The Classification of the Finite Simple Groups

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The Classification
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Daniel Gorenstein
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ABSTRACT. This is the first monograph in a series devoted to a revised proof of the classification of the finite simple groups.

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To the memory of
Pearl Solomon (1918–1978)
and
Marvin Lyons (1903–1975)
## Contents

Preface xiii

Preface to the Second Printing xv

Part I, Chapter 1: Overview 3

   Introduction to the Series 3

A. The Finite Simple Groups 6

   1. Simple groups 6
   2. $K$-groups 12

B. The Structure of Finite Groups 12

   3. The Jordan-Hölder theorem and simple groups 12
   4. The generalized Fitting subgroup and quasisimple groups 15
   5. $p'$-cores and $p$-components 19
   6. The embedding of $p'$-cores and $p$-components 20
   7. Terminal and $p$-terminal $p$-components 22
   8. $p$-constrained and $p$-solvable groups 24

C. Classifying Simple Groups 27

   9. Internal analysis: targeting local structure 27
   10. Internal analysis: passing from global to local information 29
   11. Identifying simple groups 31
   12. A capsule summary of this series 35
   13. The existing classification proof 38
   14. Simplifying the classification theorem 41

D. The Background Results 44

   15. Foundational material 44
   16. The initial assumptions 46
   17. Background Results: basic material 47
   18. Background Results: revised portions of the proof and selected papers 48

E. Sketch of the Simplified Proof 51

   19. Centralizers of semisimple elements 51
   20. Uniqueness subgroups 52
   21. The sets $L_p(G)$ and groups of even type 53
   22. Generic simple groups and neighborhoods 55
   23. The main case division 58
   24. Special simple groups 59
   25. Stages of the proof 61
   26. Generic simple groups 63
   27. The identification of $G$ 68
## CONTENTS

F. Additional Comments 72
   28. The length of the proof 72
   29. The $\mathcal{K}$-group environment 75
   30. The term $G \cong G^*$ 76
   31. Reading this series 77

Part I, Chapter 2: Outline of Proof 79
   Introduction 79
   A. The Grids 80
      1. Some basic terminology 80
      2. The uniqueness grid 83
      3. The classification grid 83
   B. The Uniqueness Grid 87
      4. 2-uniqueness subgroups 87
      5. Groups of restricted even type with $|M(S)| = 1$ 90
      6. Component preuniqueness subgroups 90
      7. The odd uniqueness theorem 92
      8. Some technical definitions 93
      9. Groups with a strongly embedded subgroup 95
     10. Theorem $M(S)$ 96
     11. Theorem $U(\sigma)$ 98
   C. The Classification Grid: Generic and Special Simple Groups 99
      12. $C_p$-groups 99
      13. $S_p$-groups and $T_p$-groups 102
      14. Groups of special odd type 103
      15. Groups of special even type 105
      16. Groups of generic type 106
   D. The Classification Grid: The Stages of the Proof 106
      17. Theorem $\mathcal{C}_1$ 106
      18. $p$-terminal $p$-components 108
      19. Centralizer of involution patterns 109
      20. Theorem $\mathcal{C}_2$ 110
      21. Theorem $\mathcal{C}_3$ 114
      22. Theorem $\mathcal{C}_4$ 114
      23. Theorem $\mathcal{C}_5$ 116
      24. Theorem $\mathcal{C}_6$ 118
      25. Vertical neighborhoods 118
      26. Theorem $\mathcal{C}_7$ 120
   E. Principal Techniques of the Proof 122
      27. Fusion 122
      28. The Bender method 123
      29. Signalizer functors and $k$-balanced groups 124
      30. $L_p^*$-balance, the $B_p$-property, and pumpups 127
      31. The signalizer functor method 128
      32. Near components, pushing up, and failure of factorization 129
      33. The amalgam method 131
      34. Local analysis at two primes 134
      35. Character theory and group order formulas 135
      36. Identification of the groups of Lie type 137
37. Properties of X-groups 138
F. Notational Conventions 139

Background References 140
Expository References 141
Glossary 148
Index 155
Preface

Though elated at the successful completion of the classification of finite simple groups in the early 1980's, Danny Gorenstein nevertheless immediately appreciated the urgency of a "revision" project, to which he turned without delay. Before long we had joined him in this effort, which is now into its second decade. Danny always kept the project driving forward with relentless energy, contagious optimism and his unique global vision of finite simple group theory. He inspired other collaborators—Richard Foote and Gernot Stroth—to contribute theorems designed to fit our revised strategy. Their work forms a vital part of this project. The conception of these volumes is unmistakably Danny's, and those who know his mathematics and his persuasive way of explaining it should recognize them everywhere. Since his death in 1992, we have tried to maintain his standards and we hope that whatever changes and additions we have made keep the vigorous spirit which was his trademark. Of course the responsibility for any stumbling or errors must remain with us. To accompany him on this mathematical journey was a privilege and as we continue, our debt to our teacher, colleague and loyal friend is hard to measure. Thanks, Danny.

This monograph is Number 1 of a projected dozen or so volumes, and contains two of the roughly thirty chapters which will comprise the entire project. Not all of the chapters are completely written at this juncture, but we anticipate that the publication process, now begun, will continue at a steady pace. In the first section of Chapter 1, we discuss the current status of the work in some more detail; later in that chapter we delineate the background results which form the foundation for all the mathematics in the subsequent volumes.

When Danny began, there was already a well-established tradition of "revisionism" in finite group theory. Indeed, beginning in the late 1960's, Helmut Bender produced a series of "revisions" whose beauty and depth profoundly influenced finite group theory. During the final decade of the classification proof, several more group-theorists showed how to develop deep mathematics while re-addressing some of the fundamental theorems in the theory of simple groups. While it is sometimes difficult to draw a distinct line between revisionist and other mathematics, the clear successes of Bender, Michel Enguehard, George Glauberman, Koichiro Harada, Thomas Peterfalvi, and Bernd Stellmacher in various revisionist projects have inspired us.

We are extremely grateful to Michael Aschbacher, Walter Feit, George Glauberman, Gary Seitz, Stephen Smith, and John Thompson for their enthusiastic support and interest in our endeavor, and specifically for having read some of the manuscripts and made valuable comments. To our collaborators Richard Foote and Gernot Stroth, our special thanks. The ideas and comments of many colleagues have been most useful and will appear in some form in the volumes to
come. Indeed, this work has ingredients of several types: there is new mathematics; there is exposition of unpublished work of our colleagues, ranging from short arguments to entire case analyses which they have generously shared with us, and which we shall acknowledge specifically as we go along; and finally there are reworkings of published papers. For now, in addition to all the people already mentioned, we must also thank Jonathan Alperin, Michel Broué, Andrew Cheremak, Michael Collins, K. M. Das, Alberto Delgado, Paul Fong, Robert Gilman, David Goldschmidt, Kensaku Gomi, Robert Griess, Robert Guralnick, Jonathan Hall, William Kantor, Martin Liebeck, Ulrich Meierfrankenfeld, Michael O’Nan, Lluis Puig, Geoffrey Robinson, Jan Saxl, Ernie Shult, Franz Timmesfeld, Jacques Tits, John Walter, Richard Weiss, and Sia K. Wong.

We are happy to acknowledge the financial support of the National Science Foundation and the National Security Agency, and the support of DIMACS and the Institute for Advanced Study. We extend our sincere thanks to our skillful and patient typists Adelaide Boullé, Lisa Magretto, Ellen Scott, Pat Toci, and Dorothy Westgate.

