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DIRECT AND INVERSE SCATTERING ON THE LINE

RICHARD BEALS
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CARLOS TOMEI
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In memory of our parents

Robert Beals
Philip Deift
Rose Deift
Enrico Tomei
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Preface

This monograph deals with the theory of linear ordinary differential operators of arbitrary order. Unlike treatments which focus on spectral theory, our treatment centers on the construction of special eigenfunctions (generalized Jost solutions) and on the inverse problem: the problem of reconstructing the operator from (minimal) data associated to the special eigenfunctions. In the second order case this program includes spectral theory and is equivalent to quantum mechanical scattering theory; the essential analysis involves only the bounded eigenfunctions. For higher order operators, although bounded eigenfunctions are again sufficient for spectral theory and quantum scattering theory, they are far from sufficient for a successful inverse theory.

The inverse theory which we develop is motivated by its applications to nonlinear wave equations in the spirit of KdV, although we feel that it is also of intrinsic interest for the theory of ordinary differential equations. Applications to spectral theory and quantum mechanical scattering theory, in addition to nonlinear wave equations, are included.
APPENDIX A

Rational approximation

Here we prove the following result used in §26.

PROPOSITION A.1 (cf. [BC1, Appendix A.2]). Let \( \theta(z) \) be a smooth, complex-valued function on \( \Sigma_k \) and on \( \Sigma_{k+1} \) (i.e., \( \theta \in C^\infty(\Sigma_k \cup \Sigma_{k+1}) \) and \( \lim_{z \to 0, z \in \Sigma_k} (d^{m'} \theta/dz^{m'}) (z), \lim_{z \to 0, z \in \Sigma_{k+1}} (d^{m''} \theta/dz^{m''}) (z) \) exist for all \( m', m'' \geq 0 \)) and suppose that

\[
(A.2) \quad \lim_{z \to 0} \frac{d^j \theta}{dz^j} (z) = 0 = \lim_{z \to 0} \frac{d^j \theta}{dz^j} (z), \quad 0 \leq j \leq N - 1,
\]

for some positive integer \( N \), and that

\[
(A.3) \quad \frac{d^j \theta}{dz^j} (z) = O(|z|^{-(K+1)}) \quad \text{as} \quad z \to \infty, \quad z \in \Sigma_k \cup \Sigma_{k+1},
\]

for some positive integer \( K \) and all \( j \geq 0 \). Then, given \( \eta > 0 \), there exists a rational function \( \chi(z) \), nonsingular on \( \Sigma_k \cup \Sigma_{k+1} \), such that

\[
(A.4) \quad \lim_{z \to 0} \frac{d^j \chi}{dz^j} (z) = 0, \quad 0 \leq j \leq N - 1,
\]

and

\[
(A.5) \quad \rho(z)^K \left| \frac{d^j}{dz^j} (\chi(z) - \theta(z)) \right| \leq \eta
\]

for \( z \in \Sigma_k \cup \Sigma_{k+1} \), and for \( 0 \leq j \leq N \).

PROOF. Let the bisector of \( \Omega_{k+1} \) point along the direction of \( \dot{w} \), \( |\dot{w}| = 1 \). Set \( \phi(z) = (z + \dot{w})^K \theta(z) \) for \( z \in \Sigma_k \cup \Sigma_{k+1} \). Then \( \phi(z) \) is smooth on \( \Sigma_k \) and on \( \Sigma_{k+1} \),

\[
(A.2)' \quad \lim_{z \to 0} \frac{d^j \phi}{dz^j} (z) = 0 = \lim_{z \to 0} \frac{d^j \phi}{dz^j} (z), \quad 0 \leq j \leq N - 1,
\]

and

\[
(A.3)' \quad \frac{d^j \phi}{dz^j} (z) = O \left( \frac{1}{|z|} \right) \quad \text{as} \quad z \to \infty, \quad z \in \Sigma_k \cup \Sigma_{k+1}
\]

for all \( j \geq 0 \).
For $z \in \Sigma_k \cup \Sigma_{k+1}$ and $\varepsilon > 0$, set

$$\phi_\varepsilon(z) = \int_{\Sigma_{k+1} \cup \Sigma_k} \phi(\zeta) \left( \frac{1}{\zeta - z - \varepsilon \omega} - \frac{1}{\zeta - z + \varepsilon \omega} \right) \frac{d\zeta}{2\pi i},$$

where the integral on $\Sigma_k$ runs from $0$ to $\infty$ and the integral on $\Sigma_{k+1}$ runs from $\infty$ to $0$.

