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Fundamental Groups of Compact Kähler Manifolds

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Fundamental Groups of Compact Kähler Manifolds

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ABSTRACT. We study the fundamental groups of compact Kähler manifolds. It is shown that these groups satisfy very strong restrictions arising from Hodge theory, and from its combination with rational homotopy theory, with L^2-cohomology, with the theory of harmonic maps and with gauge theory. We also give many examples, some of them new, of groups which arise as fundamental groups of compact Kähler manifolds, in fact, of smooth complex projective varieties; and we consider what happens when one relaxes the Kählerianity assumption on a manifold, or the smoothness assumption on a projective variety.

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Preface

"On ignore à peu près tout de quels groupes peuvent être groupes fondamentaux de variétés algébriques,... " P. Deligne (1987)[1]

This book was written because a lot can now be said about the fundamental groups of algebraic varieties and of compact Kähler manifolds. Over the last few years there has been a lot of progress on trying to understand which groups arise. These developments were the topic of the Swiss Seminar (Borel Seminar) in the Spring of 1995. To a large extent, this book is based on lectures given in the seminar, although it is not, and is not meant to be, a faithful account of the seminar. We try to explain what is currently known about the fundamental groups of compact Kähler manifolds[2] and about some closely related questions. A lot of examples are given to show that this class of groups is large and interesting. However, most of the book is devoted to proving restrictions on these groups arising from the work of Johnson–Rees, Gromov, Carlson–Toledo, Simpson and many others. In many cases, especially when good accounts do not already exist, we give complete detailed proofs; in other cases we prove special results and refer the reader to the literature for the general case. The techniques used are a mixture of topology, differential and algebraic geometry, and complex analysis.

Chapter 1, written mostly by D.K., is an introduction and overview, and it explains the context of the problem to which this book is devoted. The section on fundamental groups of compact complex surfaces contains a few new results which have not appeared elsewhere.

Chapter 2, written mostly by D.K., discusses the general problem of finding a holomorphic map inducing a given representation or homomorphism of the fundamental group of a compact Kähler manifold. In general, such a map does not exist, but there are two notable exceptions. The first one is the quotient homomorphism of the fundamental group to its first homology modulo torsion. The Albanese map is a holomorphic map realising this. The second exception is representations onto surface groups of genus at least two. We prove a theorem of Siu to the effect that if the surface group is of maximal genus for the given manifold, then the representation is induced by a surjective holomorphic map with connected fibers from the given Kähler manifold to a Riemann surface. More generally, the property of admitting some holomorphic map onto a closed hyperbolic Riemann surface is seen to be a property of the fundamental group of a compact Kähler manifold. This allows us to divide Kähler groups into so–called fibered and non–fibered groups. These concepts tie in nicely with some of the techniques and results in later chapters. We

[1] [**40**], Page 1

[2] Such groups shall be called *Kähler groups* for short.

give complete proofs based on classical arguments (Albanese map, Castelnuovo–de Franchis Theorem), which are toy versions of more delicate arguments in the same style which are used in later chapters.

Chapter 3, written mostly by J.A., applies the techniques of real homotopy theory to study Kähler groups. Rather than looking at the fundamental group itself, we look here at its real Malcev completion. This approach goes back to the work of Sullivan and of Deligne–Griffiths–Morgan–Sullivan in the 1970s, but there are a number of new results as well.

Chapter 4, written mostly by M.B., applies L^2–cohomology to prove restrictions on the fundamental groups of Kähler manifolds, following an idea of Gromov and the elaborations on it by Arapura–Bressler–Ramachandran. We give a careful account of Gromov's theorem showing that Kähler groups cannot split as free products. More generally, we show that Kähler groups have finitely many ends. These results are proved by constructing holomorphic fibrations over curves, generalising the results of Chapter 2.

Chapter 5 gives an outline of some existence theorems for harmonic maps. These are needed for the applications in Chapters 6 and 7. First we outline a proof of the theorem of Eells–Sampson, giving existence of harmonic maps in homotopy classes of maps whose target has a non–positively curved Riemannian metric. Then we explain the generalisation of this result to twisted or equivariant harmonic maps due to Corlette, Donaldson and Labourie.

Chapter 6, written mostly by D.T., applies harmonic maps to the study of Kähler groups. It begins with a proof of the Siu–Sampson Bochner formula, which implies that certain harmonic maps are in fact pluriharmonic. Combining this with the existence theorems of Chapter 5, we have a large supply of pluriharmonic maps from Kähler manifolds to negatively curved manifolds. Following Carlson–Toledo, Siu and Sampson, we prove a general factorisation theorem for such maps, which has a number of geometric corollaries. These include a proof of Siu's theorem that was proved using more classical methods in Chapter 2, and many restrictions on Kähler groups. For example, it is shown that fundamental groups of real hyperbolic manifolds of dimension at least three cannot be fundamental groups of compact Kähler manifolds. The final section of this chapter discusses geometric applications of more general harmonic maps, maps for which the target space is a negatively curved space which need not be a manifold. This more general existence theorem is not covered in Chapter 5, and we refer to the original paper by Gromov–Schoen for it.