Finally, to our wives Lisa and Myriam, to our families, and to Helen Gorenstein, we gratefully acknowledge the love and commitment by which you have endured the stresses of this long project with good humor, grace and understanding.

Richard Lyons and Ronald Solomon
June, 1994
Preface to the Second Printing

Six years have passed since the first printing of this book and sixteen since the detailed classification strategy presented in Chapter 2 was first adumbrated. At present the first four books in our series have been published and the fifth and sixth are well on their way to completion. Since the first printing, some important second-generation pieces of the classification of the finite simple groups have also appeared, in particular two volumes [BG1, P4] presenting the revised proofs by Bender, Glauberman and Peterfalvi of the Odd Order Theorem of Feit and Thompson. We have taken the Feit-Thompson Theorem as a Background Result, but now could therefore remove [FT1] from the list of Background Results (see pp. 48–49) and replace it with Peterfalvi's revision of Chapter V of that paper [P4]. In addition, Peterfalvi's treatment of the Bender-Suzuki Theorem (Part II of [P1]) has now appeared in an English version [P5]. We have taken the opportunity of this second printing to add these new references, and to correct some misprints and other minor slips in the first printing.

We are nearing a milestone in our series – the completion of our revised proof of the Classification Theorem for X-proper simple groups of odd type following the strategy carefully outlined in this volume, and in particular in our Theorems C2, C3 and C7.

During these same years there has been considerable research activity related to simple groups of even type from the so-called "unipotent" point of view, that is, by means of the analysis of the structure of 2-constrained 2-local subgroups and their interplay with one another. With the maturing of the theory of amalgams, such notable results have emerged as Stellmacher's second-generation proofs of the classifications of N-groups and thin groups, first proved by Thompson and Aschbacher, respectively. Furthermore an ambitious program proposed by Meierfrankenfeld is currently being pursued by Meierfrankenfeld, Stellmacher, Stroth, Cheremak and others; it is directed towards the classification of groups of characteristic p-type for an arbitrary prime p, excepting those satisfying uniqueness conditions akin to the existence of a strongly p-embedded subgroup. This promises to remain an area of intense activity for another decade and one can at best speculate from this distance concerning what major theorems will emerge once the dust settles. Nevertheless, since it is the amalgam approach which we have proposed for our Theorem C4, recent results have naturally had an impact on the appropriateness of the strategy presented in this volume.

In the seminal work of Thompson on N-groups, the parameter e(G) was introduced to denote the maximum value of \( m_p(H) \) as \( p \) ranges over all odd primes and \( H \) ranges over all 2-local subgroups of \( G \). A crucial division in the set of all groups of characteristic 2-type was made between those groups \( G \) with \( e(G) \leq 2 \).
and those with $e(G) \geq 3$, the former being dubbed *quasithin*\(^1\) groups. This subdivision persisted in the full classification of finite simple groups of characteristic 2-type (simple groups $G$ of 2-rank at least 3 such that $F^*(H) = O_2(H)$ for every 2-local subgroup $H$ of $G$) undertaken during the 1970's. It was however discovered by Gorenstein and Lyons in the work leading up to their Memoirs volume [GL1] that the case $e(G) = 3$ presented unique difficulties deserving special treatment. Hence [GL1] treats simple groups of characteristic 2-type with $e(G) \geq 4$, and the case $e(G) = 3$ was handled independently by Aschbacher in [A13]. Both treatments took a “semisimple” rather than a “unipotent” approach, studying $p$-signalizers and $p$-components of $p$-local subgroups for primes $p$ other than the “characteristic”, in these cases for $p > 2$.

The Quasithin Case ($e(G) \leq 2$) was subdivided further into the case $e(G) = 1$ (the “Thin Case”) and the case $e(G) = 2$. The former was treated by Aschbacher[A10], while the latter was undertaken and almost completed by G. Mason. A unified treatment of the Quasithin Case is nearing completion by Aschbacher and S. D. Smith and will be published in this AMS series. When published, the Aschbacher-Smith volumes will represent a major milestone: the completion of the first published proof of the Classification of the Finite Simple Groups. But it is also pertinent to our strategy, since Aschbacher and Smith have relaxed the hypotheses somewhat. Rather than restricting themselves to the set of all simple groups of characteristic 2-type, they have partly accommodated their work to the strategic plan outlined in this volume by having their theorem encompass the larger class of all simple groups of even type. Indeed their main theorems have the following immediate consequence, in our terminology.

**The $e(G) \leq 2$ Theorem.** Let $G$ be a finite $K$-proper simple group of even type. Suppose that every 2-local subgroup $H$ of $G$ has $p$-rank at most 2 for every odd prime $p$. Then one of the following conclusions holds:

(a) $G \in \text{Chev}(2)$, and $G$ is of twisted Lie rank at most 2, but $G$ is not isomorphic to $U_5(q)$ for any $q > 4$; or

(b) $G \cong L_4(2), L_5(2), Sp_6(2), A_9, L_4(3), U_4(3), G_2(3), M_{12}, M_{22}, M_{23}, M_{24}, J_1, J_2, J_3, J_4, HN, He, \text{ or Ru}.$

However it must be noted that the Aschbacher-Smith Theorem will not complete the classification of “quasithin” groups of even type as this term is defined in the current volume (p. 82), the reason being that during the 1980’s we chose to redefine “quasithin” to include the $e(G) = 3$ case for groups of even type. This was motivated both by the peculiar difficulties of the $e(G) = 3$ case noted above and by conversations concerning the Amalgam Method which suggested to us that treating groups with $e(G) \leq 3$ via the Amalgam Method would not be substantially more difficult than treating groups with $e(G) \leq 2$.

In any event the recent work of Aschbacher and Smith has now reopened the possibility of restoring “quasithin” more or less to its original meaning, or more precisely to the hypotheses of the $e(G) \leq 2$ Theorem above. The groups of even type with $e(G) = 3$ then could be treated by a “semisimple” strategy akin to the one for the proof of Theorem $C_6$ discussed briefly in Section 24 of Chapter 2.

Such a change in strategy would entail a different case division defining rows 4, 5 and 6 of the classification grid (p. 85), i.e., altered hypotheses for Theorems

\[^1\]The word “quasithin” has been used in a number of distinct senses by various authors. In this paragraph and the next, we stick to the original meaning $e(G) \leq 2$.\]
\( \mathcal{C}_4, \mathcal{C}_5 \) and \( \mathcal{C}_6 \) (pp. 105–106). Namely the \( e(G) \leq 2 \) Theorem would take the place of Theorem \( \mathcal{C}_4 \), and in rough terms the dichotomy \( m_p(M) \geq 4 \) or \( m_p(M) \leq 3 \) for certain 2-local subgroups would be replaced throughout by the dichotomy \( m_p(M) \geq 3 \) or \( m_p(M) \leq 2 \). Consequently there would be changes in the sets \( \mathcal{K}_4, \mathcal{K}_5 \) and \( \mathcal{K}_6 \) of target groups, the first one shrinking and the other two expanding (and all would be nonempty). In addition the uniqueness theorems \( \mathcal{M}(S) \) and \( \mathcal{U}(\sigma) \) would have to be strengthened to accommodate weakened hypotheses on the \( p \)-ranks of 2-local subgroups of \( G \). The authors are now contemplating and investigating the possibility of adopting this different approach.

