Geometric Analysis on Symmetric Spaces

Second Edition

Sigurdur Helgason
Geometric Analysis on Symmetric Spaces

Second Edition
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Sigurdur Helgason
To my Danish mathematical friends

past and present
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This book has been unavailable for some time and I am happy to follow the publisher’s suggestion for a new edition.

While a related forthcoming book, “Integral Geometry and Radon Transforms” (here denoted [IGR]) deals with several examples of homogeneous spaces in duality with corresponding Radon transforms, the present work follows the direction of the first edition and concentrates on analysis on Riemannian symmetric spaces \( X = G/K \). We develop further the theory of the Fourier transform and horocycle transform on \( X \), also taking into account tools developed by Eguchi for the Schwartz space \( S(X) \). These transforms provide the principal methods for analysis on \( X \), existence and uniqueness theorems for invariant differential equations on \( X \), explicit solution formulas, as well as geometric properties of the solutions, for example the harmonic functions and the wave equation on \( X \). On the space \( X \) there is a canonical hyperbolic system on \( X \), introduced by Semenov-Tian-Shansky, which is multitemporal in the sense that the time variable has dimension equal to the rank of \( X \). The solution has remarkable analogies to the classical wave equation on \( \mathbb{R}^n \), summarized in a table in Chapter V, §5.

My intention has been to make the exposition easily accessible to readers with some modest background in Lie group theory which by now is rather widely known. To facilitate self-study and to indicate further developments each chapter concludes with a section “Exercises and Further Results”. Solutions and references are collected at the end of the book. The harder problems are starred. Occasionally results and proofs rely on material from my previous books “Differential Geometry, Lie Groups and Symmetric Spaces” abbreviated [DS] and “Groups and Geometric Analysis”, abbreviated [GGA].

Once again I wish to express my gratitude to my friends and collaborators, Adam Korányi, Gestur Ólafsson, François Rouvière and Henrik Schlichtkrull and especially to my long-term colleague David Vogan for significant help at specified spots in the text. Finally, I thank Brett Coonley and Jan Wetzel for their invaluable help in the production and the editor Dr. Edward Dunne for his interest in the work and his patient and accommodating cooperation.

I would also like to express my thanks for the following permissions of partial quotations:

(ii) To Elsevier concerning my paper [2005].
(iii) To John Wiley and Sons concerning my paper [1998a] and my paper with Schlichtkrull [1999].

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Preface

Among Riemannian manifolds the symmetric spaces in the sense of É. Cartan form an abundant supply of elegant examples whose structure is particularly enhanced by the rich theory of semisimple Lie groups. The simplest examples, the classical 2-sphere $S^2$ and the hyperbolic plane $H^2$, play familiar roles in many fields in mathematics.

On these spaces, global analysis, particularly integration theory and partial differential operators, arises in a canonical fashion by the requirement of geometric invariance. On $\mathbb{R}^n$ these two subjects are related by the Fourier transform. Also harmonic analysis on compact symmetric spaces is well developed through the Peter-Weyl theory for compact groups and Cartan’s refinement thereof. For the noncompact symmetric spaces, however, we are presented with a multitude of new and natural problems.

The present monograph is devoted to geometric analysis on noncompact Riemannian symmetric spaces $X$. (The Euclidean case and the compact case are also briefly investigated in Chapter III, §§7–9, and Chapter IV, §5, but from an unconventional point of view). A central object of study is the algebra $D(X)$ of invariant differential operators on the space. A simultaneous diagonalization of these operators is provided by a certain Fourier transform $f \rightarrow f^\sim$ on $X$ which is the subject of Chapter III. Just as is the case with $\mathbb{R}^n$ the symmetric space $X$ turns out to be self-dual under the mentioned Fourier transform; thus range questions like the intrinsic characterization of $(C^\infty_c(X))^\sim$ in analogy with the classical Paley-Wiener theorem in $\mathbb{R}^n$ become natural and their answers useful.

Chapters II and IV are devoted to the theory of the Radon transform on $X$, particularly inversion formulas and range questions. The space $\Xi$ of horocycles in $X$ offers many analogies to the space $X$ itself and this gives rise to the study of conical functions and conical distributions on $\Xi$ which are the analogs of the spherical functions on $X$. They have interesting connections with the representation theory of the isometry group $G$ of $X$, discussed in Chapter II, §4, and in Chapter VI, §3, where the conical distributions furnish intertwining operators for the spherical principal series. In Corollary 3.9, Ch. VI, these intertwining operators are explicitly related to the above-mentioned Fourier transform on $X$.

While the Fourier transform theory in Chapter III gives rise to an explicit simultaneous diagonalization of the algebra $D(X)$, the Radon transform theory in Chapter II is considered within the framework of a general integral transform theory for dual fibrations in the sense of Chapter I, §3. This viewpoint is extremely general: two dual integral transforms arise whenever we are given two subgroups of a given group $G$. In the introduction to Chapter I we stress this point by indicating five such examples
arising in this fashion from the single group \( G = SU(1, 1) \) of the conformal maps of the unit disk, namely the X-ray transform, the horocycle transform, the Poisson integral, the Pompeiu problem, theta series, and cusp forms. When range results are considered, this viewpoint of the Poisson integral as a Radon transform offers a very interesting analogy with the X-ray transform in \( \mathbf{R}^3 \) (Chapter I, §3, No. 5).

With the tools developed in Chapters I–IV we study in Chapter V some natural problems for the invariant differential operators on \( X \), solvability questions, the structure of the joint eigenfunctions, with emphasis on the harmonic functions, as well as the solutions to the invariant wave equation on \( X \). In Chapter VI we consider in some detail the representations of \( G \) which naturally arise from the joint eigenspaces of the operators in the algebra \( D(X) \) and the algebra \( D(\Xi) \).

The length of this book is a result of my wish to make the exposition easily accessible to readers with some modest background in semisimple Lie group theory. In particular, familiarity with representation theory is not needed. To facilitate self-study and to indicate further developments each chapter is concluded with a section “Exercises and Further Results”. Solutions and references are given towards the end of the book. The harder problems are starred. Occasionally, results and proofs rely on material from my earlier books, “Differential Geometry, Lie groups, and Symmetric Spaces” and “Groups and Geometric Analysis”. In the text these books are denoted by [DS] and [GGA].

Some of the material in this book has been the subject of courses at MIT over a number of years and feedback from participants has been most beneficial. I am particularly indebted to Men-chang Hu, who in his MIT thesis from 1973 determined the conical distributions for \( X \) of rank one. His work is outlined in Chapter II, §6, No. 5–6, following his thesis and in greater detail than in his article Hu [1975]. I am also deeply grateful to Adam Korányi for his advice and generous help with the material in Chapter V, §§3–4, as explained in the notes to that chapter. Similarly, I am grateful to Henrik Schlichtkrull for beneficial discussions and for his suggestions of Proposition 8.6 in Chapter III and Corollary 5.11 in Chapter V, indicated in the text. I have also profited in various ways from expert suggestions from my colleague David Vogan. I am grateful to the National Science Foundation for support during the writing of the book.

Many people have read at least parts of the manuscript and have furnished me with helpful comments and corrections; of these I mention Fulton Gonzalez, Jeremy Orloff, An Yang, Werner Hoffman, Andreas Juhl, François Rouvière, Sönke Seifert, and particularly Frank Richter. I thank them all. Finally, I thank Judy Romvos for her expert and conscientious \( \text{TPX-setting of the manuscript.} \)

A good deal of the material in this monograph has been treated in earlier papers of mine. While subsequent consolidation has usually led to a rewriting of the proofs, texts of theorems as well as occasional proofs
have been preserved with minimal change. I thank Academic Press for permission to quote from the following journal publications of mine, listed in the bibliography: [1970a], [1976], [1980a], [1992b], [1992d], as well as the book [1962a].
SOLUTIONS TO EXERCISES

CHAPTER I

A. Radon Transform on $\mathbb{R}^n$.

A.1. By §2 (27) each $E_k \otimes p^l$ belongs to $\tilde{N}$. Conversely let $\psi \in \tilde{N}$. Let $G = M(n)$ with Haar measure $dg$, let $\xi_o$ be the hyperplane $x_n = 0$ in $\mathbb{R}^n$, and let $\hat{\psi}(g) = \psi(g \cdot \xi_o)$ for $g \in G$. For $F \in D(G)$, $\xi = h \cdot \xi_o$ put

$$\psi_F(\xi) = \int_G F(g)\psi(g^{-1}\xi)dg = \int_G F(g)\psi(g^{-1}h \cdot \xi_o)dg$$

which lies in $\mathcal{E}(\mathbb{P}^n) \cap \tilde{N} = \mathcal{N}$. Let $F$ run through a sequence $(F_i)$ with $F_i \geq 0$, $\int F_i = 1$, $\text{supp}(F_i) \to e$. Then $\psi_{F_i} \to \psi$ in $C(\mathbb{P}^n)$ so statement follows from Theorem 2.5.

A.2. For the Fourier transform $\tilde{\varphi}(s)$ we have

$$\varphi^{(k)}(p) = \frac{1}{2\pi} \int_{\mathbb{R}} \tilde{\varphi}(s)(is)^k e^{isp} ds .$$

By the definition

$$(Y_k \otimes \varphi)^\vee(x) = \frac{1}{\Omega_n} \int \varphi((x, \eta))Y_k(\eta)d\eta$$

$$= \frac{1}{\Omega_n} \frac{1}{2\pi} \int_{\mathbb{R}} \tilde{\varphi}(s) \left( \int_{S^{n-1}} e^{is(x, \eta)}Y_k(\eta)d\eta \right) ds .$$

On the other hand, we have the classical formula (see e.g. [GGA], p. 25)

$$\int_{S^{n-1}} e^{i\lambda(\eta, \omega)}Y_k(\omega)d\omega = c_{n,k}Y_k(\eta) \frac{J_{n+2k-1}(\lambda)}{\lambda^{(n/2)-1}} ,$$

where $c_{n,k} = (2\pi)^{n/2}k^n$ and $J_r$ is the Bessel function. Here we replace $k$ by $0$ and $n$ by $n + 2k$. Then we obtain

$$\int_{S^{n-1}} e^{i\lambda(\omega, \eta)}Y_k(\eta)d\eta = \left( \frac{i\lambda}{2\pi} \right)^k Y_k(\omega) \int_{S^{n+2k-1}} e^{i\lambda\zeta} d\zeta$$

Finally, we put $x = r\omega$ and get

$$(Y_k \otimes \varphi)^\vee(r\omega) = \Omega_n^{-1}(r/2\pi)^k Y_k(\omega) \int_{S^{n+2k-1}} \frac{1}{2\pi} d\zeta \int_{\mathbb{R}} \tilde{\varphi}(s)(is)^k e^{isr\zeta} ds$$

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as desired.

A. 3. We know from §2, No. 3 that $P_k(2, \cos \theta) = H_k(\sin \theta, \cos \theta)$ where $H_k(x_1, x_2)$ is the unique harmonic polynomial on $\mathbb{R}^2$ which is homogeneous of degree $k$, is invariant under $(x_1, x_2) \rightarrow (-x_1, x_2)$ and satisfies $H_k(0, 1) = 1$. Since $(x_1 + ix_2)^k$ and $(x_1 - ix_2)^k$ span the space of homogeneous $k^{th}$ degree harmonic polynomials we have

$$H_k(x_1, x_2) = \text{Re}((x_2 + ix_1)^k),$$

which gives the desired result.

A. 4. If $dk$ is the normalized Haar measure on $K$ we have

$$(\hat{f})^\vee(x) = \int_K \left( \int f(x + k \cdot y)dm(y) \right) dk$$

$$= \int dm(y) \int_K f(x + k \cdot y)dk = \int (M^{|y|}f)(x)dm(y),$$

where $(M^rf)(z)$ is the average of $f$ over $S_r(z)$. Hence

$$(\hat{f})^\vee(x) = \Omega_d \int_0^\infty (M^rf)(x)r^{d-1}dr,$$

so, using polar coordiante around $x$

$$(\hat{f})^\vee(x) = \frac{\Omega_d}{\Omega_n} \int_{\mathbb{R}^n} |x - y|^{d-n}f(y)dy$$

and now the inversion formula follows from the standard inversion of the Riesz potential, ([GGA], Ch. I, Prop. 2.38).

The statement (i) amounts to that if $V$ is a $k$-dimensional vector subspace of $\mathbb{C}^n$ then $V = k \cdot \xi_o$ for some $k \in \mathbb{U}(n)$. This is obvious by choosing a basis of $V$ orthonormal with respect to the standard Hermitian inner product $\langle \ , \rangle$ on $\mathbb{C}^n$.

Statement (ii) amounts to proving that if $W$ is a Lagrangian vector subspace of $\mathbb{R}^{2n}$ then $W = k \cdot \xi_o$ for some $k \in \mathbb{U}(n)$. It is well known that $\dim W = n$. Writing $z = x + iy, \ w = u + iv$ with $x, y, u, v \in \mathbb{R}^n$ we have

$$\langle z, w \rangle = x \cdot u + y \cdot v - i(x \cdot v - y \cdot u)$$

$$= (x, y) \cdot (u, v) - i\{ (x, y), (u, v) \}$$

so the action of $\mathbb{U}(n)$ on $\mathbb{C}^n \sim \mathbb{R}^{2n}$ preserves both the standard inner product on $\mathbb{R}^{2n}$ and the skew symmetric form $\{ \ , \}$. If $e_1, \ldots, e_n$ is an orthonormal basis of $W$ over $\mathbb{R}$ then the formula above shows, $W$ being
isotropic, that \( \langle e_i, e_j \rangle = \delta_{ij} \) so the \( e_i \) form a complex orthonormal basis of \( \mathbb{C}^n \). Viewing the standard orthonormal basis of \( \mathbb{R}^n \times 0 \) as a complex orthonormal basis of \( \mathbb{C}^n \) we see that \( W = k \cdot \xi \) for a suitable \( k \in \mathbb{U}(n) \).

