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An Introduction to CR Structures

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30 **The Markoff and Lagrange spectra,** Thomas W. Cusick and Mary E. Flahive

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AN INTRODUCTION TO CR STRUCTURES

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To my parents

Contents

Preface

In 1907 Poincaré showed that two real hypersurfaces in $\mathbb{C}^2$ are in general biholomorphically inequivalent, **[Po]**. That is, given two real analytic submanifolds of real dimension three in $\mathbb{C}^2$ there usually is no biholomorphism of one open set in $\mathbb{C}^2$ to another open set in $\mathbb{C}^2$ that takes a piece of the first submanifold onto the second. He then raised the question of finding the invariants that distinguish one real hypersurface from another. This basic question was first completely answered by Cartan, in **[Ca2]**. Cartan remarked "Je reprends la question directement comme application de ma méthode générale d'équivalence. La résolution complète du problème de Poincaré me conduit à des notions géométriques nouvelles ... ". A second solution was given by Moser in 1973. In joint work with Chern **[CM]** this was generalized, along with Cartan's original solution, to dimensions greater than two. (At about this time, Tanaka in **[Tan1]** and **[Tan2]** gave a different extension of Cartan's work to higher dimensions.)

The study of the basic problem for CR structures primarily rests on these two works. Thus, although most introductions to an area of mathematics are a synthesis from many sources, reflecting how mathematics usually develops, the present one, to a surprising degree, is not. Rather, it is in large measure an exposition of the papers of Cartan and of the joint paper of Chern and Moser.

For Cartan one needs to know something of his general method of equivalences and of the structure of the Lie group $SU(2, 1)$. So we have tried to provide this necessary background before going over the main construction. This background is also important for the part of the Chern-Moser paper that extends Cartan's work to higher dimension. For the rest of the Chern-Moser paper the problem is somewhat different. Here the approach is more straightforward but also more technically difficult. We do the lowest-dimensional case in detail. This should also make the higher-dimensional case more accessible.

An exception to this focus on the above two works is Chapter 1. Here we give the basic definitions and properties and draw from many sources in the literature and in the "folklore." Chapter 2 uses simple facts about the automorphism groups of the ball and polydisc in order to show that the Riemann Mapping Theorem does not hold in higher dimension. Then the automorphism group of the ball, $SU(2, 1)$, is computed in enough detail for future needs. The next two chapters go over the Moser normal form. In Chapter 5, we give the background necessary to understand Cartan's work. Then in Chapter 6 we present Cartan's basic construction along with two variations, one of which is the lowest-dimensional case of the construction in the Chern-Moser paper. In the next three chapters we explore some aspects of Cartan's "new geometric ideas," and also relate his invariants to those found by Moser. Finally, in Chapter 10 we consider the landmark paper of Hans Lewy **[Le2]** and the realizability problem for abstract CR structures. Thus we end with a certain historical completeness — Professor Lewy has stated that his interest in the partial differential operators associated to real hypersurfaces arose while trying to understand the papers of Cartan.

A number of exercises are included; some to help the reader clarify material for himself or herself and others to simplify the author's task. After Chapter 10 there are Notes providing additional information for some of the chapters and also indicating some of the material that would have been included if the author had not been anxious to conclude this long delayed project.

This book evolved from lecture notes for courses given at Rutgers University and Université de Grenoble. The author is grateful for these opportunities to lecture on this subject. The author is also grateful to the National Science Foundation for its support of his research, some of which has found its way into this book, and to Provost Walter Gordon of Rutgers-Camden for his support of this project. Finally, it is a pleasure to acknowledge the constant encouragement and support of Dalia Ritter.

Notes

Chapter 1

§**2.** Cartan called his geometry "pseudoconformal" in order to emphasize that what he was studying was a generalization of the theory of one complex variable (conformal geometry). The term "CR manifold" was first used in [**Gr**]. A careful discussion of the basic definitions and elementary properties may be found in [**Ta**] where the CR structures are not restricted to be of hypersurface type. There is little overlap between this book and ours; it focuses on a problem not considered here, namely, the extension problem for CR functions, and is a good introduction to this field. Everyone should know the two basic extension results: A CR function defined on the boundary of a bounded open set with connected complement extends to a function holomorphic on the open set (Bochner-Hartog Theorem) and a CR function on a strictly pseudoconvex hypersurface extends as a holomorphic function to some open set that contains the hypersurface in its boundary (Lewy Theorem).

