1. Now (i) follows from the fact that Z_1 is invariant under conjugation (Lemma 5.17). The assertion of (ii), that I_1 is the complex extension of the identity on E_1, is obvious. To prove that M_0^1 is the complex extension of a linear continuous map L_0^1 on E_1, it is by Lemma 3.11 sufficient to prove the relation

$$M_0^1(\bar{z}) = \overline{M_0^1(z)}, \qquad z \in Z_1. \tag{5.51}$$

But this relation follows from the fact that $M_0^1(z) = M_0(z)$ for $z \in Z_1$, by (5.41) and that $M_0(\bar{z}) = \overline{M_0(z)}$ since M_0 is by assumption the extension of a linear continuous map on E. That L_0^1 is completely continuous follows from Lemma 3.9 since $M_0^1(z) = M_0(z)$ is completely continuous by assumption. Finally, that $l_0^1 = I_1 - L_0^1$ is nonsingular follows also from Lemma 3.9 since otherwise, by that lemma, m_0^1 and therefore m_0 would be singular. Finally that assertion (iii) is true was already noticed in the lines directly following (5.37).

5.20. Our next goal is to prove that $m_0 = I - M_0$ is L.-S. homotopic (with a homotopy satisfying (4.12)) to either I or to a map of type I^- (see Definition 4.4). It follows from subsection 5.19 and (5.49) that m_0 is L.-S. homotopic (with the condition (4.12)) to

$$\tilde{m}_0 = \begin{pmatrix} m_0^0 & 0 \\ 0 & I_1 \end{pmatrix}, \qquad m_0^0 = I_0 - M_0^0. \tag{5.52}$$

It will therefore be sufficient to prove the corresponding assertion for the map m_0^0 which maps the finite-dimensional space Z_0 onto Z_0 (cf. Lemma 5.17). Our first step in this direction will be to prove

5.21. LEMMA. *There exists a real base $b_1, b_2, \ldots, b_N$ of Z_0 such that*

$$m_0^0(b_j) = -b_j, \qquad j = 1, 2, \ldots, N, \tag{5.53}$$

where N is the dimension of Z_0 (cf. (5.48)).

PROOF. As shown in the proof of part (ii) of Lemma 5.17 the decomposition of Z_0 given by (5.47) is direct. Therefore a basis for Z_0 is obtained by putting together the bases for each of the spaces Q_jZ_0, $j = 1, 2, \ldots, s$. But by subsection 2.17 and by assertion (iii) of Corollary 4.11

$$\dim Q_jZ_0 = \nu(\lambda_j) = \mu(\lambda_j). \tag{5.54}$$

Now let λ_0 be one of the λ_j, and let Q_0 be the corresponding Q_j. Q_0Z_0 is invariant under conjugation (see Corollary 4.11). It follows that Q_0Z_0 is the complexification of a real Banach space E_0 of the same dimension, and every base of E_0 is a real base of Z_0 (cf. Lemma 3.12). Therefore if L_0 is the linear map $E_0 \to E_0$ of which the map M_0^0 (restricted to Q_0Z_0) is the complex extension, it will for the proof of our lemma be sufficient to show that there exists a base $b_1, \ldots, b_{N_0}$ of E_0 such that

$$l_0(b_j) = -b_j, \qquad j = 1, 2, \ldots, N_0, \tag{5.55}$$

where

$$l_0(x) = x - L_0(x), \qquad x \in E_0. \tag{5.56}$$

Let n_1 be the "index of nilpotency" of the map $Q_0Z_0 \to Q_0Z_0$ given by

$$l_{\lambda_0} = \lambda_0 I_0 - L_0, \tag{5.57}$$

i.e., the smallest positive integer n for which (4.5) holds for all $x \in Q_0Z_0$. Then by a well-known theorem of linear algebra (see, e.g., [**26**, p. 111]) there exist integers $r, n_2, n_3, \ldots, n_r$ and elements $\beta_1, \beta_2, \ldots, \beta_r$ such that $n_1 \geq n_2 \geq \cdots \geq n_r$ and such that the elements

$$\beta_1, l_{\lambda_0}\beta_1, \ldots, l_{\lambda_0}^{n_1-1}\beta_1, \ldots, \beta_\rho, l_{\lambda_0}\beta_\rho, \ldots, l_{\lambda_0}^{n_\rho-1}\beta_\rho, \ldots, \beta_r, l_{\lambda_0}\beta_r, \ldots, l_{\lambda_0}^{n_r-1}\beta_r \tag{5.58}$$

form a base for E_0, while

$$l_{\lambda_0}^{n_1}\beta_1 = l_{\lambda_0}^{n_2}\beta_2 = \cdots = l_{\lambda_0}^{n_r}\beta_r = \theta. \tag{5.59}$$

Thus E_0 is the direct sum of spaces $E_{0\rho}$ with bases

$$\beta_\rho^1 = \beta_\rho, \quad \beta_\rho^2 = l_{\lambda_0}\beta^\rho, \ldots, \beta^{n_\rho} = l_{\lambda_0}^{n_\rho-1}\beta_\rho. \tag{5.60}$$

We then see from (5.58) that

$$l_{\lambda_0}\beta_\rho^1 = \beta_\rho^2, \quad l_{\lambda_0}\beta_\rho^2 = \beta_\rho^3, \ldots, l_{\lambda_0}\beta_\rho^{n_\rho-1} = \beta_\rho^{n_\rho}, \tag{5.61}$$

while by (5.59)

$$l_{\lambda_0}^{n_\rho}\beta_\rho^1 = \theta. \tag{5.62}$$

But from (5.56) and (5.57)

$$l_0 = (1 - \lambda_0)I_0 + l_{\lambda_0},$$

and therefore from (5.61), (5.62),

$$l_0\beta_\rho^1 = (1 - \lambda_0)\beta_\rho^1 + \beta_\rho^2, \ldots, l_0\beta_\rho^{n_\rho-1} = (1 - \lambda_0)\beta_\rho^{n_\rho-1} + \beta_\rho^{n_\rho},$$
$$l_0\beta_\rho^{n_\rho} = (1 - \lambda_0)\beta_\rho^{n_\rho}. \tag{5.63}$$

