Mathematical
Surveys
and
Monographs

Number 47

Rings, Modules, and Algebras in Stable Homotopy Theory

A. D. Elmendorf
I. Kriz
M. A. Mandell
J. P. May

with an Appendix by
M. Cole

American Mathematical Society

http://dx.doi.org/10.1090/surv/047

Selected Titles in This Series

47 **A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May (with an appendix by M. Cole),** Rings, modules, and algebras in stable homotopy theory, 1997

46 **Stephen Lipscomb,** Symmetric inverse semigroups, 1996

45 **George M. Bergman and Adam O. Hausknecht,** Cogroups and co-rings in categories of associative rings, 1996

44 **J. Amorós, M. Burger, K. Corlette, D. Kotschick, and D. Toledo,** Fundamental groups of compact Kähler manifolds, 1996

43 **James E. Humphreys,** Conjugacy classes in semisimple algebraic groups, 1995

42 **Ralph Freese, Jaroslav Ježek, and J. B. Nation,** Free lattices, 1995

41 **Hal L. Smith,** Monotone dynamical systems: an introduction to the theory of competitive and cooperative systems, 1995

40.2 **Daniel Gorenstein, Richard Lyons, and Ronald Solomon,** The classification of the finite simple groups, number 2, 1995

40.1 **Daniel Gorenstein, Richard Lyons, and Ronald Solomon,** The classification of the finite simple groups, number 1, 1994

39 **Sigurdur Helgason,** Geometric analysis on symmetric spaces, 1993

38 **Guy David and Stephen Semmes,** Analysis of and on uniformly rectifiable sets, 1993

37 **Leonard Lewin, Editor,** Structural properties of polylogarithms, 1991

36 **John B. Conway,** The theory of subnormal operators, 1991

35 **Shreeram S. Abhyankar,** Algebraic geometry for scientists and engineers, 1990

34 **Victor Isakov,** Inverse source problems, 1990

33 **Vladimir G. Berkovich,** Spectral theory and analytic geometry over non-Archimedean fields, 1990

32 **Howard Jacobowitz,** An introduction to CR structures, 1990

31 **Paul J. Sally, Jr. and David A. Vogan, Jr., Editors,** Representation theory and harmonic analysis on semisimple Lie groups, 1989

30 **Thomas W. Cusick and Mary E. Flahive,** The Markoff and Lagrange spectra, 1989

29 **Alan L. T. Paterson,** Amenability, 1988

28 **Richard Beals, Percy Deift, and Carlos Tomei,** Direct and inverse scattering on the line, 1988

27 **Nathan J. Fine,** Basic hypergeometric series and applications, 1988

26 **Hari Bercovici,** Operator theory and arithmetic in H^∞, 1988

25 **Jack K. Hale,** Asymptotic behavior of dissipative systems, 1988

24 **Lance W. Small, Editor,** Noetherian rings and their applications, 1987

23 **E. H. Rothe,** Introduction to various aspects of degree theory in Banach spaces, 1986

22 **Michael E. Taylor,** Noncommutative harmonic analysis, 1986

21 **Albert Baernstein, David Drasin, Peter Duren, and Albert Marden, Editors,** The Bieberbach conjecture: Proceedings of the symposium on the occasion of the proof, 1986

20 **Kenneth R. Goodearl,** Partially ordered abelian groups with interpolation, 1986

19 **Gregory V. Chudnovsky,** Contributions to the theory of transcendental numbers, 1984

18 **Frank B. Knight,** Essentials of Brownian motion and diffusion, 1981

17 **Le Baron O. Ferguson,** Approximation by polynomials with integral coefficients, 1980

16 **O. Timothy O'Meara,** Symplectic groups, 1978

15 **J. Diestel and J. J. Uhl, Jr.,** Vector measures, 1977

14 **V. Guillemin and S. Sternberg,** Geometric asymptotics, 1977

13 **C. Pearcy, Editor,** Topics in operator theory, 1974

12 **J. R. Isbell,** Uniform spaces, 1964

(*Continued in the back of this publication*)

Rings, Modules, and Algebras in Stable Homotopy Theory

Mathematical
Surveys
and
Monographs
Volume 47

Rings, Modules, and Algebras in Stable Homotopy Theory

A. D. Elmendorf
I. Kriz
M. A. Mandell
J. P. May

with an Appendix by
M. Cole

American Mathematical Society

Kriz and May were supported in part by NSF Grants, and Cole was supported by NSF and GAANN Fellowships.

2000 *Mathematics Subject Classification.* Primary 19D99, 55N20, 55P42;
Secondary 19L99, 55N22, 55T25.

Library of Congress Cataloging-in-Publication Data

Rings, modules, and algebras in stable homotopy theory / A. D. Elmendorf... [et al.] ; with an appendix by M. Cole.

p. cm. — (Mathematical surveys and monographs ; v. 47)

Includes bibliographical references and index.

ISBN 0-8218-0638-6 (hard : alk. paper); ISBN 978-0-8218-4303-1 (soft : alk. paper)

1. Homotopy theory. 2. Groups of points. I. Elmendorf, Anthony D., 1952–. II. Series: Mathematical surveys and monographs ; no. 47.

QA612.7.R56 1996　　96-35999
514′.24—dc20　　CIP

∞ The paper used in this book is acid-free and falls within the guidelines
established to ensure permanence and durability.
Visit the AMS home page at `http://www.ams.org/`

10 9 8 7 6 5 4 3 2 1　　12 11 10 09 08 07

Abstract

Let S be the sphere spectrum. We construct an associative, commutative, and unital smash product in a complete and cocomplete category $\mathscr{M}_S$ of "S-modules" whose derived category $\mathscr{D}_S$ is equivalent to the classical stable homotopy category. This allows a simple and algebraically manageable definition of "S-algebras" and "commutative S-algebras" in terms of associative, or associative and commutative, products $R \wedge_S R \longrightarrow R$. These notions are essentially equivalent to the earlier notions of A_∞ and E_∞ ring spectra, and the older notions feed naturally into the new framework to provide plentiful examples. There is an equally simple definition of R-modules in terms of maps $R \wedge_S M \longrightarrow M$. When R is commutative, the category $\mathscr{M}_R$ of R-modules also has an associative, commutative, and unital smash product, and its derived category $\mathscr{D}_R$ has properties just like the stable homotopy category.

Working in the derived category $\mathscr{D}_R$, we construct spectral sequences that specialize to give generalized universal coefficient and Künneth spectral sequences. Classical torsion products and Ext groups are obtained by specializing our constructions to Eilenberg-Mac Lane spectra and passing to homotopy groups, and the derived category of a discrete ring R is equivalent to the derived category of its associated Eilenberg-Mac Lane S-algebra.

We also develop a homotopical theory of R-ring spectra in $\mathscr{D}_R$, analogous to the classical theory of ring spectra in the stable homotopy category, and we use it to give new constructions as MU-ring spectra of a host of fundamentally important spectra whose earlier constructions were both more difficult and less precise.

Working in the module category $\mathscr{M}_R$, we show that the category of finite cell modules over an S-algebra R gives rise to an associated algebraic K-theory spectrum KR. Specialized to the Eilenberg-Mac Lane spectra of discrete rings, this recovers Quillen's algebraic K-theory of rings. Specialized to suspension spectra $\Sigma^\infty(\Omega X)_+$ of loop spaces, it recovers Waldhausen's algebraic K-theory of spaces.