Chapter 7, written mostly by K.C., is an introduction to the non–Abelian Hodge theory of Corlette and Simpson. This uses the existence theorems for harmonic maps in Chapter 5. There is a detailed discussion of the Riemann surface case due to Hitchin. This motivates the general case, the details of which are often omitted and replaced by references to the original papers. Some applications to fundamental groups are given following Simpson. At the end we present Reznikov's recent proof of the Bloch conjecture.

Chapter 8, written mostly by D.T., gives a number of very non–obvious examples of groups which occur as Kähler groups, in fact, as fundamental groups of smooth complex projective varieties. This includes non–Abelian nilpotent groups, and some groups which are not residually finite. Some of the examples in this chapter have not appeared elsewhere.

There are two appendices summarising standard material used throughout the book. The first appendix, written by J.A. and D.K., presents some generalities about projective completions of finitely generated groups, and the second one serves as a reference for results in Hodge theory.

The main topic *not* covered in this book is the so–called Shafarevich conjecture[3]. At the end of his book [**114**], Shafarevich raised the question whether the universal covering of a smooth complex algebraic variety has to be holomorphically convex, and one can ask the same question for arbitrary compact Kähler manifolds. Over the last few years, the question of Shafarevich has led to intense activities in algebraic geometry. These are obviously related to the study of Kähler groups, but they tend to have a different flavour from the material presented in this book. We refer the reader to the recent monograph by Kollár [**81**] for an overview of this topic. See also [**13**], [**79**].

The 1995 Borel Seminar was organised by D. Kotschick and M. Burger with the cooperation of N. A'Campo and J. Amorós. It met ten times for a total of twenty–five lectures given by N. A'Campo, J. Amorós, M. Burger, F. Campana, J. Carlson, K. Corlette, D. Kotschick, F. Labourie, S. Maier, D. Toledo, A. Valette and K. Zuo. The lecturers at a preparatory meeting in December 1994 were F. Catanese and D. Kotschick. Financial support was provided by the IIIe Cycle Romand de Mathématiques, the Swiss National Fund and the University of Basle.

We are grateful to all the speakers, and to the other participants, for their valuable contributions to the seminar, and, by extension, to our understanding of the topic of this book.

J.A., M.B., K.C., D.K. and D.T.
December 1995

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[3]In the seminar this was discussed in lectures of F. Campana and K. Zuo.

APPENDIX A

Pro group theory

This Appendix collects the basic constructions and properties of projective completions of groups used in this book.

1. Definitions of group completions

Categorically, group completions are defined as left adjoint functors of inclusions.

We denote by $\mathcal{P}$ a subcategory of the category of groups $\mathcal{G}r$ and by $i\colon \mathcal{P} \hookrightarrow \mathcal{G}r$ its inclusion functor. For example, $\mathcal{P}$ could be the category of finite or of nilpotent groups, with morphisms their group homomorphisms. We will denote by $Hom_{\mathcal{P}}$ the morphisms of the category $\mathcal{P}$.

The defining property of the left adjoint functor $\cdot^{\mathcal{P}}\colon \mathcal{G}r \to \mathcal{P}$, if it exists, is that for every pair of objects $\Gamma \in \mathcal{G}r$, $\Delta \in \mathcal{P}$, there is a bijection

$$Hom_{\mathcal{G}r}(\Gamma, i(\Delta)) \overset{\cong}{\longleftrightarrow} Hom_{\mathcal{P}}(\Gamma^{\mathcal{P}}, \Delta) \ , \tag{12}$$

which is natural in both arguments.

Definition A.1. Let $\mathcal{G}r$ be the category of groups, and $\mathcal{P} \subset \mathcal{G}r$ a subcategory such that the inclusion functor $i\colon \mathcal{P} \hookrightarrow \mathcal{G}r$ has a left adjoint functor

$$\cdot^{\mathcal{P}} : \mathcal{G}r \longrightarrow \mathcal{P} \ .$$

The *$\mathcal{P}$-completion of a group* Γ is the homomorphism

$$\eta^{\mathcal{P}} : \Gamma \longrightarrow \Gamma^{\mathcal{P}} \ ,$$

which is the element of $Hom_{\mathcal{G}r}(\Gamma, i(\Gamma^{\mathcal{P}}))$ corresponding to the identity of $\Gamma^{\mathcal{P}}$ under the adjointness bijection (12).

There is an equivalent definition of the $\mathcal{P}$–completion in terms of a universal property:

Definition A.2. Let Γ be a group. The *$\mathcal{P}$–completion* of Γ is a group homomorphism

$$\eta^{\mathcal{P}} : \Gamma \longrightarrow \Gamma^{\mathcal{P}} \ ,$$

where $\Gamma^{\mathcal{P}}$ is an object of $\mathcal{P}$ having the universal property for homomorphisms from Γ to groups in $\mathcal{P}$.