**A. 5.** From [GGA], Ch. I, Theorem 2.20 we have \( \text{supp}(T) \subset \overline{B_A(0)} \). For \( \epsilon > 0 \) let \( f \in \mathcal{D}(X) \) have \( \text{supp}(f) \subset B_{A-\epsilon}(0) \). Then \( \text{supp}(\hat{f}) \subset \overline{B_{A-\epsilon}} \) where

\[
\beta_R = \{ \xi : d(o, \xi) < R \}.
\]

Also by the inversion formula \( cf = (\Lambda \hat{f})^\vee \), since \( \Lambda \) is now a differential operator,

\[
T(cf) = T((\Lambda \hat{f})^\vee) = \hat{T}(\Lambda \hat{f}) = 0
\]

so \( \text{supp}(T) \cap \overline{B_{A-\epsilon}(0)} = \emptyset \).

**A. 6.** We have with a constant \( c \)

\[
(\check{\varphi} \ast f)(x) = c \int_{\mathbb{R}^n} \left( \int_{S^{n-1}} \varphi(w, (w, x) - (w, y)) \, dw \right) f(y) \, dy
\]

\[
= \int_{S^{n-1}} \left( \int_{\mathbb{R}} \varphi(w, (w, x) - p) \hat{f}(w, p) \, dp \right) \, dw
\]

\[
= c \int_{S^{n-1}} (\varphi \ast \hat{f})(w, (w, x)) \, dw = (\varphi \ast \hat{f})^\vee(x)
\]

(Natterer, [1986], p. 14).

**B. Homogeneous Spaces. Grassmann Manifolds.**

**B. 1.** For (ii) we may take \( x_2 = x_o \) and write \( x_1 = g_1K, \xi = \gamma H \). Then

\[
x_o, \xi \text{ incident } \iff \gamma h = k \quad (\text{some } h \in H, k \in K)
\]

\[
x_1, \xi \text{ incident } \iff g_1 k_1 = \gamma h_1 \quad (\text{some } h_1 \in H, k_1 \in K).
\]

Thus if \( x_o, x_1 \) are incident to \( \xi \) we have \( g_1 = kh^{-1}h_1k_1^{-1} \). Conversely, if \( g_1 = k'h'k'' \) we put \( \gamma = k'h' \) and then \( x_o, x_1 \) are incident to \( \xi = \gamma H \).

For (iii) suppose first \( KH \cap HK = K \cup H \). Let \( x_1 \neq x_2 \) in \( X \). Suppose \( \xi_1 \neq \xi_2 \) in \( \Xi \) both incident to \( x_1 \) and \( x_2 \). Let \( x_i = g_iK, \xi_j = \gamma_jH \). Since \( x_i \) is incident to \( \xi_j \) there exist \( k_{ij} \in K, h_{ij} \in H \) such that

\[
g_i k_{ij} = \gamma_j h_{ij} \quad i = 1, 2; \quad j = 1, 2.
\]

By eliminating \( g_i \) and \( \gamma_j \) we obtain

\[
k_{21}^{-1} k_{21} h_{21}^{-1} h_{11} = h_{22}^{-1} h_{12} k_{12}^{-1} k_{11}.
\]
This being in \( KH \cap HK \) it lies in \( K \cup H \). If the left hand side is in \( K \), then
\[
g_2K = \gamma_1 h_{21} K = \gamma_1 h_{11} K = g_1 K
\]
which is a contradiction. Similarly, if the mentioned left hand side is in \( H \) we have \( k_{21}^{-1} k_{21} \in H \) which gives the contradiction \( \gamma_2 H = \gamma_1 H \).

Conversely, suppose \( KH \cap HK \neq K \cup H \). Then there exist \( h_1, h_2, k_1, k_2 \) such that \( h_1 k_1 = k_2 h_2 \) and \( h_1 k_1 \notin K \cup H \). Put \( x_1 = h_1 K, \xi_2 = k_2 H \). Then \( x_o \neq x_1, \xi_o \neq \xi_2 \), yet both \( \xi_o \) and \( \xi_2 \) are incident to both \( x_o \) and \( x_1 \).

**B. 2–3.** For the first statement see [GGA], Cor. 4.10, Ch. II. For the other suppose the generators \( D_i = D_{P_i} \) were not algebraically independent. Let
\[
P = \Sigma a_{n_1} \ldots n_\ell x_1^{n_1} \ldots x_\ell^{n_\ell}
\]
be a nonzero polynomial such that \( P(D_1, \ldots, D_\ell) = 0 \). Let \( d_i = \text{degree} \ (P_i) \) and \( N = \max(\Sigma d_i n_i) \), the maximum taken over the set of \( \ell \)-tuples \( (n_1, \ldots, n_\ell) \) for which \( a_{n_1} \ldots n_\ell \neq 0 \). We write the polynomial
\[
S = \Sigma a_{n_1} \ldots n_\ell P_1^{n_1} \ldots P_\ell^{n_\ell}
\]
as the sum \( S = Q + R \), where
\[
Q = \sum_{\Sigma d_i n_i = N} a_{n_1} \ldots n_\ell P_1^{n_1} \ldots P_\ell^{n_\ell}
\]
and degree \( (R) < N \). Also \( Q \neq 0 \) by assumption. Consider the operator
\[
\Sigma a_{n_1} \ldots n_\ell D_1^{n_1} \ldots D_\ell^{n_\ell} - D_S
\]
whose order is \( < N \) ([GGA], p. 287). This operator equals \( 0 - D_Q - D_R \) which by the definition in Exercise B2 has order \( N \). This gives the desired contradiction.

**B. 10.** Method of Helgason [1957] or [GGA], Ch. V, Lemma 2.6. First show that it suffices to compute
\[
\int_{U(n)} |v_{ij}|^2 |v_{k\ell}|^2 dV
\]
and that this integral is given by

(i) \( (n(n+1))^{-1} \) if \( (i, j) \) and \( (k, \ell) \) are either in the same row or the same column (not both).

(ii) \( 2(n(n+1))^{-1} \) if \( (i, j) = (k, \ell) \)
(iii) \((n^2 - 1)^{-1}\) if \((i,j)\) and \((k,\ell)\) are neither in the same row nor the same column. See also Faraut-Korányi [1993], p. 237.

B. 11. The proof is obtained by expanding in a Fourier series on \(T^2\) (also observed by Gindikin).

B. 12. If \(U/K\) has rank one see [GGA], Ch. I, Cor. 4.19. If \(U/K\) has higher rank the result is immediate from Exercise 11 as pointed out by Grinberg.

B. 13. \(\tilde{d}\) is a \(K\)-orbit containing \((1,0)\) so equals \(B\). Also \(H \cdot o\) is two-dimensional so equals \(D\).

CHAPTER II

A. The Spaces \(X = G/K\) and \(\Xi = G/MN\).

A. 1. If \(kN \subset NK\) then \(k \cdot \xi_o \subset \xi_o\) so \(k \subset M\) by text. If \(nK \subset KN\) then \(n \cdot o\) belongs to each horocycle through \(o\). If \(n \neq e\), \(n \cdot o = ka \cdot o\) \((a \neq e)\). But \(k \cdot \xi_o\) does not contain \(ka \cdot o = n \cdot o\).

Let \(g = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}\) be the usual Iwasawa decomposition of \(g = \mathfrak{sl}(2, \mathbb{R})\) (as before Lemma 4.9). Let \(g = \mathfrak{m} + \mathfrak{n} + \mathfrak{q}\) where \(\mathfrak{q}\) is \(MN\)-invariant. Let \(H \in \mathfrak{a}\) have the component \(H_1\) in \(\mathfrak{q}\). Then \([H_1, \mathfrak{n}] \subset \mathfrak{n}\) is a contradiction.

A. 2. Use (4) \(\S 3\).

A. 3. Recall proof of Lemma 4.9 (ii).

A. 4. Consider \(V = \mathbb{C}^{n+1}\) with the Hermitian form

\[
\langle y, w \rangle = y_0 \bar{w}_0 - y_1 \bar{w}_1 - \cdots - y_n \bar{w}_n
\]

and put \(V^+ = \{y \in \mathbb{C}^{n+1} : \langle y, y \rangle > 0\}\). The Hermitian hyperbolic space can be taken as \(V^+ / \mathbb{C}^*\). With non-homogeneous coordinates \(z_i = y_i / y_o\), \(V^+ / \mathbb{C}^*\) is identified with the ball

\[
B^+ = \{z \in \mathbb{C}^n : |z_1|^2 + \cdots + |z_n|^2 < 1\}
\]

and the unitary action \(U(1, n) = U(V)\) on \(V\) induces the action of the projective group \(\mathbf{PU}(V)\) on \(B^+\) \((\mathbf{SU}(1, n)\) mod its center, cf. [DS], X, Exercise D1). Let \(\pi : V \rightarrow V / \mathbb{C}^*\) be the natural map. Choose \(\ell^* \in \partial B^+\) and choose \(y^* \neq 0\) on \(\ell^*\). The Iwasawa subgroup \(N\) (the unipotent radical of the isotropy group \(\mathbf{PU}(V)_{\ell^*}\)) viewed as a subgroup of \(\mathbf{SU}(1, n)\) fixes \(y^*\) and hence also the function

\[
d_y^*(\ell) = \frac{\langle y^*, y \rangle}{\langle y, y \rangle^{\frac{1}{2}}} , \quad y \in \pi^{-1}(\ell), \quad \ell \in B^+.
\]
Thus the equation $d_{y^*} = c$, that is,

$$|\langle y^*, y \rangle|^2 = |\langle y, y \rangle|^2 c^2$$

is a horocycle. In non-homogeneous coordinates this is

$$|1 - z_1^* \bar{z}_1 - \cdots - z_n^* \bar{z}_n|^2 = (1 - |z_1|^2 - \cdots - |z_n|^2) \frac{c^2}{|y^*_o|^2}$$

which is an ellipsoid in the Euclidean metric. A $\textbf{PU}(V)$-invariant metric on $B^+$ is given by (cf. Mostow [1973], p. 136)

$$d(w, y) = \cosh^{-1} \left( \frac{|\langle w, y \rangle|}{\langle w, w \rangle^{\frac{1}{2}} \langle y, y \rangle^{\frac{1}{2}}} \right)$$

so the sphere $S_r(\pi(w))$ is

$$\frac{|\langle w, y \rangle|}{|\langle y, y \rangle|^2} = \frac{1}{\langle w, w \rangle^{\frac{1}{2}}} \text{ch } r.$$ 

Let $w \to y^*, r \to \infty$ with $\langle w, w \rangle^{\frac{1}{2}} \text{ch } r = c$ (where $\langle y^*, y^* \rangle = 0$). Then the sphere converges to the horocycle above.

Another verification in terms of the notation of [DS], IX, (§3 and Exercise B4). The horocycle $\tilde{N} \cdot o$ is given by

$$(w_1, w_2) = \left( \frac{2it - |z|^2}{2(1 - it) + |z|^2}, \frac{-2 \bar{z}}{2(1 - it) + |z|^2} \right)$$

and therefore equals the ellipsoid

$$2|w_1 + \frac{1}{2}|^2 + |w_2|^2 = \frac{1}{2}.$$ 

Similarly the horocycle $N \cdot o$ equals

$$2|w_1 - \frac{1}{2}|^2 + |w_2|^2 = \frac{1}{2}.$$ 

Let

$$a_r = \begin{pmatrix} \text{ch } r & 0 & \text{sh } r \\ 0 & 1 & 0 \\ \text{sh } r & 0 & \text{ch } r \end{pmatrix}.$$ 

Then the sphere $S_r(o)$ equals $K a_r \cdot o$ which is given by

$$|z_1|^2 + |z_2|^2 = \text{th}^2 r.$$ 

The image $a_r \cdot S_r(0)$ is by [DS], IX, Exercise B4 given by

$$a_r \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = (z_1 \text{sh } r + \text{ch } r)^{-1} \begin{pmatrix} z_1 \text{ch } r + \text{sh } r \\ z_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$
so the equation for $a_r \cdot S_r(o)$ is
\[
(1 + \text{th}^2 r)|w_1|^2 - \text{thr}(w_1 + \tilde{w}_1) + |w_2|^2 = 0.
\]
Thus as $r \to \infty$ the sphere $a_r S_r(0)$ converges to the horocycle (\text{*}).