Replacing our ground ring S by a commutative S-algebra R, we define R-algebras and commutative R-algebras in terms of maps $A \wedge_R A \longrightarrow A$, and we show that the categories of R-modules, R-algebras, and commutative R-algebras are all topological model categories. We use the model structures to study Bousfield localizations of R-modules and R-algebras. In particular, we prove that KO and KU are commutative ko and ku-algebras and therefore commutative S-algebras.

We define the topological Hochschild homology R-module $THH^R(A; M)$ of A with coefficients in an (A, A)-bimodule M and give spectral sequences for the calculation of its homotopy and homology groups. Again, classical Hochschild homology and cohomology groups are obtained by specializing the constructions to Eilenberg-Mac Lane spectra and passing to homotopy groups.

Contents

APPENDIX A

Twisted half-smash products and function spectra

by Michael Cole

1. Introduction

Let U, U' be universes, $\mathscr{I}(U, U')$ the space of linear isometries $U \longrightarrow U'$, A a space, $\alpha : A \longrightarrow \mathscr{I}(U, U')$ a map, and let $E \in \mathscr{S}U$, $E' \in \mathscr{S}U'$ be spectra. The twisted half-smash product $\alpha \ltimes E \in \mathscr{S}U'$ and the twisted function spectrum $F[\alpha, E') \in \mathscr{S}U$ are fundamental constructions which underly the foundations of stable homotopy theory that were introduced in [38] and developed further in this book. These functors specialize to the change of universe functors that are necessary to define internal smash products and function spectra, and they provide a simple way to prove that the different internal smash products so obtained are canonically and coherently equivalent upon passage to the stable category.

In connection with the research presented in this book, the theory of twisted half-smash products has undergone major clarification and sharpening of the results originally given in [38]. In particular, the improved "untwisting theorem", given as Theorem 5.5 below, has led to corresponding improvements of the theorems concerning the homotopy invariance properties of $\alpha \ltimes E$ and $F[\alpha, E')$ in the variable α — the crucial point being that for tame spectra E, the homotopical properties of $\alpha \ltimes E$ depend only on the homotopical properties of the space A rather than on the particular map α. In particular, it is vital to the theory that for tame spectra E the functor $\alpha \ltimes E$ is well-behaved homotopically before passage to the stable category.

We present new definitions of twisted half-smash products and function spectra which have considerable advantages, both conceptually and expositionally, compared to the definitions in [38]. In particular, our definitions do not require choosing arbitrary cofinal sequences of representations and therefore our treatment avoids the technical complications concerning the behavior of colimits found in the approach of [38]. We shall recast much of the material of [38, VI§§1-3] in a simpler, more user-friendly, form.

The treatment of twisted half-smash products and function spectra in [38] is written in the equivariant context, with a compact Lie group G acting on all spaces and spectra in sight. We present our definitions and theorems nonequivariantly here and note that virtually everything that we say carries over mutatis mutandis to the equivariant case. We will point out the small exceptions to this at the end.

We give preliminary definitions and constructions in Sections 2-4, reaching the definition of twisted half-smash products and function spectra in Section 5. That section also gives a simple proof of the untwisting theorem, which describes an isomorphism $\alpha \ltimes E \cong E \wedge A_+$ when the spectrum E is a finite desuspension of a space. Our proof relies on an idea of Neil Strickland. We give some formal properties of our functors in Section 6 and consider their homotopical properties in Section 7. In Section 8 we prove a cofibration theorem that plays an important technical role in the theory of S-modules. We shall derive it from the untwisting theorem. We conclude in Section 9 with a brief discussion of the equivariant versions of our constructions; more details will appear in [16].

2. The category $\mathscr{S}(U';U)$

To define our constructions we introduce a category $\mathscr{S}(U';U)$. An object $\mathscr{E} \in \mathscr{S}(U';U)$ is a family of spectra $\mathscr{E}_V \in \mathscr{S}U'$, one for each indexing space $V \subset U$. We require isomorphisms $\rho_{V,W} : \Sigma^{W-V}\mathscr{E}_W \longrightarrow \mathscr{E}_V$, $V \subset W$, that satisfy the evident transitivity relation. Here we are mixing universes since we are suspending the spectrum $\mathscr{E}_W$ which is indexed on U' by the indexing space $W - V$ which is in U, but in any case $\Sigma^{W-V}\mathscr{E}_W$ just means the usual smash product of the spectrum $\mathscr{E}_W$ with the space S^{W-V}. We think of $\mathscr{E}$ as the spectrum $\mathscr{E}_0 \in \mathscr{S}U'$, an object stable with respect to suspension by indexing spaces in U', but also equipped with a choice of compatible desuspensions by indexing spaces in U.

EXAMPLE 2.1. If $U' = U$ and X is a space, we get an associated object $\mathscr{E}(X) \in \mathscr{S}(U;U)$ by setting $\mathscr{E}(X)_V = \Sigma^\infty_V X$ and considering the canonical natural isomorphisms $\Sigma^{W-V}\Sigma^\infty_W X \cong \Sigma^\infty_V X$ as structure maps.

EXAMPLE 2.2. Generalizing Example 2.1, let $f : U \longrightarrow U'$ be a linear isometry and X a space. The specification $\mathscr{E}_f(X)_V = \Sigma^\infty_{fV} X$ defines an object $\mathscr{E}_f(X) \in \mathscr{S}(U';U)$ with structure maps given by $\Sigma^{W-V}\Sigma^\infty_{fW}X \cong \Sigma^{fW-fV}\Sigma^\infty_{fW}X \cong \Sigma^\infty_{fV}X$.

We will prove shortly that a map $\alpha : A \longrightarrow \mathscr{I}(U,U')$ gives rise to an object $\mathscr{M}\alpha \in \mathscr{S}(U';U)$ for which $\mathscr{M}\alpha_0 \cong \Sigma^\infty A_+$. The construction is natural in spaces over $\mathscr{I}(U,U')$. If α is the constant map at $f \in \mathscr{I}(U,U')$ then $\mathscr{M}\alpha$ is the object discussed in Example 2.2 with $X = A_+$.

We will define a smash product

$$\wedge : \mathscr{S}(U';U) \times \mathscr{S}U \longrightarrow \mathscr{S}U'$$

and a function spectrum

$$F(-,-) : \mathscr{S}(U';U)^{\mathrm{op}} \times \mathscr{S}U' \longrightarrow \mathscr{S}U$$

satisfying an adjunction isomorphism

$$\mathscr{S}U'(\mathscr{E} \wedge E, E') \cong \mathscr{S}U(E, F(\mathscr{E}, E')), \tag{2.3}$$

where $E \in \mathscr{S}U$, $E' \in \mathscr{S}U'$, $\mathscr{E} \in \mathscr{S}(U';U)$. For a map $\alpha : A \longrightarrow \mathscr{I}(U, U')$ and spectra $E \in \mathscr{S}U$, $E' \in \mathscr{S}U'$ we will define

$$\alpha \ltimes E = \mathscr{M}\alpha \wedge E$$

and

$$F[\alpha, E') = F(\mathscr{M}\alpha, E')$$

and show that this agrees with the definitions of [38].

3. Smash products and function spectra

We first record the following obvious fact.

PROPOSITION 3.1. *For an object $\mathscr{E} \in \mathscr{S}(U';U)$ and a space $X \in \mathscr{T}$ there is a smash product $\mathscr{E} \wedge X \in \mathscr{S}(U';U)$ defined by $(\mathscr{E} \wedge X)_V = \mathscr{E}_V \wedge X$ and the evident structure maps. There is an adjunction isomorphism*

$$\mathscr{S}(U';U)(\mathscr{E} \wedge X, \mathscr{D}) \cong \mathscr{T}(X, \mathscr{S}(U';U)(\mathscr{E}, \mathscr{D}))$$

where the morphism set $\mathscr{S}(U';U)(\mathscr{E}, \mathscr{D})$ is given the evident topology and basepoint.