In other words, $\eta^{\mathcal{P}}$ has the following property: every group homomorphism $f\colon \Gamma \longrightarrow \Delta$, where Δ is in $\mathcal{P}$, factors uniquely through $\eta^{\mathcal{P}}$ to give a commutative diagram

$$\begin{array}{ccc} \Gamma & \xrightarrow{\eta^{\mathcal{P}}} & \Gamma^{\mathcal{P}} \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle f^{\mathcal{P}}} & \\ \Delta & & \end{array}$$

The universal property implies that the universal object it defines is functorial in Γ and is unique up to canonical isomorphism. However, a universal object with the postulated property may not exist, as in the following example:

EXAMPLE A.3. As $\mathbb{Z}$ has finite quotients of arbitrarily large order, there cannot exist any finite group through which all the quotient homomorphisms factor.

By Definition A.1, the existence of the $\mathcal{P}$-completion is equivalent to the existence of a left adjoint for the inclusion functor $i\colon \mathcal{P} \hookrightarrow \mathcal{G}r$. There is a categorical sufficient condition for this. In the situation at hand, the condition is very simple:

PROPOSITION A.4. *If $\mathcal{P}$ is a subcategory of the category of groups closed under projective limits, then the inclusion functor $i\colon \mathcal{P} \to \mathcal{G}r$ has a left adjoint.*

PROOF. Using the universal property, rather than the categorical definition, one constructs the objects $\Gamma^{\mathcal{P}}$ for $\mathcal{P}$ closed under projective limits as a projective limit of homomorphisms $\Gamma \to \Delta \in \mathcal{P}$. □

The fact that not every subcategory defines a completion is remedied by considering the projective closure of a category, e.g. *pro*–finite or *pro*–nilpotent groups and completions. Nevertheless, the $\mathcal{P}$-completion $\eta^{\mathcal{P}}\colon \Gamma \to \Gamma^{\mathcal{P}}$ is determined by the homomorphisms from Γ to any cofinal set of objects in $\mathcal{P}$, so the pro–finite or pro–nilpotent completions are determined by their universal properties with respect to homomorphisms of Γ into finite or nilpotent groups. In fact, we may and do define the $\mathcal{P}$-completion of a group for *any* subcategory $\mathcal{P}$ of groups as the left adjoint of the inclusion functor of its projective closure in the category of all groups.

Here are some examples of $\mathcal{P}$-completions:

EXAMPLE A.5. It is customary to denote the *pro–finite completion* $\eta^{\text{finite}}\colon \Gamma \to \Gamma^{\text{finite}}$ by

$$\hat{\eta}\colon \Gamma \longrightarrow \hat{\Gamma}\ .$$

When Γ is the fundamental group of an algebraic variety, $\hat{\Gamma}$ is its *algebraic fundamental group.*

EXAMPLE A.6. The *pro-l-finite completion* is the completion with respect to the subcategory of pro–l–groups, i.e., projective limits of finite groups of order l^n, where l is any prime. It is denoted as

$$\eta_l^{\wedge}\colon \Gamma \longrightarrow \Gamma_l^{\wedge}$$

and may be of interest for a motivic study of fundamental groups.

EXAMPLE A.7. The *nilpotent completion*

$$\eta^{nilp}\colon \Gamma \longrightarrow \Gamma^{nilp}$$

corresponds to taking for $\mathcal{P}$ the category $Nilp$ of pro–nilpotent groups.

EXAMPLE A.8. The *torsion–free nilpotent completion* corresponds to the case when $\mathcal{P}$ is the projective completion $Nilp_0$ of the full subcategory of torsion–free nilpotent groups and will be denoted by[1]

$$\eta_0^{nilp}\colon \Gamma \longrightarrow \Gamma_0^{nilp}\ .$$

[1]This notation conflicts with [**23**], where our Γ_0^{nilp} is denoted Γ^{nilp}, which is more naturally reserved for the previous example.

EXAMPLE A.9. The *k–unipotent completion* is obtained by taking for $\mathcal{P}$ the category U_k of projective limits of k–unipotent groups, where k is any field of characteristic zero. This is denoted by

$$\eta \otimes k : \Gamma \longrightarrow \Gamma \otimes k \ .$$

When $k = \mathbb{R}$ and Γ is the fundamental group of a smooth manifold, $\Gamma \otimes \mathbb{R}$ is its *de Rham fundamental group.*

EXAMPLE A.10. The *algebraic hull*[2] of Γ

$$\eta^{alg} : \Gamma \longrightarrow \Gamma^{alg}$$

is obtained by taking for $\mathcal{P}$ the projective completion of the category of affine algebraic groups over $\mathbb{C}$.