Differentiating with respect to $z$ and using (A.2)', we obtain

$$\frac{d^j \phi_\varepsilon}{dz^j}(z) = \int_{\Sigma_{k+1} \cup \Sigma_k} \frac{d^j \phi}{d\zeta^j}(\zeta) \left( \frac{1}{\zeta - z - \varepsilon \omega} - \frac{1}{\zeta - z + \varepsilon \omega} \right) \frac{d\zeta}{2\pi i}$$

for $0 \leq j \leq N$.

Standard computations for the Poisson integral together with (A.3)' and the identity

$$\int_{\Sigma_{k+1} \cup \Sigma_k} \left( \frac{1}{\zeta - z - \varepsilon \omega} - \frac{1}{\zeta - z + \varepsilon \omega} \right) \frac{d\zeta}{2\pi i} = 1$$

now prove that, for sufficiently small positive $\varepsilon$,

$$\sup_{z \in \Sigma_k \cup \Sigma_{k+1}} \left| \frac{d^j \phi_\varepsilon}{dz^j}(z) - \frac{d^j \phi}{dz^j}(z) \right| < \eta$$

for $0 \leq j \leq N$.

Set

$$P(\zeta, z, \varepsilon) = \frac{1}{2\pi i} \left( \frac{1}{\zeta - z - \varepsilon \omega} - \frac{1}{\zeta - z + \varepsilon \omega} \right),$$

and let $\hat{\zeta}_k, \hat{\zeta}_{k+1}$ be unit vectors pointing outward along $\Sigma_k, \Sigma_{k+1}$, respectively. Given $m$, set

$$\phi_{\varepsilon, m}(z) = \frac{1}{m} \sum_{s=1}^{m^2} \phi\left( \frac{s}{m} \hat{\zeta}_k \right) P\left( \frac{s}{m} \hat{\zeta}_k, z, \varepsilon \right) - \frac{1}{m} \sum_{s=1}^{m^2} \phi\left( \frac{s}{m} \hat{\zeta}_{k+1} \right) P\left( \frac{s}{m} \hat{\zeta}_{k+1}, z, \varepsilon \right).$$
Then, for $0 \leq j \leq N$ and fixed $\varepsilon > 0$,

$$\left| \frac{d^j \phi_{\varepsilon,m}}{dz^j} (z) - \frac{d^j \phi_{\varepsilon}}{dz^j} (z) \right| \leq \left| \int_{\Sigma_k \cup \Sigma_{k+1}} \phi(\zeta) \frac{d^j}{dz^j} P(\zeta, z, \varepsilon) d\zeta \right|

+ \sum_{s=1}^{m^2} \int_{\{(s/m)\zeta_k \}} \left( \phi(\zeta) \frac{d^j}{dz^j} P(\zeta, z, \varepsilon) - \phi\left(\frac{s}{m} \zeta_k\right) \frac{d^j}{dz^j} P\left(\frac{s}{m} \zeta_k, z, \varepsilon\right) \right) d\zeta

+ \sum_{s=1}^{m^2} \int_{\{(s/m)\zeta_k+1 \}} \left( \phi(\zeta) \frac{d^j}{dz^j} P(\zeta, z, \varepsilon) - \phi\left(\frac{s}{m} \zeta_{k+1}\right) \frac{d^j}{dz^j} P\left(\frac{s}{m} \zeta_{k+1}, z, \varepsilon\right) \right) d\zeta

\leq \text{const.} \left( \sup_{|\zeta| > m} \left| \phi(\zeta) \right| \right) \int_{\Sigma_k \cup \Sigma_{k+1}} \frac{1}{((\zeta - z)^2 - \varepsilon_2 \hat{\omega}^2)} d\zeta