Although not obvious from the definitions, it can be shown that there is a "function object" $F(X, \mathscr{D}) \in \mathscr{S}(U';U)$ such that the adjunction extends in the expected way.

We now define the object $F(\mathscr{E}, E')$.

DEFINITION 3.2. For objects $\mathscr{E} \in \mathscr{S}(U';U)$ and $E' \in \mathscr{S}U'$ the spectrum $F(\mathscr{E}, E') \in \mathscr{S}U$ is given by $F(\mathscr{E}, E')(V) = \mathscr{S}U'(\mathscr{E}_V, E')$. Abbreviating $\mathscr{S}' = \mathscr{S}U'$, the structure maps are given by the sequences of isomorphisms

$$\mathscr{S}'(\mathscr{E}_V, E') \cong \mathscr{S}'(\Sigma^{W-V}\mathscr{E}_W, E') \cong \mathscr{S}'(\mathscr{E}_W, \Omega^{W-V}E') \cong \Omega^{W-V}\mathscr{S}'(\mathscr{E}_W, E').$$

For $\mathscr{E} \in \mathscr{S}(U';U)$ and $E \in \mathscr{S}U$, our definition of $\mathscr{E} \wedge E \in \mathscr{S}U'$ is dictated by the desired adjunction (2.3). Recall that for a spectrum $E \in \mathscr{S}U$ and prespectrum $D \in \mathscr{P}U$, the morphism set $\mathscr{P}U(D, E)$ may be described by

$$\mathscr{P}U(D, E) = \lim_{V \subset U} \mathscr{T}(DV, EV), \tag{3.3}$$

where the limit is taken over the maps

$$\mathscr{T}(DW, EW) \longrightarrow \mathscr{T}(\Sigma^{W-V}DV, EW) \cong \mathscr{T}(DV, \Omega^{W-V}EW) \cong \mathscr{T}(DV, EV).$$

It follows that

$$\begin{aligned} \mathscr{S}U(E, F(\mathscr{E}, E')) &= \lim \mathscr{T}(EV, F(\mathscr{E}, E')V) \\ &= \lim \mathscr{T}(EV, \mathscr{S}U'(\mathscr{E}_V, E')) \\ &= \lim \mathscr{S}U'(\mathscr{E}_V \wedge EV, E') \\ &= \mathscr{S}U'(\operatorname{colim} \mathscr{E}_V \wedge EV, E'), \end{aligned}$$

where the colimit is taken over the maps

$$\mathscr{E}_V \wedge EV \cong \Sigma^{W-V}\mathscr{E}_W \wedge EV \cong \mathscr{E}_W \wedge \Sigma^{W-V}EV \longrightarrow \mathscr{E}_W \wedge EW. \tag{3.4}$$

Hence our definition of $\mathscr{E} \wedge E$:

DEFINITION 3.5. If $\mathscr{E} \in \mathscr{S}(U';U)$ and $E \in \mathscr{S}U$, then $\mathscr{E} \wedge E \in \mathscr{S}U'$ is the spectrum $\operatorname{colim}_{V\subset U} \mathscr{E}_V \wedge EV$ where the colimit is taken over the maps (3.4).

We have contrived our definitions to make the following true.

PROPOSITION 3.6. *There is an adjunction isomorphism*

$$\mathscr{S}U'(\mathscr{E} \wedge E, E') \cong \mathscr{S}U(E, F(\mathscr{E}, E')).$$

The following result is easy.

PROPOSITION 3.7. *If $\mathscr{E} \in \mathscr{S}(U';U)$, $E \in \mathscr{S}U$, and $X \in \mathscr{T}$ there are natural isomorphisms*

$$(\mathscr{E} \wedge E) \wedge X \cong \mathscr{E} \wedge (E \wedge X) \cong (\mathscr{E} \wedge X) \wedge E.$$

In practice if a spectrum E is the spectrification LD of a prespectrum D, it is often useful to describe $\mathscr{E} \wedge E$ as a colimit involving the spaces DV rather than the spaces EV. A simple adjunction argument together with (3.3) proves the following result.

PROPOSITION 3.8. *If $D \in \mathscr{P}U$ is a prespectrum and $\mathscr{E} \in \mathscr{S}(U';U)$ then*

$$\mathscr{E} \wedge LD \cong \operatorname{colim}_{V\subset U} \mathscr{E}_V \wedge DV$$

where the colimit is taken over the maps (3.4) (replacing E by D).

In particular, if E is the desuspension spectrum $\Sigma^\infty_V X$ of a space X, this has the following consequence.

PROPOSITION 3.9. *For a fixed indexing space $V \subset U$ there is an isomorphism*

$$\mathscr{E} \wedge \Sigma^\infty_V X \cong \mathscr{E}_V \wedge X$$

that is natural in $\mathscr{E}$ and X.

PROOF. By Proposition 3.8 we see that

$$\mathscr{E} \wedge \Sigma^\infty_V X \cong \operatorname{colim}_{W\subset U} \mathscr{E}_W \wedge \Sigma^{W-V} X.$$

But clearly, for $V \subset W$ the structure map

$$\mathscr{E}_V \wedge X \cong \Sigma^{W-V}\mathscr{E}_W \wedge X \cong \mathscr{E}_W \wedge \Sigma^{W-V} X$$

is an isomorphism. Hence the colimit stabilizes at V and the claim follows. □

4. The object $\mathscr{M}\alpha \in \mathscr{S}(U';U)$

Let $\alpha : A \longrightarrow \mathscr{I}(U, U')$ be a map. For any $V \subset U$ and $V' \subset U'$, let $A_{V,V'}$ denote the pullback

$$\begin{tikzcd} A_{V,V'} \arrow[rr] \arrow[d] & & \mathscr{I}(V,V') \arrow[d] \\ A \arrow[r, "\alpha"'] & \mathscr{I}(U,U') \arrow[r] & \mathscr{I}(V,U') \end{tikzcd}$$

Thus $A_{V,V'}$ is the set of $a \in A$ such that $\alpha(a)V \subset V'$. Note that $A_{V,V'}$ will often be empty (for example, if V' is too small). Note that $A_{0,V'} = A$ for any V'. If $V \subset W$ then $A_{W,V'} \subset A_{V,V'}$. If $V' \subset W'$ then $A_{V,V'} \subset A_{V,W'}$ and, for fixed V, $A = \bigcup_{V' \subset U'} A_{V,V'}$. In particular, if A is compact, or more generally has compact image, then, for fixed V, we see that $A = A_{V,V'}$ for large V'.

Now for any V, V' let $\eta(\alpha)_{V,V'}$ denote the vector bundle over $A_{V,V'}$ with total space

$$E(\eta(\alpha)_{V,V'}) = \{(a, v) \in A_{V,V'} \times V' \mid v' \perp \alpha(a)V\}.$$

Let $T\alpha_{V,V'}$ be the Thom space of $\eta(\alpha)_{V,V'}$. In the case that $A_{V,V'} = \emptyset$ then $\eta(\alpha)_{V,V'}$ is the empty bundle $\emptyset \longrightarrow \emptyset$ and, by convention, the Thom space is a single point.