A very desirable property of a completion functor $\cdot^{\mathcal{P}}$ is *idempotence*, i.e., $\cdot^{\mathcal{P}} \circ i \circ \cdot^{\mathcal{P}} = \cdot^{\mathcal{P}}$, or the slightly stronger property that $\cdot^{\mathcal{P}} \circ i = Id_{\mathcal{P}}$, that is, $\Gamma^{\mathcal{P}} \cong \Gamma$ naturally for all $\Gamma \in \mathcal{P}$. The latter admits a neat description in our case:

LEMMA A.11. *Let $\mathcal{P}$ be a group subcategory closed under projective limits. Then $\eta^{\mathcal{P}}$ is a natural isomorphism $\Gamma^{\mathcal{P}} \cong \Gamma$ for all $\Gamma \in \mathcal{P}$ if and only if $\mathcal{P} \subset \mathcal{G}r$ is a full subcategory, i.e., $Hom_{\mathcal{P}}(\Gamma, \Delta) = Hom_{\mathcal{G}r}(\Gamma, \Delta)$ for all $\Gamma, \Delta \in \mathcal{P}$.*

PROOF. The completion $\cdot^{\mathcal{P}}$ exists, and by its definition we have

$$Hom_{\mathcal{G}r}(\Gamma, i(\Delta)) \overset{\cong}{\longleftrightarrow} Hom_{\mathcal{P}}(\Gamma^{\mathcal{P}}, \Delta)$$

for every $\Gamma \in \mathcal{P}$.

If $\Gamma^{\mathcal{P}} \cong \Gamma$ for all $\Gamma \in \mathcal{P}$, this is the definition of fullness for $\mathcal{P}$. Conversely, if $\mathcal{P}$ is full, setting $\Gamma^{\mathcal{P}} \cong \Gamma$ for all $\Gamma \in \mathcal{P}$, the identity of $\Gamma \in \mathcal{P}$ satisfies the universal property defining $\eta^{\mathcal{P}}$. □

Looking at the above examples, it turns out that the completions A.5–A.8 are idempotent, but the k–unipotent completion and the algebraic hull are not, because of algebraicity conditions on the morphisms of k–unipotent and affine complex–algebraic groups.

When $\eta^{\mathcal{P}}$ is not an isomorphism, it may still be injective or surjective. More generally, it is interesting to study its kernel and cokernel. The case when the completion is injective is of particular importance.

DEFINITION A.12. A group Γ is *residually $\mathcal{P}$* if its completion $\eta^{\mathcal{P}} : \Gamma \to \Gamma^{\mathcal{P}}$ is injective.

Being residually $\mathcal{P}$ is equivalent to the following condition on Γ: for each $g \in \Gamma$ which is not the neutral element there exists a homomorphism $f : \Gamma \to \Delta$ with $\Delta \in \mathcal{P}$ such that $f(g) \neq e$.

EXAMPLE A.13. The group $\mathbb{Z}$, and in general torsion–free Abelian groups, are residually $\mathcal{P}$ in all of the previous examples. The same holds for finite rank free groups, although the proof is less straightforward.

In Chapter 8, we use the following criterion for residual finiteness:

---

[2]Some authors use the notation π_1^{alg} for the algebraic fundamental group of Example A.5. This should not be confused with the algebraic hull.

THEOREM A.14 (Miller [**92**]). *Consider an extension of groups*

$$1 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 1$$

such that

1. *the groups A and C are residually finite, and*
2. *A is finitely generated and has trivial center.*

Then B is residually finite.

We conclude this section with some easy examples of $\mathcal{P}$-completions:

EXAMPLE A.15. 1. $\Gamma = \{1\} \Longrightarrow \hat{\Gamma} = \Gamma_0^{\mathrm{nilp}} = \Gamma \otimes k = \{1\}$,
2. $|\Gamma| < \infty \Longrightarrow \hat{\Gamma} = \Gamma$ and $\Gamma_0^{\mathrm{nilp}} = \Gamma \otimes k = \{1\}$,
3. $\Gamma = \mathbb{Z} \Longrightarrow \hat{\Gamma} = \prod_{p \text{ prime}} \mathbb{Z}_p$, $\Gamma_0^{\mathrm{nilp}} = \Gamma = \mathbb{Z}$ and $\Gamma \otimes k = k$,
4. Γ finitely generated Abelian $\Longrightarrow \hat{\Gamma} = (\hat{\mathbb{Z}})^{\mathrm{rank}(\Gamma)} \times \mathrm{Tor}(\Gamma)$, $\Gamma_0^{\mathrm{nilp}} = \Gamma/_{\mathrm{Tor}}$ and $\Gamma \otimes k = \Gamma \otimes_{\mathbb{Z}} k$ (usual tensor product, see Example A.17).

2. Nilpotent completions

The nilpotent, torsion–free nilpotent and k–unipotent completions described in the above examples are in fact strongly related to each other. They can all be obtained by taking limits of simple projective systems associated with the lower central series. In many cases they can even be computed explicitly.