We now define for $x \in E_{0\rho}$ and $t \in [0,1]$ a homotopy $l_t = l(x,t)$ by setting

$$\begin{aligned} l_t\beta_\rho^1 &= [(1-\lambda_0)(1-t)+(-1)t]\beta_\rho^1 + (1-t)\beta_\rho^2, \\ &\vdots \\ l_t\beta_\rho^{n_\rho-1} &= [(1-\lambda_0)(1-t)+(-1)t]\beta_\rho^{n_\rho-1} + (1-t)\beta_\rho^{n_\rho}, \\ l_t\beta_\rho^{n_\rho} &= [(1-\lambda_0)(1-t)+(-1)t]\beta_\rho^{n_\rho}. \end{aligned} \tag{5.64}$$

Obviously (5.63) and (5.64) agree for $t = 0$, while for $t = 1$ by (5.64)

$$l_1\beta_\rho^j = -\beta_\rho^j, \qquad j = 1,2,\ldots,n_\rho. \tag{5.65}$$

Recalling that $\lambda_0 > 1$ we see that $[(1-\lambda_0)(1-t)+(-1)t]$ is the linear convex combination of the points -1 and $1-\lambda_0 < 0$, and therefore does not contain the point 0. This shows that l_t is not singular for all $t \in [0,1]$. Thus l_0 and l_1 are homotopic and (5.65) proves our lemma

5.22. We will now prove the assertion stated in subsection 5.20 as our "next goal." By subsection 5.20 and Lemma 5.21 this assertion obviously follows from the following lemma.

5.23. LEMMA. *Let Z_0 be a complex Banach space of finite dimension N which is the complexification of a real Banach space E_0. Let m_0^0 be a nonsingular linear map $Z_0 \to Z_0$. Suppose there exists a real base $b_1, b_2, \ldots, b_N$ in Z_0 for which* (5.54) *holds. Then m_0^0 is L.-S. homotopic if* (4.12) *holds to the identity I if N is even and to a map of type I^- if N is odd.*

PROOF. If N is even, we set $N = 2p$ and define $m_0^0(x,t)$ for $t \in [0,1]$ and for $i = 1,2,\ldots,p$ by

$$\begin{aligned} m_0^0(b_{2i-1},t) &= b_{2i-1}\cos(1-t)\pi - b_{2i}\sin(1-t)\pi, \\ m_0^0(b_{2i},t) &= b_{2i-1}\sin(1-t)\pi + b_{2i}\cos(1-t)\pi. \end{aligned} \tag{5.66}$$

Then $m_0^0(z,0) = m_0^0(z)$ by (5.54), while $m_0^0(z,1)$ sends b_j into b_j for $j = 1,2,\ldots,N$, i.e., $m_0^0(z,1)$ is the identity map. Moreover $m_0^0(z,t)$ is not singular, the determinant of (5.66) being 1.

If N is odd, we change our notation by denoting the given base of Z_0 by $b_0, b_1, \ldots, b_N$. We then set $m_0^0(b_0,t) = -b_0$, while $m_0^0(b_j,t)$ for $j = 1,2,\ldots,N = 2p$ is given by (5.66). We thus obtain a nonsingular homotopy $m_0^0(z,t)$ connecting the map $m_0^0(z)$ given by $m_0^0(b_j) = -b_j$ for $j = 0,1,\ldots,N$ with the map given by $m_0^0(b_0,1) = -b_0$, $m_0^0(b_j,1) = b_j$, $j = 1,2,\ldots,N$. The latter map is by definition of the I^- type.

5.23. We thus proved the assertion of subsection 5.20. It is clear that the identity map I on Z is not homotopic to a map m of type I^- since for such m the equality $j(m) = j(I)$ would hold (see 5.10). But by (4.21) and Lemma 4.5 this leads to the contradiction $+1 = -1$. Thus to finish the proof of Theorem 4.7 it remains to prove that two maps of the I^- type are L.-S. homotopic.

5.24. LEMMA. *Let $m_0(z) = z - M_0(z)$ and $M_1(z) = z - M_1(z)$ be linear L.-S. maps $Z \to Z$ of type I^- which are complex extensions of the real maps l_0 and l_1 resp. Then there exists an L.-S. homotopy $m(z,t) = z - M(z,t)$ connecting m_0 with m_1 and satisfying the condition* (4.12).

PROOF. Again let E denote the real Banach space of which Z is the complexification. l_0 and l_1 are then L.-S. linear maps $E \to E$ of type I^-. It is easily seen that it will be sufficient to show that l_0 and l_1 are L.-S. homotopic.

By Definition 4.4 there exists for $i = 0, 1$ a direct decomposition $E = E_1^i \dotplus E_2^i$, where E_1^i is a one-dimensional subspace, and there exists a corresponding unique representation for $x \in E$

$$x = x_1^i + x_2^i, \qquad x_1^i \in E_1^i,\ x_2^i \in E_2^i,$$

such that

$$l_0(x) = -x_1^0 + x_2^0 = x - 2x_1^0, \qquad l_1(x) = x - 2x_1^1. \tag{5.67}$$

Now let e^0 and e^1 be unit elements in E_1^0 and E_1^1 resp.:

$$\|e^0\| = \|e^1\| = 1. \tag{5.68}$$

Then by (5.67)

$$\begin{aligned} l_0(e^0) &= -e^0, & l_1(e^1) &= -e^1, \\ l_0(x) &= x - 2\alpha(x)e^0, & l_1(x) &= x - 2\beta(x)e^1, \end{aligned} \tag{5.69}$$

where α and β are real-valued continuous functionals on E satisfying

$$\alpha(e^0) = \beta(e^1) = 1. \tag{5.70}$$

For the proof that the maps (5.69) are L.-S. homotopic, we distinguish three cases:

(A) $e^0 = e^1$;

(B) $e^0 = -e^1$;

(C) e^0 and e^1 are linearly independent.