OBSERVATION 4.1. For fixed V, $\{T\alpha_{V,V'}\}$ is a prespectrum indexed over U', which we will denote by $\mathscr{T}\alpha_V$. The structure maps $\Sigma^{W'-V'}T\alpha_{V,V'} \longrightarrow T\alpha_{V,W'}$ are induced by the evident vector bundle morphisms

$$\eta(\alpha)_{V,V'} \oplus (W' - V') \cong \eta(\alpha)_{V,W'}|_{A_{V,V'}} \longrightarrow \eta(\alpha)_{V,W'}.$$

OBSERVATION 4.2. For fixed V' we have maps $\Sigma^{W-V}T\alpha_{W,V'} \longrightarrow T\alpha_{V,V'}$ that are induced by the evident vector bundle morphisms

$$\eta(\alpha)_{W,V'} \oplus (W - V) \cong \eta(\alpha)_{V,V'}|_{A_{W,V'}} \longrightarrow \eta(\alpha)_{V,V'}.$$

Thus for each $V \subset U$ we have a prespectrum $\mathscr{T}\alpha_V \in \mathscr{P}U'$ and we have maps of prespectra $\Sigma^{W-V}\mathscr{T}\alpha_W \longrightarrow \mathscr{T}\alpha_V$ that satisfy the evident transitivity condition. Let $\mathscr{M}\alpha_V \in \mathscr{S}U'$ be the spectrification of $\mathscr{T}\alpha_V$. Then we have maps of spectra $\Sigma^{W-V}\mathscr{M}\alpha_{W'} \longrightarrow \mathscr{M}\alpha_V$ that satisfy the transitivity condition.

PROPOSITION 4.3. *The maps $\Sigma^{W-V}\mathscr{M}\alpha_W \longrightarrow \mathscr{M}\alpha_V$ are isomorphisms of spectra. Hence $\mathscr{M}\alpha$ is an object in the category $\mathscr{S}(U';U)$.*

PROOF. For a compact $K \subset A$ let $\alpha|_K$ be the composite $K \longrightarrow A \xrightarrow{\alpha} \mathscr{I}(U, U')$. Since, as always, we are working in the category of compactly generated spaces, A is topologized as the union of its compact subspaces and it follows that in the category of spaces over $\mathscr{I}(U, U')$, $\alpha = \operatorname{colim}_K \alpha|_K$ where the colimit is taken over all compact $K \subset A$. All constructions we have made are natural in spaces over $\mathscr{I}(U, U')$, and they commute with filtered colimits. In particular, $\mathscr{M}\alpha_V = \operatorname{colim}_K \mathscr{M}(\alpha|_K)_V$ and, of course, Σ^{W-V} commutes with colimits. Hence it suffices to show that the map $\Sigma^{W-V}\mathscr{M}\alpha_W \longrightarrow \mathscr{M}\alpha_V$ is an isomorphism when A is compact.

Assume then that A is compact. Fix $V \subset W \subset U$ and choose V' large enough that $A_{V,V'} = A_{W,V'} = A$. Then for $V' \subset W' \subset U'$ the structure maps $\Sigma^{W'-V'} T\alpha_{V,V'} \longrightarrow T\alpha_{V,W'}$ are isomorphisms. Hence $\mathscr{T}\alpha_V$ agrees cofinally with the desuspension prespectrum $\{\Sigma^{W'-V'} T\alpha_{V,V'}\}$ and thus $\mathscr{M}\alpha_V \cong \Sigma^\infty_{V'} T\alpha_{V,V'}$. Similarly $\mathscr{M}\alpha_W \cong \Sigma^\infty_{V'} T\alpha_{W,V'}$ and we have the sequence of isomorphisms

$$\Sigma^{W-V}\mathscr{M}\alpha_W \cong \Sigma^{W-V}\Sigma^\infty_{V'} T\alpha_{W,V'} \cong \Sigma^\infty_{V'}\Sigma^{W-V} T\alpha_{W,V'} \cong \Sigma^\infty_{V'} T\alpha_{V,V'} \cong \mathscr{M}\alpha_V.$$

□

LEMMA 4.4. *There is an isomorphism $\mathscr{M}\alpha_0 \cong \Sigma^\infty A_+$ that is natural in α.*

PROOF. It is immediate from the definitions that $\eta(\alpha)_{0,V'}$ is the trivial bundle $A \times V'$ over A and hence $T\alpha_{0,V'} \cong \Sigma^{V'} A_+$. □

Together with the fact that $\Sigma^V \mathscr{M}\alpha_V \cong \mathscr{M}\alpha_0$, this leads one to suspect that $\mathscr{M}\alpha_V$ is isomorphic to a shift desuspension of the space A_+. That turns out to be the case and, in our version of twisted half-smash products, this is the explanation for the untwisting theorem which we will discuss in the next section.

LEMMA 4.5 (UNTWISTING). *Let $V \subset U$ and $V' \subset U'$ be indexing spaces such that $V \cong V'$ and let $\alpha : A \longrightarrow \mathscr{I}(U, U')$ be a map. There is an isomorphism $\mathscr{M}\alpha_V \cong \Sigma^\infty_{V'} A_+$ that is natural in α.*

PROOF. Let $\mathscr{O}(U')$ be the orthogonal group of linear isometric isomorphisms $U' \longrightarrow U'$ that are the identity off a finite subspace. Since the restriction map $r' : \mathscr{O}(U') \longrightarrow \mathscr{I}(V', U')$ is a bundle over the contractible paracompact space $\mathscr{I}(V', U')$, it admits a section $s' : \mathscr{I}(V', U') \longrightarrow \mathscr{O}(U')$. Fixing an isomorphism $f : V \longrightarrow V'$, we obtain a homeomorphism $f^* : \mathscr{I}(V', U') \longrightarrow \mathscr{I}(V, U')$. Hence we have a composite

$$s = s' \circ (f^*)^{-1} : \mathscr{I}(V, U') \longrightarrow \mathscr{O}(U')$$

with the property that $s(g)(fv) = gv$ for any $g \in \mathscr{I}(V, U')$ and $v \in V$.

Now $\alpha : A \longrightarrow \mathscr{I}(U, U')$ passes to a map $\alpha : A \longrightarrow \mathscr{I}(V, U')$ which we also write as α. For $W' \subset U'$, let $A_{[V,W']}$ denote $\emptyset$ unless $V' \subset W'$, in which case $A_{[V,W']}$ denotes the pullback

$$\begin{tikzcd} A_{[V,W']} \arrow[rr] \arrow[d] & & \mathscr{O}(W') \arrow[d] \\ A \arrow[r, "\alpha"'] & \mathscr{I}(V, U') \arrow[r, "s"'] & \mathscr{O}(U'). \end{tikzcd}$$

Thus when $V' \subset W'$, $A_{[V,W']}$ is the set of all $a \in A$ such that $s(a) \in \mathscr{O}(W')$. Note that $A_{[V,W']} \subset A_{V,W'}$ and $A = \bigcup_{W' \subset U'} A_{[V,W']}$. Let $\zeta(\alpha)_{[V,W']}$ denote the trivial bundle $A_{[V,W']} \times (W' - V')$. Taking Thom spaces, we obtain a prespectrum $\mathscr{T}\alpha_{[V]} \in \mathscr{S}U'$ with

$$\mathscr{T}\alpha_{[V]}V' = T(\zeta(\alpha)_{[V,W']}) \cong \Sigma^{W'-V'}(A_{[V,W']})_+$$

and the obvious structure maps. Let $\mathscr{M}\alpha_{[V]}$ be the spectrification of $\mathscr{T}\alpha_{[V]}$. Note that $\mathscr{M}\alpha_{[V]}$ is functorial in α and commutes with filtered colimits of spaces over $\mathscr{I}(U, U')$. We will show that 1) $\mathscr{M}\alpha_{[V]} \cong \Sigma^{\infty}_{V'} A_+$ and 2) $\mathscr{M}\alpha_{[V]} \cong \mathscr{M}\alpha_V$, naturally in α.