Richard Lyons and Ronald Solomon
August, 2000
Reference [P4] has been added in the Second Printing. It can be used as a substitute Background Reference for [FT1].

Expository References


The page contains references to various works on group theory, particularly focusing on fields such as finite groups, Lie groups, and their applications. Some key references include:

- **Be5**: Finite groups with dihedral Sylow 2-subgroups, J. Algebra 70 (1981), 216–228.
- **BuWi1**: N. Burgoyne and C. Williamson, On a theorem of Borel and Tits for finite Chevalley groups, Arch. Math. 27 (1976), 489–491.
- **FT1**: W. Feit and J. G. Thompson, Solvability of groups of odd order, Pacific J. Math. 13 (1963), 775–1029.
- **FinFr1**: L. Finkelstein and D. Frohardt, Simple groups with a standard 3-component of type $A_n(2)$, with $n \geq 5$, Proc. London Math. Soc. (3) 43 (1981), 385–424.
- **FinS1**: L. Finkelstein and R. Solomon, A presentation of the symplectic and orthogonal groups, J. Algebra 60 (1979), 423–438.
- **FinS2**: Finite simple groups with a standard 3-component of type $Sp_{2n}(2)$, $n \geq 4$, J. Algebra 59 (1979), 466–480.
Finite groups with a standard 3-component isomorphic to $\Omega^\pm(2m,2)$, $m \geq 5$, $F_4(2)$ or $E_6(2)$, $n = 6, 7, 8$, J. Algebra 73 (1981), 70–138.


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C. C. Sims, The existence and uniqueness of Lyons’ group, 138–141 in [SGH1].


S. D. Smith, Large extraspecial subgroups of widths 4 and 6, J. Algebra 58 (1979), 251–280.

Added in Second Printing:


### GLOSSARY

<table>
<thead>
<tr>
<th>PAGE</th>
<th>SYMBOL</th>
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<tr>
<td>90</td>
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<tr>
<td>97</td>
<td>≥</td>
</tr>
<tr>
<td>4–5</td>
<td>[I₁], [I₂], [I_A], [I_G], [II_A], [II_G], [II_P], [II_S]</td>
</tr>
<tr>
<td>60, 101</td>
<td>2Aₙ, (2)Aₙ</td>
</tr>
<tr>
<td>101</td>
<td>2Dₙ(q)</td>
</tr>
<tr>
<td>101</td>
<td>3U₄(3)</td>
</tr>
<tr>
<td>101</td>
<td>4L₃(4)</td>
</tr>
<tr>
<td>101</td>
<td>[A₁ × A₂]K (central extension)</td>
</tr>
<tr>
<td>101</td>
<td>A⁺K (central extension)</td>
</tr>
<tr>
<td>81</td>
<td>Alt</td>
</tr>
<tr>
<td>6, 8</td>
<td>Aₙ(q)</td>
</tr>
<tr>
<td>7, 8</td>
<td>Aₙ(q), 2Aₙ(q), Aₙ⁺(q), Aₙ⁻(q)</td>
</tr>
<tr>
<td>130</td>
<td>A(T)</td>
</tr>
<tr>
<td>60, 104</td>
<td>B₂, B₂-group</td>
</tr>
<tr>
<td>7, 8, 10</td>
<td>Bₙ(q), 2B₂(2ⁿ)</td>
</tr>
<tr>
<td>116</td>
<td>B_p²(G)</td>
</tr>
<tr>
<td>24</td>
<td>B_p(X)</td>
</tr>
<tr>
<td>7, 8</td>
<td>Cₙ(q)</td>
</tr>
<tr>
<td>9, 11</td>
<td>C₀₁, C₀₂, C₀₃</td>
</tr>
<tr>
<td>100</td>
<td>C_p, C_p-group</td>
</tr>
<tr>
<td>102</td>
<td>C_p</td>
</tr>
<tr>
<td>100</td>
<td>C_p', C_p'-group</td>
</tr>
<tr>
<td>81</td>
<td>Chev, Chev(p)</td>
</tr>
<tr>
<td>90</td>
<td>C(G,S)</td>
</tr>
<tr>
<td>101</td>
<td>C_Kp</td>
</tr>
<tr>
<td>22</td>
<td>C(K, x)</td>
</tr>
<tr>
<td>63</td>
<td>C_x</td>
</tr>
<tr>
<td>22</td>
<td>Cₓ(A/B)</td>
</tr>
<tr>
<td>14</td>
<td>Cₓ(V)</td>
</tr>
<tr>
<td>139</td>
<td>C_Y</td>
</tr>
<tr>
<td>64, 124</td>
<td>Δₓ(B)</td>
</tr>
<tr>
<td>7, 8, 10</td>
<td>Dₙ(q), 2Dₙ(q), 3D₄(q), Dₙ⁺(q), Dₙ⁻(q)</td>
</tr>
<tr>
<td>7, 8, 10</td>
<td>Eₙ(q), 2E₆(q), E₆⁺(q), E₆⁻(q)</td>
</tr>
<tr>
<td>14</td>
<td>E_p²</td>
</tr>
<tr>
<td>124</td>
<td>E_p(X), E_p²(X)</td>
</tr>
<tr>
<td>139</td>
<td>E(X), E_k(X)</td>
</tr>
<tr>
<td>17</td>
<td>E(X)</td>
</tr>
</tbody>
</table>