Case (A). We set

$$\gamma_t(x) = (1-t)\alpha(x) + t\beta(x), \qquad t \in [0,1], \tag{5.71}$$

and

$$l_t(x) = x - 2\gamma_t(x)e^0. \tag{5.72}$$

It is clear that for $t = 0$ and $t = 1$ this definition agrees with the one given by (5.69). It remains to verify that l_t is nonsingular for each $t \in [0,1]$. If this were not true, then $l_{\bar t}(\bar x) = \theta$ for some $\bar t \in [0,1]$ and $\bar x \in E$ different from θ, i.e., by (5.72), $\bar x = ae^0$ for some real $a \neq 0$. Since $l_{\bar t}$ is linear, we see that $l_{\bar t}(e^0) = \theta$, and therefore from (5.72) that $e^0(1 - 2\gamma_{\bar t}(e^0)) = 0$. Thus

$$2\gamma_{\bar t}(e^0) = 1. \tag{5.72a}$$

But since $e^0 = e^1$ by assumption, we see from (5.70) and (5.71) that $\gamma_{\bar t}(e^0) = (1-\bar t)\alpha(e^0) + \bar t\beta(e^1) = 1$ in contradiction to (5.72a).

Case (B). This case reduces to case (A) since, by (5.69), $l_1(x) = x - 2\tilde{\beta}(x)e^0$ with $\tilde{\beta}(x) = \beta(-x)$, and since $\tilde{\beta}(e^0) = \beta(-e^0) = \beta(e^1) = 1$.

Case (C). Let E_2 be the two-dimensional subspace of E spanned by e^0 and e^1, and let E_3 be a complementary subspace to E_2 such that every $x \in E$ has the unique representation

$$x = x_2 + x_3, \qquad x_2 \in E_2,\ x_3 \in E_3. \tag{5.73}$$

Now let $\tilde{r}_t$ be the rotation in E_2 determined by

$$\begin{aligned} \tilde{r}_t(e^0) &= e^0 \cos(t\pi/2) + e^1 \sin(t\pi/2), \\ \tilde{r}_t(e^1) &= -e^0 \sin(t\pi/2) + e^1 \cos(t\pi/2). \end{aligned} \qquad 0 \le t \le 1. \tag{5.74}$$

We note that

$$\tilde{r}_1(e^0) = e^1. \tag{5.75}$$

We extend $\tilde{r}_t$ to a map r_t on E by setting, for $x \in E$ and $t \in [0,1]$,

$$r_t(x) = x - P_2(x) + \tilde{r}_t(P_2(x)), \tag{5.76}$$

where P_2 is the projection $x = x_2 + x_3 \to x_2$ (cf. (5.73)). This is indeed an extension of $\tilde{r}_t$ since $x = P_2(x)$ for $x \in E_2$. (5.76) is obviously an L.-S. map. To show that it is nonsingular we will prove that r_t^{-1} exists: let $y = r_t(x)$. Since $P_2^2 = P_2$ and since $r_t(P_2(x)) \in E_2$, we see from (5.76) that $P_2(y) = \tilde{r}_t(P_2(x))$, and since $\tilde{r}_t$ obviously has an inverse, it follows that

$$P_2(x) = \tilde{r}_t^{-1}(P_2(y)). \tag{5.77}$$

On the other hand, if P_3 is the projection $x = x_2 + x_3 \to x_3$, we see from (5.76) that $P_3(x) = P_3(y)$. Thus by (5.77) $x = P_2(x) + P_3(x) = \tilde{r}_t^{-1}(P_2(y)) + P_3(y)$ which shows that the inverse r_t^{-1} exists. We therefore may define for $x \in E$ and $t \in [0,1]$

$$l(x,t) = x - 2\alpha(r_t^{-1}(x))r_t(e^0) \tag{5.78}$$

with α as in (5.69). Then

$$l(x,0) = l_0(x), \tag{5.79}$$

since r_0 is the identity map. Moreover by (5.75)

$$l(x,1) = x - 2\alpha(r_1^{-1}(x))e^1 = x - 2\alpha_1(x)e^1, \tag{5.80}$$

where

$$\alpha_1(x) = \alpha(r_1^{-1}(x)). \tag{5.81}$$

To show that $l_0(x)$ and $l(x,1)$ are L.-S. homotopic we have, because of (5.80), only to show that $l(x,t)$ is not singular. If this were not true, then by (5.78)

$$x = 2\alpha(r_t^{-1}(x))r_t(e^0) \tag{5.82}$$

for some $x \in E$ with $x \neq \theta$ and some $t \in [0,1]$. Thus $x = ar_t(e^0)$ with $a \neq 0$. Substitution in (5.82) and cancelling a shows that $1 = 2\alpha(r_t^{-1}r_t(e_0)) = 2\alpha(e^0)$ which contradicts (5.70). Now we want to prove that l_0 and l_1 (see (5.69) and (5.70)) are L.-S. homotopic. Since we just proved that l_0 and $l(\cdot,1)$ are

L.-S. homotopic, it remains to show that l_1 and $l(\cdot, 1)$ are L.-S. homotopic. But comparing (5.80) with the definition (5.69) of l_1 we see that we are in case (A) provided that

$$\alpha_1(e^1) = 1. \tag{5.83}$$

Now by (5.81), (5.75) and (5.70), $\alpha_1(e^1) = \alpha(r_1^{-1}(e^1)) = \alpha(\tilde{r}_1^{-1}(e^1)) = \alpha(e^0) = 1$.

This finishes the proof of the linear homotopy theorem.

§6. Two multiplication theorems for the indices

6.1. THEOREM. *Let E be a real Banach space, and l_0 and l_1 be two nonsingular L.-S. maps $E \to E$. Then*

$$j(l_0 l_1) = j(l_0) \cdot j(l_1). \tag{6.1}$$

PROOF. Let I be the identity map on E and let l^- be a fixed L.-S. map of type I^- on E. Then by Lemma 4.5

$$j(I) = +1, \qquad j(l^-) = -1. \tag{6.2}$$

By the linear homotopy theorem each of the maps l_0 and l_1 is L.-S. homotopic to exactly one of the maps I and l^-. Denoting L.-S. homotopy by the symbol "$\sim$" there are four cases:

$$\begin{array}{lll} \text{(i)} & l_0 \sim I, & l_1 \sim I; \\ \text{(ii)} & l_0 \sim I, & l_1 \sim l^-; \\ \text{(iii)} & l_0 \sim l^-, & l_1 \sim I; \\ \text{(iv)} & l_0 \sim l^-, & l_1 \sim l^-. \end{array} \tag{6.3}$$