For 1) we observe that $\mathscr{M}\alpha_{[V]} = \operatorname{colim}_K \mathscr{M}(\alpha|_K)_{[V]}$ where the colimit is taken over all compact $K \subset A$. Hence it suffices to establish 1) when A is compact. But for compact A, $A_{[V,W']} = A$ for W' large enough, so we see that $\mathscr{T}\alpha_{[V]}$ agrees cofinally with the desuspension prespectrum $\{\Sigma^{W'-V'} A_+\}$. The conclusion follows.

For 2) we construct bundle morphisms $\zeta(\alpha)_{[V,W']} \longrightarrow \eta(\alpha)_{V,W'}$. Recall that

$$E(\eta(\alpha)_{V,W'}) = \{(a, w') \in A_{V,W'} \times W' \mid w' \perp \alpha(a)V\}.$$

We define a map

$$\theta_{W'} : A_{[V,W']} \times (W' - V') \longrightarrow E(\eta(\alpha)_{V,W'})$$

by $\theta_{W'}(a, w') = (a, s(\alpha(a))(w'))$. The properties of s ensure that these maps are well-defined and induce a map of prespectra $\theta : \mathscr{T}\alpha_{[V]} \longrightarrow \mathscr{T}\alpha_V$ on passage to Thom spaces. To show that the resulting map of spectra $\theta : \mathscr{M}\alpha_{[V]} \longrightarrow \mathscr{M}\alpha_V$ is an isomorphism, the colimit argument again lets us reduce the problem to the case when A is compact. But for compact A, with W' large enough so that $A_{[V,W']} = A$, the bundle map $\theta_{W'}$ is an isomorphism. Therefore the map of prespectra $\theta : \mathscr{T}\alpha_{[V]} \longrightarrow \mathscr{T}\alpha_V$ is cofinally a spacewise isomorphism, and the conclusion follows. $\square$

Observe that the isomorphism $\mathscr{M}\alpha_V \cong \Sigma^{\infty}_{V'} A_+$ depends on the choice of f and of the section s'. The idea of exploiting such a section, although proposed in the context of a very different proof of the untwisting theorem, is due to Neil Strickland.

5. Twisted half-smash products and function spectra

DEFINITION 5.1. For $\alpha : A \longrightarrow \mathscr{I}(U, U')$, $E \in \mathscr{S}U$, and $E' \in \mathscr{S}U'$, the twisted half-smash product $\alpha \ltimes E \in \mathscr{S}U'$ is the spectrum $\mathscr{M}\alpha \wedge E$ and the twisted function spectrum $F[\alpha, E') \in \mathscr{S}U$ is the spectrum $F(\mathscr{M}\alpha, E')$.

PROPOSITION 5.2. *The above definitions agree with the definitions of* [38].

PROOF. Consider first the case of A compact. Then for a given $V \subset U$ one can find $V' \subset U'$ large enough that $\mathscr{M}\alpha_V \cong \Sigma^{\infty}_{V'} T\alpha_{V,V'}$. Thus

$$F[\alpha, E')V = \mathscr{S}U'(\mathscr{M}\alpha_V, E') \cong \mathscr{S}U'(\Sigma^{\infty}_{V'} T\alpha_{V,V'}, E') \cong F(T\alpha_{V,V'}, E'V'),$$

which is the definition of [38]. Therefore, by uniqueness of adjoints, the definitions of $\alpha \ltimes E$ agree for compact A. For general A one simply observes that our constructions and the definitions of [38] behave "properly" with respect to colimits. Specifically,

$$\alpha \ltimes E = \operatorname{colim}_K \alpha|_K \ltimes E \quad \text{and} \quad F[\alpha, E') = \lim_K F[\alpha|_K, E'),$$

where K runs through the compact subspaces of A. $\square$

The difference between our approach and the approach of [38] is this: The definitions in [38] of $\alpha \ltimes E$ and $f[\alpha, E')$ are simple and concrete when A is compact, but they depend on arbitrary choices of cofinal sequences of indexing spaces in U and U'. Hence naturality in α is difficult to prove. To get around this, the authors of [38] develop an elaborate theory of "connections" that allows them to show that the definitions of $\alpha \ltimes E$ and $F[\alpha, E')$ are independent of the choices involved and, therefore, that the constructions are natural in α. Only after proving all this is it possible to define $\alpha \ltimes E$ for non-compact A as the relevant spectrum-level colimit and $F[\alpha, E')$ as the limit. The advantage of our treatment is that the objects $\alpha \ltimes E$ and $F[\alpha, E')$ are defined once and for all for arbitrary A and naturality in α is immediate from the definitions.

We show in the following pair of results that twisted half-smash products and function spectra generalize change of universe functors and are in a sense built up out of them.

PROPOSITION 5.3. *If $f : U \longrightarrow U'$ is a linear isometry regarded as a map $* \longrightarrow \mathscr{I}(U, U')$ then $f \ltimes E = f_*E$ and $F[f, E') = f^*E'$, where $f_* : \mathscr{S}U \longrightarrow \mathscr{S}U'$ and $f^* : \mathscr{S}U' \longrightarrow \mathscr{S}U$ are the standard change of universe functors. In particular, if $U = U'$ and* id *denotes the identity map then* $\mathrm{id} \ltimes E \cong E$.

PROOF. For any $V \subset U$ we have $T\alpha_{V,fV} \cong \{f\}_+ \cong S^0$ and thus $\mathscr{M}f_V = \Sigma^\infty_{fV} S^0$. Therefore

$$F[f, E')V = \mathscr{S}U'(\Sigma^\infty_{fV} S^0, E') \cong F(S^0, E'(fV)) \cong E'(fV) = (f^*E')V.$$

The structure maps work properly and thus $F[f, E') = f^*E'$. The fact that $f \ltimes E = f_*E$ follows by the uniqueness of adjoints, or by an easy inspection. □

One should think of $\alpha \ltimes E$ intuitively as the union over $a \in A$ of the spectra $\alpha(a)_*E$, suitably topologized. Similarly, $F[\alpha, E')$ should be thought of as a suitably structured object that arises from the collection of spectra $\{\alpha(a)^*E'\}$. More precisely, we have the following result [38, VI.2.7], which admits an easy proof in our setup.

