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Structural Properties of Polylogarithms

Leonard Lewin
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Preface

As editor of this monograph on polylogarithms I would like to take the liberty of commencing with a few personal reminiscences. I first encountered the dilogarithm function many years ago in high school; it was a fascinating discovery for me, and it initiated a romance that has lasted almost sixty years. For the dilogarithm, the transition from its standing as a curious mathematical oddity to its current status as an important element in the fabric of modern mathematical structure began about fifteen years ago with Bloch's studies on its applications in algebraic K-theory and algebraic geometry. Since then, the pace of discovery has quickened dramatically. In 1980, when I was in the throes of completing my "Polylogarithms and Associated Functions," I became dimly aware that the handful of peculiar numerical identities that had been known since the time of Euler and Landen were, in fact, just the tip of an iceberg of unlimited extent. Thus emerged the new discoveries on cyclotomic equations and their related polylogarithmic "ladders"—a nomenclature that came to me in a dream, after much chewing over of other, more artificial, verbal constructs. Ten years of development in this arena, conducted mostly by the methods of classical analysis with the help of number-crunching computers, ran parallel with other, and more important, discoveries in diverse branches of abstract algebra and algebraic geometry. The confluence of these two streams of thought in the last few years, due to the work of several mathematicians, but particularly to studies of Browkin in Poland and Zagier in Germany, has lead to the present synthesis which I have tried to present in this timely, I hope, monograph.

One of the biggest problems has been the pace of new research; it is obviously extremely difficult to produce a book that is current when new discoveries are taking place all the time and making already-written material partially outdated—though it is also a sign of a very flourishing field when things go this way. During the approximately twelve months that the book has been in active preparation many new discoveries were made. I have endeavored to keep the material up-to-date by the last minute inclusion of two appendices: one on a special workshop on polylogarithms held in November 1990; and one on very recent discoveries on the relation of functional equations to polylogarithmic ladders, Dedekind's zeta function; and including the remarkable discovery by D. Zagier and H. Gangl at the Max-Planck-Institut für Mathe-

matik of a two-variable functional equation for the hexalogarithm—the first significant advance in this area since Kummer's work of 150 years ago. In my earlier (1958) book on dilogarithms, talking about the difficulty of making much further progress in this area, I had written "But the complexity of the present results makes a completely new approach imperative if much progress is to be made." It is now clear what this new approach is entailing: on the one hand the structural analysis arising from algebraic K-theory and related fields; and on the other the extensive use of computers, both for high precision numerical work and also for machine computation using symbolic logic. It is doubtful that many of the new and interesting formulas could have been found by hand alone; powerful computer programs are becoming almost as important as mathematical skills and the ability to generate new constructive conjectures.

This book could not have been written without the splendid help and cooperation of the several contributors who gave generously of their time and effort. Many helpful suggestions and contacts were made. I would particularly like to thank Richard Hain for his assistance in the compilation of the bibliography, Don Zagier for his extensive up-to-date appendix, and Han Sah and Robert MacPherson for their report on the recent polylogarithm workshop.

Authors have very individual styles of writing and it is not practical, for the purpose of uniform presentation, to constrain them into one common pattern of text organization. Even so, I think the overall volume has not suffered from any ensuing tendency to be "patchy," and I hope that, the disparate contributions notwithstanding, the material as a whole is sufficiently coherent to give the entire work the integrity that I, as editor, have sought.

Most authors have written their chapters in the absence of knowing in detail what others were writing. This has given rise to a small amount of redundancy which I have not thought fit to try to remove; I do not think the work has suffered in any way from this. Rather, it has been interesting to see how similar ideas have arisen independently and received corresponding treatment. The whole subject is now in a state of rapid transition; even as I write, new discoveries vie for admission. With reluctance I have had to call a halt to the inclusion of a flood of new material. It will be fascinating to see what further developments the coming decade will bring. Don Zagier once wrote that "the dilogarithm is the only mathematical function with a sense of humor." As this subject matures and gets more important, and more serious, I hope it manages to retain its once light-hearted beginnings. Its ability over the years to attract and hold the interest of so many mathematicians, many of them of the finest caliber, has been outstanding. I hope that its capacity for fruitful exploration will continue unabated for a long time to come.

Leonard Lewin

January 1991

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