**A. 5.** First reduce the problem to the case $X = H^2$ as follows. Let $X_\alpha$ be a root vector in the Lie algebra of $N$ and let $G_\alpha$ denote the analytic subgroup of $G$ with Lie algebra $R X_\alpha + R \theta X_\alpha + R[X_\alpha, \theta X_\alpha]$. Then $G_\alpha \cdot o$ is a totally geodesic submanifold of $X$ isometric to $H^2$ and the horocycle $\exp t X_\alpha \cdot o$ in $G_\alpha \cdot o$ equals $(G_\alpha \cdot o) \cap (N \cdot o)$. This reduces the problem to $H^2$ with metric
\[
ds^2 = \frac{dx^2 + dy^2}{y^2}, \quad y > 0,
\]
where the geodesics are the semicircles
\[
\gamma_{u,r} : \quad x = u + r \cos \theta, \quad y = r \sin \theta, \quad 0 < \theta < \pi.
\]
We have
\[
\tilde{f}(\gamma_{u,r}) = \int_0^\pi f(u + r \cos \theta, r \sin \theta)(\sin \theta)^{-1} d\theta
\]
so taking $\xi$ as the line $y = 1$ our assumption amounts to
\[
\int_{\gamma_{u,r}} \frac{f(x,y)}{y} dw = 0, \quad r < 1,
\]
where $dw$ is the Euclidean arc element. The rapid decrease of $f$ implies that $f(x,y)/y$ extends to a smooth function $F$ on $\mathbb{R}^2$ by $F(x,y) = f(x,|y|)/|y|$. Then
\[
(* \quad \int_{S_r(x)} F(s) dw(s) = 0 \quad x \in \mathbb{R}, \ r < 1.
\]
This implies for the corresponding disk $B_r(x)$
\[
\int_{B_r(x)} F(u,v) dudv = 0,
\]
whence
\[
\int_{B_r(o)} (\partial_1 F)(x + u, v) du \ dv = 0
\]
with $\partial_1 = \partial/\partial u$. Using the divergence theorem on the vector field $F(x + u, v)\partial/\partial u$ we get
\[
\int_{S_r(o)} F(x + u, v) u dw(u,v) = 0.
\]
Combining this with (*) we deduce

\[
(\star\star) \quad \int_{S_r(x)} F(s) s_1 dw(s) \quad s = (s_1, s_2).
\]

Iterating the implication \((*) \implies (\star\star)\) we obtain

\[
\int_{S_r(x)} F(s) P(s_1) dw(s) = 0
\]

where \(P\) is any polynomial so we get the desired conclusion \(f \equiv 0\) on the strip \(0 < y < 1\).

**A. 7.** Because of Theorem 2.9 it suffices to prove that the convolution algebra \(C_c^0(MN)\) of \(M\)-bi-invariant functions in \(C_c(MN)\) is commutative. This result from Korányi [1980] follows (for \(m_{2\alpha} \neq 1\)) from Kostant’s theorem (Exercise D3 below) which implies that for each \(n \in N\) there exists an \(m \in M\) such that \(mnm^{-1} = n^{-1}\). Thus \(f(n) = f(n^{-1})\) for \(f \in C_c^0(MN)\) which implies the commutativity. For the case \(m_{2\alpha} = 1\) see [GGA], Ch. IV, Exercise B10.

**A. 8.** With the customary notation we have (as \(m^*k(\hat{n})M = k(\hat{n}(m^*\hat{n}))M\),

\[
\int_{\tilde{N}} F(k(\hat{n})M) e^{-2\rho(H(\hat{n}))} d\hat{n} = \int_{K/M} F(kM) dk_M
\]

\[
= \int_{\tilde{N}} F(k(\hat{n}(m^*\hat{n}))M) e^{-2\rho(H(\hat{n}))} d\hat{n},
\]

and since by §6, \(H(\hat{n}) = H(\hat{n}(m^*\hat{n})) + B(m^*\hat{n})\), this integral equals

\[
\int_{\tilde{N}} F(k(J\hat{n})M) e^{-2\rho(H(J\hat{n}))} e^{-2\rho(B(m^*\hat{n}))} \frac{d\hat{n}}{d(J\hat{n})} d(J\hat{n}),
\]

proving the result.

**A. 9.** The vector \(v\) is in the center of \(\mathfrak{t}_o\) so is fixed under \(Ad_{G_o}(K_o)\); also \(e\) is in the highest root space so, \(Ad_{G_o}\) being spherical, \(e\) is \(M_o\)-fixed. By computation

\[
Ad(a_t)v = v + 3 \text{ sht} \begin{pmatrix} sht i & 0 & -\text{chti} \\ 0 & 0 & 0 \\ \text{chti} & i & 0 \end{pmatrix}
\]

\[
Ad(a_t)e = e^{2t} \begin{pmatrix} i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & -i \end{pmatrix}.
\]
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Put
\[
v_o = \begin{pmatrix} -\frac{i}{2} & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -\frac{i}{2} \end{pmatrix}, \quad v_1 = \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 & 0 & -i \\ i & 0 & 0 \end{pmatrix}.
\]

Then the curve
\[
t \mapsto \text{Ad}(a_t)\nu = v_o + \frac{3}{2}\text{ch}2t \quad v_1 + \frac{3}{2}\text{sh}2t \quad v_2
\]
lies in the intersection of \(X_o\) with the plane \((s_1, s_2) \mapsto v_o + s_1v_1 + s_2v_2\).

A. 10. Consider \(a_o\) as in [DS], Cor. 7.6, Ch. VIII. The geodesic \(\text{Ad}(\exp t(X_\gamma + X_{-\gamma}))\nu\) is easily computed and lies in the plane
\[
(s_1, s_2) \mapsto v + s_1(X_\gamma - X_{-\gamma}) + s_2H_\gamma.
\]

B. Conical Functions.

Part (i) is immediate from Theorem 4.8. For (ii) recall that by Corollary 4.13, \(-s^*\mu\) is the highest weight of the contragredient \(\pi'_\psi\). For \(m^*\) we choose
\[
m^* = \begin{pmatrix} 0 & \ldots & 0 & \epsilon \\ 0 & \ldots & 1 & 0 \\ \vdots \\ 1 & 0 & 0 & 0 \end{pmatrix}
\]
where \(\epsilon = \pm 1\), the sign determined by \(\det(m^*) = 1\). Also \(M\) consists of the diagonal matrices \(m\) with diagonal elements \(\pm 1\) satisfying \(\det(m) = 1\). If \(g \in \tilde{N}M\tilde{A}N\), \(g = \tilde{n}(g)m(g)\exp B(g)n_B(g)\) then by [DS], IX, Exercise A2, the diagonal matrix \(\exp B(g)\) has entries
\[
\exp B(g)_{ii} = \left| \frac{\Delta_i(g)}{\Delta_{i-1}(g)} \right|,
\]
where \(\Delta_i(g) = \det((g\ell m)_{1 \leq \ell, m \leq i})\) with \(g = (g\ell m)\). By Theorem 4.7
\[
\psi(m^*g \cdot \xi_o) = \psi(m^*\tilde{n}(g)\exp B(g) \cdot \xi_o)
\]
\[
= (\pi_\psi(\exp(-B(g))\tilde{n}(g)^{-1}(m^*)^{-1})e, e')
\]
\[
= (\pi_\psi((m^*)^{-1})e, \pi_\psi(\exp(B(g))e')
\]
\[
= \psi(\xi^*)e^{-s^*(B(g))}.
\]
Now if \(h = m^*g\) so \(g = (m^*)^{-1}h\) then \(|\Delta_i(g)| = |D_i(h)|\) so the desired formula for \(\psi(h \cdot \xi_o)\) follows.

The conical functions in this case are related to “conical polynomials” studied in the book by Faraut and Korányi [1993].
C. Hyperbolic Space; Inversion and Support Theorems.

C. 1. (i) By orthogonality with the geodesics, the horocycles are the 
$(n - 1)$-spheres tangential to the boundary $|x| = 1$. The induced metric on
the horocycle is flat. This is obvious for example in the upper half-space
model where $N \cdot o$ is a horizontal plane.

(ii) We see that if $\xi_o = N \cdot o$ and $dq$ the volume element on $\xi_o$ then
\[
(f^\wedge)(g \cdot o) = \int dk \int f(gk \cdot q)dq = \int_k \left[ M^{d(o,q)} f \right](p)dq,
\]
where $(M^r f)(p)$ is the average of $f$ over $S_r(p)$. Thus
\[
(f^\wedge)(p) = \Omega_{n-1} \int_0^\infty (M^r f)(p)\rho^{n-2}d\rho,
\]
where $r = d(o,q)$ ($d$ = distance in $H^n$) and $\rho = d'(o,q)$ ($d'$ = distance on
horocycle).

It suffices to prove $\rho = \sinh r$ when $q$ is in the $x_1x_n$-plane so we are in
the two-dimensional case. From [GGA] p. 36 ($R$ and $N \cdot o$ are isometric
under $x \rightarrow \frac{x}{x+1}$) we see that
\[
r = \frac{1}{2} \log \left( \frac{1+|x|}{1-|x|} \right) \quad \rho = |x|.
\]
The first formula means
\[
\frac{|x|}{|x+1|} = \tanh r \quad \text{or} \quad \rho(1+\rho^2)^{-\frac{1}{2}} = \tanh r
\]
so $\rho = \sinh r$. Hence
\[
(f^\wedge)(p) = \Omega_{n-1} \int_0^\infty (M^r f)(p)\sinh^{n-2}r \cosh \rho dr.
\]

(v) (vi) Since the area of $S_r(p)$ is proportional to $\sinh^{n-1}(2r)$ the formula
in (v) follows from [GGA], Ch. II, Prop. 5.26. For (vi) we can write
\[
(f^\wedge)(p) = \frac{1}{2}\Omega_{n-1} \int_0^\infty (M^r f)(p)\sinh(2r)\sinh^{n-3}(r)dr.
\]

(vi)–(vii) Let $F(r) = (M^r f)(p)$, let $\Delta_r = \Delta(L)$ and assume $k$ even
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> 0. Then

\[
\int \text{sh}^k \text{rsh}(2r) \Delta_r F(r) dr = (k + 2)(k - 2n + 4) \int_0^\infty F(r) \text{sh}^k \text{rsh}(2r) dr
\]

\[
+ k(k - n + 2) \int_0^\infty F(r) \text{sh}^{k-2} \text{rsh}(2r) dr.
\]

If \( k = 0 \) this should be

\[
-2(n - 2) \left( \int_0^\infty 2F(r) \text{sh}(2r) dr + F(0) \right).
\]

Proof. By the Darboux equation, \( L \) applied to \((\bar{f})^\wedge(p)\) amounts to the application of \( \Delta(L) = \Delta_r \) to \( F(r) \). Now

\[
\int_0^\infty \text{sh}^k \text{rsh}(2r) \left( \frac{d^2 F}{dr^2} + 2(n - 1) \coth(2r) \frac{dF}{dr} \right) dr = \left[ \text{sh}^k r \text{ sh}(2r) F' \right]_0^\infty
\]

\[
- \int_0^\infty F' [\text{sh}^k r \text{ ch}(2r) 2 + k \text{sh}^{k-1} \text{ rch} \text{ sh}(2r) - 2(n - 1) \text{sh}^k r \text{ ch}(2r)] dr
\]

\[
= 2(n - 2) \int_0^\infty \text{sh}^k r \text{ ch}(2r) F' dr - \frac{k}{2} \int_0^\infty \text{sh}^{k-2} r \text{ sh}^2(2r) F'' dr
\]

\[
= 2(n - 2) \left\{ \left[ \text{sh}^k r \text{ ch}(2r) F \right]_0^\infty \right. \}
\]

\[
- \int_0^\infty F [2\text{sh}(2r) \text{sh}^k r + k \text{sh}^{k-1} \text{ rch} \text{ sh}(2r)] dr \}
\]

\[
- \frac{k}{2} \left\{ \left[ \text{sh}^{k-2} r \text{ sh}^2(2r) F \right]_0^\infty \right. \}
\]

\[
- \int_0^\infty F [\text{sh}^{k-2} r 4\text{sh}(2r) \text{ch}(2r) + (k - 2) \text{sh}^{k-3} r \text{ ch} \text{ r sh}^2(2r)] dr \}
\]

\[
= -2(n - 2) \int_0^\infty F [2\text{sh}(2r) \text{sh}^k r + \frac{k}{2} \text{sh}^{k-2} r \text{sh}(2r) + k \text{sh}^k r \text{ sh}(2r)] dr
\]

\[
+ \frac{k}{2} \int_0^\infty F [4\text{sh}^{k-2} \text{rsh}2r + 8\text{sh}^k \text{rsh}(2r) + (k - 2)(2\text{sh}^{k-2} r + 2\text{sh}^k r) \text{sh}(2r)] dr
\]
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\[ = \int_0^\infty F \sinh^k \cosh(2r) dr \{ (k + 2)(k - 2n + 4) \} dr \]

\[ + \int_0^\infty F(r) \sinh^{k-2} \cosh(2r) \{ k(k - n + 2) \} dr \]

This means

\[ (L + (k + 2)(2n - k - 4)) \int_0^\infty F(r) \sinh^k \cosh(2r) dr \]

\[ = -(n - k - 2)k \int_0^\infty F(r) \sinh^{k-2} \cosh(2r) dr. \]

By iteration, \( k = n - 3, n - 5, \ldots \), we obtain

\[ (L + (n - 1)(n - 1)) \ldots (L + 2(2n - 4)) \{ \tilde{f} \}^\nu = (-1)^{n-1} \Omega_{n-1}(n - 2)! f. \]

For a different inversion method see Gelfand, Graev and Vilenkin [1966], Ch. V, §2.

C. 2. Use [GGA], Ch. IV, Exercise C3 (for the case of a hyperbolic space) and combine with [GGA], Ch. I, Lemma 4.4. (For full details see Helgason [1980b]).