Then

$$\begin{aligned} l_0 l_1 &\sim I \cdot I = I \quad \text{in case (i)}, \\ l_0 l_1 &\sim I \cdot l^- = l^- \quad \text{in case (ii)}, \\ l_0 l_1 &\sim l^- I = l^- \quad \text{in case (iii)}, \\ l_0 l_1 &\sim l^- l^- = I \quad \text{in case (iv)}. \end{aligned}$$

By Theorem 4.6 and by (6.2) this implies

$$\begin{aligned} j(l_0 l_1) &= +1 \quad \text{in case (i)}, \\ j(l_0 l_1) &= -1 \quad \text{in case (ii)}, \\ j(l_0 l_1) &= -1 \quad \text{in case (iii)}, \\ j(l_0 l_1) &= +1 \quad \text{in case (iv)}. \end{aligned} \tag{6.4}$$

On the other hand we see from (6.3), from Theorem 4.6 and from (6.2) that

$$\begin{array}{lll} j(l_0) = 1, & j(l_1) = 1 & \text{in case (i)}, \\ j(l_0) = 1, & j(l_1) = -1 & \text{in case (ii)}, \\ j(l_0) = -1, & j(l_1) = +1 & \text{in case (iii)}, \\ j(l_0) = -1, & j(l_1) = -1 & \text{in case (iv)}. \end{array} \tag{6.5}$$

Comparison of (6.4) with (6.5) makes the assertion (6.1) evident.

6.2. THEOREM. *Let l be a nonsingular L.-S. map $E \to E$. Let $E = E_1 \dotplus E_2$ such that every $h \in E$ has the unique representation*

$$h = h_1 + h_2, \qquad h_1 \in E_1,\ h_2 \in E_2. \tag{6.6}$$

Let l_1 and l_2 be the restrictions of l to E_1 and E_2 resp., and suppose that

$$l(h) = \begin{pmatrix} l_1(h_1) & 0 \\ 0 & l_2(h_2) \end{pmatrix}. \tag{6.7}$$

Then

$$j(l) = j(l_1) \cdot j(l_2). \tag{6.8}$$

PROOF. It is clear that l_1 and l_2 are linear nonsingular L.-S. maps $E_1 \to E$ and $E \to E_2$ resp. Thus the right member of (6.8) is defined. Now let

$$m_1 = \begin{pmatrix} l_1 & 0 \\ 0 & I_2 \end{pmatrix}, \qquad m_2 = \begin{pmatrix} I_1 & 0 \\ 0 & l_2 \end{pmatrix}, \tag{6.9}$$

where I_1 and I_2 are the identity maps on E_1 and E_2 resp. Then $l = m_1 m_2$. Therefore, by Theorem 6.1, $j(l) = j(m_1) \cdot j(m_2)$. It remains to verify that

$$j(l_1) = j(m_1), \qquad j(l_2) = j(m_2). \tag{6.10}$$

By Definition 4.3 of the index j, it will for the proof of the first of these equalities be sufficient to show:

(i) a real number $\lambda \neq 0$ is an eigenvalue of $M_1 = I - m_1$ if and only if λ is an eigenvalue of $L_1 = I_1 - l_1$;

(ii) for such an eigenvalue $\lambda \neq 0$

$$\mu_{l_1}(\lambda) = \mu_{m_1}(\lambda), \tag{6.11}$$

where $\mu_{l_1}(\lambda)$ and $\mu_{m_1}(\lambda)$ denote the generalized multiplicity of λ as an eigenvalue of L_1 and M_1 resp. (see Definition 4.2).

Proof of (i). Let $\lambda \neq 0$ be an eigenvalue of $M_1 = I - m_1$. Then there exists an $h \neq 0$ such that $(I - m_1)h = \lambda h$ or by (6.9) and (6.6) such that

$$L_1 h_1 = \lambda h_1 \tag{6.12}$$

and $0h_2 = \lambda h_2$. This implies $h_2 = \theta$ since $\lambda \neq 0$. Thus $h = h_1$, and (6.12) shows that λ is an eigenvalue of L_1. The converse follows even more easily.

Proof of (ii). Let $\lambda \neq 0$ be an eigenvalue of L_1 and therefore, by (i), an eigenvalue of $M_1 = I - m_1$. By (6.9)

$$\lambda I - M_1 = \begin{pmatrix} \lambda I_1 - L_1 & 0 \\ 0 & \lambda I_2 \end{pmatrix},$$

and for any positive integer n

$$(\lambda I - M_1)^n = \begin{pmatrix} (\lambda I_1 - L_1)^n & 0 \\ 0 & \lambda^n I_2 \end{pmatrix}.$$

From this, from (6.6) and from $\lambda \neq 0$ it follows that for $h \in E$, $(\lambda I - M_1)^n h = \theta$ if and only if $(\lambda I_1 - L_1)^n h_1 = \theta$ and $h_2 = \theta$. This implies (6.11) by Definition 4.2, and therefore, by Definition 4.3, proves the first of our assertions (6.10). The second one is proved correspondingly.

6.3. COROLLARY TO THEOREM 6.2. *Let E, E_1, E_2, h, h_1, h_2 be as in Theorem 6.2, and for $i = 1, 2$ let π_i be the projection $h = h_1 + h_2 \to h_i$. Let m be a nonsingular L.-S. map $E \to E$. Let m_1 denote the restriction of $\pi_1 m$ to E_1, and let m_2 denote the restriction of $\pi_2 m$ to E_2. We assume*

$$\pi_2 m(E_1) = \theta. \tag{6.13}$$

Then

$$j(m) = j(m_1) \cdot j(m_2). \tag{6.14}$$

PROOF. Let $m_{12}(h_2) = \pi_1 m(h_2)$. Then

$$m(h) = \begin{pmatrix} m_1(h_1) & m_{12}(h_2) \\ \theta & m_2(h_2) \end{pmatrix}. \tag{6.15}$$

We note first that m_2 is nonsingular, for otherwise $m_2(h_2) = 0$ for some $h_2 \neq \theta$. Then $m(h) = 0$ for $h = h_2$ since then $h_1 = \theta$ and $m_{12}(h_2) = \pi_1 m(h_2) = \theta$. This contradicts the assumed nonsingularity of m. Similarly one sees that m_1 is not singular.