148
18 $\Phi(X)$
9, 11 $F_1, F_2, F_3, F_5$
7, 8, 10 $F_4(q), 2^*F_4(2^n), 2F_4(2')$
9, 11 $F_{i22}, F_{i23}, F_{i24}$
95 $\mathfrak{F}M9$
6 $F_q$
16 $F(X)$
17 $F^*(X)$
66 $\gamma(G)$
132 $\Gamma, \Gamma(\alpha)$
82 $\Gamma_{P,2}(X)$
82 $\Gamma_{P,k}(X)$
27 $G^*$
109–121 $G \approx G^*$
66, 121 $G_0(N)$
7, 8, 10 $G_2(q), 2^*G_2(3^n)$
71 $G_\alpha, G^*_\alpha$
6 $GL_n(q)$
102 $S_p$
32–33 $G(q), \tilde{G}(q)$
7 $GU_n(q)$
9, 11 $He$
101 $\mathcal{H}_p, \mathcal{H}_p, \mathcal{H}^*_p$
9, 11 $HS$
55, 81 $T_p(G)$
55, 103 $T^*(G), T^2(G)$
13 $Inn(X)$
14 $Int, Int(x)$
139 $\mathcal{I}(X)$
102 $J_0(p), J_1(p), J_2(p)$
9, 11 $J_1, J_2, J_3, J_4$
26, 130 $J(P)$
90 $[K]$
53, 81 $\mathfrak{X}$
110 $\mathfrak{X}^{(2)*}$
86 $\mathfrak{X}^{(i)}, i = 0, \ldots, 7$
53, 81 $\mathfrak{X}_p$
12 $\mathfrak{X}$-group
12 $\mathfrak{X}$-proper
6, 8 $L_n(q) = L^+_n(q)$
8 $L^+_n(q)$
55, 103 $L^*_p(G)$
53, 81 $L^*_p(G)$
126 $L^*_p(G; A)$
20 $L^*_{p'}(X)$
21 $\tilde{L}^*_{p'}(X)$
127 $L^*_{p'}(X)$
7 $L(q)$
<table>
<thead>
<tr>
<th>Page</th>
<th>Entry</th>
</tr>
</thead>
<tbody>
<tr>
<td>139</td>
<td>$\mathcal{L}(X)$</td>
</tr>
<tr>
<td>139</td>
<td>$L(X)$</td>
</tr>
<tr>
<td>9, 11</td>
<td>$L_Y$</td>
</tr>
<tr>
<td>135</td>
<td>$m_{2,p}(G)$</td>
</tr>
<tr>
<td>9, 11</td>
<td>$M_{11}, M_{12}, M_{22}, M_{23}, M_{24}$</td>
</tr>
<tr>
<td>9, 11</td>
<td>$M_{c}$</td>
</tr>
<tr>
<td>123</td>
<td>$M \rightsquigarrow N$</td>
</tr>
<tr>
<td>30, 81</td>
<td>$m_{p}(A)$</td>
</tr>
<tr>
<td>58, 60, 82</td>
<td>$\mathcal{M}(S), \mathcal{M}_I(S)$</td>
</tr>
<tr>
<td>139</td>
<td>$m(X)$</td>
</tr>
<tr>
<td>65</td>
<td>$N$</td>
</tr>
<tr>
<td>97</td>
<td>$N(M), N^*(M)$</td>
</tr>
<tr>
<td>19</td>
<td>$N_X(Q)$</td>
</tr>
<tr>
<td>139</td>
<td>$N_Y$</td>
</tr>
<tr>
<td>26</td>
<td>$\Omega_1(B)$</td>
</tr>
<tr>
<td>117</td>
<td>$\Omega_3^-(3)$-type</td>
</tr>
<tr>
<td>9, 11</td>
<td>$O'N$</td>
</tr>
<tr>
<td>24</td>
<td>$O_{p, p}(X)$</td>
</tr>
<tr>
<td>19</td>
<td>$O^p(X), O^p(X), O^\pi(X)$</td>
</tr>
<tr>
<td>29</td>
<td>$Out(X)$</td>
</tr>
<tr>
<td>19</td>
<td>$O(X), O_p(X), O_p'(X), O_\pi(X)$</td>
</tr>
<tr>
<td>19</td>
<td>$\pi, \pi'$</td>
</tr>
<tr>
<td>32</td>
<td>$\Pi$</td>
</tr>
<tr>
<td>7, 8, 101</td>
<td>$P\Omega_{n}^{\pm}(q), P\Omega_{n}(q), (P)\Omega_{n}(q)$</td>
</tr>
<tr>
<td>6, 8</td>
<td>$PSL_n(q)$</td>
</tr>
<tr>
<td>7, 8</td>
<td>$PSp_n(q)$</td>
</tr>
<tr>
<td>7, 8</td>
<td>$PSU_n(q)$</td>
</tr>
<tr>
<td>98</td>
<td>$Q(M_p)$</td>
</tr>
<tr>
<td>9, 11</td>
<td>$Ru$</td>
</tr>
<tr>
<td>37, 82</td>
<td>$\sigma(G)$</td>
</tr>
<tr>
<td>107</td>
<td>$\sigma^*(G)$</td>
</tr>
<tr>
<td>58, 83</td>
<td>$\sigma_0(G)$</td>
</tr>
<tr>
<td>120</td>
<td>$\sigma_9(G)$</td>
</tr>
<tr>
<td>118</td>
<td>$\sigma_T(G)$</td>
</tr>
<tr>
<td>32</td>
<td>$\Sigma, \Sigma^+, \Sigma^-$</td>
</tr>
<tr>
<td>6</td>
<td>$\Sigma_n$</td>
</tr>
<tr>
<td>94</td>
<td>$S^p(X), S^p_n(X)$</td>
</tr>
<tr>
<td>29</td>
<td>$S(G)$</td>
</tr>
<tr>
<td>6</td>
<td>$SL_n(q)$</td>
</tr>
<tr>
<td>101</td>
<td>$(S)L_n(q)$</td>
</tr>
<tr>
<td>81</td>
<td>$Spor$</td>
</tr>
<tr>
<td>7</td>
<td>$SU_n(q)$</td>
</tr>
<tr>
<td>9, 11</td>
<td>$Suz$</td>
</tr>
<tr>
<td>8, 10</td>
<td>$Sz(2^n)$</td>
</tr>
<tr>
<td>30, 124–125</td>
<td>$\Theta, \Theta(G; (a)), \Theta(G; A), \Theta_{k+\frac{1}{2}}, \Theta_{k+\frac{1}{2}}(G; A)$</td>
</tr>
<tr>
<td>102</td>
<td>$T_p$</td>
</tr>
<tr>
<td>8</td>
<td>$U_n(q)$</td>
</tr>
<tr>
<td>139</td>
<td>$\overline{X}, \bar{X}, X^*$</td>
</tr>
<tr>
<td>Page</td>
<td>Glossary Term</td>
</tr>
<tr>
<td>------</td>
<td>---------------</td>
</tr>
<tr>
<td>101</td>
<td>$</td>
</tr>
<tr>
<td>26</td>
<td>$Z(J(P))$</td>
</tr>
<tr>
<td>6,8</td>
<td>$Z_p$</td>
</tr>
<tr>
<td>88</td>
<td>$\mathbb{Z}, \mathbb{Z}_X$</td>
</tr>
<tr>
<td>115</td>
<td>2-amalgam $G^*$-type</td>
</tr>
<tr>
<td>114-115</td>
<td>2-amalgam type, 2-amalgam type $(M_1, M_2; S)$</td>
</tr>
<tr>
<td>111</td>
<td>2-central $G^*$-type</td>
</tr>
<tr>
<td>88</td>
<td>2-central involution</td>
</tr>
<tr>
<td>135</td>
<td>2-local $p$-rank</td>
</tr>
<tr>
<td>111</td>
<td>2-maximal $G^*$-type mod cores</td>
</tr>
<tr>
<td>98</td>
<td>${2, p}$-uniqueness subgroup</td>
</tr>
<tr>
<td>110</td>
<td>2-terminal $G^*$-type</td>
</tr>
<tr>
<td>114</td>
<td>2-terminal $L\mathcal{T}_2$-type</td>
</tr>
<tr>
<td>82</td>
<td>2-uniqueness subgroup</td>
</tr>
<tr>
<td>64</td>
<td>3/2-balance</td>
</tr>
<tr>
<td>120</td>
<td>3/2-balanced type</td>
</tr>
<tr>
<td>13</td>
<td>$A$-composition factor, length, series</td>
</tr>
<tr>
<td>118,121</td>
<td>algebraic automorphism</td>
</tr>
<tr>
<td>93</td>
<td>almost $p$-constrained $p$-component preuniqueness subgroup</td>
</tr>
<tr>
<td>18</td>
<td>almost simple group</td>
</tr>
<tr>
<td>94</td>
<td>almost strongly $p$-embedded subgroup</td>
</tr>
<tr>
<td>131-132</td>
<td>amalgam method: $A_\alpha, Q_\alpha, S_\alpha, X_\alpha, Z_\alpha$</td>
</tr>
<tr>
<td>125</td>
<td>associated $(k + \frac{1}{2})$-balanced functor</td>
</tr>
<tr>
<td>96</td>
<td>associated module of a near component</td>
</tr>
<tr>
<td>44-50</td>
<td>Background Results, Background References</td>
</tr>
<tr>
<td>119</td>
<td>base of a neighborhood</td>
</tr>
<tr>
<td>34</td>
<td>$(B, N)$-pair, split $(B, N)$-pair</td>
</tr>
<tr>
<td>34</td>
<td>Borel subgroup</td>
</tr>
<tr>
<td>24</td>
<td>$B_p$-property</td>
</tr>
<tr>
<td>33,34</td>
<td>Bruhat decomposition: $B, H, N, R, U, V, X_\alpha, W, h_\alpha(t), n_\alpha(t)$</td>
</tr>
<tr>
<td>33</td>
<td>Cartan subgroup</td>
</tr>
<tr>
<td>46,109</td>
<td>centralizer of involution pattern</td>
</tr>
<tr>
<td>90</td>
<td>characteristic 2-core</td>
</tr>
<tr>
<td>136</td>
<td>characteristic power</td>
</tr>
<tr>
<td>16</td>
<td>characteristic subgroup</td>
</tr>
<tr>
<td>25</td>
<td>characteristic $p$-type</td>
</tr>
<tr>
<td>32-33</td>
<td>Chevalley commutator formula</td>
</tr>
<tr>
<td>7</td>
<td>Chevalley group</td>
</tr>
<tr>
<td>13</td>
<td>chief factor, series</td>
</tr>
<tr>
<td>6</td>
<td>classical group</td>
</tr>
<tr>
<td>17,81</td>
<td>component</td>
</tr>
<tr>
<td>13</td>
<td>composition factor, length, series</td>
</tr>
<tr>
<td>87</td>
<td>control of (strong) $G$-fusion (in $T$)</td>
</tr>
<tr>
<td>91</td>
<td>control of rank 1 (or rank 2) fusion</td>
</tr>
<tr>
<td>16</td>
<td>covering group</td>
</tr>
<tr>
<td>112</td>
<td>doubly transitive $G^*$-type</td>
</tr>
<tr>
<td>95-96</td>
<td>doubly transitive of Suzuki type</td>
</tr>
<tr>
<td>55,81</td>
<td>even type</td>
</tr>
<tr>
<td>122</td>
<td>extremal conjugation</td>
</tr>
</tbody>
</table>
26 failure of factorization module
19 Fitting length, series
16 Fitting subgroup
39 four-group
18 Frattini subgroup
17 generalized Fitting subgroup
58, 106 generic, generic type
106 generic even type
106 generic odd type
25 Hall $\sigma$-subgroup
46, 109 involution fusion pattern
124 $k$-balance, weak $k$-balance
125 $k$-balanced signalizer functor, weakly $k$-balanced signalizer functor
125 $(k + \frac{1}{2})$-balance
17 layer
104 $L\mathcal{B}_2$-type
105 $L\mathcal{C}_p$-type
94 $L\mathcal{C}_p$-uniqueness subgroup
32 Lie rank (twisted or untwisted)
128 local balance
126 local $(k + \frac{1}{2})$-balance
126 local $k$-balance, weak local $k$-balance
19 local subgroup
21 $L_p'$-balance
127 $L_p^*$-balance
57–58 $\mathcal{L}_p$-generic type
58, 103 $\mathcal{L}_p$-special type, $\mathcal{L}_2$-special type
104 $\mathcal{L}\mathcal{T}_2$-type
105 $\mathcal{L}\mathcal{T}_p$-type
34 monomial subgroup
96, 98 near component (of $Y$): linear, alternating, standard
97 near component 2-local uniqueness subgroup
64, 119 neighbor, $(y, I)$-neighbor
65, 119, 120 neighborhood: level, split, vertical
8, 9 notation for simple groups
58–59, 81 odd type
101 $p$-central element of order $p$
20 $p$-component
91 $p$-component preuniqueness hypothesis
90 $p$-component preuniqueness subgroup
20 $p$-constrained group
19 $p'$-core
20 $p$-layer
19 $p$-local subgroup
107 ${p, q}$-parabolic type
89 proper 2-generated core
121 proper semisimple type
24 $p$-solvable group
64 $p$-source
Glossary