For any two elements $a, b \in \Gamma$, their commutator is denoted by $[a, b]$. The commutator of two subgroups $G, H \subset \Gamma$ is defined to be the subgroup of Γ generated by the commutators $[a, b]$ with $a \in G, b \in H$. The *lower central series* of a group Γ is defined recursively by

$$\Gamma_1 = \Gamma\,, \qquad \Gamma_{n+1} = [\Gamma_n, \Gamma]\,.$$

We can naturally assign to Γ a tower of nilpotent quotients

$$\ldots \longrightarrow \Gamma/\Gamma_3 \longrightarrow \Gamma/\Gamma_2\ . \tag{13}$$

It is immediate to check that homomorphisms from Γ to nilpotent groups of nilpotency class n factor through Γ/Γ_{n+1}. Thus, the tower (13) is cofinal for homomorphisms from Γ to all nilpotent groups. Therefore, its projective limit is the *nilpotent completion*:

$$\Gamma^{nilp} = \varprojlim \Gamma/\Gamma_n$$

and the homomorphism η^{nilp} is induced by the projections $\Gamma \to \Gamma/\Gamma_n$.

The torsion–free nilpotent completion can be computed in a similar manner. For a nilpotent group H, the torsion elements form a normal subgroup, which is finite if H is finitely generated. Thus, by dividing by this subgroup, we can naturally associate to H a torsion–free nilpotent group $H/_{Tor}$ and to every group Γ a tower of torsion–free nilpotent groups

$$\ldots \longrightarrow (\Gamma/\Gamma_3)/_{Tor} \longrightarrow (\Gamma/\Gamma_2)/_{Tor}\,. \tag{14}$$

As in the case of the nilpotent completion, the torsion–free nilpotent completion of Γ is the projective limit of this tower, together with the homomorphism induced by the projections of Γ onto the finite stages of the tower.

The k–unipotent completion also admits a characterisation in terms of the lower central series, but the details are more involved. In the case of *finitely presentable* groups, there is an alternative description in terms of the group algebra.

Let Γ be a finitely presentable group, and k a field of characteristic zero. The *group algebra* $k\Gamma$ has an augmentation k–algebra homomorphism

$$\begin{aligned}\varepsilon\colon k\Gamma &\longrightarrow k\\ g\in\Gamma &\longmapsto 1\ .\end{aligned}$$

Denote by $J = \ker\varepsilon$ the *augmentation ideal*, and by $\widehat{k\Gamma} = \varprojlim k\Gamma/J^n$ the J–adic completion of the group algebra[3]. The *completed group algebra* $\widehat{k\Gamma}$ has a coproduct defined by

$$\Delta(g) = g\otimes g$$

for all $g\in\Gamma$, which makes it into a complete Hopf algebra. As such, it has two subsets of distinguished elements:

1. The *group-like elements*

$$\mathcal{G}(\widehat{k\Gamma}) = \{x\in 1+\hat{J}\mid \Delta(x) = x\otimes x\}\,,$$

which form a group with respect to the multiplication in the algebra.

2. The *primitive elements*

$$\mathcal{P}(\widehat{k\Gamma}) = \{x\in\hat{J}\mid \Delta x = x\otimes 1 + 1\otimes x\}\,,$$

which form a Lie algebra over k with the bracket defined by commutation in the algebra.

For the proof of the following Proposition, we refer to [**103**]:

PROPOSITION A.16 (Quillen). *For a finitely presentable group Γ, its k–unipotent completion is*

$$\Gamma\otimes k\cong\mathcal{G}(\widehat{k\Gamma})\ ,$$

with the morphism $\eta\otimes k$ induced by the natural inclusion $\Gamma\hookrightarrow k\Gamma$.

EXAMPLE A.17. If Γ is an Abelian group, the above construction yields the ordinary tensor product, i.e., $\Gamma\otimes k\cong\Gamma\underset{\mathbb{Z}}{\otimes}k$, which is easily seen to have the universal property for homomorphisms to unipotent groups. This justifies the notation for the k–unipotent completions.

For any field k of characteristic zero, there is a well–known equivalence between unipotent groups and nilpotent Lie algebras over k. This equivalence is given, for instance, by the Baker–Campbell–Hausdorff formula, and it is compatible with projective limits. This induces a categorical equivalence between pro–k–unipotent groups and pro–k–nilpotent Lie algebras. It is often convenient to work with the Lie algebra, rather than with the group.

DEFINITION A.18. Let Γ be a group. The *k-Malcev algebra* of Γ, denoted by $\mathcal{L}(\Gamma,k)$, is the pro–k–nilpotent Lie algebra of the pro–k–unipotent group $\Gamma\otimes k$.

When the group Γ is finitely presentable, its Malcev algebra can also be characterised in terms of the completed group algebra as the Lie algebra of primitive elements:

$$\mathcal{L}(\Gamma,k)\cong\mathcal{P}(\widehat{k\Gamma})\ .$$

[3]This notation has nothing to do with the pro–finite completion. In particular, $\widehat{k\Gamma}$ and $k\hat{\Gamma}$ are entirely different algebras.