Now let for $t \in [0, 1]$

$$m^t(h) = \begin{pmatrix} m_1(h_1) & t m_{12}(h_2) \\ \theta & m_2(h_2) \end{pmatrix}. \tag{6.16}$$

We prove that for each t this map is not singular by showing that for arbitrary $k \in E$ the equation

$$m^t(h) = k \tag{6.17}$$

has a solution. Now if $k = k_1 + k_2$ with $k_1 \in E_i$, then by (6.16) the equation (6.17) is equivalent to

$$m_1(h_1) + t m_{12}(h_2) = k_1, \qquad m(h_2) = k_2.$$

Since m_1 and m_2 are nonsingular, this system obviously has a solution. We thus see from (6.16) and (6.15) that $m = m^1$ is L.-S. homotopic to the map

$$m^0 = \begin{pmatrix} m_1(h_1) & \theta \\ \theta & m_2(h_2) \end{pmatrix}.$$

Consequently by Theorem 4.6, $j(m) = j(m^0)$. But $j(m^0) = j(m_1) \cdot j(m_2)$ by Theorem 6.2. This proves the assertion (6.14).

APPENDIX B

Proof of the Sard-Smale Theorem 4.4 of Chapter 2

1. THE THEOREM OF SARD. *Let R^n and R^m be real Euclidean spaces of finite dimension n and m resp. Let U be an open subset of R^n, and let $f \in C^r(U): U \to R^m$, where r is a positive integer satisfying*

$$r > \max(0, n - m). \tag{1}$$

Then the set of critical values of f (see Definition 17 in Chapter 1) is of Lebesgue measure 0.

For a proof of and further literature on Sard's theorem we refer the reader to [**1**, §15].

2. COROLLARY TO SARD'S THEOREM. *Let E^n and E^m be (real) Banach spaces of finite dimension n and m resp. Let Ω be a bounded open set in E^n and $f \in C^r(\overline{\Omega})$ (cf. §1.16) where r is a positive integer satisfying (1). Then the set of points in E^m which are regular values for f (see §1.17) is dense in E^m.*

This is an obvious consequence of Theorem 1 since on the one hand a subset of R^m of measure 0 contains no open set and therefore its complement contains a point in every open subset of R^m and is thus dense in R^m, while on the other hand every finite-dimensional Banach space is linearly isomorphic to a Euclidean space of the same dimension (see, e.g., [**18**, p. 245]).

3. The proof given below for the Sard-Smale theorem is an adaptation to the simpler Banach *space* case of the proof given in [**1**, §16] for Smale's generalization of Sard's theorem to certain Banach *manifolds*. Using the preceding corollary we will prove a "local version" of the Sard-Smale theorem in Lemma 4 for maps of a very special form. The lemma in §5 will allow us to show that the local theorem holds also for the more general maps treated in §6. The latter result implies the Sard-Smale theorem as shown in §7.

4. LEMMA. *Let Π_1, K, and K^* be Banach spaces, where K is finite dimensional and where*

$$\dim K^* = \dim K - p, \tag{2}$$

with p being a positive integer. Let Π *and* Σ *be the Banach spaces defined as the direct products*

$$\Pi = \Pi_1 \dot{+} K, \qquad \Sigma = \Pi_1 \dot{+} K^* \tag{3}$$

(cf. §1.3). Let V *be a bounded open subset of* Π. *Let* ψ *be a* C^s*-map* $V \to \Sigma$ *where* s *is an integer satisfying*

$$s \geq p + 1. \tag{4}$$

We suppose moreover that the map $\psi\colon V \to W = \psi(V)$ *is of the special form*

$$\psi(\varsigma) = \varsigma_1 + \chi(\varsigma), \tag{5}$$

where

$$\varsigma = \varsigma_1 + \varsigma_2, \qquad \varsigma_1 \in \Pi_1, \ \varsigma_2 \in K, \tag{6}$$

and where the map $\chi\colon V \to K^*$ *is completely continuous.*

Now if ς^0 *is a point of* V, *then there exists an open neighborhood* V_0 *of* ς^0 *such that*

$$V_0 \subset \overline{V}_0 \subset V \tag{7}$$

and such that the set $R(\overline{\psi}_0)$ *of those points of* Σ *which are regular values for the restriction* $\overline{\psi}_0$ *of* ψ *to* V_0 *is dense and open.*

PROOF. Let $\varsigma^0 = \varsigma_1^0 + k^0$, $\varsigma_1^0 \in \Pi_1$, $k^0 \in K$. Since ς^0 is an element of the open set V, there obviously exist positive numbers ε_1 and ε_2 such that the "rectangular" neighborhood $V_0 = V_1 \times V_2$ satisfies (7) if

$$V_1 = \{\varsigma_1 \in \Pi_1 \mid \|\varsigma_1 - \varsigma_1^0\| < \varepsilon_1\}, \qquad V_2 = \{k \in K \mid \|k - k^0\| < \varepsilon_2\}. \tag{8}$$

We will show first that the set $R(\psi_0)$ of regular values for the restriction ψ_0 of ψ to V_0 is dense in Σ. For this purpose we consider an arbitrary point $\overline{\sigma} = \overline{\varsigma}_1 + \overline{k}^*$ in Σ where $\overline{\varsigma}_1 \in \Pi_1$ and $\overline{k}^* \in K^*$. We have to show that every neighborhood N of $\overline{\sigma}$ contains a point which is a regular value for ψ_0. Now this is obvious if the equation

$$\psi_0(\varsigma) = \overline{\sigma} = \overline{\varsigma}_1 + \overline{k}^* \tag{9}$$

has no solution in V_0 since then, by definition, $\overline{\sigma}$ is a regular value of ψ_0. Suppose now (9) has a solution $\tilde{\varsigma}_1 + \tilde{k}$, $\tilde{\varsigma}_1 \in V_1$, $\tilde{k} \in V_2$. Then by (5) and (9), $\tilde{\varsigma}_1 = \overline{\varsigma}_1$ and

$$\chi(\overline{\varsigma}_1 + \tilde{k}) = \overline{k}^*. \tag{10}$$

Now for $\overline{\varsigma}_1 \in V_1$ we define a map $\chi_{\overline{\varsigma}_1}\colon V_2 \subset K$ into K^* by setting

$$\chi_{\overline{\varsigma}_1}(k) = \chi(\overline{\varsigma}_1 + k). \tag{11}$$