23, 108  $p$-terminal $p$-component
63  $p$-terminal $S_p$-pair
118  $p$-terminal $S_F$-type

22, 127  pumpup: vertical, trivial, diagonal, proper
82  $p$-uniqueness subgroup
26  quadratic chief factor
25  quadratic $F_pX$-module

16, 81  quasisimple group
117  quasisymplectic type

60, 82  quasithin, quasithin type, quasithin case
34  reduced monomial subgroup
10  Ree group

136  regular
95  restricted even type
32  root subgroup, root system
25  $\sigma$-subgroup
21  Schreier property
18  Schur multiplier
12  section
16  semisimple group
64  semisimple neighbor
121  semisimple type
124  signalizer functor: $A$-signalizer functor, solvable, closed, complete
13  simple group
109  solvable component, solvable $p$-component
13  solvable group

66, 121  span of $N$
58  special, special type
105  special even type
103  special odd type
53  standard component
91  standard preuniqueness subgroup

32-33  Steinberg presentation, relations
10  Steinberg variation

43, 89  strongly closed subgroup
93  strongly closed type $p$-component preuniqueness subgroup

30–31, 82  strongly embedded subgroup
52, 82  strongly $p$-embedded subgroup
88  strongly $\mathbb{Z}$-embedded subgroup

52, 94  strong $p$-uniqueness subgroup
64  subterminal $(x, K)$-pair
10  Suzuki group
117  symplectic pair: faithful, trivial
116  symplectic type

23, 81  terminal component

26, 130  Thompson subgroup
34  Tits system
7  twisted group, untwisted group
17  universal covering group
33 universal version
89 weakly $\mathbb{Z}$-embedded subgroup
33 Weyl group
116 wide $\mathcal{L}_p$-type
93 wreathed $p$-component preuniqueness subgroup
71 $Y$-compatible
65 $(y, I)$-neighborhood
117 $\mathbb{Z}_6 \times \mathbb{Z}_2$-neighborhood
INDEX

2-central involution 88
2-local $p$-rank 135
3/2-balanced functor 43, 64–65

algebraic automorphism 118, 121
almost strongly $p$-embedded subgroup, see uniqueness subgroups
Alperin, J. 39, 41
alternating group $A_n$ 6, 32, 36
as $C_2$, $T_2$, or $S_2$-group 103
amalgam method 5–6, 26, 39, 41, 43, 60–61, 105, 131–133
associated graph 132–133
Artin, E. 11
Aschbacher, M. 18, 30, 37, 39–43, 45–48, 50, 53, 89, 99, 125, 129, 130
Aschbacher $\chi$-block 39, 41, 53
associated $(k + \frac{1}{2})$-balanced functor 125
Atlas of Finite Groups 45, 50, 139