The k–Malcev algebra of Γ may be presented as a projective limit, analogous to that of the nilpotent completion Γ^{nilp},

$$\cdots \to \mathcal{L}_3(\Gamma, k) \to \mathcal{L}_2(\Gamma, k) \to \mathcal{L}_1(\Gamma, k),$$

where $\mathcal{L}_n(\Gamma, k)$ is the *n–step nilpotent k-Malcev algebra* of Γ: it may be defined as the k–Malcev algebra of the group Γ/Γ_{n+1}, or alternatively when Γ is finitely presentable as the quotient $\mathcal{P}(\widehat{k\Gamma})/\left(\mathcal{P}(\widehat{k\Gamma}) \cap \hat{J}^{n+1}\right)$. Also when Γ is finitely presentable, the n–step Malcev algebras are finite-dimensional, and $\mathcal{L}_n(\Gamma, k)$ determines the limit algebra $\mathcal{L}(\Gamma, k)$ when n is sufficiently large.

To end this Section, here is a convenient homological characterisation of homomorphisms inducing isomorphisms of $\mathbb{Q}$–unipotent completions:

THEOREM A.19 (Stallings [**125**]). *Let $f\colon \Gamma \longrightarrow \Delta$ be a homomorphism between finitely presentable groups with $f_i\colon H^i(\Gamma, \mathbb{Q}) \longrightarrow H^i(\Delta, \mathbb{Q})$ bijective for $i \leq 1$ and injective for $i = 2$. Then $f \otimes \mathbb{Q}\colon \Gamma \otimes \mathbb{Q} \longrightarrow \Delta \otimes \mathbb{Q}$ is an isomorphism.*

As we will see in the following section, this theorem is actually valid with coefficients in any field k of characteristic zero.

3. Comparison of nilpotent completions

In this Section we compare the different nilpotent completions of a finitely presentable group Γ. In general, one has the following result:

LEMMA A.20. *If $\mathcal{P}' \subset \mathcal{P} \subset \mathcal{G}r$ are nested subcategories, then there are functorial homomorphisms*

$$\Gamma^{\mathcal{P}} \longrightarrow \Gamma^{\mathcal{P}'}$$

which, when factorised through $\eta^{\mathcal{P}'}$, yield canonical isomorphisms

$$(\Gamma^{\mathcal{P}})^{\mathcal{P}'} \xrightarrow{\cong} \Gamma^{\mathcal{P}'} .$$

This is clear, because the homomorphisms of Γ to groups in $\mathcal{P}$ contain the homomorphisms to groups in $\mathcal{P}'$. Formally, one can check the required universal property by a diagram chase.

REMARK A.21. It is a consequence of the lemma that, if $\Gamma^{\mathcal{P}} \cong \Delta^{\mathcal{P}}$, then a fortiori $\Gamma^{\mathcal{P}'} \cong \Delta^{\mathcal{P}'}$ for all subcategories $\mathcal{P}' \subset \mathcal{P}$.

This Lemma and Remark apply to the nilpotent completions because of the following inclusions:

$$U_k \subset Nilp_0 \subset Nilp .$$

Some properties of the functorial homomorphisms associated with these inclusions are given by the following Lemmata.

LEMMA A.22. *For every finitely presentable group Γ, the natural homomorphism $\Gamma^{nilp} \longrightarrow \Gamma_0^{nilp}$ is surjective and has a pro–finite kernel.*

PROOF. The natural homomorphisms between the finite stages of the towers (13) and (14) are surjective and have finite kernels. Passing to the projective limits proves the statement. $\square$

LEMMA A.23. *The natural homomorphism $\Gamma_0^{nilp} \to \Gamma \otimes k$ is injective.*

PROOF. As every group is a projective limit of finitely generated groups, it is enough to prove the claim for finitely generated Γ. By the same reduction, one only needs to prove that every finitely generated torsion–free nilpotent group embeds in a k–unipotent group. This is done for example in an Appendix to [**103**]. □

The image of the homomorphism $\Gamma_0^{nilp} \to \Gamma \otimes k$ is, morally speaking, an integral form of $\Gamma \otimes k$, as the following lemma on the nilpotent steps of the tower shows.

LEMMA A.24. *If Γ is a finitely presentable nilpotent group, then every element $g \in \Gamma \otimes \mathbb{Q}$ has the property that there exists an n such that*

$$g^n \in \Gamma_0^{nilp} \subset \Gamma \otimes \mathbb{Q} \ .$$

PROOF. See the Appendix to [**103**]. □

The categories of pro–k–unipotent groups are partially ordered by the field extensions $K|k$, with the inclusion $U_k \subset U_K$ given by extension of scalars. The relation between the k–unipotent completions for different k is similarly given:

LEMMA A.25. *If Γ is finitely presentable, given a field extension $K|k$, there are K-Lie algebra isomorphisms*

$$(\mathcal{L}(\Gamma, k)) \underset{k}{\otimes} K \cong \mathcal{L}(\Gamma, K) \ ,$$

which are natural in Γ.

This is *not* a direct consequence of the universal property and categorical diagram chasing; it follows from properties of unipotent groups. For its proof, we refer to [**71**] or to [**66**][4].

Lemma A.25 implies that there are natural isomorphisms

$$\mathcal{L}(\Gamma, k) \cong \mathcal{L}(\Gamma, \mathbb{Q}) \underset{\mathbb{Q}}{\otimes} k \ .$$

Thus all the k–unipotent completions are obtained from the $\mathbb{Q}$–unipotent completion by extension of scalars.