+ \sum_{s=1}^{m^2} \frac{1}{2m^2} \left( \sup_{\zeta \in \{(s/m)\zeta_k \}} \left| \frac{d}{d\zeta} \left( \phi(\zeta) \frac{d^j}{dz^j} P(\zeta, z, \varepsilon) \right) \right| \right)

+ \sup_{\zeta \in \{(s/m)\zeta_{k+1} \}} \left| \frac{d}{d\zeta} \left( \phi(\zeta) \frac{d^j}{dz^j} P(\zeta, z, \varepsilon) \right) \right|.

Using (A.3), we find that, for fixed $\varepsilon > 0$,

$$\left| \frac{d^j \phi_{\varepsilon,m}}{dz^j} (z) - \frac{d^j \phi_{\varepsilon}}{dz^j} (z) \right| \leq \text{const.} \left( \sup_{|\zeta| > m} \left| \phi(\zeta) \right| \right)

+ \frac{1}{m} \int_{\Sigma_k \cup \Sigma_{k+1}} \left| d\zeta \right| \frac{1}{((\zeta - z)^2 - \varepsilon_2 \hat{\omega}^2)} < \eta,

\text{for all } z \in \Sigma_k \cup \Sigma_{k+1}, \text{ provided } m \text{ is chosen sufficiently large.}

Finally, set

$$\chi(z) = \frac{1}{(z + \hat{\omega})^K} \left\{ \phi_{\varepsilon,m}(z) - \sum_{j=0}^{N-1} \frac{z^j \frac{d^j \phi_{\varepsilon,m}}{dz^j}(0)}{j!} \right\} / \left[1 - (\gamma z)^N\right],$$

where $\gamma$ is chosen so that $\chi(z)$ has no singularities on $\Sigma_k \cup \Sigma_{k+1}$.

Clearly $\chi(z)$ is a rational function, $\lim_{z \to 0} (d^j \chi/dz^j) = 0$, $0 \leq j \leq N - 1$, and

$$\rho(z)^K \frac{d^j}{dz^j} (\chi(z) - \theta(z)) = \rho^K(z) \left( \frac{d^j}{dz^j} \left\{ \left[(-\sum_{j=0}^{N-1} \frac{z^j \frac{d^j \phi_{\varepsilon,m}}{dz^j}(0)}{j!})/\left[1 - (\gamma z)^N\right]\right] \right\} \right)

+ \rho^K(z) \left( \frac{d^j}{dz^j} \left[ \frac{\phi_{\varepsilon,m}(z) - \phi_{\varepsilon}(z)}{(z + \hat{\omega})^k} \right] \right)

+ \rho^K(z) \left( \frac{d^j}{dz^j} \left[ \frac{\phi_{\varepsilon}(z) - \phi(z)}{(z + \hat{\omega})^k} \right] \right).$$
The second and third terms are dominated by $\eta$, by (A.8) and (A.9). Moreover, (A.8) and (A.9) also imply that $|\langle d^j \phi_{\varepsilon,m} \rangle d z^j(0)| \leq 2\eta$ for $0 \leq j \leq N - 1$, by (A.2)', so the first term is also dominated by $\eta$. 

**Remark A.10.** Requiring (A.3) for all $j \geq 0$ is clearly too much. It is enough that $\vartheta(z)$ be $C^{N+1}$ and that (A.3) hold for $0 \leq j \leq N + 1$. 
APPENDIX B

Some Formulas

Here we collect some useful algebraic formulas. When some indication of the derivation is given at the first appearance of the formula, we cite that appearance. Otherwise we give a derivation here.