Background References 47–50, 140
Background Results 59, 63, 79, 87, 104, 118
listed, 44–50
balance, $k$-balance, see group; also see signalizer functor
bar convention 18, 139
Baumann, B. 39, 131
Bender, H. 16–17, 48–50, 123
Bender method 30, 38, 43, 60, 62, 104, 110, 123, 134
Blackburn, N. 46, 47
$(B, N)$-pair 34
split 34
recognition of rank 1 36–37, 39, 49, 50, 63, 113, 138
recognition of rank 2 37, 63, 111, 113, 115, 137–138
Bombieri, E. 49
Borel, A. 25
$B_p$-property 24, 28, 40–41, 62, 123, 127
partial 30, 36, 42, 64
Brauer, R. 51
theory of blocks 38, 46, 50, 62
defect groups of 2-rank at most 3 50
Brauer-Suzuki theory of exceptional characters 38, 135
Burnside, W. 29–30
building 34, 73, 138
Carter, R. 45, 47

centralizer
  of element of odd prime order \( p \) 35, 41, 42, 51, 54–56
  of element of prime order \( p \) 108
  of semisimple element 51, 54–56

centralizer of involution pattern 46, 59, 77, 109–110
character theory 31, 46, 50, 60, 62, 104, 108, 135–137
  ordinary vs. modular 50

Chevalley, C. 3

Chevalley groups, see groups of Lie type

chief factor, series 13–14

classical groups 6ff., see also groups of Lie type

Classification Grid 79, 83, 85, 99–121

Classification Theorem, see Theorems
  Theorems \( \mathcal{C}_1-\mathcal{C}_7 \) 104–106

component (see also \( p \)-component) 17, 51, 81
  standard 53, 91–92
  terminal 23, 42, 53, 81, 90–92, 108
  solvable 51, 67, 109

composition factor, length, series 12
  \( A \)-composition factor, length, series 13

computer 35, 45, 68

cell control
  of 2-locals 129
  of fusion 87, 122
  of rank 1 or 2 fusion 91–92

Conway, J. 11

core 20; see also \( p' \)-core
  elimination 40, 43, 60, 110–111

covering group 16
  universal 17, 33
  notation for 101

\( \mathcal{C}_p \)-groups 54, 57, 81, 99–101
  as pumpups 101–102

\( \mathcal{C}_{p'} \)-groups 95, 100

Curtis, C. 35

Das, K. M. 35, 71

Delgado, A. 37

Dickson, L. 7

Dieudonné, J. 47

double transitivity of Suzuki type 95–96

Enguehard, M. 49–50

Expository References 47, 141–146

Feit, W. 46, 47, 48, 107–108

Finkelstein, L. 35

Fischer, B. 11, 39–40
  transpositions 11, 39
Fitting length, series 19
Fitting subgroup 16
    generalized Fitting subgroup 17, 123
Foote, R. 5, 38, 53, 98
Frattini subgroup 18
Frobenius, G. 29
Frobenius group 107
Frohardt, D. 35
fusion 29, 60, 62, 63, 104, 120, 122
    extremal conjugation 122

\[ G \cong G^* \quad 59, 62–63, 67–68, 76–77, 83 \]
\[ G^* = A_n \]
    \( n \leq 12 \) 109, 114
    \( n \geq 13 \) 121
\[ G^* \text{ of Lie type} \]
    of large Lie rank 116, 121
    of small Lie rank
        of characteristic 2 95–96, 115
        of characteristic 2 and Lie rank 1 95–96
        of odd characteristic 110, 112, 113
\[ G^* \text{ sporadic} \quad 109 \]
general local group theory 45–48
generalized Fitting subgroup 17, 123
geometry associated with a finite group 35, 73–74
Gilman, R. 35, 39, 41
Glauberman, G. 21, 38, 39, 48–50, 124, 130
Goldschmidt, D. 39, 43, 49, 125
Gomi, K. 37
Gorenstein, D. 29, 38, 39, 41, 46, 47–50, 99, 124, 126, 127
\( S_p \)-groups 57–58, 63, 103
Griess, R. L. 17, 35, 41, 45
group
    almost simple 18
    alternating, see alternating group
    covering 16
        universal 17, 33
        notation for 101
    \( k \)-balanced 124–125, 129
        \((k + \frac{1}{2})\)-balanced 125, 129
        locally balanced 128
        locally \( k \)-balanced, \( k + \frac{1}{2} \)-balanced 126
        weakly \( k \)-balanced, weakly locally \( k \)-balanced 124, 126
\( \mathcal{K} \)-proper 12, 21
nilpotent 15–16
of Lie type, see groups of Lie type
\( p \)-constrained 20
perfect 16
\( p \)-solvable 24
quasisimple 16, 53, 81
semisimple 16
simple, table of 8–10
solvable 13, 16, 73–74
sporadic, see sporadic group
group order formulas 50, 135–137
groups in \( \mathcal{C}_p \), see \( \mathcal{C}_p \)-groups
groups in \( \mathcal{C}_p' \), see \( \mathcal{C}_p' \)-groups
groups in \( \mathcal{S}_p \), see \( \mathcal{S}_p \)-groups
groups in \( \mathcal{T}_p \), see \( \mathcal{T}_p \)-groups
groups of 3/2-balanced type 120–121
groups of characteristic 2-type 55, 98, 116–117
groups of characteristic \( p \)-type 25, 116–117
groups of even type 36, 53–55, 57–59, 81, 87, 98
groups of restricted even type 58–59, 82, 90, 95, 99, 102, 106
groups of \( G^* \)-type (\( G^* \) a target group)
groups of doubly transitive \( G^* \)-type, \( G^* = L_3(p^n), U_3(p^n), 2G_2(3^n) \) 112
groups of 2-amalgam \( G^* \)-type, \( G^* \in \mathcal{K}(4) \) 115
groups of 2-central \( G^* \)-type, \( G^* \in \mathcal{K}^{(2)} \) of Lie type 111
groups of 2-maximal \( G^* \)-type mod cores, \( G^* \in \mathcal{K}^{(2)*} \) 111
groups of 2-terminal \( G^* \)-type, \( G^* \in \mathcal{K}^{(2)*} \) 110
groups of generic type 35–36, 55–59, 63–68, 79, 106, 118–121
groups of generic even type 106
groups of generic odd type 106
groups of \( L_p \)-generic type 57–59, 61
groups of semisimple type 121
groups of proper semisimple type 121
groups of \( GF(2) \)-type 40–43
groups of large sporadic type 37–38, 61
groups of Lie type 6ff., 32ff., 45–49
as \( (B, N) \)-pair 34
Borel subgroup of 33
Bruhat decomposition of 34
Cartan subgroup of 33
Chevalley group 3
Chevalley commutator formula 32
Dynkin diagram of 32, 71
generation of 48, 65–66
Lie rank of
  twisted 32
  untwisted 32
monomial subgroup of 34
  reduced 34
parabolic subgroup of 26, 62
rank 1 subgroup of 33, 35, 71
Ree group 3, 10, 49, 50
root subgroup of 32
root system 32
  fundamental system 32
Schur multiplier of 45, 48
semisimple element of 51
Steinberg variation 7
Steinberg presentation of, relations 33
Suzuki group 3, 10
universal version of 33
and universal covering group 33
untwisted, see Chevalley group
Weyl group of 33, 35
groups of odd order
  groups of odd order uniqueness type 107
  groups of \{p,q\}-parabolic type 107–108
  groups of \sigma^*(G)-uniqueness type 107
groups of odd type 58–59, 81, 86
groups of quasithin type 82, 105, 114–116
  groups of 2-amalgam type 114–115
thin subcase 37, 41
  groups of \mathcal{L}_2-special type 58–60, 103–104
    groups of \mathcal{L}_2\mathcal{B}_2-type 61–63, 83, 104, 110–113, 123
    \mathcal{S}L_2(q)-subcase 61–63
    groups of \mathcal{L}\mathcal{I}_2-type 104, 114
      groups of 2-terminal \mathcal{L}\mathcal{I}_2-type 114
  groups of \mathcal{L}_p-special type 58–60, 105–106
    groups of \mathcal{L}\mathcal{E}_p-type 105, 116–118
      groups of quasisymplectic \mathcal{L}\mathcal{E}_p-type 117
      groups of wide \mathcal{L}\mathcal{E}_p-type 116, 133–135
  groups of \mathcal{L}\mathcal{J}_p-type 106, 118
    groups of \mathcal{L}_p-terminal \mathcal{L}\mathcal{J}_p-type 118
  groups of odd order uniqueness type 107
  groups of \{p,q\}-parabolic type 107–108
  groups of \sigma^*(G)-uniqueness type 107
  groups of special even type 105, 106, 114–118
  groups of special odd type 103–104, 106–108, 110–114
groups with specified 2-structure
  groups of 2-rank at most 2 36, 135
  groups of 2-rank at most 3 50
  groups with semidihedral or wreathed Sylow 2-subgroups 76, 136–137
    characteristic power 136–137
  Brauer group order formula for regular groups 137

