

MATHEMATICAL Surveys and Monographs

Volume 42

Free Lattices

Ralph Freese
Jaroslav Ježek
J. B. Nation



American Mathematical Society

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Open Problems

Below are the open problems which appeared in the text.

PROBLEM 1.25 (page 22). *Which unary polynomials on free lattices are fixed point free? For which unary polynomials f does $\bigvee f^i(a)$ exist for all a ?*

PROBLEM 5.28 (page 107). *Characterize those ordered sets which can be embedded into a free lattice.*

PROBLEM 5.68 (page 133). *Which lattices (and in particular which countable lattices) are sublattices of a free lattice?*

PROBLEM 10.17 (page 198). *Does there exist a quadruple $a < c_1 \leq c_2 < b$ of elements of a free lattice $\mathbf{FL}(X)$ such that the intervals c_1/a and b/c_2 are both infinite and every element of b/a is comparable with either c_1 or c_2 ?*

PROBLEM 10.18 (page 198). *Describe all meet reducible elements a of $\mathbf{FL}(X)$ such that every element above a is comparable with a canonical meetand of a .*

PROBLEM 10.22 (page 201). *Which completely join irreducible elements $a \in \mathbf{FL}(X)$ satisfy $\kappa(a) = a^\partial$? Are there infinitely many for fixed X ?*

PROBLEM 10.23 (page 201). *Is the element a from (1) on page 200 the only element of $\mathbf{FL}(X)$ which is invariant under the automorphisms of $\mathbf{FL}(X)$ and satisfies $\kappa(a) = a^\partial$?*

PROBLEM 10.25 (page 202). *Is every middle element of $\mathbf{FL}(X)$ above a minimal one?*

PROBLEM 11.7 (page 218). *Can one decide if an ordered set of size n is a lattice in time faster than $O(n^{5/2})$? Can the various data structures mentioned in Theorem 11.6 be computed in time faster than $O(n^{5/2})$?*

PROBLEM 11.40 (page 250). *Is there a polynomial time algorithm which decides if $w \in \mathbf{FL}(\mathbf{P})$ is completely join irreducible?*

PROBLEM 11.41 (page 250). *Is there a polynomial time algorithm which decides if a finitely presented lattice is finite?*

PROBLEM 11.42 (page 251). *Is there an algorithm to decide if a finitely presented lattice is weakly atomic? of finite width?*

PROBLEM 11.43 (page 251). *Is there a polynomial time algorithm which decides if a finitely presented lattice is projective?*

PROBLEM 12.27 (page 274). *Does every finitely generated lattice variety have a finite, convergent AC term rewrite system? What about every variety generated by a finite lower (or upper) bounded lattice?*