The proof that a real analytic almost complex structure is complex can be found, for instance, in [**KN**, volume 2]. The original Newlander-Nirenberg Theorem required smoothness of class $C^{2n+\lambda}$ for manifolds of dimension $2n$. This was improved to $C^{1+\lambda}$ in [**NW**]. There are now several very different proofs of the Newlander-Nirenberg Theorem; for instance, those of Malgrange [**Ma**], [**Ni2**], Kohn [**FK**], and Webster [**We2**]. The one-dimensional version of the Newlander-Nirenberg Theorem is the existence of conformal coordinates. The standard references are [**Be**] and [**Che**].

§**3.** Theorem 1 holds in a formal sense for C^∞ functions and hypersurfaces. That is, if $f = f(z, \overline{z}, u)$ is CR on

$$\operatorname{Im} W = v(z, \overline{z}, u)$$

then there exists a formal power series

$$F = \sum a_{lk} z^l w^k$$

such that the Taylor coefficients at the origin for $f(z, \overline{z}, u)$ come from the formal power series for $F(z, u + iv(z, \overline{z}, u))$. This can be proved by adapting to formal power series the proof of Theorem 1.

§**4.** There are several proofs of Theorem 2 in the literature. The earliest seems to be [**AH**] where, however, it is remarked, "Although we were unable to find a proof... in the literature, (the) theorem seems to have been known for a long time." The present author has known each of the proofs given in Chapter One long enough to be unsure of their origins, but he believes that he learned the first from H. Rossi and the second from C. LeBrun. Realizability results in the absence of analyticity involve subtle questions of partial differential equations. Some of these can be avoided if the CR structure is part of a compact CR manifold: A compact strictly pseudoconvex CR manifold of dimension greater than three is locally realizable [**BdM**]. The restriction to dimensions greater than three is essential. The nonrealizable structures of dimension three constructed in Chapter Ten can, in a trivial manner, be extended to strictly pseudoconvex structures on the sphere.

However, a compact CR manifold may be realizable in the neighborhood of each of its points without being globally realizable. There are two interesting and relatively simple examples of such structures on S^3. We have seen that

$$L = w\frac{\partial}{\partial \overline{z}} - z\frac{\partial}{\partial \overline{w}}$$

globally defines the CR structure on $S^3 \subset \mathbb{C}^2$. For $0 \leq t < 1$, the operator $L + t\overline{L}$ also defines a CR structure on S^3. This structure is real analytic, therefore, locally realizable. Indeed, there is a global immersion $S^3 \subset \mathbb{C}^3$ that realizes this structure. But for $t \neq 0$ there is no embedding: any function $f(z, \overline{z}, w, \overline{w})$ on S^3 that is annihilated by $L + t\overline{L}$ for some t, $0 < t < 1$, is even, $f(z, \overline{z}, w, \overline{w}) = f(-z, -\overline{z}, -w, -\overline{w})$. This result is due to [**Bu**] who adapted an example in [**Ros1**]. For a new proof and generalizations see [**Fal**].

This example is strictly pseudoconvex at every point of the sphere. Our second example is strictly pseudoconvex at no point. A CR structure is called Levi flat if the Levi form is identically zero. It is easy to see that any Levi flat CR structure is locally realizable. On the other hand, any compact hypersurface in a Euclidean space has a point at which it is strictly convex. So it follows from the proof of Lemma 4 that any compact hypersurface in $\mathbb{C}^2$ has a point at which it is strictly pseudoconvex. Thus it only remains to construct a Levi flat CR structure on S^3 in order to have a locally, but not globally, realizable structure. A Levi flat structure provides a foliation of S^3 with two-dimensional leaves. So we start with one such foliation. See, for instance, [**La**] for a discussion of the Reeb foliation. This foliation uses the decomposition of S^3 into two solid tori joined along their common torus boundary. We use this common boundary to define an orientation on each leaf. We then place any metric on S^3 and let the J operator on any leaf be rotation by ninety degrees. This gives us the desired Levi flat CR structure.

Note that this structure induces a complex structure on each leaf and also that the foliation has a compact leaf, namely, the common torus boundary. So any global CR map into any $\mathbb{C}^n$ must map this torus to a point (and indeed any global CR function must be a constant on all of S^3). This is a second reason why the CR structure is not globally realizable.

§**6.** There are even strictly pseudoconvex CR structures, with the symmetry of Reinhardt hypersurfaces, which have the constants as the only global CR functions [**Ba**]. In Chapter Ten, we will find a CR structure on which the constants are even the only local CR functions. Such a structure, of course, is not locally realizable at any of its points.