Hall, M. 11
Hall, P. 16, 25
Hall \sigma-subgroup
Harada, K. 11, 39, 89
Hayashi, M. 37
Holt, D. 89
Hunt, D. 49
Huppert, B. 46, 47
identification of simple groups see recognition of simple groups
involution fusion pattern 46, 60, 62, 109, 110
Isaacs, I. M. 46, 47
isomorphism question 11
Janko, Z. 11, 49
$\mathcal{K}$-groups 5, 12
  theory of 12, 45, 48, 75–76, 138–139
$\mathcal{K}$-proper group 12, 75–76, 79
Klinger, K. 38, 116–117
Klinger-Mason method 61, 105, 116–118
layer, see also $p$-layer 17
Leech lattice 11, 35
linear group 18
local subgroup, see also $p$-local subgroup 19
  general structure of 27–28
$L_{p'}$-balance 21, 127–128
  analogue for near components 96
  analogue for two primes 134
$L_{p^*}$-balance 127–128
Lyons, R. 41, 48–50, 99
Mason, G. 37–38, 41, 116–117
Mathieu, E. 11
maximal subgroup 60, 62–63, 123
McBride, P. 30, 49, 124
Meierfrankenfeld, U. 130
modules
  failure of factorization 26, 130, 133
  quadratic 25–26, 123, 130
near component 50, 96–97, 129–131, 133
  alternating 96–97, 130
  associated module of 96
  linear 96–97, 130, 131
  standard 98
  type $G_2(3^n)$ 97
neighbor 42, 64–65, 71, 119
semisimple 64–65
neighborhood 55–58, 59, 63–68, 118–120
  base of 119
  example of 56–57, 68–70
  level 66, 121
  span of 66–67, 121
  vertical 65, 76–77, 118–120
  $Z_6 \times Z_2$- 61, 117
$N$-group 38–39, 73
Niles, R. 39, 50
notation 139
INDEX

Odlyzko, A. 49
O'Nan, M. 18, 45, 49, 138
outer automorphisms 18
overall strategy of proof 35–38, 42–43

Parts of the series 4–5, 59, 77–78, 80
Part II 38, 52–53
Part III 36
Part IV 37
Part V 37, 38

p-central p-element 101
p-component 20
  p-terminal 22ff., 63, 108–109
    pumping up to 23, 108–109
  solvable 64, 109
p-component preuniqueness hypothesis 91–92
p-component preuniqueness subgroup, see uniqueness subgroups
p-component uniqueness theorems 30–31, 38, 53, 65, 90–92, 118
p'-core 19; see also core
  elimination 23–24, 37, 61, 120
  embedding of p'-core of p-local subgroups 21, 127–128
p-layer 20
p-local subgroup 19
  embedding of p'-core of 20–21, 127–128
  embedding of p-layer of 21–24, 127
perfect central extension, see covering group
permutation group 18, 35
  doubly transitive 74, 112–113, 138
    of Suzuki type 95–96
      split (B, N)-pair of rank 1 95–96, 112–113, 138
  highly transitive 11, 35
    of rank 3 11
    representation as 31
Peterfalvi, T. 48–49, 96, 107, 138
Phan, K.-W. 35, 71
presentation 31–35
  of classical groups 35
  of symmetric groups 32, 36, 68–70
Steinberg presentation of groups of Lie type 32, 33, 36, 51, 118
    à la Curtis-Tits 34, 67, 70
    à la Gilman-Griess 35, 67, 71
p-source 64
pumpup 22, 127
  as C_p, T_p, or G_p-group 103, 114
  diagonal, proper, trivial, or vertical 22, 127
quasisimple group 16, 53, 81
quasithin 5, 37, 41, 43, 60–61, 82, 105, 114–116
recognition of simple groups 31–35
  groups of Lie type 32–35, 42, 49, 137–138
alternating groups 32
sporadic groups 35
recognition of symmetric groups 68–70

Ree, R. 10
Ree groups 3, 10, 49, 50
Rowley, P. 37

Schreier, O. 21
Schreier property 21, 24, 29
Schur, I. 17
Schur multiplier 18, 45, 48, 139
Scott, L. 18
section 12
Seitz, G. 48
\(\sigma(G)\) 37–38, 52–53, 55, 58–59, 61, 82, 86, 92–93, 105, 131
\(\sigma_0(G)\) 58–61, 83, 86, 105–106, 116, 118, 131
Sibley, D. 48
signalizer functor 29–30, 124ff.
  closure 30, 124
  closed 124
  complete 124
  \(k\)-balanced, weakly \(k\)-balanced 124–126
  solvable 124
  trivial 124, 128
simple groups, table of 8–10
simplicity criteria 29–31
Sims, C. 39, 45
  conjecture 74
Smith, F. 41
Smith, S. 41
Solomon, R. 35, 39, 53
sporadic groups 3, 6, 9, 11, 87, 100
  as target group 77, 87
  as \(S_p\)-group 103
background properties of 44–48, 50
existence and uniqueness of 46–47, 50, 72
individual groups:
  Baby Monster \(F_2 = BM\) 11, 38, 87, 100
  Conway groups \(Co_1 = \cdot 1, Co_2 = \cdot 2, Co_3 = \cdot 3\) 11, 38, 42, 72, 87, 100
  Fischer 3-transposition groups \(Fi_{22}, Fi_{23}, Fi'_{24}\) 11, 38, 87, 100
  Fischer-Griess Monster \(F_1 = M\) 3, 11, 38, 42, 47, 61, 72, 87, 100
  Harada-Norton \(F_5 = HN\) 11, 67–68, 87, 100
  Held \(He = HHM\) 11, 87, 100
  Higman-Sims \(HS\) 11, 87, 100
  Janko \(J_1, J_2 = HJ, J_3 = HJM, J_4\) 3, 11, 87, 100
  Lyons-Sims \(Ly\) 11, 45, 87, 100
  Mathieu \(M_{11}, M_{12}, M_{22}, M_{23}, M_{24}\) 11, 42, 46, 72, 87, 100
McLaughlin \textit{Mc} 11, 87, 100
O’Nan-Sims \textit{O’N} 11, 87, 100
Rudvalis \textit{Ru} 11, 87, 100
Suzuki \textit{Suz} 11, 87, 100
Thompson \textit{F}_3 = Th 11, 87, 100

standard form problems 38
Steinberg, R. 3, 7, 10, 17, 45, 47, 48
Stellmacher, B. 37, 50
strongly closed subgroup 61
strongly embedded subgroup, see uniqueness subgroups
strongly \( p \)-embedded subgroup, see uniqueness subgroups
strongly \( Z \)-embedded subgroup, see uniqueness subgroups
Stroth, G. 5, 37, 38, 43, 53, 98–99, 107, 130
subterminal pair 64–66
Suzuki, M. 10, 38, 46, 47, 49
Suzuki groups \( Sz(2^n) \) 3, 10
symmetric group \( \Sigma_n \) 6
symplectic type 2-group 116
symplectic pair 117
faithful, of \( \Omega_8^-(3) \)-type, trivial 117