From (2) and (4) we see that this map satisfies the assumption of the Corollary 2 (with $E^m = K^*$, $E^n = K$). Therefore by that corollary every neighborhood $N_2 \subset K^*$ of $\overline{k}^*$ contains a regular value k^* for $\chi_{\overline{\varsigma}_1}$. We will show that for such k^* the point $\sigma = \overline{\varsigma}_1 + k^* \in \Sigma$ is a regular value for ψ_0. This will obviously prove our assertion that every neighborhood N of $\overline{\sigma} = \overline{\varsigma}_1 + \overline{k}^*$ contains a regular value for ψ_0. Now if the equation

$$\chi_{\overline{\varsigma}_1}(k) = k^* \tag{12}$$

has no solution $k \subset V_2$, then the equation

$$\psi_0(\varsigma) = \bar{\varsigma}_1 + k^* \tag{13}$$

has no solution $\varsigma \in V_0$ as is seen from (5) and (11). Thus in this case $\bar{\varsigma}_1 + k^*$ is a regular value for ψ_0.

Suppose now (12) has a solution $k \in V_2$. Then since k^* is a regular value for $\chi_{\bar{\varsigma}_1}$, the differential $D\chi_{\bar{\varsigma}_1}(k;\kappa)$, $\kappa \in K$, is nonsingular. On the other hand, $\varsigma = \bar{\varsigma}_1 + k$ is the corresponding solution of (13), and with $h = \lambda + \kappa$, $\lambda \in \Pi_1$, $\kappa \in K$, we see from (5) that

$$D\psi_0(\varsigma; h) = \begin{pmatrix} \lambda & \theta \\ D\chi(\varsigma;\lambda) & D\chi(\varsigma;\kappa) \end{pmatrix}. \tag{14}$$

But $D\chi(\varsigma;\kappa) = D\chi_{\bar{\varsigma}_1}(k;\kappa)$ by (11). This shows that the differential (14) is not singular and therefore that $\bar{\varsigma}_1 + k^*$ is a regular value for ψ_0 for every solution $\varsigma \in V_1 \times V_2$ of (13).

This finishes the proof that $R(\psi_0)$ is dense in Σ. Now let U_0 be a rectangular neighborhood of ς^0 whose closure $\overline{U}_0$ is a subset of V_0. As follows directly from the definition of "regular value," the relation $\overline{U}_0 \subset V_0$ implies that the set of regular values for the restriction $\overline{\psi}_0$ of ψ to $\overline{U}_0$ contains the set $R(\psi_0)$ and is therefore dense in Σ. In other words, changing our notation, we can choose the rectangular neighborhood V_0 of ς^0 satisfying (7) in such a way that $R(\overline{\psi}_0)$ is dense in Σ.

To prove our lemma it remains to show that $R(\overline{\psi}_0)$ is open or, what is the same, that its complement, the set $S(\overline{\psi}_0)$ of singular values for $\overline{\psi}_0$, is closed. Now it follows directly from the definition of "singular value" that $S(\overline{\psi}_0) = \overline{\psi}_0(S_1)$ if S_1 denotes the set of singular *points* of $\overline{\psi}_0$ in $\overline{V}_0$. But to prove that S_1 is closed, it will be sufficient to show that S_1 is closed as follows from the special form (5) of ψ (cf. the proof of part (ii) of §1.10). The proof that S_1 is closed is quite similar to the proof of part (ii) of Lemma 19 in Chapter 1: let $\varsigma^1, \varsigma^2, \ldots$ be a convergent sequence of points in S_1. We have to prove that

$$\bar{\varsigma} = \lim_{i\to\infty} \varsigma^i \in S_1.$$

Since $\varsigma^i \in S_1 \subset \overline{V}_0$ and since $\overline{V}_0$ is a closed set, we see that

$$\bar{\varsigma} \in \overline{V}_0 \subset V.$$

Now if $\bar{\varsigma} \notin S_1$ the differential $D\psi(\bar{\varsigma}; h)$ would not be singular, i.e., $\bar{\varsigma} \subset V$ would be a regular point for ψ. But then, by the continuity of the differential and by Lemma 5.3 of Appendix A, all points of some neighborhood of $\bar{\varsigma}$ would be regular points for ψ. This contradicts the fact that every neighborhood of $\bar{\varsigma}$ contains infinitely many of the singular points ς^i.

5. LEMMA. *Let Π be a Banach space, let E be a subspace of Π of finite codimension p, let Z be an open bounded subset of Π, and let $\phi = \phi(z)$ be an*

L.-S. map $\overline{Z} \to E$. *We suppose that* $\phi \in C^s(\overline{z})$ *for some integer* $s \geq 1$. *Let* z_0 *be a point of* Z *and*

$$l_0(\eta) = D\phi(z_0; \eta). \tag{15}$$

Moreover, let K *denote the kernel of* l_0, *i.e.,* $K = \{\eta \in \Pi \mid l_0(\eta) = \theta\}$. *Then the following assertions* (i) *and* (ii) *are made*:

(i) K *is finite dimensional, and there exists a subspace* Π_1 *of* Π *such that*

$$\Pi = \Pi_1 \dot{+} K. \tag{16}$$

Moreover if $R \subset E$ *is the range of* l_0, *then there exists a finite-dimensional subspace* K^* *of* E *with the properties:*

$$\text{(a)} \quad E = R \dot{+} K^*; \qquad \text{(b)} \quad \dim K^* = \dim K - p. \tag{17}$$