D. Conical Distributions.

D. 1. (Sketch) To see first that the theorem is local let \( \{ V_\alpha \}_{\alpha \in A} \) be a locally finite covering of \( V \) by coordinate neighborhoods and \( 1 = \sum \phi_\alpha \) a partition of 1 subordinate to this covering. Then \( T = \sum \phi_\alpha (T|V_\alpha) \) where each restriction \( T|V_\alpha \) is assumed to have the indicated representation with distributions \( \tilde{T}_{n_1, \ldots, n_p, \alpha} \) on \( V_\alpha \). In order to move the \( \phi_\alpha \) past the \( X_i \) over on \( \tilde{T}_{n_1, \ldots, n_p, \alpha} \) we repeatedly use the formula \( \phi(Xf) = X(f\phi) - fX\phi \). For the local version of the theorem let \( \exp tX_i \) be a local 1-parameter group of local diffeomorphisms of a neighborhood of \( w \in W \) in \( V \). Then

\[ (X_1X_2\phi)(v) = \left\{ \frac{d}{dt_1}(X_2\phi)(\exp(-tX_1) \cdot v) \right\}_{t_1=0} \]

\[ = \left\{ \frac{d}{dt_1} \frac{d}{dt_2} \phi(\exp(-t_2X_2) \exp(-t_1X_1) \cdot v) \right\}_{t_1=t_2=0} \]

and if \( \partial_i = \partial/\partial t_i \),

\[ ((X_1^{n_1} \ldots X_p^{n_p})(\phi))(v) = \{ \partial_1^{n_1} \ldots \partial_p^{n_p} \phi(\exp(-t_pX_p) \ldots \exp(-t_1X_1) \cdot v) \}_{t=0}. \]

Schwartz' theorem representing \( T \) in terms of the \( \partial_i \) (Schwartz [1966], Th. XXXV) therefore gives the result of the exercise.
D. 2. Let \( \mathfrak{g}^\alpha \) be the subalgebra generated by \( \mathfrak{g}_\alpha \) and \( \mathfrak{g}_{-\alpha} \). Then \( \mathfrak{g}^\alpha \) is semisimple of real rank one and

\[
\mathfrak{g}^\alpha = \mathfrak{g}_{-2\alpha} + \mathfrak{g}_{-\alpha} + \mathfrak{g}_\alpha + \mathfrak{g}_{2\alpha} + (\mathfrak{g}^\alpha)_0
\]

([DS], IX, §2). Let \( e_j \in \mathfrak{g}_{j\alpha} \). Then \( e_j, \theta e_j \) and \( w = [e_j, \theta e_j] \) span \( \mathfrak{sl}(2, \mathbb{C}) \) which operates on \( (\mathfrak{g}^\alpha)^G \). By [GGA], Appendix, Cor. 1.5, \( \mathfrak{g}_{j\alpha} \subset [(\mathfrak{g}^\alpha)^G, e_j] \) so

\[
[(\mathfrak{g}^\alpha)_0, e_j] = \mathfrak{g}_{j\alpha}.
\]

A fortiori \( [m + a, e_j] = \mathfrak{g}_{j\alpha} \) so the orbit \( M \cdot e_j \) has codimension 1 so if the sphere is connected it must be \( M \cdot e_j \) (cf. Kostant [1975], Ch. II).

D. 3. (Sketch following Wallach [1973] and Lepowsky [1975].)

(a) \( \mathfrak{g} = \mathfrak{g}_{2\alpha} + \mathfrak{g}_{-\alpha} + \mathfrak{g}_0 + \mathfrak{g}_\alpha + \mathfrak{g}_{2\alpha} \quad \mathfrak{g}_0 = m + a. \) Select \( X \in \mathfrak{g}_\alpha, Y = -\theta X \in \mathfrak{g}_{-\alpha} \), such that the vector \( H = [X, Y] \in a \) satisfies

\[
[H, X] = 2X, \quad [H, Y] = -2Y.
\]

The algebra \( s = RX + RY + RH \) is isomorphic to \( \mathfrak{sl}(2, \mathbb{R}) \) and \( \pi = ad_{\mathfrak{g}}|s \) is a representation of \( s \) on \( \mathfrak{g} \). Deduce from [GGA], Appendix, Lemma 1.2 (ii) that since the eigenvalues of \( ad \) \( H \) on \( \mathfrak{g} \) are 0, ±2, ±4 the dimensions of the irreducible components of \( \pi \) can only be 1, 3 or 5.

(b) Let \( \mathfrak{g}^i \) denote the sum of all the \( (2i + 1) \)-dimensional irreducible components of \( \mathfrak{g} \) and put

\[
\mathfrak{g}_j^i = \mathfrak{g}^i \cap \mathfrak{g}_{j\alpha} \quad (0 \leq i \leq 2, -2 \leq j \leq 2).
\]

Then

\[
\mathfrak{g}^i = \oplus_j \mathfrak{g}_j^i, \mathfrak{g}_{\pm 2\alpha} = \mathfrak{g}_{\pm 2}, \mathfrak{g}_{\pm \alpha} = \mathfrak{g}_{\pm 1} \oplus \mathfrak{g}_{\pm 2}, \mathfrak{g}_0 = \mathfrak{g}_0^0 \oplus \mathfrak{g}_0^1 \oplus \mathfrak{g}_0^2,
\]

and the decomposition

\[
\mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{g}^1 \oplus \mathfrak{g}^2
\]

is both \( B^- \) and \( B_0 \)-orthogonal.

(c) Using

\[
[X_\alpha, X_{-\alpha}] - B(X_\alpha, X_{-\alpha})A_\alpha \in m,
\]

show that

\[
\mathfrak{g}_2^0 \subset m, \quad \mathfrak{g}_0^1 = RA_\alpha \oplus (\mathfrak{g}_0^1 \cap m), \quad \mathfrak{g}_0^2 \subset m.
\]

Let \( m_i = \mathfrak{g}_i \cap m \ (i = 0, 1, 2) \). The \( m_0 \) is the Lie algebra of \( M_0 \).

(d) For \( Z \in \mathfrak{g} \) put \( Z^* = [X, Z], \quad Z^{**} = (Z^*)^*, Z_* = [Y, Z], \quad Z_{**} = (Z_*)^* \). Prove that if \( Z \in m_2 \),

\[
(Z^{**})_* = 4Z^*, \quad (Z_*^*)^* = 4Z_*, \quad (Z^*)^*_* = 6Z, \quad (Z_*)^*_* = 6Z
\]
and deduce for $Y, Z \in m_2$
\[
[Y, Z^{**}] = [Y^{**}, Z] = \frac{2}{3}[Z^*, Y^*], \quad [Y, Z]^{**} = \frac{2}{3}[Y^*, Z^*].
\]

(e) Given $Z \in g$, let $Z_1$ be the component in $g^i$ in the decomposition $g = g^0 \oplus g^i \oplus g^2$. Then if $Y, Z \in m_2$,
\[
[Y, Z]_1 = 0, \quad [[Y, Z]_0 + 2[Y, Z]_2, Z^{**}] = 0.
\]

(f) Suppose $Y, Z \in m_2$ and $B_\theta(Y^{**}, Z^{**}) = 0$. Then
\[
[Y^{**}, Z_{**}] = -[Z^{**}, Y_{**}] \in m, \quad [Y^{**}, Z_{**}]_1 = 0
\]
and
\[
[Y^*, Z_{**}] = -6[Y, Z]_*, \quad [[Y, Z]_0 Z^{**}] = \frac{\langle \alpha, \alpha \rangle}{g} B(Y, \theta Y) Y^{**}.
\]

(g) Let $U \in g_{2\alpha}$ and select $Z \in m_2$ such that $U = Z^{**}$. Let $V$ be in the orthocomplement (for $B_\theta$) of $U$ in $g_{2\alpha}$ and select $Y \in m_2$ such that $Y^{**} = V$. Deduce from (f) that $[W, U] = V$ for some $W \in m_o$ and consequently $M_o \cdot U$ fills up a sphere in $g_{2\alpha}$.

**D. 4.** For the existence of $S_\Psi$, one can just repeat the proof of Prop. 4.4. Part (a) is obvious. For Part (b) we have by the definition of $\Psi_o$, Lemma 3.1 and Cor. 6.2,
\[
\Psi_o(\varphi) = \int_{\Xi} (\varphi - \varphi_o)(\xi)e^{\rho(\log a(\xi))}d\xi
\]
\[
= \int_{\tilde{N}_A} (\varphi - \varphi_o)(\tilde{n}a \cdot \xi_o)e^{-\rho(\log a + B(m^* \tilde{n}))}e^{2\rho(\log a)}da \, d\tilde{n}.
\]
Now take $\varphi$ of the form $\varphi(\tilde{n}a \cdot \xi_o) = f(\tilde{n})g(a)$ where $\int g(a)e^{\rho(\log a)}da = 1$. Then (b) follows.

**D. 5.** (i) Use Theorem 4.1 and Corollary 6.2. (ii) Use Cor. 6.2. (iii) Use the $M$-invariance of $S$ and $S_\Psi$. (iv) Prove
\[
(u^2 + v^2)D_\delta \otimes T_0 \in \text{Con}(D'_0)
\]
as an intermediary result. (vi) With the particular $g$ chosen one finds (with $f^{n-1}(\tilde{n}) = f(\tilde{n}(\tilde{n}))$)
\[
\Psi((f \otimes g)^{n-1}) = (S + c\Delta \delta)f^{n-1}
\]
and for the particular choice of $f$, $(\Delta \delta)(f^{n-1}) = 0$. Thus $h(s) = S(f^{n-1}) - S(f)$ and the contradiction $h'(0) \neq 0$ is obtained by an elementary computation.
D. 6. Solution is similar to that of Exercise D5. For (iv) it is useful to remark the following. Let
\[
g = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} \in \text{SU}(2,1)
\]
and
\[
\sigma = \begin{pmatrix} p & -\bar{q} & 0 \\ q & \bar{p} & 0 \\ 0 & 0 & 1 \end{pmatrix} \in K
\]
such that \( k(g)M = \sigma M \). Then
\[
p = \left( g_{11} + g_{13} \right) / \left( g_{31} + g_{33} \right), \quad q = \left( g_{21} + g_{23} \right) / \left( g_{21} + g_{33} \right)
\]
and \( k(\bar{n}(n\bar{n}))M = k(n\bar{n})M \).

E. The Heisenberg Group.

E. 1.–E. 2. See Faraut and Harzallah [1987].

E. 3. The homogeneity and the left invariance are obvious. Since \( d(g, e) = \| g \| \) only the inequality \( \| g_1 g_2 \| \leq \| g_1 \| + \| g_2 \| \) remains to be proved and this just involves the Schwarz inequality (Cygan [1981], Korányi [1983] or Faraut and Harzallah [1987]).

For E. 4, E. 5, and E. 6, see Cowling [1982], Folland [1973] and Korányi [1982b]. For an exposition of these results see Faraut and Harzallah [1987]. Much of the theory is generalized to \( \tilde{N} \) for \( G/K \) of rank one in Cowling, Dooley, Korányi and Ricci [1992].

CHAPTER III

A. Differential Operators.

A. 1. We have for \( k \in K, \, g \in G, \, n \in N, \, a \in A \)

\[
\eta_\lambda(kgn) = \eta_\lambda(g), \quad \eta_\lambda(ga) = e^{(i\lambda-\rho)(\log a)}\eta_\lambda(g).
\]

In the decomposition
\[
\mathbf{D}(G) = (t\mathbf{D}(G) + \mathbf{D}(G)n) \oplus \mathbf{D}(A)
\]
let \( D \rightarrow D_A \) denote the projection of \( \mathbf{D}(G) \) onto \( \mathbf{D}(A) \). If \( T \in t, \, X \in n \) and \( D_1, D_2 \in \mathbf{D}(G) \) we have

\[
D_1 X \eta_\lambda = 0, \quad (TD_2 \eta_\lambda)(e) = 0, \quad (D \eta_\lambda)(e) = (D A \eta_\lambda)(e),
\]
and if \( f \in \mathcal{D}(G) \) is right invariant under \( K \),

\[
\int_G (TD_2 \eta_\lambda)(g)f(g)dg = \int_G (D_2 \eta_\lambda)((-T)f)(g)dg = 0.
\]

Hence

\[
\int_G (D \eta_\lambda)(g)f(g)dg = \int_G (D_A \eta_\lambda)(g)f(g)dg
\]

\[
= (D_A \eta_\lambda)(e) \int_G \eta_\lambda(g)f(g)dg = (D \eta_\lambda)(e) \int_G \eta_\lambda(g)f(g)dg.
\]

A. 2. See Helgason [1992a].

B. Rank One Results.

B. 1. By the Fourier expansion for a \( F \in \mathcal{E}(K/M) \) (see e.g. [GGA], Ch. V, §3, (13)) we have

\[
F(e) = \sum_{\delta \in \hat{K}_M} d(\delta) \int_k \tilde{F}(k) \sum_{i=1}^{d(\delta)} \langle \delta(k)v_i, v_i \rangle dk
\]

where \( \tilde{F}(k) = F(kM) \), \( (v_i) \) is an orthonormal basis of \( V_\delta \) such that \( v = v_1 \) span \( V_\delta^M \). Replacing \( k \) by \( km \) and integrating over \( M \) the sum over \( i \) can be restricted to \( i = 1 \).

D. The Compact Case.

D. 1. (i) By calculation \( (xt_\theta x^{-1})_1 = \cos \theta \). Alternatively, note that \( u \rightarrow xux^{-1} \) is a rotation fixing \( t_0 \) and \( i \pi \). (ii) The area of a sphere in \( S^3 \) of radius \( \theta \) is a constant multiple of \( \sin^2 \theta \). (iii) Calculate \( \lim_{\theta \to 0} F_f(\theta)/\theta \). (iv) The basis \( z^p w^q (p + q = \ell) \) diagonalizes \( \pi_\ell(t_\theta) \) giving the formula for \( \chi_\ell(t_\theta) \). Then note that by (ii) and the fact that \( F_f \) is odd,

\[
\chi_\ell(f) = \int_U f(u)\chi_\ell(u)du = \frac{1}{4\pi} \int_0^{2\pi} (e^{-i\theta} - e^{i\theta}) F_f(\theta)\chi_\ell(t_\theta)d\theta
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} F_f(\theta) e^{-i(\ell+1)\theta} d\theta.
\]

Part (v) follows from the fact that \( \chi_\ell \) has \( L^2 \) norm on \( U \) equal to 1 as a result of (ii) and (iii). Part (vi) follows from (iv). For (vii) suppose \( \pi \in \hat{U} \) is not of the form \( \pi_\ell \); using (vi) on \( f = \text{Trace}(\pi) \) we get a contradiction.
D. 2.  If $k \in K$ we have

$$(\tilde{\varphi} \ast \tilde{f})(u) = \int \tilde{\varphi}(uv^{-1})\tilde{f}(v)dv = \int \tilde{\varphi}(uvk^{-1})\tilde{f}(vk)dv$$

which by averaging over $K$ becomes

$$\tilde{\varphi}(u) \int \tilde{\varphi}(v^{-1})\tilde{f}(v)dv.$$

The generalization follows from [GGA], Proposition 2.4 in Ch. IV.