(B.1) \[ \Lambda^{-1}_z J_z \Lambda_z = z J(z); \] see (2.10).
(B.2) \[ J(\alpha z) = \alpha^{-1} J(z); \] see (9.3).
(B.3) \[ \Lambda_{\alpha z} = \Lambda_z; \] see (9.4).
(B.4) \[ R(J_z)^* R = J_z; \] see (9.8).
(B.5) \[ R J(z)^* R = J(\tilde{z}); \] see (9.10).
(B.6) \[ \Lambda(z)^* \Lambda(z) = n I; \] see (9.12).
(B.7) \[ R \Lambda(z) R = \Lambda(\tilde{z}) J(\tilde{z}); \] see (9.13).
(B.8) \[ \pi_{j+2} = \pi_j, \quad \pi_{\alpha z} = \pi_z; \] see (12.4).
(B.9) \[ \pi_z J_+(z) \pi_z = J_-(z); \]

this is essentially the definition of the period 2 permutation matrix \( \pi_z, z \in \Sigma; \) see Definition 11.11.

(B.10) \[ R \pi_z R = \pi_z; \]

in view of (B.8) this need only be established for \( z \in \Sigma_0 = i \mathbb{R}_+ \) and \( z \in \Sigma_1. \) Now the ordering of roots in \( \Omega_{j+n} = -\Omega_j \) is the opposite of the ordering in \( \Omega_j, \) so \( R \pi_0 R = \pi_n \) and \( R \pi_1 R = \pi_{n+1} = \pi_{n-1}, \) as desired.

(B.11) \[ R = \pi_j \pi_{j+1} \cdots \pi_{j+n-1} = \pi_{j+n-1} \pi_{j+n-2} \cdots \pi_j; \]

in fact, the product on the right is a permutation matrix which converts (vertical vectors) from the \( \Omega_j \) ordering to the \( \Omega_{j+n} = -\Omega_j \) ordering, which is opposite. The other product is the inverse, and \( R^{-1} = R. \)

The next identity refers to the global order operators introduced in Definition 38.22.

(B.12) \[ R \Lambda_g = \Lambda_g J_g; \]

in fact,

\[ (R \Lambda_g)_{jk} = (\Lambda_g)_{j-n-1,j} = \alpha^{(n-j)k} \]

\[ = \alpha^k = \alpha^{(j-1)k} \alpha^k = (\Lambda_g)_{jk} \alpha^k = (\Lambda_g J_g)_{jk}. \]
Recall that

$$\Pi = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & 0 & 1 \\ 1 & 0 & \cdots & 0 \end{pmatrix} = J_1.$$  

(B.13) \hspace{1cm} R\Pi^*R = \Pi; \hspace{0.5cm} \text{take } z = 1 \text{ in (B.5)}.

(B.14) \hspace{1cm} \Pi = \Lambda(z)J(z)\Lambda(z)^{-1};

in fact,

$$[\Pi\Lambda(z)]_{jk} = \Lambda(z)_{j+1,k} = \alpha_k^j = \alpha_k\alpha_k^{j-1}$$

$$= \alpha_k\Lambda(z)_{j,k} = [\Lambda(z)J(z)]_{jk}.$$  

(B.15) \hspace{1cm} \Pi = \Lambda_gJ_g\Lambda_g^{-1};

this is essentially the same as the preceding calculation:

$$(\Pi\Lambda_g)_{jk} = (\Lambda_g)_{j+1,k} = \alpha^j = \alpha^{(j-1)k}\alpha^k$$

$$= [\Lambda_gJ_g]_{jk}.$$  

Similarly,

(B.16) \hspace{1cm} \Pi = \Lambda_g^{-1}RJ_gR\Lambda_g;

in fact

$$(RA_g\Pi)_{jk} = (\Lambda_g\Pi)_{n-j+1,k} = (\Lambda_g)_{n-j+1,k-1}$$

$$= \alpha^{(n-j)(k-1)} = \alpha^{j(k-1)} = \alpha^j\alpha^{-jk}$$

$$= \alpha^j\alpha^{(n-j)k} = \alpha^j(\Lambda_g)_{n-j+1,k}$$

$$= (JRA_g)_{jk}.$$  

Finally, we recall Definition 12.11, which amounts to

(B.17) \hspace{1cm} j \sim j + 1 \hspace{0.5cm} \text{at } z \text{ if } (\pi_x)_{j,j+1} \neq 0.
References


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