By Remark A.21, $\Gamma \otimes k$ is a coarser invariant of Γ than Γ_0^{nilp}. The comparison between the two completions is easier if one is given a homomorphism $f \colon \Gamma \to \Delta$ with good properties, such as those induced by proper holomorphic maps.

LEMMA A.26. *Let $f \colon \Gamma \to \Delta$ be a homomorphism between finitely presentable groups.*

(i) *If $Im(f) \subset \Delta$ has finite index, then $f \otimes k$ is an isomorphism if and only if f_0^{nilp} is injective and has finite index image.*

(ii) *If f is surjective, then $f \otimes k$ is an isomorphism if and only if f_0^{nilp} is.*

PROOF. By the extension of scalars property of Lemma A.25 it suffices to prove the case of $k = \mathbb{Q}$.

(i) Let $f \otimes \mathbb{Q}$ be an isomorphism. As $\Gamma_0^{nilp} \to \Gamma \otimes \mathbb{Q}$ is injective, the map f_0^{nilp} must also be injective. Moreover, the fact that $\mathrm{Im}(f) \subset \Delta$ has finite index implies that the induced homomorphisms

$$(\Gamma/\Gamma_n)/_{Tor} \longrightarrow (\Delta/\Delta_n)/_{Tor}$$

[4]We owe this reference to R. Hain.

have images with finite index. Consequently, there is a tower of surjective homomorphisms from the finite quotient $\Delta/\mathrm{Im}(f)$ to the projective system of quotients

$$\cdots \longrightarrow ((\Delta/\Delta_n)/_{Tor})\,/\mathrm{Im}(f) \longrightarrow \cdots$$

These homomorphisms extend to a surjection from $\Delta/\mathrm{Im}(f)$ to its projective limit $\Delta_0^{nilp}/\mathrm{Im}(f_0^{nilp})$, which must therefore be finite.

Conversely, assume that f_0^{nilp} is injective and almost surjective. This implies that all the finite steps of the projective system

$$(f_0^{nilp})_n : (\Gamma/\Gamma_n)/_{Tor} \longrightarrow (\Delta/\Delta_n)/_{Tor}$$

must also be injective and almost surjective homomorphisms. Consider now the projective system of maps

$$(f \otimes \mathbb{Q})_n : (\Gamma/\Gamma_n) \otimes \mathbb{Q} \longrightarrow (\Delta/\Delta_n) \otimes \mathbb{Q}\ .$$

By Lemma A.24 every element of $\ker(f \otimes \mathbb{Q})_n$ has a power in $\ker(f_0^{nilp})_n = \{1\}$. The groups $(\Gamma/\Gamma_n) \otimes \mathbb{Q}$ are torsion free, so the homomorphisms $(f \otimes \mathbb{Q})_n$ must be injective. Moreover, every element of $(\Delta/\Delta_n) \otimes \mathbb{Q}$ has a power in $(\Delta/\Delta_n)/_{Tor}$, thus also a possibly higher power in $\mathrm{Im}(f_0^{nilp})_n \subset \mathrm{Im}(f \otimes \mathbb{Q})_n$. As $(f \otimes \mathbb{Q})_n$ is a homomorphism of $\mathbb{Q}$-unipotent groups, this means that it is onto.

The completion $f \otimes \mathbb{Q}$ is the projective limit of the isomorphisms $(F \otimes \mathbb{Q})_n$, so it must also be an isomorphism.

(ii) can be proved analogously. □

Although the categorical inclusions might lead one to expect the opposite, the extension of scalars property of Lemma A.25 shows that field extensions $K|k$ coarsen the isomorphism type of k–unipotent completions. This is illustrated by the following example, which is Remark II.2.15 in [**104**]:

EXAMPLE A.27. For any field k, the *Heisenberg algebra* of dimension 3 with coefficients in k is the Lie algebra

$$\mathfrak{h}_3(k) = \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{gl}(3,k) \right\}.$$

The $\mathbb{Q}$–Lie algebras $\mathfrak{h}_1 = \mathfrak{h}_3(\mathbb{Q}[\sqrt{2}])$ and $\mathfrak{h}_2 = \mathfrak{h}_3(\mathbb{Q}) \times \mathfrak{h}_3(\mathbb{Q})$ are not isomorphic, but the $\mathbb{R}$ Lie algebras $\mathfrak{h}_1 \underset{\mathbb{Q}}{\otimes} \mathbb{R}$ and $\mathfrak{h}_2 \underset{\mathbb{Q}}{\otimes} \mathbb{R}$ are.

APPENDIX B

A glossary of Hodge theory

The most fundamental tool in the study of the topology of Kähler manifolds is Hodge theory. In this appendix we collect a few basic facts of this theory, referring to the standard textbooks, e.g. [**55**], for proofs and further discussion.