Tanaka, Y. 37
target group \( (G^*) \) 27, 54–55, 86–87
the 8 families \( \mathcal{K}^{(i)} \) \( (0 \leq i \leq 7) \) 86
\( A_n, n \leq 12 \) 77, 111–112, 114–116
\( A_n, n \geq 13 \) 63–68, 76–77
\( \Sigma_n \) 68–70
in \( K^{(0)} \) 89
of Lie type 76–77
\( L_3^\pm(q), PSp_4(q), G_2(q), 3D_4(q), L_4^\pm(q), q \) odd 61–63
of large Lie rank 63–68, 71, 76
of Lie rank 1 77, 110–113
of small Lie rank 77, 110–116
quasithin 77
sporadic 77, 87
large sporadic 116–117

Theorems
Alperin-Brauer-Gorenstein-Walter classification of groups of 2-rank 2 39, 41
Alperin-Goldschmidt conjugation theorem 97, 122
Aschbacher classical involution theorem 41–43
Aschbacher proper 2-generated core theorem 89
Aschbacher uniqueness case theorem 43, 92
Aschbacher-Bender-Suzuki strongly \( Z \)-embedded subgroup theorem 88
Aschbacher-Gilman component theorem 39, 40, 52, 75
generalized to odd primes 75–76
Baumann-Glauberman-Niles theorem 50, 130–131
revision of 50
Bender \( F^* \)-theorem 17, 138
Bender-Suzuki strongly embedded subgroup theorem 30–31, 33, 36, 38, 52
for $\mathcal{K}$-proper groups 75
Bender-Thompson signalizer lemma 116
Bender uniqueness theorem 123
Brauer-Suzuki quaternion theorem 52
Burnside $p^{a}q^{b}$-theorem 30
Classification Theorem 6, 79
    comparison of new and old proofs 41–44, 55, 63, 72, 74–76, 98
    four-part division 58–59
    main logic 106
    stages of the proof 61, 106–121
Theorems $\mathcal{C}_{1}$–$\mathcal{C}_{7}$ 104–106
Curtis-Tits theorem 35, 63, 67, 113, 115
    variations for classical groups 35, 63, 113, 115
Feit-Thompson theorem, see Odd Order Theorem
Fong-Seitz classification of split $(B,N)$-pairs of rank 2 63, 138
Gilman-Griess recognition theorem for groups of Lie type 35, 67, 71
Glauberman $ZJ$-theorem 26, 38
Glauberman $Z^{*}$-theorem 30, 31, 36, 38, 43, 48, 79, 122, 135
global $C(G,S)$-theorem 38, 53
Goldschmidt strongly closed abelian 2-subgroup theorem 43, 89
Gorenstein-Harada sectional 2-rank at most 4 theorem 39, 42, 43
Gorenstein-Walter dihedral Sylow 2-subgroup classification 50, 74
    revision by Bender and Glauberman 50, 74
Hall-Higman Theorem B 26
Holt’s Theorem 89
Jordan-Hölder Theorem 12ff.
    local $C(X,T)$-theorem 50, 96–97, 129–131
Mason Quasithin Theorem 37, 41
Odd Order Theorem 30, 31, 36, 38–39, 48, 79, 104, 134, 135
    revision of 48, 74, 123
$p$-component uniqueness theorems 30–31, 38, 53, 65, 90–92, 118
Schur-Zassenhaus theorem 24
signalizer functor theorems 49, 124
Solomon maximal 2-component theorem 39
Theorem $\mathcal{C}_{1}$ 104
    stages 1–4 106–108
Theorem $\mathcal{C}_{2}$ 104
    stages 1–4 110–113
Theorem $\mathcal{C}_{3}$ 104
    stages 1–3 113
Theorem $\mathcal{C}_{4}$ 105
    stages 1–4 114–116
Theorem $\mathcal{C}_{5}$ 105
    stages 1–4 116–118
Theorem $\mathcal{C}_{6}$ 106
    stages 1–2 118
Theorem $\mathcal{C}_{7}$ 106
    stages 1–5 118–121
Theorem $\mathcal{M}(S)$ 5, 90, 99, 129
INDEX

stages 1–3 97–98
Theorem PS 92
Theorems PU₁, PU₂, PU₃ 92
Theorem SA 89, 97
Theorem SE 89–90
stages 1–2 95–96
Theorem SF 89
Theorem SZ 88–89
Theorem TS 92, 114
Theorem U(σ) 5, 86, 92, 106–108, 129, 131
stages 1–3 98–99
Corollary U(σ) 92
Theorem U(2) 89, 97, 106
Thompson A × B-lemma 21
Thompson dihedral lemma 134
Thompson factorization theorem 26, 130
Thompson N-group classification theorem 38–40, 43
Thompson order formula 114, 136
Thompson replacement theorem 26, 130
Thompson transfer lemma 55, 104, 122, 128
Three subgroup lemma 17
Timmesfeld root involution theorem 42–43
Tits classification of spherical buildings of rank at least 3 73
Walter classification of groups with abelian Sylow 2-subgroups 74
revision by Bender 74
Thompson subgroup 26, 130
tightly embedded subgroup 39, 43
Timmesfeld, F. G. 37, 39, 40
Tits, J. 10, 25, 34, 137
Tits system, see (B, N)-pair
T_p-groups 57, 102–103, 129
as pumpups 101–103
transfer 29, 31, 74, 107
triality 10

uniqueness grid 79, 83, 84, 87–89
uniqueness case 5, 38, 43, 53
uniqueness subgroups

{2, p}-uniqueness subgroup 53, 98–99
2-uniqueness subgroup 82, 87–89
p-component preuniqueness subgroup 90–93
controlling rank 1 or 2 fusion 91–92
of strongly closed type 93
standard 91–92
wreathed 93
p-uniqueness subgroup, p odd 52–53, 82
strong p-uniqueness subgroup 52–53, 58, 63, 68, 82–83, 92–96, 99, 106
LₐCₚ-uniqueness subgroup 94
near component 2-local uniqueness subgroup 97
strongly embedded subgroup 30–31, 75, 82, 87–89, 95–96
strongly $p$-embedded subgroup 31, 82, 91–92
   almost strongly $p$-embedded subgroup 91–94
   almost $p$-constrained 93
   of strongly closed type 93
   wreathed 93
strongly $\mathbb{Z}$-embedded subgroup 87–89, 96
weakly $\mathbb{Z}$-embedded subgroup 89

Walter, J. H. 29, 38–39, 124, 126, 127
Ward, H. 49
weak closure method 43
weakly $\mathbb{Z}$-embedded subgroup, see uniqueness subgroups
Wong, W. 35

Yoshida, T. 29