D. 3.  The dual of the symmetric space $G/K$ is now $(U \times U)/U^*$ where the diagonal $U^*$ is isomorphic to $K$. Formula (24) in §9 gives

$$d(\mu) = \left\{ \prod_{\alpha \in \Sigma^+} \frac{\langle \mu + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle} \right\}^2.$$

Here $d(\mu)$ is the degree of the irreducible representation $\tau_\mu$ of $U \times U$ which has a fixed vector under the diagonal group $U^*$ and highest weight $\mu$. The irreducible representations $\tau$ of $U \times U$ are of the form

$$\tau(u_1, u_2) = \pi_1(u_1) \otimes \pi_2(u_2)$$

where $\pi_1, \pi_2 \in \hat{U}$ (cf. Weil [1940], §17). Here $\tau$ has a fixed vector under $U^*$ if and only if there is a nonzero vector $A \in V_1 \otimes V_2$ such that

$$\pi_1(u) \otimes \pi_2(u)A = A, \quad u \in U,$$

$V_i$ being the representation space of $\pi_i$. This means for the tensor product $\pi_1 \otimes \pi_2$

$$(\pi_1 \otimes \pi_2)(u)A = A.$$

Because of the identification $V_1 \otimes V_2 = \text{Hom}(V'_2, V_1)$ $A$ is a linear transformation of $V'_2$ into $V_1$ so this equation amounts to $\pi_1(u)A\tilde{\pi}_2(u^{-1}) = A$ which means $\pi_1$ and $\pi_2$ contragredient, i.e., $\pi_1 \sim \pi, \pi_2 \sim \tilde{\pi}$. Thus $\mu = (\nu, -s\nu)$ where $\nu$ is the highest weight of $\pi$ (relative to a maximal abelian subalgebra $t \subset u$) and $s$ is the “maximal” Weyl group element. Considering the relationship between the root system $\Delta(u^e, t^e)$ and the restricted root system of $u \times u$ with respect to $t^e = \{(H, -H) : H \in t\}$ ([DS], Ch. VII, §4), where each restricted root has multiplicity 2. Note also for the Killing forms

$$B_{u\times u}((H, -H), (H', H')) = 2B_u(H, H').$$
Thus
\[
\prod_{\alpha \in \Sigma^+} \frac{\langle \mu + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle} = \prod_{\beta > 0} \frac{\langle \nu + \rho_0, \beta \rangle}{\langle \rho_0, \beta \rangle}
\]
where on the left \( \langle , \rangle \) refers to \( B_u \times u \), on the right to \( B_u, \beta \) runs over the positive roots in \( \Delta(u^c, t^c) \) and \( \rho_0 \) half their sum. Since \( d(\mu) = d(\nu)^2 \) the formula above gives the formula for \( d(\nu) \) the degree of \( \pi \).

E. The Flat Case.


E. 3. We have
\[
(M^y M^x f)(z) = \int \int f(z + \ell \cdot x + k \cdot y) dk \, d\ell
\]
\[
= \int \int f(z + \ell \cdot x + \ell k \cdot y) dk \, d\ell = \int (M^{x+k} y f)(z) dk.
\]
Here we take \( x = re_n, y = se_n \) where \( e_n = (0, \cdots, 1) \). Then the last integral is constant for \( k \) in the subgroup fixing \( e_n \) so the integral equals
\[
\frac{1}{\Omega_n} \int_{S^{n-1}(0)} (M^{x+s w} f)(z) dw.
\]
Letting \( \theta \) denote the angle between \( e_n \) and \( w \) we integrate this last integral with \( w \) first varying in the section of \( S^{n-1}(0) \) with the plane \( (e_n, y) = \cos \theta \). Since
\[
|x + sw|^2 = r^2 + s^2 - 2rs \cos \theta
\]
this gives the second expression for \( (M^y M^x f)(z) \). The last is obtained by the substitution \( t = (r^2 + s^2 - 2rs \cos \theta)^\frac{1}{2} \). (For a different proof see John [1955], p. 80; see also Ågesrøn [1937]).

F. The Noncompact Case.

F. 1. If \( \lambda \in \mathfrak{a}^* \) then \( |c(\lambda)|^2 = c(\lambda)c(-\lambda) = c(s\lambda)c(-s\lambda) \).

F. 2. The formula
\[
\int_G f(g) \varphi_{-\lambda}(g) dg = \int_A F_f(a) e^{-i\lambda \log a} da
\]
converts the statement into an analogous one for the exponentials \( e^{i\lambda} \) for which it is obvious.

F. 4. Clearly \( \varphi \times f \in L^2(X)^2 \). If \( F \in L^2(X)^2 \) is orthonal to all \( \varphi \times f \) then
\[
\int_{\mathfrak{a}^*} \tilde{F}(\lambda) \tilde{\varphi}(\lambda) \tilde{f}(\lambda) |c(\lambda)|^{-2} d\lambda = 0.
\]
Since the functions $\tilde{\varphi}$ form a uniformly dense subalgebra of $C_\alpha(a^*/W)$ and since $\tilde{f}$ is analytic on $a^*$, $F = 0$ a.e.

For the general case let $F \in L^2(X)$ be orthogonal to $f^\tau(g)$ for all $g \in G$. Then

$$\int_X F^\tau(g)(x)f^\tau(h)(x)dx = 0 \quad g, h \in G.$$ 

Here we can replace $F^\tau(g)$ and $f^\tau(h)$ by their $K$-averages $(F^\tau(g))^k$ and $(f^\tau(h))^k$. Integrating against $\varphi(h)$ ($\varphi \in D^k(G)$) then gives $(F^\tau(g))^k = 0$ by the first part. Hence $F = 0$ a.e.

**F.5.** Let $A_\delta(g) \in a$ be given by

$$g \in N_\delta \exp A_\delta(g)K$$

and as in (3) §1 put $A_\delta(gk, km) = A_\delta(k^{-1}g)$. Then

$$A_\delta(gK, km) = sA(gK, km_\delta M).$$

The $\rho$ which corresponds to $N_\delta$ is $s\rho$ so the formula

$$f_\delta(\lambda, km) = \tilde{f}(s^{-1}\lambda, km_\delta M)$$

follows easily.

**CHAPTER IV**

1. Writing $h$ in $G$ as $h = kan$ according to the Iwasawa decomposition and using the $K$-invariance of $f_2$ we have

$$(f_1 \times f_2)(g \cdot o) = \int_G f_1(gh^{-1} \cdot o)f_2(h \cdot o)dh$$

$$= \int_{AN} f_1(gm^{-1}a^{-1} \cdot o)f_2(an \cdot o)e^{2\rho \log a}da \, dn.$$ 

Hence

$$(f_1 \times f_2)(k_1a_1 \cdot \xi_o) = \int_{AN}(f_1 \times f_2)(k_1a_1n_1 \cdot o)dn,$$

$$\int_{AN} \left( \int_{N} f_1(k_1a_1n_1a^{-1} \cdot o)dn_1 \right)f_2(an \cdot o)e^{2\rho \log a}da \, dn$$

Interchanging $n_1$ and $a^{-1}$ in the inner integral cancels out the factor $e^{2\rho \log a}$ so the expression reduces to

$$\int_{A} \tilde{f}_1(k_1a_1a^{-1} \cdot \xi_o)\tilde{f}_2(a \cdot \xi_o)da$$
as desired. Since \(*\) is commutative whereas \(\times\) is not the \(K\)-invariance condition cannot be dropped.

2. Because of the \(K\)-invariance of \(\varphi\) we write \(\varphi(H)\) instead of \(\varphi(k \exp H \cdot \xi_o)\). Then by Ch. II, §3, (56),
\[
(f \times \tilde{\varphi})(x) = \int_G f(g \cdot o) \int_B \varphi(A(g^{-1} \cdot x, b)) e^{2\rho(A(g^{-1} \cdot x, b))} db \, dg.
\]
Using loc. cit. (47) and (51) this becomes
\[
\int_G f(g \cdot o) \int_B \varphi(A(x, g(b)) - A(g \cdot o, g(b))) e^{2\rho(A(x, g(b)))} db \, dg
= \int_{K/M} e^{2\rho(A(x,kM))} \int_{G/K} f(g \cdot o) \varphi(A(x, kM) - A(g \cdot o, kM)) \, dg \, dk.
\]
Now use the formula
\[
\int_{AN} F(kan \cdot o) \, da \, dn = \int_{G/K} F(kg \cdot o) \, dg \, K = \int_{G/K} F(g \cdot o) \, dg \, K
\]
on the function \(F(y) = f(y) \varphi(A(x, kM) - A(y, kM))\) whereby our integral over \(G/K\) becomes
\[
\int_{AN} f(kan \cdot o) \varphi(A(x, kM) - \log a) \, da \, dn = (\tilde{f} \times \varphi)(k \exp A(x, kM))
\]
Substituting and using (56) again this gives
\[
(f \times \varphi)(x) = (\tilde{f} \star \varphi)^\vee
\]
as stated.

3. By definition
\[
(A\mu_h)(F) = \mu_h(A^* F) = \int_K F(\exp H(hk)) e^{-\rho(H(hk))} \, dk
\]
and \(\{H(hk) : k \in K\} = C(h) ([GGA], \text{Ch. IX, Theorem 10.5}).

5. (i) The Fourier series (20) §3 converges in the topology of \(\mathcal{E}'(\mathbb{R} \times S^1)\) so
\[
\sigma(\psi) = \sum_n \sigma(\psi_{-n}(t) \otimes e^{-in\theta}) = \sum_n (e^{2\pi \sigma_n}(\psi_{-n}).
\]
(ii) By Theorems 2.4 and 3.4, if \(\sigma \in \mathcal{E}'(\Xi)\) then the following conditions are equivalent:
(a) $\sigma \in \mathcal{E}'(H^2)^\wedge$.
(b) $\sigma(\psi) = 0$ for each $\psi \in \mathcal{E}(\Xi)$ satisfying
\[ D_n(e^t \psi_n) \text{ is odd} \quad (n \in \mathbb{Z}) \]
where $D_n$ denotes $(D + 1) \cdots (D + 2|n| - 1)$, $D = d/dt$.
(c) $\sigma(\psi) = 0$ for each $\psi \in \mathcal{E}(\Xi)$ satisfying
\[ e^t \psi_n \in (D_n^* \mathcal{E}'(R))^\perp \quad (n \in \mathbb{Z}) \]
\* denoting adjoint and subscript $e$ indicating “even”, and $\perp$ denoting annihilator.

If $\sigma \in \mathcal{E}'(\Xi)$ is such that $\sigma_n$ has the form in (ii) then $e^t \sigma_n = D_n^* \tau_n$ where $\tau_n \in \mathcal{E}'_e(R)$. If $\psi \in \mathcal{E}(\Xi)$ satisfies (1) then
\[ (e^{2t} \sigma_n)(\psi_{-n}) = (D_n^* \tau_n)(e^t \psi_n) = 0 \]
so $\sigma(\psi) = 0$ by (i). Thus by (b) we have $\sigma \in \mathcal{E}'(H^2)^\wedge$.

On the other hand, suppose $\sigma \in \mathcal{E}'(\Xi)$ satisfies (c), that is
\[ \sigma(\psi) = 0 \quad \text{whenever} \quad e^t \psi_n \in (D_n^* \mathcal{E}'(R))^\perp \quad (n \in \mathbb{Z}). \]

Fix $k \in \mathbb{Z}$ and use this on the function $\psi(\xi_{t,\theta}) = \psi_{-k}(t)e^{-ik\theta}$. Then $\sigma(\psi) = 0$ implies $(e^{2t} \sigma_k)(\psi_{-k}) = 0$, that is $(e^t \sigma_k)(e^t \psi_{-k}) = 0$. This means that $e^t \sigma_k$ belongs to the double annihilator $(D_k^* \mathcal{E}'_e(R))^\perp\perp$, which equals $D_k^* \mathcal{E}'_e(R)$, this latter space being closed in $\mathcal{E}'_e(R)$ (cf. Theorem 2.16 in Ch. I). Since $k \in \mathbb{Z}$ was arbitrary this shows property (ii) for $\sigma$.

**CHAPTER V**

1. By the symmetry of $L$
\[ \int_X (L u)(x)e^{2\rho(A(x,b))}dx = 0 \]
so the conditions are necessary. For the sufficiency, consider the Fourier transform
\[ \tilde{f}(\lambda, b) = \int_X f(x)e^{-i\lambda \rho(A(x,b))}dx. \]
The conditions amount to $\tilde{f}(\pm i\rho, b) = 0$ so $\tilde{f}(\lambda, b)$ is divisible by $\langle \lambda, \lambda \rangle + \langle \rho, \rho \rangle$ and the quotient is holomorphic of uniform exponential type and satisfies (3) in Ch. III, §5. By the Paley-Wiener theorem, $u$ exists.