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List of Symbols

Notation	Page(s)	Description
$A \subseteq B, A \subset B$	7	Set inclusion, proper set inclusion.
$a \vee b, \bigvee S$	8	Join, least upper bound.
$a \wedge b, \bigwedge S$	8	Meet, greatest lower bound.
$\mathbf{L} = \langle L, \vee, \wedge \rangle$	8	A lattice on the set L .
$J(\mathbf{L})$	8	The join irreducible elements of L .
$M(\mathbf{L})$	8	The meet irreducible elements of L .
$a \prec b, b \succ a$	8	a is covered by b , b covers a .
b/a	9	The interval from a to b .
$\mathbf{L}[C]$	9	\mathbf{L} with a convex set doubled.
λ	9	Natural homomorphism from $\mathbf{L}[C]$ onto \mathbf{L} .
$\ker f$	56	The kernel of f .
$t^{\mathbf{L}}(a_1, \dots, a_n), t^{\mathbf{L}}$	10	Interpretation of a term t in \mathbf{L} .
$\mathbf{FL}(X), \mathbf{FL}(n)$	10	Free lattice.
$\mathbf{FV}(X)$	11	Relatively free lattice in \mathbf{V} over X .
$\mathbf{2}$	11	The two element lattice.
$\underline{x} = \bigwedge (X - \{x\})$	13	Atoms of $\mathbf{FL}(X)$.
$\bar{x} = \bigvee (X - \{x\})$	13	Coatoms of $\mathbf{FL}(X)$.
(W)	13	Whitman's condition.
(W+)	13	A modification of Whitman's condition.
$A \ll B, C \gg D$	15	Join and meet refinement.
$(SD_{\vee}), (SD_{\wedge})$	18	Join and meet semidistributivity.
$\mathbf{H}, \mathbf{S}, \mathbf{P}_u, \mathbf{V}$	25	Class closure operators.
$\beta(a), \beta_h(a)$	27	The least preimage of a .
$\alpha(a), \alpha_h(a)$	27	The largest preimage of a .
A^{\wedge}, A^{\vee}	28	Meet closure of A , join closure of A .
$\beta_k(a), \beta_{k,h}(a)$	28	Step k in computing $\beta(a)$.
$\alpha_k(a), \alpha_{k,h}(a)$	29	Step k in computing $\alpha(a)$.
$\mathcal{C}(a)$	29	Nontrivial join covers of a .
$\mathcal{M}(a)$	30	Minimal nontrivial join covers of a .
$D_k(\mathbf{L})$	31	Elements such that $\beta(a) = \beta_k(a)$.
$D(\mathbf{L})$	31	$\bigcup_{k \in \omega} D_k(\mathbf{L})$.
$\rho(a)$	31	D-rank of a .
$a D b$	39	Join dependency relation.
R_{θ}, S_{θ}	40	Join irreducibles with $\langle a, a_* \rangle \in \theta, J(\mathbf{L}) - R_{\theta}$.
\mathbf{QL}	41	Ordered set isomorphic to $J(\mathbf{Con L})$.

$\mathcal{M}^*(a)$	45	Join covers $U \in \mathcal{M}(a)$ with $\bigvee U$ minimal in \mathbf{L} .
$J(w)$	48, 72	J -closed set determined by w .
u_*	49	Lower cover of completely join irreducible u .
v^*	49	Upper cover of completely meet irreducible v .
$a A b, a B b, a C b$	51	Dependency relations on a semidistributive lattice.
Λ	57	The lattice of lattice varieties.
$V(\varepsilon)$	60	Variety of all lattices satisfying ε .
$\kappa(w)$	67	$\kappa_{\mathbf{FL}(X)}(w)$.
$M(w)$	72	M -closed set determined by w (dual to $J(w)$).
$\mathbf{L}^\vee(w)$	74	Finite, lower bounded, subdirectly irreducible lattice associated with w .
$\mathbf{L}^\wedge(w)$	76	Finite, upper bounded, subdirectly irreducible lattice associated with w .
w_\dagger	76	Lower cover of w in $\mathbf{L}^\vee(w)$.
w^\dagger	76	Upper cover of w in $\mathbf{L}^\wedge(w)$.
$K(w)$	79	$\{v \in J(w) : w_\dagger \vee v \not\leq w\}$.
$\mathbf{FL}(\mathbf{P})$	104, 249	Free lattice over an ordered set or partial lattice; finitely presented lattice.
σ_u, μ_u	136	Endomorphisms of $\mathbf{FL}(X)$.
$\overline{\mathbf{N}}_5, \overline{\mathbf{N}}_5(k)$	161	Covers labelled pentagons.
$C(m, k)$	162	m chains of length and k chains of length one with a common top.
a^∂	201	The dual of a .
$O(f(x))$	205	Big oh notation.
$a \leftarrow 5$	206	Assignment in computer programs.
E_{\leq}, E_{\prec}	208	Edge sets of an ordered set.
e_{\leq}, e_{\prec}	208	Cardinalities of the edge sets.
$p \rightarrow q, r \rightarrow_R t$	256	Term rewrite rule.
$\text{nf}(w), \text{nf}_R(w)$	256	Normal form (for the TRS R).

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