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Free Lattices

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Open Problems

Below are the open problems which appeared in the text.

PROBLEM 1.25 (page 22). *Which unary polynomials on free lattices are fixed point free? For which unary polynomials f does $\bigvee f^i(a)$ exist for all a?*

PROBLEM 5.28 (page 107). *Characterize those ordered sets which can be embedded into a free lattice.*

PROBLEM 5.68 (page 133). *Which lattices (and in particular which countable lattices) are sublattices of a free lattice?*

PROBLEM 10.17 (page 198). *Does there exist a quadruple $a < c_1 \leq c_2 < b$ of elements of a free lattice $\mathbf{FL}(X)$ such that the intervals c_1/a and b/c_2 are both infinite and every element of b/a is comparable with either c_1 or c_2?*

PROBLEM 10.18 (page 198). *Describe all meet reducible elements a of $\mathbf{FL}(X)$ such that every element above a is comparable with a canonical meetand of a.*

PROBLEM 10.22 (page 201). *Which completely join irreducible elements $a \in \mathbf{FL}(X)$ satisfy $\kappa(a) = a^{\partial}$? Are there infinitely many for fixed X?*

PROBLEM 10.23 (page 201). *Is the element a from* (1) *on page 200 the only element of $\mathbf{FL}(X)$ which is invariant under the automorphisms of $\mathbf{FL}(X)$ and satisfies $\kappa(a) = a^{\partial}$?*

PROBLEM 10.25 (page 202). *Is every middle element of $\mathbf{FL}(X)$ above a minimal one?*

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PROBLEM 12.27 (page 274). *Does every finitely generated lattice variety have a finite, convergent AC term rewrite system? What about every variety generated by a finite lower (or upper) bounded lattice?*

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List of Symbols

Notation	**Page(s)**	**Description**
$A \subseteq B$, $A \subset B$	7	Set inclusion, proper set inclusion.
$a \vee b$, $\bigvee S$	8	Join, least upper bound.
$a \wedge b$, $\bigwedge S$	8	Meet, greatest lower bound.
$\mathbf{L} = \langle L, \vee, \wedge \rangle$	8	A lattice on the set L.
$\mathrm{J}(\mathbf{L})$	8	The join irreducible elements of L.
$\mathrm{M}(\mathbf{L})$	8	The meet irreducible elements of L.
$a \prec b$, $b \succ a$	8	a is covered by b, b covers a.
b/a	9	The interval from a to b.
$\mathbf{L}[C]$	9	$\mathbf{L}$ with a convex set doubled.
λ	9	Natural homomorphism from $\mathbf{L}[C]$ onto $\mathbf{L}$.
$\ker f$	56	The kernel of f.
$t^{\mathbf{L}}(a_1, \ldots, a_n)$, $t^{\mathbf{L}}$	10	Interpretation of a term t in $\mathbf{L}$.
$\mathbf{FL}(X)$, $\mathbf{FL}(n)$	10	Free lattice.
$\mathbf{F}_{\mathcal{V}}(X)$	11	Relatively free lattice in $\mathcal{V}$ over X.
$\mathbf{2}$	11	The two element lattice.
$\underline{x} = \bigwedge(X - \{x\})$	13	Atoms of $\mathbf{FL}(X)$.
$\overline{x} = \bigvee(X - \{x\})$	13	Coatoms of $\mathbf{FL}(X)$.
(W)	13	Whitman's condition.
(W+)	13	A modification of Whitman's condition.
$A \ll B$, $C \gg D$	15	Join and meet refinement.
$(\mathrm{SD}_\vee)$, $(\mathrm{SD}_\wedge)$	18	Join and meet semidistributivity.
$\boldsymbol{H}$, $\boldsymbol{S}$, $\boldsymbol{P}_u$, $\boldsymbol{V}$	25	Class closure operators.
$\beta(a)$, $\beta_h(a)$	27	The least preimage of a.
$\alpha(a)$, $\alpha_h(a)$	27	The largest preimage of a.
$A^\wedge$, $A^\vee$	28	Meet closure of A, join closure of A.
$\beta_k(a)$, $\beta_{k,h}(a)$	28	Step k in computing $\beta(a)$.
$\alpha_k(a)$, $\alpha_{k,h}(a)$	29	Step k in computing $\alpha(a)$.
$\mathcal{C}(a)$	29	Nontrivial join covers of a.
$\mathcal{M}(a)$	30	Minimal nontrivial join covers of a.
$\mathrm{D}_k(\mathbf{L})$	31	Elements such that $\beta(a) = \beta_k(a)$.
$\mathrm{D}(\mathbf{L})$	31	$\bigcup_{k \in \omega} \mathrm{D}_k(\mathbf{L})$.
$\rho(a)$	31	D–rank of a.
$a\ D\ b$	39	Join dependency relation.
R_θ, S_θ	40	Join irreducibles with $\langle a, a_* \rangle \in \theta$, $\mathrm{J}(\mathbf{L}) - R_\theta$.
$\mathbf{Q_L}$	41	Ordered set isomorphic to $\mathrm{J}(\mathbf{Con\ L})$.

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