3. (i) See Helgason [1976], (Theorem 8.1); another proof is in Dadok [1979].
   (ii) See Helgason [1973b] and Eguchi [1979a].
5. See deRham [1955], Ch. V.

4.6–7. See Theorems 5.3, 6.1–6.3 in Helgason [1964a].

8. One has to verify

\[ \frac{d^2}{d\theta^2} (\gamma(\theta)) = \delta - \frac{1}{2\pi}, \quad \int_{-\pi}^{\pi} \gamma(\theta) d\theta = 0 \]

and using \((d^2/d\theta^2)(|\theta|) = 2\delta\) this is a simple matter.

10. (i) If \(T \in \mathfrak{u}, [T, U_i] = \sum_j c_{ij} U_j\) where \((c_{ij})\) is skew symmetric. Hence

\[ [T, \omega] = \sum_i [T, JU_i] U_i + JU_i [T, U_i] \]
\[ = \sum_i c_{ij} (JU_j) U_i + \sum_i c_{ij} (JU_i) U_j \]
\[ = \sum_i c_{ij} (JU_j) U_i - \sum_i c_{ij} (JU_j) U_i = 0. \]

Similarly,

\[ [JT, \omega] = \Sigma [JT, JU_i] U_i + \sum_i JU_i [JT, U_i] \]
\[ = - \sum_i c_{ij} U_j U_i + \sum_i c_{ij} (JU_i) (JU_j) \]
\[ = \frac{1}{2} \sum c_{ij} (U_i U_j - U_j U_i) + \frac{1}{2} \sum c_{ij} ((JU_i) (JU_j) - (JU_j) (JU_i)) \]
\[ = \frac{1}{2} \sum c_{ij} [U_i, U_j] + \frac{1}{2} \sum c_{ij} [JU_i, JU_j] = 0. \]

This proves (i). For (ii) observe that \(\omega\) annihilates all \(C^\infty\) functions \(f\) on \(G\) which are right invariant under \(K\). Thus if \(\omega u = f\) we find a contradiction by averaging over right translations by \(K\).

11. (From a discussion with Schlichtkrull). Let \(\nu : D(G) \rightarrow E(X)\) be the homomorphism (from Ch. III, §10) given by the action of \(G\) on \(X\). Then \(T\) commutes with each \(\nu(D)\) so by (1) loc. cit. \(TZ = ZT\) for each \(Z \in Z(G/K)\). Let \(D \in D(G/K)\). By Theorem 10.1 in Ch III, \(DZ_1 = Z_2\) for some \(Z_1 \neq 0, Z_2 \in Z(G/K)\). Then \(TDZ_1 = TZ_2, \ DZ_1 = Z_2T\) so \((TD - DT)(Z_1 f) = 0\) for \(f \in E(X)\). By the surjectivity of \(Z_1\) (Theorem 1.4) we conclude \(TD = DT\).

12. The first statement is immediate from the theorem quoted. For the necessity of the condition and for the compact case see Helgason [1992a].
13. The equation holds for all $f$ of the form $f(kan) = f_1(k)f_2(a)f_3(n)$, hence for all $f$.

14. Suppose first $f$ holomorphic on all of $D$. Since the rotations $z \rightarrow e^{i\theta}z$ belong to the center of $K$ we have (replacing $f$ by $f^\tau(k)$)

$$f(0) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}z) d\theta = \frac{1}{2\pi} \int_0^{2\pi} \int f(e^{i\theta}k \cdot z) d\theta = \int f(k \cdot z) dk.$$ 

Applying this to the composite function $f \circ g \ (g \in G_0)$ we see that $f$ satisfies the mean value theorem (Cor. 2.2) so is harmonic. This argument can be localized since Cor. 2.2 can.

15. We have by (38) in §4,

$$A_\circ \cdot b_{\Gamma_1} = \left\{ \sum_{\gamma \in \Gamma - \Gamma_1} \tanh y_\gamma X_{-\gamma} + b_{\Gamma_1} : y_\gamma \in \mathbb{R} \right\}$$

proving (i). Part (ii) follows from the fact that the Weyl group consists of all signed permutations.

17. See Proposition 5.2 in Helgason [1987]. The flat case is proved in Menzala and Schonbeck [1984] on the basis of the spherical support theorem [GGA], Ch. I, Lemma 2.7.

18. It suffices to prove this for $b = eM$ and then the function $v$ is $N$-invariant. If $D \in D(G/K)$ then $\Delta_N(D)$, the $N$-radial part of $D$, is given by $\Delta_N(D) = e^\rho \Gamma(D) \circ e^{-\rho}$ ([GGA], Ch. II, Cor. 5.19); the statement about $v$ is then easily verified.

19. For this we use the transmutation property

(1) $A(D)D\varphi = \Gamma(D)A\varphi, \quad \varphi \ - \text{invariant},$

(Ch. IV, Theorem 4.1) and the Darboux equation

(2) $D_{gK} \left( \int_K f(gk \cdot x) \, dk \right) = D_x \int_K f(gk \cdot x) \, dk.$

Putting

$$f^\circ(x) = \int_K f(k \cdot x) \, dk,$$

we have

$$F(gK, \log a) = e^{\rho(\log a)} \int_N (f^\tau g^{-1})^\circ(\alpha n \cdot o) \, dn.$$
Applying $\Gamma(D)_a$ and using (1) this becomes

$$e^{\rho(\log a)} \int_N D(f_\tau(x^{-1}))^k(\alpha_n \cdot o) \, dn$$

which by (2) becomes

$$e^{\rho(\log a)} \int_N \left(D_{gK} \int_K f(gk\alpha_n \cdot o) \, dk \right) \, dn = (DF)(gK,a).$$

CHAPTER VI

1. Using a $K$-invariant Laplace-Beltrami operator on $K/M$ we see that each joint eigenspace $E$ is finite-dimensional. Let $E = \bigoplus_i E_i$ be the direct decomposition into irreducible subspaces. Pick $f_i \in E_i$ such that $f_i(eM) = 1$ and $f_i$ is $M$-invariant. Then each $f_i$ is a spherical function and $Df_i = \chi(D)f_i$, where the homomorphism $\chi : D(K/M) \rightarrow \mathbb{C}$ is the same for all $i$. Using [GGA], Ch. IV, Cor. 2.3 we find that all $f_i$ coincide so $E$ is irreducible.

Taking $K = \text{SU}(2)$, $M = e$, each joint eigenspace has to contain a character $\chi$ of $K$. If $T$ is a maximal torus with Lie algebra spanned by a vector $H$ it is easily seen that $H\chi$ is not a constant multiple of $\chi$ (cf. e.g. [GGA], Ch. V, Ex. A7).

2. This is a basic step in Bruhat’s analysis [1956] §6) of the principal series for $G$. By Schur’s lemma (for unitary representations) (i) is equivalent to the statement that all bounded linear operators $A : K_\lambda \rightarrow K_\lambda$ commuting with all $\tau_\lambda(g)$ ($g \in G$) are scalars. Let $A$ be one such operator, consider the sesquilinear form

$$B(\varphi, \psi) = \int_{K/M} \varphi(kMN) \text{conj}(A\psi)(kMN) \, dk_m$$

and the form

$$\tilde{B}(f, g) = B(f^k, g^k) \quad f, g \in \mathcal{D}(G),$$

where

$$f^k(xMAN) = \int_{MAN} f(xamn)e^{(-i\lambda + \rho)(\log a)} \, dm \, da \, dn.$$ 

Then

$$\tilde{B}(f^{L(x)R(p_1)}, g^{L(x)R(p_2)}) = e^{-(i\lambda + \rho)(\log a_1)}e^{(i\lambda - \rho)(\log a_2)} \tilde{B}(f, g).$$
and by the Schwartz kernel theorem (Hörmander [1983], Ch. V)

\[ \tilde{B}(f, g) = \int_{G \times G} f(x) \text{conj}(g(y)) d\tilde{T}(x, y), \]

where \( \tilde{T} \in \mathcal{D}'(G \times G) \). Then

\[ \tilde{T}^L(x, x)R(p_1, p_2) = e^{(i\lambda + \rho)(\log a_1)} e^{(-i\lambda + \rho)(\log a_2)} \tilde{T}, \]

where \( L(x, x)R(p_1, p_2) \) denotes the diffeomorphism \((u, v) \mapsto (xu, xv)\) of \( G \times G \). Consider the diffeomorphism \( \varphi : (x, y) \mapsto (y^{-1}x, y) \) of \( G \times G \). Then, if \( h \in \mathcal{D}(G \times G) \), we have by the left invariance of \( \tilde{T} \)

\[ \tilde{T}^\varphi(h) = \tilde{T}(h^\varphi)^{-1} = \tilde{T}((h^\varphi)^{-1}L(z, z)). \]

However

\[ (h^\varphi)^{-1}L(z, z)(x, y) = h^\varphi^{-1}(z^{-1}x, z^{-1}y) = h(y^{-1}x, y) \]

so,

\[ (h^\varphi)^{-1}L(z, z) = (h^L(e, z))^{-1}. \]

Thus

\[ \tilde{T}^\varphi(h) = \tilde{T}^\varphi(h^L(e, z)) \]

so

\[ \tilde{T}^\varphi(h) = \int_G \int_G h(x, y) dS(x) dy, \]

where \( S \in \mathcal{D}'(G) \). Since \( \varphi^{-1}(x, y) = (yx, y) \) this implies

\[ \tilde{T}(f \otimes g) = \int_G \int_G f(yx)g(y) dS(x) dy. \]

Using the homogeneity of \( \tilde{T} \) under \( R(p_1, p_2) \) we obtain the homogeneity condition for \( S \) in (ii). The converse follows by reversing the steps. All the commuting operators \( A \) are proportional if and only if the corresponding \( S \) is proportional and then they must be the example stated.


4. Using Lemma 3.10 and (39) we see quickly that \( \Psi^*_{\lambda, e} = \Psi_{-\lambda, e} \). Thus if \( \varphi \in \mathcal{D}(\Xi) \)

\[ (\varphi \times \Psi_{\lambda, e})(kaMN) = \Psi_{-\lambda, e}(\varphi \circ \tau(ka)) \]

\[ = e^{(i\lambda - \rho)(\log a)} \int_A \varphi(kcMN)e^{(-i\lambda + \rho)(\log c)} dc. \]
Taking \( \varphi(kaMN) = \beta(kM)\gamma(a) \) the result follows.

5. In the solution below \( C_i \) and \( C'_i \) denote compact sets and \( \overset{o}{\to} A \) denotes the interior of a set \( A \). Let \( C_1 \overset{o}{\to} C_2 \subset \Xi \), let \( D_{C_1}(\Xi) \) denote the set of \( \varphi \in D(\Xi) \) with support in \( C_1 \), and let \( C'_i \subset G \) satisfy \( \pi(C'_i) = C_i \), \( C'_i \subset (C'_2)^o \subset G \), \( \pi : G \rightarrow G/MN \) being the natural mapping. Let \( C_o \) be a compact neighborhood of \( e \) in \( MN \) and put \( \tilde{C}_i = C'_i C_o \) \( (i = 1, 2) \). Let \( f_1 \in D(G) \) be \( \geq 0 \) on \( G \), \( > 0 \) on \( \tilde{C}_1 \), and \( \text{supp}(f_1) \subset \tilde{C}_2 \). Then the function

\[
    f(g) = \begin{cases} 
        f_1(g) \frac{\varphi(\pi(g))}{f_1(\pi(g))} & \text{if } \pi(g) \in C_1 \\
        0 & \text{if } \pi(g) \notin C_1
    \end{cases}
\]

satisfies \( \bar{f} = \varphi \) (cf. (36) §3). Also \( \varphi \rightarrow f \) is a continuous mapping of \( D_{C_1}(\Xi) \) into \( D_{C'_2}(G) \). Thus by (37) in §3, \( \Psi \times \eta \) is a distribution. For the last part one must show

\[
    \int \psi(\xi)(\varphi \times \eta^*)(\xi)d\xi = \int (\psi \times \eta)(\xi)\varphi(\xi)d\xi.
\]

Let \( f_1 \in D(G) \) satisfy \( \bar{f}_1 = \psi \). Then this last equation amounts to

\[
    \int_{\tilde{G}} f_1(g) \int_{\tilde{G}} f(gh^{-1})d(\eta^*) \sim (h)dg = \int_{\tilde{G}} f(g) \int_{\tilde{G}} f_1(gh^{-1})d\bar{\eta}(h)dg.
\]

However, \( (\eta^*) \sim = \bar{\eta} \) so this last equation is obvious.
BIBLIOGRAPHY

ABOUELZ, A.
ABOUELZ, A. and EL FOURCHI, O.
ADIMURTI, KUMARESAN, S.
AGRANOVSKI, M. L.
AGRANOVSKI, M. L., KUCHMENT, P. and QUINTO, E.T.
AGRANOVSKI, M. L. and QUINTO, E.T.
AGRANOVSKI, M. L. and QUINTO, E.T.
ANDERSON, R., and CAMPORESI, R.
ANKER, J-PH.
BIBLIOGRAPHY

ARThUR, J.

ÁSGEIRSSON, L.

ASTENGO, F., CAMPORESI, R. and DI BLASIO, B.

BADERTSCHER, E.

BADERTSCHER, E., and KOORNWINDER, T. H.

BADERTSCHER, E. and REIMANN, H.M.

BAGCI, S., and SITARAM, A.

BAN, VAN DEN, E.P.

BAN, VAN DEN, E.P., and SCHLICHTKRULL, H.


BARKER, W. H.

BARLET, D., and CLERC, J. L.