Chapter 3

The basic technique in this chapter is to equate coefficients in Taylor expansions. This was already outlined by Poincaré who commented that to find the invariants of a hypersurface one could use "des calculs qui peuvent être longs mais qui restent élémentaires." What is surprising, besides the amount of careful work necessary to carry out this "elementary" computation, is that, as we see in Chapter 4, this computation reveals an underlying geometric structure. Further, this same structure appears in a completely different approach to the problem (Chapters 6 to 9).

The usefulness of weights in this computation is a reflection of the fact that Q is invariant under the map $(z, w) \to (tz, t^2 w)$ where t is any nonzero real number. The existence of this dilation plays an important role also in other analytic properties of strictly pseudoconvex structures. See, for instance [**BFG**].

Chapter 5

There is a holomorphic version of the Frobenius Theorem which arises from the usual statement and the usual proof simply by assuming that all functions, vector fields, etc. are holomorphic in all their arguments. This version was used in Chapter 1 to establish that all C^ω CR structures are realizable.

There is also a complex version of the Frobenius Theorem. Here the functions, vector fields, etc., are complex-valued but not holomorphic. This is a completely different result and is patterned on the Newlander-Nirenberg Theorem. See [**Ni1**] and [**Ho2**].

Chapter 6

It follows from Cartan's construction that a sufficiently smooth diffeomorphism of real analytic, strictly pseudoconvex CR structures must be real analytic. For let M and M' be real analytic CR structures and let ϕ be a diffeomorphism of M to M'. If ϕ is at least seven times differentiable, then it may be lifted to a map of the geometric bundles B and B'. This lift satisfies a real analytic system of equations of Frobenius form and so is itself real analytic. Thus ϕ is also real analytic. Much stronger results are known. These are related to "reflection principles." See **[BR]** and the references cited therein.

There is a way to directly relate the constructions of Cartan and Moser. Let M be the hypersurface, B the bundle of initial data for the normal form map, and ω_{MC} the Maurer-Cartan connection for $\mathrm{SU}(2, 1)$. Fix some point $p \in M$ and some point $b \in B$ in the fiber over p. Consider the mapping to normal form that corresponds to b. This provides an osculation of M to Q and of B to $\mathrm{SU}(2, 1)$. The connection ω_{MC} can then be transferred from the fiber of $\mathrm{SU}(2, 1)$ over the origin to the fiber of B over p. See **[Ja1]**.

Chapter 7

Every C^∞ complex structure is equivalent to a C^ω complex structure. This is not the case for CR structures, as may be shown using the relative invariant r. For it is not difficult to construct some M that has $r = 0$ on an open set without r being identically zero. In any other coordinate system, r would have this same property. But if the structure on M were analytic, then r would be analytic; and this is impossible. (M can be given explicitly in normal form **[Fa]**.) However, every realizable CR structure, strictly pseudoconvex or not, is given by the boundary values of analytic functions **[HJ]**. This is a partial converse to the fact that any C^ω structure is realizable.

Chapter 8

The chains on the Reinhardt hypersurfaces R_B have another interesting property. Consider the chains at some point $P_1 = (z_0, w_0)$. For an open set of initial directions the projection $z(t)$ of the chain engulfs the origin. Each such chain contains the point $P_2 = (z_0, e^{2\pi i B} w_0)$ which, for B not an integer, is different from the original point. Thus the two points P_1 and P_2 may be connected by an infinite family of distinct chains. These chains correspond to some subset of the chains on Q that pass through a given point.

Chapter 10

There are now in partial differential equations very general nonsolvability results, which are completely divorced from the several complex variables framework of Lewy's original example. See, for instance, **[Ho1]** and **[NT]**. Also, some very simple examples have been found in two dimensions **[Ni2]**. From the point of view of CR structures, the most important remaining solvability question is that of realizing five-dimensional strictly pseudoconvex manifolds. An interesting recent result here is **[NR]** in which it is shown that, unlike in higher dimensions, there is no "homotopy operator" in dimension five. There is also a new class of nonrealizability results for dimension three **[Ro]**.

Finally, time and other constraints precluded the inclusion of several additional topics which might belong in a book such as this. Two which would have been especially appropriate are the Fefferman bundle, from which the chains are realized as the projections of the null geodesics of a Lorentz metric, and Webster's construction of the Cartan curvature, in analogy with the construction of the Weyl conformal curvature from a Riemannian metric. For the first, see [**Fef**] and [**BDS**]; for the second [**We1**]. Of course, there is also an extensive literature on the analysis of the Lewy operators. A good place to start would be the survey article [**BFG**] which, in addition, treats some of the topics in this book.

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Subject Index