PROPOSITION 5.4. *Let $\alpha : \mathscr{I}(U, U')$ be a map and let $E \in \mathscr{S}U$ and $E' \in \mathscr{S}U'$ be spectra. A map $\xi : \alpha \ltimes E \longrightarrow E'$ determines and is determined by maps $\xi(\alpha) : E \longrightarrow \alpha(a)^*E'$ for points $a \in A$ such that the functions*

$$\zeta_{V,V'} : T\alpha_{V,V'} \wedge EV \longrightarrow E'V'$$

specified by $\zeta_{V,V'}((a, v') \wedge y) = \sigma(\xi(a)(y) \wedge v')$ for $a \in A_{V,V'}$, $v' \in V' - \alpha(a)V$, and $y \in EV$ are continuous, where σ denotes the structure map

$$\sigma_{\alpha(a)V,V'} : \Sigma^{V' - \alpha(a)V} E'(\alpha(a)V) \longrightarrow E'V'.$$

PROOF. Since for any $V \subset U$, $\mathscr{M}\alpha_V$ is the spectrification of $\mathscr{T}\alpha_V = \{T\alpha_{V,V'}\}$ we can write $\mathscr{M}\alpha_V$ as the spectrum level colimit

$$\mathscr{M}\alpha_V = \mathrm{colim}_{V' \subset U'} \Sigma^\infty_{V'} T\alpha_{V,V'}.$$

Therefore

$$\begin{aligned}\mathscr{S}U'(\alpha \ltimes E, E') &= \mathscr{S}U'(\operatorname{colim}_{V\subset U} \mathscr{M}\alpha_V \wedge EV, E') \\ &= \lim_{V\subset U} \mathscr{S}U'(\mathscr{M}\alpha_V \wedge EV, E') \\ &= \lim_{V\subset U} \mathscr{S}U'(\operatorname{colim}_{V'\subset U'} \Sigma^\infty_{V'} T\alpha_{V,V'} \wedge EV, E') \\ &= \lim_{V\subset U} \lim_{V'\subset U'} \mathscr{S}U'(\Sigma^\infty_{V'} T\alpha_{V,V'} \wedge EV, E') \\ &= \lim_{V\subset U} \lim_{V'\subset U'} \mathscr{T}(T\alpha_{V,V'} \wedge EV, E'V').\end{aligned}$$

This gives the connection between maps of spectra $\alpha \ltimes E \longrightarrow E'$ and families of maps $\zeta_{V,V'} : T\alpha_{V,V'} \wedge EV \longrightarrow E'V'$.

By restriction to points $a \in A$, a given map $\xi : \alpha \ltimes E \longrightarrow E'$ induces maps $\alpha(a)_* E \longrightarrow E'$ which are adjoint to maps $\xi(a) : E \longrightarrow \alpha(a)^* E'$. By projection from the double limit, ξ induces the map $\zeta_{V,V'} : T\alpha_{V,V'} \wedge EV \longrightarrow E'V'$, which may be checked to have the claimed description.

Conversely, given maps $\xi(a) : E \longrightarrow \alpha(a)^* E'$ satisfying the hypotheses, the maps $\zeta_{V,V'} : T\alpha_{V,V'} \wedge EV \longrightarrow E'V'$ specified by $\zeta_{V,V'}((a,v')\wedge y) = \sigma(\xi(a)(y)\wedge v')$ are easily checked to be compatible with the maps over which the double limit is taken, and therefore they specify a map of spectra $\alpha \ltimes E \longrightarrow E'$. □

The following untwisting theorem is an important sharpening of the result originally given in [38] and is vital to the theory of S-modules. The original proof of Elmendorf, Kriz, May, and Mandell, although not long or difficult, is rather technical in that it relies heavily on Elmendorf's category of spectra [20].

THEOREM 5.5 (UNTWISTING). *Let $V \subset U$ and $V' \subset U'$ be indexing spaces such that $V \cong V'$, let $\alpha : A \longrightarrow \mathscr{I}(U, U')$ be a map, and let X be a based space. There are isomorphisms*

$$\alpha \ltimes \Sigma^\infty_V X \cong A_+ \wedge \Sigma^\infty_{V'} X$$

and

$$\Omega^\infty_V F[\alpha, E') \cong F(A_+, \Omega^\infty_{V'} E')$$

that are natural in α and X.

PROOF. By Proposition 3.9 there is an isomorphism $\mathscr{E} \wedge \Sigma^\infty_V X \cong \mathscr{E}_V \wedge X$ that is natural in spaces $X \in \mathscr{T}$ and objects $\mathscr{E} \in \mathscr{S}(U'; U)$. Thus

$$\alpha \ltimes \Sigma^\infty_V X = \mathscr{M}\alpha \wedge \Sigma^\infty_V X \cong \mathscr{M}\alpha_V \wedge X.$$

By the untwisting lemma, $\mathscr{M}\alpha_V \cong \Sigma^\infty_{V'} A_+$, so it follows that

$$\alpha \ltimes \Sigma^\infty_V X \cong (\Sigma^\infty_{V'} A_+) \wedge X \cong A_+ \wedge \Sigma^\infty_{V'} X.$$

The isomorphism $\Omega^\infty_V F[\alpha, E') \cong F(A_+, \Omega^\infty_{V'} E')$ follows by uniqueness of adjoints. □

6. Formal properties of twisted half-smash products

We here give three basic formal properties of twisted half-smash products and function spectra. They have been used over and over in the text of the book. The first is an easy direct consequence of Proposition 3.7.

PROPOSITION 6.1. *For a map $\alpha : A \longrightarrow \mathscr{I}(U,U')$, spectra $E \in \mathscr{S}U$ and $E' \in \mathscr{S}U'$ and a based space X, there are natural isomorphisms*

$$(\alpha \ltimes E) \wedge X \cong \alpha \ltimes (E \wedge X)$$

and

$$F[\alpha, F(X,E')) \cong F(X, F[\alpha, E')).$$

Moreover, if Y is an unbased space and we denote by $\alpha \times Y$ the map

$$A \times Y \xrightarrow{\pi} A \xrightarrow{\alpha} \mathscr{I}(U,U'),$$

there are natural isomorphisms

$$(\alpha \times Y) \ltimes E \cong (\alpha \ltimes E) \wedge Y_+ \cong \alpha \ltimes (E \wedge Y_+).$$

The following two results relate twisted half-smash products to the naturality properties of spaces of linear isometries.

PROPOSITION 6.2. *Let $\alpha : A \longrightarrow \mathscr{I}(U,U')$ and $\beta : B \longrightarrow \mathscr{I}(U',U'')$ be maps and let $\beta \times_c \alpha$ denote the composite*

$$B \times A \xrightarrow{\beta\times\alpha} \mathscr{I}(U',U'') \times \mathscr{I}(U,U') \xrightarrow{c} \mathscr{I}(U,U''),$$

where c denotes composition. Then there are isomorphisms

$$(\beta \times_c \alpha) \ltimes E \cong \beta \ltimes (\alpha \ltimes E)$$

and

$$F[\beta \times_c \alpha, E'') \cong F[\alpha, F[\beta, E''))$$

that are natural in $E \in \mathscr{S}U$ and $E'' \in \mathscr{S}U''$.

PROPOSITION 6.3. *Let $\alpha : A \longrightarrow \mathscr{I}(U_1,U_1')$ and $\beta : B \longrightarrow \mathscr{I}(U_2,U_2')$ be maps and let $\alpha \times_\oplus \beta$ denote the composite*

$$A \times B \xrightarrow{\alpha\times\beta} \mathscr{I}(U_1,U_1') \times \mathscr{I}(U_2,U_2') \xrightarrow{\oplus} \mathscr{I}(U_1 \oplus U_2, U_1' \oplus U_2').$$

There is a natural isomorphism

$$(\alpha \times_\oplus \beta) \ltimes (E_1 \wedge E_2) \cong (\alpha \ltimes E_1) \wedge (\beta \ltimes E_2),$$

where $\wedge$ denotes the external smash product of spectra.