BARUT, A. D. and RACZKA, R.

BEERENDS, R.J.


BENABDALLAH, A-I., and ROUVIÈRE, F.

BERENSTEIN, C. and ZALCMAN, L.

BERENSTEIN, C., and SHAHSHAHANI, M.
BERENSTEIN, C., and CASADIO-TARABUSI, E.
BERENSTEIN, C., CASADIO-TARABUSI, E. and KURUSA, A.
BERLINE, N., and VÉRGENE, M.
BETORI, W., FARAUT, J., and PAGLIACCI, M.
The horocycles of a tree and the Radon transform (preprint).
BOCHNER, S.
BOMAN, J.
BOMAN, J. and QUINTO, E. T.
BONAMI, A., BURACZEWSKI, D., DAMEK, E., HULANICKI, A., PENNEY, R. and TROJAN, B.
BOTT, R.
BOURBACI, N.
BOUSSEJRA, A. and INTISSAR, A.
BRANSON, T., and OLAFSSON, G.

BRANSON, T.P., OLAFSSON, G., and SCHLICHTKRULL, H.


BRAY, W. O. and SOLMON, D.C.


BRUHAT, F.


BURACZEWSKI, D.


CAMPOLI, O.


CAMPORESI, R.


CARLEMAN, T.


CARTAN, É.


CARTAN, H., and GODEMENT, R.


CASSELMAN, W., and MILICIC, D.


CERÉZO, A., and ROUVIERE, F.


CHAMPETIER, C., and DELORME, P.


CHANG, W.


CHERN, S. S.

CHEVALLEY, C.

CLERC, J. L.

CLERC, J. L., EYMARD, P., FARAUT, J., RAÏS, M., and TAKAHASHI, R.

CLOZEL, L., and DELORME, P.

COHN, L.

CORMACK, A.M., and QUINTO, T.

COWLING, M.

COWLING, M., and KORÁNYI, A.

COWLING, M., DOOLEY, A. H., KORÁNYI, A., and RICCI, F.

COWLING, M., SITARAM, A. and SUNDARI, M.

CYGAN, J.

DADOK, J.

DAVIDSON, M.G., ENRIGHT, T. J., and STANKE, R. J.

DEBIARD A., and GAVEAU, B.
DEITMAR, A.

DE RHAM, G

DELORME, P.

DELORME, P., and FLENSTED-JENSEN, M.

DIEUDONNÉ, J.

DJUK, VAN, G.

DIXMIER, J.

DORAN, R.S., and VARADARAJAN, V.S. (Eds.)

DUFLO, M.


DUFLO, M., and RAÏS, M.

DUFLO, M., and WIGNER, D.

DUISTERMAAT, J. J.

DUISTERMAAT, J. J., KOLK, J. A. C., and VARADARAJAN, V. S.

EBATA, M., EGUCHI, M., KOIZUMI, S. and KUMAHARA, K.

EGUCHI, M.


EGUCHI, M., HASHIZUME, M., and OKAMOTO, K.
BIBLIOGRAPHY

EGUCHI, M., and KOWATA, A.

EGUCHI, M., and KUMAHARA, K.

EGUCHI, M. and OKAMOTO, K.

EHRENPREIS, L.


EHRENPREIS, L., and MAUTNER, F.

EYMARD, P.

EYMARD, P. and LOHOUÉ, N.

FARAH, S. B., and KAMOUN, L.

FARAUT, J.


FARAUT, J., and HARZALLAH, K.


FARAUT, J., and KORÁNYI, A.


FELIX, R.

FLENSTED-JENSEN, M.
1972  Paley-Wiener theorems for a differential operator connected with symmetric spaces. Ark. Mat. 10 (1972), 143-162.


FLENSTED-JENSEN, M. and KOORNWINDER, T.


FOLLAND, G. B.


FUGLEDE, B.


FURSTENBERG, H.


GANGOLLI, R.


GANGOLLI, R. and VARADARAJAN, V.S.


GÅRDING, L.


GELFAND, I. M.


GELFAND, I. M., GINDIKIN, S. G., and GRAEV, M. I.


GELFAND, I. M. and GRAEV, M. I.


GELFAND, I. M., GRAEV, M. I., and SHAPIRO, S.J.


GELFAND, I. M., GRAEV, M. I., and VILENKIN, N.

GELFAND, I. M. and NAIMARK, M. A.
GELFAND, I. M., and RAIKOV, D.A.
1943 Irreducible unitary representations of locally compact groups. Mat. Sb. 13 (1943), 301-316.
GELLER, D. and STEIN, E. M.
GILBERT, J.E., and MURRAY, M.A.M.
GINDIKIN, S. G.
GINDIKIN, S. G., and KARPELEVICH, F. I.
GLOBEVIK, J.
GODEMENT, R.
GODIN, P.
GONZALEZ, F.
1990a Bi-invariant differential operators on the complex motion group and the range of the d-plane transform on C^n. Contemp. Math. 113 (1990), 97-110.
BIBLIOGRAPHY


GONZALEZ, F., and HELGASON, S.


GONZALEZ, F., and KAKEHI, T.


GONZALEZ, F., and QUINTO, E. T.


GONZALEZ, F., and ZHANG, J.


GOODEY, P., and WEIL, W.


GOODMAN, R.


GOODMAN, R. and WALLACH, N.


GRINBERG, E.


GRINBERG, E. and QUINTO, E.T.


GRINBERG, E. and RUBIN, B.


GROSS, K., and KUNZE, R.


GUARIE, D.


GUILLEMIN, V.


GUILLEMIN, V. and STERNBERG, S.


GÜNTHER, P.

HARDY, G.H.

HARISH-CHANDRA

HARZALLAH, K.

HASHIZUME, M., MINEMURA, K., and OKAMOTO, K.

HECKMAN, G., and SCHLICHTKRULL, H.

HELGAISON, S.


HILGERT, J. 1993 Radon transform on half planes via group theory. In Tanner and Wilson [1994].


JOHNSON, K. and KORÁNYI, A.


JOHNSON, K. D., and WALLACH, N.


KAKEHI, T.


KAKEHI, T. and TSUKAMOTO, C.


KARPELEVICH, F. I.


KASHIWARA, M., KOWATA, A., MINEMURA, K., OKAMOTO, K., OSHIMA, T., and TANAKA, M.


KASHIWARA, M. and OSHIMA, T.


KASHIWARA, M. and SCHMID, W.

BIBLIOGRAPHY

KAWAZOE, T.

KELLEY, J. L.

KNAPP, A.W.

KNAPP, A.W. and STEIN, E. M.

KOLK, J. and VARADARAJAN, V. S.

KOORNWINDER, T.H.

KORÁNYI, A.

KORÁNYI, A. and MALLIAVIN, P.

KORÁNYI, A., and WOLF, J. A.

KOSTANT, B.


KOSTANT, B., and RALLIS, S.

KÖTHE, G.

KOUFANY, K., and ZHANG, G.

KOWATA, A., and OKAMOTO, K.

KOWATA, A., and TANAKA, M.

KRÖTZ, B. and ÖLAFSSON, G.

KRÖTZ, B., ÖLAFSSON, G. and STANTON, R.

KUCHMENT, P. A.


KUNZE, R., and STEIN, E.

KURUSA, A.


LANGLANDS, R.

LASALLE, M.


LAX, P., and PHILLIPS, R. S.
LEE, C.Y.
LEPOWSKY, J.
LEWIS, J. B.
LÉVY-BRUHL, P.
LIMIC, N., NIDERLE, J., and RACZKA, R.
LINDAHL, L.-Å.
LIONS, J. L.
LIONS, J. L., and MAGENES, E.
LOOMIS, L. H.
LOHOÜÉ, N. and RYCHENER, T.
LOWDENSLAGER, D.
LUDWIG, D.
MACKEY, G.W.
1952 Induced representations of locally compact groups, I. Ann. of Math. 55 (1952), 101-139.
1953 Induced representations of locally compact groups, II. Ann. of Math. 58 (1953), 193-221.
MADYCH, W. R., and SOLMON, D.C.
MALGRANGE, B.
MANO, G.
MAUTNER, F. I.

MAYER-LINDENBERG, F.

MAZZEO, R.R. and VASY, A.

MEANEY, C.

MENZALA, G.P. and SCHONBECK, T.

MICHELSON, H. L.

MIZONY, M.

MOHANTY, P., RAY, S.K., SARKAR, R.P. and SITARAM, A.

MOORE, C. C.

MOSTOW, G.D.

NATTERER, F.

NELSON, E.

ODA, H.

ØRSTED, B.

OKAMOTO, K.

ÓLAFSSON, G., and QUINTO, E. (Eds.)

ÓLAFSSON, G., and PASQUALE, A.

ÓLAFSSON, G., and SCHLICHTKRULL, H.

2008 Fourier series on compact symmetric spaces. (preprint)

2008 Representation theory, Radon transform and the heat equation on a Riemannian symmetric space. (preprint)
OLEVSKY, M.

ORLOFF, J.


OSHIMA, T., and SEKIGUCHI, J.

PALAMODOV, V., and DENISJUK, A.

PALEY, R., and WIENER, N.

PARTHASARATHY, K. R., RANGA RAO, R., and VARADARAJAN, V.S.

PASQUALE, A.

PENNEY, R.

PESENSON, I.


PHILLIPS, R. S., and SHAHSHAHANI, M.

POISSON, S.D.

QUINTO, E. T.


RADER, C.

RADON, J.

RAÏS, M.


RAUCH, J. and WIGNER, D.

REVUZ, A.

RICHTER, F.


ROSSMANN, W.

ROUVIÈRE, F.


1994a Transformations de Radon. Lecture Notes, Université de Nice, Nice, France, 1994.

BIBLIOGRAPHY

RUBIN, B.

RUDIN, W.

SARKAR, R.P. and SENGUPTA, J.

SARKAR, R.P. and SITARAM, A.

SCHIFFMANN, G.

SCHIMMING, R. and SCHLICHTKRULL, H.

SCHLICHTKRULL, H.

SCHLICHTKRULL, H., and STETKÆR, H.

SCHMID, W.

SCHWARTZ, G. and ZHU, C.-B.

SCHWARTZ, L.

SEMENOV-TJAN-SHANSKI, M. A.

SEMYANISTY, V. I.
BIBLIOGRAPHY

SEN Gupta, J.

Serre, J.-P.

Shahshahani, M.


Shahshahani, M., and Sitaram, A.

Sherman, T.


Shimeno, N.


Shimura, G.


Sitaram, A.


Sitaram, A. and Sundari, M.

Sjögren, P.


Solomon, D. C.


Solomatina, L. E.

SPEH, B., and VOGAN, D.

STANTON, R. J.

STANTON, R. J. and TOMAS, P. A.


STEIN, E.M.


STEIN, E.M., and WEISS, G.

STENZEL, M.B.

STETKÆR, H.


STRASBURGER, A.

STRICHARTZ, R. S.


BIBLIOGRAPHY
BIBLIOGRAPHY


STROHMAIER, A.

SUGIURA, M.

SULANKE, R.

TAKAHASHI, R.

TAKEUCHI, M.

TANNER, E.A. and WILSON, R. (Eds.)

TAYLOR, M.E.

TEDONE, O.
1898 Sull' integrazione dell'equazione $\frac{\partial^2 f}{\partial t^2} - \Sigma \frac{\partial^2 f}{\partial x_i^2} = 0$. *Ann. Mat.* **1** (1898), 1-24.

TERRAS, A.

THANGAVELU, S.


THOMAS, E.G.F.

TITCHMARSH, E. C.

TITS, J.

TORASSO, P.
BIBLIOGRAPHY

TRÈVES, F.

TRIMÈCHE, K.

TROMBI, P., and VARADARAJAN, V. S.

VARADARAJAN, V.S

VILENKIN, N.

VILENKIN, N., and KLIMYK, A.U.

VOGAN, D.

VOGAN, D. and WALLACH, N.

VRETARE, L.


WALLACH, N.


WARNER, G.

WAWRZYNCZYK, A.


WEIL, A.


WIEGERINCK, J. J. O. O.

WIGNER, D.
WILLIAMS, F.L.

WILLIAMS, G.D.

WOLF, J. A.


YANG, A.

ZHANG, G.

ZHELOBENKO, D. P.

ZHU, CHEN-BO.