Thus every $y \in E$ *has the unique representation*

$$y = r + k^* = \pi_1(y) + \pi_2(y), \qquad r \in R, \ k^* \in K^*, \tag{18}$$

where π_1 *and* π_2 *are the projections* $y \to r$ *and* $y \to k^*$ *resp.*

(ii) *Let* Σ *be the direct sum of* Π_1 *and* K^* (*see part* (ii) *of* §1.3):

$$\Sigma = \Pi_1 \dot{+} K^* \tag{19}$$

such that each point $\sigma \in \Sigma$ *has the unique representation*

$$\sigma = \sigma_1 + \sigma_2 = \pi_1^\Sigma(\sigma) + \pi_2^\Sigma(\sigma), \qquad \sigma_1 \in \Pi_1, \ \sigma_2 \in K^*, \tag{20}$$

where π_1^Σ *and* π_2^Σ *are the projections* $\sigma \to \sigma_1$ *and* $\sigma \to \sigma_2$ *resp. It is asserted: there exists a linear L.-S. isomorphism* h *of* E *onto* Σ *and an L.-S.* C^s *isomorphism* h_1 *of some neighborhood* $U \subset Z \subset \Pi$ *of* z_0 *onto some neighborhood* $V \subset \Pi$ *of* $h_1(z_0)$ *such that the map*

$$\psi(\varsigma) = h\phi(h_1^{-1}(\varsigma)), \qquad \varsigma \in V, \tag{21}$$

is of the form (5) *where* $\varsigma, \varsigma_1, \varsigma_2$ *and* χ *are as described in the lines following* (5).

PROOF. (i) That K is finite dimensional follows from Lemmas 12 and 18 in Chapter 1, and the existence of a Π_1 satisfying (16) follows from Lemma 4 in Chapter 1. Since ϕ is an L.-S. map whose domain is the subset $\overline{Z}$ of Π and whose range lies in the subspace E of Π, it may also be considered as an L.-S. map $\overline{Z} \to \Pi$. Therefore, by Lemma 12 of Chapter 1 there exists a finite-dimensional subspace K_1^* of Π such that

$$\Pi = R \dot{+} K_1^*. \tag{22}$$

Now $E \cap R = R$ since $R \subset E$. Therefore intersecting both members of (22) with E we see that the finite-dimensional space $K^* = E \cap K_1^*$ satisfies (17a). Since p is the codimension of E in Π, comparison of (22) with (17a) shows that $\dim K^* = \dim K_1^* - p$. But by §1.12, $\dim K_1^* = \dim K$. This proves (17b).

Proof of (ii). We note first that the restriction of l_0 to Π_1 maps Π_1 onto R. Indeed, if r is a given element of R, there exists by definition of R a $z \in \Pi$ such that $l_0(z) = r$. But by (16)

$$z = z_1 + k, \qquad z_1 \in \Pi_1, \ k \in K. \tag{23}$$

Since $l_0(k) = \theta$ by definition of K, we see that $l_0(z_1) = r$. Thus l_0 as a map $\Pi_1 \to R$ is a nonsingular L.-S. map Π_1 onto R. It follows from the argument given for the proof of part (v) of §1.12 that this map has an inverse $m\colon R \to \Pi_1$ which is a nonsingular L.-S. map. Now every $y \in E$ has the decomposition (18). If for $r = \pi_1(y)$, $k^* = \pi_2(y)$ we define

$$h(y) = m(r) + k^*, \tag{24}$$

we obtain a map of E onto the space Σ defined by (19). Since $m(r)$ is linear and L.-S. and since K^* is finite dimensional, it follows that h is a linear nonsingular L.-S. map.

Using the decomposition (16) we set for $z = z_1 + k \in Z$, $z_1 \in \Pi_1$, $k \in K$,

$$h_1(z) = m\pi_1\phi(z) + k. \tag{25}$$

Since $\phi(z) \in E$, it follows from (18) that $\pi_1\phi(z) \in R$. Therefore the first term of the right member of (25) is an element of Π_1, and $h_1(z) \in \Pi$ by (16). Moreover since ϕ and m are L.-S. mappings and since K is finite dimensional, we see from (25) that h_1 is an L.-S. map.

We prove next that h_1 maps some open neighborhood U of z_0 onto some open neighborhood V of $h_1(z_0)$ and that $h_1^{-1} \in C^s(V)$. For this purpose it will by the inversion theorem in §1.21 be sufficient to show that the differential of h_1 at z_0 is the identity map on Π, i.e., that

$$Dh_1(z_0; \eta) = \eta \tag{26}$$

for all

$$\eta = \eta_1 + \kappa \in \Pi, \qquad \eta_1 \in \Pi_1,\ \kappa \in K \tag{27}$$

(cf. (16)). Now since m and π_1 are linear, we see from (25) that

$$Dh_1(z; \eta) = m\pi_1 D\phi(z; \eta) + \kappa \tag{28}$$

for all $z \in Z$. But for $z = z_0$ we see from (15) and from (27) that $D\phi(z_0; \eta) = l_0(\eta) = l_0(\eta_1)$ since $\kappa \in K$ and, therefore, $l_0(\kappa) = \theta$. Since R is the range of l_0, we see from (18) that $\pi_1 D\phi(z_0, \eta) = \pi_1 l_0(\eta_1) = l_0(\eta_1)$. But $l_0(\eta_1)$ is the restriction of l_0 to Π_1 and, by definition, m is its inverse. Therefore $m\pi_1 D\phi(z_0; \eta) = ml_0(\eta_1) = \eta_1$. Thus by (28) $Dh_1(z_0; \eta) = \eta_1 + \kappa$, which by (27) proves (26).

Then let U and V be sets of the properties stated above. We set

$$\varsigma = h_1(z). \tag{29}$$

If z varies over U, then ς varies over V. Since $\varsigma \in \Pi$ we may write $\varsigma = \varsigma_1 + \varsigma_2$, $\varsigma_1 \in \Pi_1$, $\varsigma_2 \in K$ (cf. (16)). Then by (25) and (20)

$$\varsigma_1 = m\pi_1\phi(z) \in \Pi_1, \qquad \varsigma_2 = k \in K. \tag{30}$$

Now if

$$\psi(\varsigma) = h\phi h_1^{-1}(\varsigma), \qquad \varsigma \in V, \tag{31}$$

then with $z = h_1^{-1}(\varsigma)$ by (24), (18) and (30)

$$\psi(\varsigma) = h\phi(z) = m\pi_1\phi(z) + \pi_2\phi(z) = \varsigma_1 + \pi_2\phi h_1^{-1}(\varsigma).$$

With $\chi(\varsigma) = \pi_2 \phi h_1^{-1}(\varsigma)$ this equality shows that ψ is of the asserted form (5). Obviously $\chi(\varsigma)$ is bounded and lies in K^*. Since K^* is finite dimensional, we see that χ is completely continuous. Thus assertion (ii) of Lemma 4 is proved.