PROOF OF PROPOSITION 6.2. For $\mathscr{D} \in \mathscr{S}(U'';U')$ and $\mathscr{E} \in \mathscr{S}(U';U)$, we define the composition smash product $\mathscr{D}\wedge_c\mathscr{E} \in \mathscr{S}(U'';U)$ by $(\mathscr{D}\wedge_c\mathscr{E})_V = \mathscr{D}\wedge\mathscr{E}_V$. It is elementary to check that

$$(\mathscr{D} \wedge_c \mathscr{E}) \wedge E \cong \mathscr{D} \wedge (\mathscr{E} \wedge E)$$

for $E \in \mathscr{S}U$. Thus it suffices to show that $\mathscr{M}(\beta \times_c \alpha) \cong \mathscr{M}\beta \wedge_c \mathscr{M}\alpha$.

We begin by constructing bundle morphisms

$$\eta(\beta)_{V',V''} \times \eta(\alpha)_{V,V'} \longrightarrow \eta(\beta \times_c \alpha)_{V,V''}.$$

By definition,

$$E(\eta(\beta)_{V',V''}) = \{(b, v'') \in B_{V',V''} \times V'' \mid v'' \perp \beta(b)V'\},$$

$$E(\eta(\alpha)_{V,V'}) = \{(a, v') \in A_{V,V'} \times V' \mid v' \perp \alpha(a)V\},$$

and

$$E(\eta(\beta \times_c \alpha)_{V,V''}) = \{(b, a, v'') \in (B \times A)_{V,V''} \times V'' \mid v'' \perp \beta(b)\alpha(a)V\}.$$

It is easily seen that $B_{V',V''} \times A_{V,V'} \subset (B \times A)_{V,V''}$. Consider the map

$$E(\eta(\beta)_{V',V''}) \times E(\eta(\alpha)_{V,V'}) \longrightarrow E(\eta(\beta \times_c \alpha)_{V,V''})$$

that takes $(b, v'') \times (a, v')$ to $(b, a, v'' + \beta(b)v')$. Passing to Thom spaces, we obtain maps

$$T\beta_{V',V''} \wedge T\alpha_{V,V'} \longrightarrow T(\beta \times_c \alpha)_{V,V''}$$

which define a map of prespectra

$$\mathscr{T}\beta_{V'} \wedge T\alpha_{V,V'} \longrightarrow \mathscr{T}(\beta \times_c \alpha)_V$$

and therefore a map of spectra

$$\mathscr{M}\beta_{V'} \wedge T\alpha_{V,V'} \longrightarrow \mathscr{M}(\beta \times_c \alpha)_V.$$

One checks that, for $V' \subset W'$, the diagram

$$\begin{tikzcd} \mathscr{M}\beta_{V'} \wedge T\alpha_{V,V'} \arrow[rr] \arrow[dr] & & \mathscr{M}\beta_{W'} \wedge T\alpha_{V,W'} \arrow[dl] \\ & \mathscr{M}(\beta \times_c \alpha)_V & \end{tikzcd}$$

commutes, where the top row is the composite isomorphism

$$\begin{aligned} \mathscr{M}\beta_{V'} \wedge T\alpha_{V,V'} &\cong \Sigma^{W'-V'}\mathscr{M}\beta_{W'} \wedge T\alpha_{V,V'} \\ &\cong \mathscr{M}\beta_{W'} \wedge \Sigma^{W'-V'}T\alpha_{V,V'} \cong \mathscr{M}\beta_{W'} \wedge T\alpha_{V,W'}. \end{aligned}$$

Thus we obtain a map of spectra

$$(\mathscr{M}\beta \wedge_c \mathscr{M}\alpha)_V = \mathscr{M}\beta \wedge \mathscr{M}\alpha_V \cong \operatorname{colim}_{V' \subset U'} \mathscr{M}\beta_{V'} \wedge T\alpha_{V,V'} \longrightarrow \mathscr{M}(\beta \times_c \alpha)_V.$$

One checks that these maps are compatible with the structure maps of our objects and so define a morphism $\theta : \mathscr{M}\beta \wedge_c \mathscr{M}\alpha \longrightarrow \mathscr{M}(\beta \times_c \alpha)$ in the category $\mathscr{S}(U'';U)$.

To show that θ is an isomorphism, we invoke the familiar colimit argument. We have

$$\mathscr{M}\beta \wedge_c \mathscr{M}\alpha = \operatorname{colim}_{K,L} \mathscr{M}\beta|_L \wedge_c \mathscr{M}\alpha|_K$$

and

$$\mathscr{M}(\beta \times_c \alpha) = \operatorname{colim}_{K,L} \mathscr{M}(\beta \times_c \alpha)|_{L\times K},$$

where the colimits are taken over all compact $K \subset A$ and all compact $L \subset B$. Thus it suffices to show that θ is an isomorphism when A and B are compact. In that case, for a fixed V, we may choose V' large enough that $A_{V,V'} = A$ and we may then choose V'' large enough that $B_{V',V''} = B$. It follows that $(B \times A)_{V,V''} = B \times A$ and the map $T\beta_{V',V''} \wedge T\alpha_{V,V'} \longrightarrow T(\beta \times_c \alpha)_{V,V''}$ is a homeomorphism. Therefore

$$\begin{aligned}
(\mathscr{M}\beta \wedge_c \mathscr{M}\alpha)_V = \mathscr{M}\beta \wedge \mathscr{M}\alpha_V &\cong \mathscr{M}\beta \wedge \Sigma^\infty_{V'} T\alpha_{V,V'} \\
&\cong \mathscr{M}\beta_{V'} \wedge T\alpha_{V,V'} \cong (\Sigma^\infty_{V''} T\beta_{V',V''}) \wedge T\alpha_{V,V'} \\
&\cong \Sigma^\infty_{V''} T\beta_{V',V''} \wedge T\alpha_{V,V'} \cong \Sigma^\infty_{V''} T(\beta \times_c \alpha)_{V,V''} \\
&\cong \mathscr{M}(\beta \times_c \alpha)_V. \quad \square
\end{aligned}$$

PROOF OF PROPOSITION 6.3. For $\mathscr{D} \in \mathscr{S}(U_1'; U_1)$ and $\mathscr{E} \in \mathscr{S}(U_2'; U_2)$ we define the external direct sum smash product $\mathscr{D} \wedge_\oplus \mathscr{E} \in \mathscr{S}(U_1' \oplus U_2'; U_1 \oplus U_2)$ by

$$(\mathscr{D} \wedge_\oplus \mathscr{E})_{V_1 \oplus V_2} = \mathscr{D}_{V_1} \wedge \mathscr{E}_{V_2}.$$

Since the set $\{V_1 \oplus V_2\}$ is cofinal in $U_1 \oplus U_2$ this specifies an object in the category $\mathscr{S}(U_1' \oplus U_2'; U_1 \oplus U_2)$. It is easily checked that, for spectra $E_i \in \mathscr{S}U_i$,

$$(\mathscr{D} \wedge_\oplus \mathscr{E}) \wedge (E_1 \wedge E_2) \cong (\mathscr{D} \wedge E_1) \wedge (\mathscr{E} \wedge E_2).$$

Thus it suffices to show that $\mathscr{M}\alpha \wedge_\oplus \mathscr{M}\beta \cong \mathscr{M}(\alpha \times_\oplus \beta)$.