ZORICH, A. V.
1991 Inversion of horospherical integral transform on Lorentz group and on some other real semisimple Lie groups. RIMS, Kyoto, 1991, 1-37.
Symbols Frequently Used

Ad: adjoint representation of a Lie group, 5
ad: adjoint representation of a Lie algebra, 5
$A(r)$: spherical area, 420, 484
$A(g)$: component in $g = n \exp A(G)k$, 86, 99
$A(B)$: space of analytic functions on $B$, 530
$A'(B)$: space of analytic functionals (hyperfunctions) on $B$, 530
$a, a^c, a^*, a^c_*$: abelian subspaces and their duals, 61
$a'$: 68
$a_c^*(\delta)$: subset of $a^c_*$, 237
$a^+, a^*_+$: Weyl chambers in $a$ and $a^*$, 61, 202
$'\alpha$: transpose, 29
$A_\lambda$: vector in $a^c$, corresponding to $\lambda$, 61
$A(x, b)$: composite distance, 99
$A_0$: projection from $p$ to $a$, 289
$A$: Abel transform, 381
$A^*$: dual Abel transform, 382
$A_r$: space of analytic vectors, 416
$B_r(p), B^r(p)$: open ball with radius $r$, center $p$, 3
$B$: Killing form, 61
$\mathcal{B}$: set of bounded spherical functions, 341
$\beta_B$: ball in $\Xi$, 364
$BC(G)$: space of bounded continuous functions on $G$, 336
$Cl$: closure, 3
conj: complex conjugate, 93
$C^n$: complex $n$-space, 298
$C_0$: special set, 6
$C(X)$: space of continuous functions of $X$, 3
$C_c(X)$: space of continuous functions of compact support, 3
$C_0^b(G)$: space of $K$-bi-invariant functions in $C_c(G)$, 80
$C_K(X)$: space of continuous functions with support in $K$, 3
$C_0(X)$: space of continuous functions vanishing at $\infty$, 3
$C^\infty(X), C_0^\infty(X)$: set of differentiable functions, set of differentiable functions of compact support, 4
$c(\lambda)$: Harish-Chandra's c-function, 90
c_s(\lambda): partial c-function, 141
c_s: generalized c-functions, 234
$C^+, -C, +C, C^-$: closures of Weyl chambers and their duals, 129
$\Gamma, \hat{\Gamma}$: isomorphisms of differential operators, 74
$\Gamma_{s, \lambda}$: intertwining operator, 240
$\Gamma_X(\lambda)$: Gamma function for $X$, 284

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\begin{itemize}
  \item \( \partial_t \): partial derivative, 3
  \item \( \delta \): density, 213
  \item \( \mathcal{D}(X) \): \( C_c^\infty(X) \), 4
  \item \( \mathcal{D}'(X) \): set of distributions on \( X \), 4
  \item \( \mathcal{D}_\lambda \): eigenspace, 76
  \item \( \mathcal{D}_K(X) \): set of \( f \in \mathcal{D} \) with support in \( K \), 4
  \item \( \mathcal{D}_{H}(\mathbb{P}^n) \): subspace of \( \mathcal{D}(\mathbb{P}^n) \), 11
  \item \( \mathcal{D}_{h}(\mathbb{G}(d,n)) \): subspace of \( \mathcal{D}(\mathbb{G}(d,n)) \), 45
  \item \( \mathcal{D}^{\bullet}(X) \): space of \( K \)-commuting functions, 273
  \item \( \mathcal{D}_{K}^{\bullet}(X) \): \( K \)-finite functions of type \( \delta \), 273
  \item \( \mathcal{D}^{\bullet}(X), \mathcal{D}'_{h}(X) \): space of \( K \)-invariant elements in \( \mathcal{D}(X), \mathcal{D}'(X) \), 207, 381
  \item \( \mathcal{D}^{\bullet}(G), \mathcal{D}'_{h}(G) \): space of \( K \)-bi-invariant members of \( \mathcal{D}(G), \mathcal{D}'(G) \), 90
  \item \( \mathcal{D}(G) \): set of left-invariant differential operators on \( G \), 70
  \item \( \mathcal{D}_{H}(G) \): subalgebra of \( \mathcal{D}(G) \), 70, 71
  \item \( \mathcal{D}(G/H) \): set of \( G \)-invariant differential operators on \( G/H \), 71, 75
  \item \( \mathcal{D}_{W}(A) \): \( W \)-invariants in \( \mathcal{D}(A) \), 70
  \item \( \mathcal{D}(X), \mathcal{D}(\Xi) \): invariant operators on \( X, \Xi \), 70, 71
  \item \( d(\delta) \) or \( d_\delta \): dimension (= degree) of a representation, 14
  \item \( \Delta(D) \): radial part of \( D \), 70
  \item \( \Delta_{MN}(D), \Delta_{K}(D), \Delta_{N}(D) \): radial parts of \( D \), 75, 70
  \item \( \Delta(g^e, h^e) \): set of roots, 128
  \item \( d_\lambda(\lambda), e_\lambda(\lambda) \): factors in \( c_\lambda(\lambda) \), 142
  \item \( \mathcal{E}(M) \): set of all differential operators on \( M \), 36
  \item \( \mathcal{E}(X) \): \( C^\infty(X) \), 4
  \item \( \mathcal{E}'(X) \): space of distributions of compact support, 4
  \item \( \mathcal{E}^{\bullet}(X), \mathcal{E}'_{h}(X) \): space of \( K \)-invariant elements in \( \mathcal{E}(X), \mathcal{E}'(X) \), 207, 381
  \item \( \mathcal{E}_{\lambda}, \mathcal{E}(\lambda), \mathcal{E}_{h}^{\infty}(\lambda), \mathcal{E}^{*}, \mathcal{E}_{x}^{\infty}, \mathcal{E}_{\lambda, \delta} \): eigenspaces, 76, 229, 282, 531
  \item \( \mathcal{E}^{\bullet}(G) \): space of \( K \)-bi-invariant members of \( \mathcal{E}(G) \), 381
  \item \( \mathcal{E}_{K} \): eigenspace of Laplacian, 11
  \item \( e_{\lambda, \delta} \): plane wave eigenfunction, 99
  \item \( F(a, b; c; z) \): hypergeometric function, 328
  \item \( f \to \hat{f} \): map from \( C_c(G) \) to \( C_c(G/H) \), 26, 155
  \item \( f^\delta \): \( K \)-commuting function, 266
  \item \( \mathcal{F} \): spherical transform, 220
  \item \( \mathcal{F}(X) \): function space, 376
  \item \( g^\varphi, T^\varphi, D^\varphi \): images of \( g \in \mathcal{E}(M), T \in \mathcal{D}'(M) \), operator \( D \) under \( \varphi \), 4
  \item \( \varphi_\lambda \): spherical function, 76, 86
  \item \( \Phi_{\lambda, \delta} \): generalized spherical function, 228
  \item \( G_0 \): a group of linear transformations of \( X_0 \), 285
  \item \( \mathcal{G}(d,n), \mathcal{G}_{d,n} \): manifolds of \( d \)-planes, 39, 41
  \item \( \mathcal{H}^{A}(\mathbb{C}^n), \mathcal{H}_{W}(\mathcal{A}_c^*), \mathcal{H}(\mathcal{A}^* \times B)_{W} \): exponential type, 261, 275, 567
  \item \( \text{Hom}(V, W) \): space of linear transformations of \( V \) into \( W \), 273
  \item \( \mathcal{H}^n \): hyperbolic space, 50
  \item \( H_t, H(p) \): space of harmonic polynomials, 16, 230
\end{itemize}
$\mathcal{H}_\lambda$: Hilbert space inside $\mathcal{E}_\lambda(X), \mathcal{E}_\lambda(p)$, 284, 309, 552
$\mathcal{H}^\delta(a^*)$: special holomorphic functions on $a^*_\varsigma$, 275
$\mathcal{H}$: Hilbert transform, 6, 390
$\eta_\lambda$: $K$-fixed vector in $\mathcal{E}_\lambda(\Xi)$, 243
$H(g)$: component in $g = k \exp H(g)n$, 99
Im: imaginary part, 261
$I(E)$: space of invariant polynomials on $E$, 229
$I^*$: Riesz potential, 6
$I(X)$: group of isometries of $X$, 50
$\mathcal{J}_\lambda^2(G)$: $K$-bi-invariant Schwartz space, 220
$I_{\lambda,s}$: intertwining integral, 244
$I_{\lambda,s}$: normalized intertwining operator, 245
$J$: inversion, 162
$J_\delta(\lambda)$: polynomial matrix, 287
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$\tilde{K}, \tilde{K}_M$: unitary dual and subset, 227, 370
$K_\lambda$: Hilbert space inside $D'_\lambda(\Xi)$, 548
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$\mathfrak{f}$: algebra in Cartan decomposition, 77
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$L^p(X)$: space of $f$ with $|f|^p \in L^1(X)$, 433
$L = L_X$: Laplace-Beltrami operator on $X$, 5
$L(g) = L_g$: left translation by $g$, 5
$l(\delta)$: dimension, 228
$\Lambda$: operator on $P$, 7, on $\Xi$, 93, weight lattice, 240
$i$: orthocomplement of $\frak{m}$ in $\mathfrak{f}$, 71
$\Lambda_0$: operator on $\Xi_0$, 390
$M_p$: the tangent space to a manifold $M$ at $p$, 3
$m^*$: element, 64
$M^\pi$: mean-value operator, 77, 484
$M(n)$: group of isometries of $R^n$, 1
$m$: centralizer of $a$ in $\mathfrak{f}$, 61
$\mathfrak{M}$: set of continuous homomorphisms, 339
$\mathfrak{M}(B)$: space of measures on $B$, 439
$N$: kernel of dual transform, 13, 367
$n$: part of Iwasawa decomposition, 61
$O(n), O(p, q)$: orthogonal groups, 1, 352
$\Omega_n$: area of $S^{n-1}$, 9
$P^n$: set of hyperplanes in $R^n$, 8
$P_l$: space of homogeneous polynomials of degree $l$, 16
$P_\lambda, P_\lambda^*$: Poisson transform, 300, 100
$P^\delta(\lambda)$: inverse of $Q^\delta(\lambda)$, 236
$\mathfrak{P}$: set of positive definite spherical functions, 340
$\pi(\lambda)$: product of roots, 91, 154
p: part of a Cartan decomposition, 61
$Q^\delta(\Lambda)$: polynomial matrix, 232
$\mathcal{R}$: ring of functions on $A^+$, 234
$R$: modified Radon transform, 220
$\mathbb{R}^n$: real $n$-space, 1
$\mathbb{R}^+$: set of reals $\geq 0$, 3
Re: real part, 90
$R_g$ or $R(g)$: right translation by $g$, 5
Res: residue, 6
$p, p_0, p^*$: half sum of roots, 61, 323
$s_*$: element, 64
$S^*$: element, 7
$S_r(p)$: sphere of radius $r$ and center $p$, 3
$S(\mathbb{R}^n)$: space of rapidly decreasing functions on $\mathbb{R}^n$, 5
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$S^*(\mathbb{R}^n), S_0(\mathbb{R}^n)$: subspaces of $S(\mathbb{R}^n)$, 10
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$S_{\lambda, s}, S'_{\lambda, s}$: distributions on $B$, 136, 142
$S(D)$: 214
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$S_H(\mathbb{P}^n)$: subspace of $S(\mathbb{P}^n)$, 8, 11
$\text{sh } x$: sinh $x$, 2
$\sigma(F, G)$: weak topology, 29
$\sigma(a)$: diffeomorphism of $\Xi$, 107
$\Sigma(g, a)$: set of restricted roots, 129
$\sigma_R$: sphere in $\Xi$, 364
$\Sigma, \Sigma^+, \Sigma_0^+, \Sigma_+^+, \Sigma_*$: sets of restricted roots, 61, 90, 129, 138
$^tA$: transpose of $A$, 29
$\text{Tr}(A)$: trace of $A$, 14
$T_\lambda, \tau_\lambda, \tilde{\tau}_\lambda$: eigenspace representations, 77, 284
$\tau$: homomorphism of $D(G)$, 232
$\tau(x)$: translation on $G/H$, 5
$\Theta$: Cartan involution, 61
$U(n)$: unitary group, 53
$V_\delta$: representation space of $\delta$, 227
$V_\delta^M$: space of fixed vectors under $\delta(M)$, 228
$W$: Weyl group, 61, 323
$\Xi$: dual space, 62
$\Xi$: special spherical function, 214
$\Xi^*$: open orbit in $\Xi$, 64
$\Xi_0$: space of horocycle planes, 387
$\xi^*$: origin in $\Xi^*$, 64
SYMBOLS FREQUENTLY USED

\( \xi(x, b) \): horocycle determined by \( x \) and \( b \), 99
\( \Psi_{\lambda,s}, \Psi'_{\lambda,s} \): conical distributions, 135, 142
\( \Psi_{\lambda,\delta} \): generalized Bessel function, 289
\( \mathbb{Z}, \mathbb{Z}^+ \): the integers, the nonnegative integers, 3
\( \mathbb{Z}(G) \): center of \( D(G) \), 322
\( \mathbb{Z}(G/K) \): image of \( \mathbb{Z}(G) \) in \( D(G/K) \), 322
\( \sim \): Fourier transform, spherical transform, lift of functions, distributions, 4, 77, 155, 198
\( \vee \): Radon transform, incidence, 1, 31
\( \check{\vee} \): Dual Radon transform, incidence, 1, 31
\( \ast, \times \): convolutions, adjoint operation, pullback, star operator, Fourier transform, 6, 9, 26, 80, 82, 96, 137, 200, 557
\( \oplus \): direct sum, 527
\( \otimes \): tensor product, 12, 108, 112
\( \langle, \rangle \): inner product, 29
\( \mathfrak{K}, E^\mathfrak{K} \): space of \( K \)-invariants in \( E \), 86, 90
\( \Box \): operator, 8, 97
\( \Box_p \): operator on \( G(p, n) \), 41
\( - \): closure, 3, restriction, 116
\( \circ \): interior, 3
\( \bot \): annihilator, 16
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This book gives the first systematic exposition of geometric analysis on Riemannian symmetric spaces and its relationship to the representation theory of Lie groups. The book starts with modern integral geometry for double fibrations and treats several examples in detail. After discussing the theory of Radon transforms and Fourier transforms on symmetric spaces, inversion formulas, and range theorems, Helgason examines applications to invariant differential equations on symmetric spaces, existence theorems, and explicit solution formulas, particularly potential theory and wave equations. The canonical multitemporal wave equation on a symmetric space is included. The book concludes with a chapter on eigenspace representations—that is, representations on solution spaces of invariant differential equations. Known for his high-quality expositions, Helgason received the 1988 Steele Prize for his earlier books Differential Geometry, Lie Groups and Symmetric Spaces and Groups and Geometric Analysis. Containing exercises (with solutions) and references to further results, this revised edition would be suitable for advanced graduate courses in modern integral geometry, analysis on Lie groups, and representation theory of Lie groups.