Let (X, ω) be a compact Kähler manifold. We use the following notation:

CONVENTION B.1. (i) J denotes the complex structure $TX \to TX$ and its transpose $T^*X \to T^*X$,
(ii) the d^c operator is defined as $d^c = J^{-1}dJ$,
(iii) the Laplacian of a map is denoted by $\Delta f = - * d_\nabla * df$, where d_∇ is the covariant derivative induced by the metrics on the domain and target manifolds.

The Kähler form ω is closed by definition, and thus defines a class in $H^2(X, \mathbb{C})$. As ω^n is a non–zero multiple of the volume form, we have

$$[\omega^k] \neq 0 \in H^{2k}(X, \mathbb{C}) ,$$

for all $k \leq n$, and thus:

FACT B.2. If X is a compact Kähler manifold, its Betti numbers of even degree are non–zero.

DEFINITION B.3. A (pure) *Hodge structure* of weight k consists of a free Abelian group $H_{\mathbb{Z}}$ of finite rank, together with a decomposition of its complexification:

$$H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} = \oplus_{p+q=k} H^{p,q} , \tag{15}$$

with the property that

$$H^{p,q} = \overline{H^{q,p}}$$

for all p, q.

Alternatively, a Hodge structure can be described as a decreasing filtration of $H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$

$$H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} = F^0 \supset F^1 \supset \ldots \supset F^k \tag{16}$$

such that

$$H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} = F^p \oplus \overline{F^{k-p+1}} .$$

The existence of the filtration is clearly equivalent to Definition B.3 because (15) leads to (16) by setting

$$F^p = H^{k,0} \oplus \ldots \oplus H^{p,k-p} ;$$

and conversely, (16) leads to (15) by setting

$$H^{p,q} = F^p \cap \overline{F^q} .$$

On a compact Kähler manifold, the equality

$$\Delta_d = 2\Delta_\partial = 2\Delta_{\bar\partial}$$

of Laplacians implies that the type decomposition of differential forms passes to cohomology and defines a Hodge structure of weight k on $H^k(X,\mathbb{Z})/_{\text{torsion}}$. In fact, the integral lattice does not play any role in this book, and we could just as well think of a Hodge structure as a type decomposition or filtration as above, defined on the complexification of a real vector space.

The Dolbeault isomorphism gives a natural interpretation of the Hodge decomposition of $H^k(X,\mathbb{C})$, by identifying the space of harmonic forms of type (p,q) with the sheaf cohomology group $H^q_{\bar\partial}(\Omega^p_X)$. In particular, the spaces of holomorphic forms inject into the cohomology.

The existence of a pure Hodge structure of odd weight on a lattice implies that the rank of the lattice is even. Thus:

FACT B.4. If X is a compact Kähler manifold, its Betti numbers of odd degree are even:

$$b_{2m+1}(X) \equiv 0 \pmod 2 .$$

On many occasions we use the following deeper result, which one can think of as a sharpening of Fact B.4:

THEOREM B.5 (Hard Lefschetz Theorem). *If (X,ω) is a compact Kähler manifold of complex dimension n, then multiplication by the cohomology class of ω defines an isomorphism*

$$\omega^k \colon H^{n-k}(X,\mathbb{C}) \longrightarrow H^{n+k}(X,\mathbb{C}) ,$$

for every $k \in \{1,\dots n\}$.

COROLLARY B.6. *For all $k \leq n$, the following bilinear pairing is non-degenerate:*

$$\begin{aligned} H^k(X,\mathbb{C}) \times H^k(X,\mathbb{C}) &\longrightarrow \mathbb{C} \\ (\alpha,\beta) &\longmapsto \int_X \alpha\wedge\beta\wedge\omega^{n-k} . \end{aligned}$$

PROOF. By Theorem B.5, multiplication by ω^{n-k} is an isomorphism if $k<n$. Then the claim follows from Poincaré duality. In the middle dimension $k=n$, the claim is just Poincaré duality. □

When $k=1$, the Corollary is very useful in the study of Kähler groups. As we have seen in Example 1.20, the above pairing passes to the group cohomology of $\pi_1(X)$, which immediately leads to strong restrictions.

The operator $d^c = J^{-1}dJ$ is the real restriction of the operator $d^c = i(\bar\partial - \partial)$ on the Dolbeault complex $\mathcal{E}^*_X \otimes \mathbb{C}$. It satisfies the following identities:

(i) nilpotence: $(d^c)^2 = 0$,
(ii) commutativity with d^c: $dd^c = -d^cd = 2i\partial\bar\partial$,
(iii) Laplacian: $\Delta_{d^c} = \Delta_d = 2\Delta_\partial = 2\Delta_{\bar\partial}$.

The Laplacian identity implies the isomorphism of cohomology algebras $H^*(X,\mathbb{R}) = H^*_d(X) \cong H^*_{d^c}(X)$.

A crucial property of d^c is:

LEMMA B.7 (dd^c Lemma). *Let X be a compact Kähler manifold, and let α be a differential form on X such that $\alpha = d\gamma$. Then there exists a form β such that $\alpha = dd^c\beta$.*

As J and dd^c commute, the operators d and d^c are actually interchangeable in the statement of the Lemma, this yields a "$d^c d$ Lemma".

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