6. LEMMA. *With the notation used in Lemma 5 we suppose that the assumptions of that lemma are satisfied. In addition we strengthen the assumption that $\phi \in C^s(\overline{Z})$ with $s \geq 1$ by requiring that*

$$s \geq p + 1. \tag{32}$$

It is asserted that corresponding to each point $z_0 \in Z$ there exists an open set U_0 which in addition to satisfying

$$z_0 \in U_0 \subset \overline{U}_0 \subset Z \tag{33}$$

has the following property: if $\overline{\phi}_0$ is the restriction of ϕ to $\overline{U}_0$, then the set $R(\overline{\phi}_0)$ of points in E which are regular values for $\overline{\phi}_0$ is dense and open.

PROOF. It follows from Lemma 5 together with the assumption (32) that all assumptions of Lemma 4 are satisfied. Consequently, the point

$$\varsigma^0 = h_1(z_0) \in \Pi \tag{34}$$

has an open neighborhood V_0 with the properties asserted in that lemma. We claim that then the set

$$U_0 = h_1^{-1}(V_0) \tag{35}$$

has the properties asserted in the present lemma. In the first place U_0 is open and $\overline{U}_0 = h_1^{-1}(\overline{V}_0)$ as follows from Theorem 2.3 in Chapter 5. Next the following two assertions hold:

(a) for $y_0 \in E$ the equation $\phi(z) = y_0$ has a solution $z \in \overline{U}_0$ if and only if for $\sigma_0 = h(y_0) \in \Sigma$ the equation $\psi(\varsigma) = \sigma_0$ has a solution $\varsigma \in \overline{V}_0$;

(b) for $z_0 \in \overline{U}_0$ the differential $D\phi(z_0; \cdot)$ is not singular if and only if the differential $D\psi(h_1(z_0); \cdot)$ is not singular.

(a) follows directly from the relation (21) between ψ and ϕ and the properties of h and h_1.

(b) holds for the same reasons in conjunction with the chain rule. But (a) and (b) together imply that $R(\overline{\phi}_0) = h^{-1}R(\overline{\psi}_0)$.

Now $R(\overline{\psi}_0)$ is dense and open by Lemma 4. Therefore $h^{-1}R(\overline{\psi}_0)$ is dense and open since h^{-1} is a linear one-to-one L.-S. map Σ onto E.

7. *Proof of the Sard-Smale theorem 4.4 of Chapter 2.* Let y_0 be an arbitrary point of E. We have to prove that every neighborhood (in E) of y_0 contains a regular value for ϕ. Now if the equation

$$\phi(z) = y_0, \qquad y_0 \in E - \phi(\partial Z), \tag{36}$$

has no solution, then, by definition, y_0 is a regular value for ϕ and every neighborhood of y_0 contains a regular value, e.g., y_0.

Suppose now that the set C_0 of roots of (36) is not empty. C_0 is compact by Lemma 10 in Chapter 1, and every z_0 has a neighborhood U_0 of the property asserted in Lemma 6. As z_0 varies over C_0, these neighborhoods form an open covering of this compact set. Consequently there exists a finite subcovering, say, $U_0^1, U_0^2, \dots, U_0^q$ such that (cf. (33)) for $i = 1, 2, \dots, q$

$$U_0^i \subset \overline{U}_0^i \subset Z; \qquad C_0 \subset U = \sum_{i=1}^{q} U_0^i. \tag{37}$$

If $\overline{\phi}_i$ is the restriction of ϕ to $\overline{U}_0^i$, then by Lemma 6 the set $R(\overline{\phi}_i)$ of regular values for $\overline{\phi}_i$ is dense and open in E. Consequently by Baire's density theorem (see e.g. [**2**; p. 108, Theorem V′]) the set

$$\Delta = \bigcap_{i=1}^{q} R(\overline{\phi}_i) \tag{38}$$

is dense in E, and therefore in every neighborhood of y_0. We show next the existence of an open neighborhood N_0 of y_0 such that

$$\phi^{-1}(N_0) \subset U. \tag{39}$$

Indeed, otherwise, there would exist a sequence of positive numbers δ_ν converging to 0, of points $y_\nu \in B(y_0, \delta_\nu)$, and of points x_ν such that

$$\phi(x_\nu) = y_\nu, \qquad x_\nu \in \overline{Z}, \tag{40}$$

but

$$x_\nu \notin U. \tag{41}$$

Now since the y_ν converge to y_0 and since ϕ is an L.-S. map, it is easy to see that a subsequence of the x_ν converges to a point x_0 which by (40) satisfies $\phi(x_0) = y_0$, i.e.,

$$x_0 \in \phi^{-1}(y_0) = C_0. \tag{42}$$

But by (41)

$$x_0 \notin U \tag{43}$$

since U is open. But by (42) and (37), $x_0 \in C_0 \subset U$ which contradicts (43).

Then let N_0 be a neighborhood of y_0 satisfying (39), and let $\Delta_1 = \Delta \cap N_0$. Δ_1 is not empty since Δ is dense and N_0 open in E. We now show that every point y_1 of Δ_1 is a regular value for ϕ. If the equation

$$\phi(z) = y_1 \tag{44}$$

has no solutions, there is nothing to prove. But if it has a solution, then every solution lies in U by (39), and therefore, by (37), in some of the sets U_0^i, say for $i = 1, 2, \dots, q_1$ with $i \le q_1 \le q$. Consider now for such i a solution $z \in U_0^i$. Then $\phi(z) = \phi_i(z) = y_1 \in \Delta = \bigcap_{j=1}^{q} R(\phi_j) \subset R(\phi_i)$, and $D\phi(z; h)$ is not singular.

This finishes the proof that every point of $N_0 \cap \Delta$ is a regular value for ϕ, and therefore it finishes the proof of the Sard-Smale theorem since an N_0 satisfying (39) can be chosen as a subset of a given neighborhood of y_0.

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