By definition,

$$E(\eta(\alpha)_{V_1,V_1'}) = \{(a, v_1') \in A_{V_1,V_1'} \times V_1' \mid v_1' \perp \alpha(a)V_1\},$$

$$E(\eta(\beta)_{V_2,V_2'}) = \{(b, v_2') \in B_{V_2,V_2'} \times V_2' \mid v_2' \perp \beta(b)V_2\},$$

and

$$E(\eta(\alpha \times_\oplus \beta)_{V_1\oplus V_2, V_1'\oplus V_2'}) =$$
$$\{(a, b, v_1', v_2') \in (A \times B)_{V_1\oplus V_2, V_1'\oplus V_2'} \times (V_1' \oplus V_2') \mid v_1' + v_2' \perp \alpha(a)V_1 \oplus \beta(b)V_2\}.$$

It follows immediately that there is a bundle isomorphism

$$\eta(\alpha)_{V_1,V_1'} \times \eta(\beta)_{V_2,V_2'} \cong \eta(\alpha \times_\oplus \beta)_{V_1\oplus V_2, V_1'\oplus V_2'}.$$

Thus we have isomorphisms of Thom spaces

$$T\alpha_{V_1,V_1'} \wedge T\beta_{V_2,V_2'} \cong T(\alpha \times_\oplus \beta)_{V_1\oplus V_2, V_1'\oplus V_2'}$$

and therefore an isomorphism of prespectra

$$\mathscr{T}\alpha_{V_1} \wedge \mathscr{T}\beta_{V_2} \cong \mathscr{T}(\alpha \times_\oplus \beta)_{V_1\oplus V_2}.$$

It follows that

$$(\mathscr{M}\alpha \wedge_{\oplus} \mathscr{M}\beta)_{V_1 \oplus V_2} \cong \mathscr{M}\alpha_{V_1} \wedge \mathscr{M}\beta_{V_2} \cong \mathscr{M}(\alpha \times_{\oplus} \beta)_{V_1 \oplus V_2}.$$

□

7. Homotopical properties of $\alpha \ltimes E$ and $F[\alpha, E')$

We now turn our attention to the homotopy preservation properties of our constructions. It is evident from the definitions that $\alpha \ltimes E$ and $F[\alpha, E')$ preserve homotopies in E and E' and homotopies over $\mathscr{I}(U, U')$ in α. However, it is vital to the theory that many homotopical properties of $\alpha \ltimes E$ and $F[\alpha, E')$ depend only on the homotopical properties of the space A.

PROPOSITION 7.1. *If A has the homotopy type of a CW complex, then the functor $\alpha \ltimes E$ preserves CW homotopy types in E and the functor $F[\alpha, E')$ preserves weak equivalences in E'.*

PROOF. It is a standard categorical observation that the two assertions are equivalent since $\alpha \ltimes -$ and $F[\alpha, -)$ are an adjoint pair of functors. Let $E_1' \longrightarrow E_2'$ be a weak equivalence in $\mathscr{S}U'$. Then each $E_1'V' \longrightarrow E_2'V'$ is a weak equivalence of spaces. Since A has CW homotopy type, $F(A_+, E_1'V') \longrightarrow F(A_+, E_2'V')$ is a weak equivalence for any $V' \subset U'$. It follows from the untwisting theorem that $F[\alpha, E_1') \longrightarrow F[\alpha, E_2')$ is a spacewise weak equivalence and therefore a weak equivalence of spectra. □

COROLLARY 7.2. *If $\alpha : A \longrightarrow \mathscr{I}(U, U')$ is a map and A has CW homotopy type then the functors $\alpha \ltimes -$ and $F[\alpha, -)$ pass to an adjoint pair of functors on the stable categories.*

As always, a functor such as $\alpha \ltimes E$ that does not preserve weak equivalences in E is defined on the stable category by first replacing E with a CW approximation. We can strengthen Proposition 6.1 by obtaining a CW structure on $\alpha \ltimes E$ from CW structures on A and E.

PROPOSITION 7.3. *Let $\alpha : A \longrightarrow \mathscr{I}(U, U')$ be a map, where A is a CW complex with skeletal filtration $\{A^n\}$. Let E be a CW spectrum with skeletal filtration $\{E^n\}$ and sequential filtration $\{E_n\}$. Then $\alpha \ltimes E$ is a CW spectrum with skeletal filtration*

$$(\alpha \ltimes E)^n = \bigcup_{p+q=n} (\alpha|_{A^p}) \ltimes E^q, \ n \in \mathbb{Z},$$

and sequential filtration

$$(\alpha \ltimes E)_n = \bigcup_{p+q=n} (\alpha|_{A^p}) \ltimes E_q, \ n \geq 0.$$

PROOF. We induct up the sequential filtration. Since $E_0 = *$, $(\alpha \ltimes E)_0 = *$. Assume that $(\alpha \ltimes E)_{n-1}$ is a CW spectrum. Let $\varepsilon : D^p \longrightarrow A$ be a p-cell with restriction σ to S^{p-1}. Let $CS^{r-1} \longrightarrow E_q$ be an r-cell of E_q with attaching map $S^{r-1} \longrightarrow E_{q-1}$, where $p + q = n$ and $r \in \mathbb{Z}$. Then $(\alpha \ltimes E)_n$ is obtained from $(\alpha \ltimes E)_{n-1}$ by attaching "twisted cells" of the form $(\alpha \circ \varepsilon) \ltimes CS^{r-1}$ along attaching maps

$$(\alpha \circ \varepsilon) \ltimes S^{r-1} \cup (\alpha \circ \sigma) \ltimes CS^{r-1} \longrightarrow (\alpha \ltimes E)_{n-1}.$$

It follows from the untwisting theorem and inspection on the space level that the pair $((\alpha \circ \varepsilon) \ltimes CS^{r-1}, (\alpha \circ \varepsilon) \ltimes S^{r-1})$ is isomorphic to the pair (CS^{p+r-1}, S^{p+r-1}). Therefore $(\alpha \ltimes E)_n$ is constructed from $(\alpha \ltimes E)_{n-1}$ by attaching genuine cells along cellular maps and is thus a CW spectrum. $\square$

Recall that a prespectrum D is said to be Σ-cofibrant if the structure maps $\Sigma^{W-V} DV \longrightarrow DW$ are cofibrations. A spectrum is Σ-cofibrant if it is isomorphic to LD for some Σ-cofibrant prespectrum D. A spectrum is tame if it is homotopy equivalent to a Σ-cofibrant spectrum. Roughly speaking, tame spectra are to spacewise homotopy equivalences of spectra as spectra with CW homotopy type are to weak equivalences of spectra. Thus, if D is tame and $f : E_1 \longrightarrow E_2$ is a map of spectra such that each $fV : E_1V \longrightarrow E_2V$ is a homotopy equivalence, then

$$f_* : h\mathscr{S}U(D, E_1) \longrightarrow h\mathscr{S}U(D, E_2)$$

is a bijection. It follows formally that a spacewise homotopy equivalence of tame spectra is a genuine homotopy equivalence. Pursuing the analogy further, the cylinder construction KE may be thought of as a "Σ-cofibrant approximation" to the spectrum E, and there is a map $KE \longrightarrow E$ that is a spacewise homotopy equivalence. Categorically, it is generally true that if $L : \mathscr{S}U \longrightarrow \mathscr{S}U'$ and $R : \mathscr{S}U' \longrightarrow \mathscr{S}U$ are a left-right adjoint pair of functors, then L preserves tameness if and only if R preserves spacewise homotopy equivalences.

THEOREM 7.4. *Let $\phi : A \longrightarrow B$ be a homotopy equivalence, let $\beta : B \longrightarrow \mathscr{I}(U, U')$ be a map, and let $\alpha : A \longrightarrow \mathscr{I}(U, U')$ be the composite $\beta \circ \phi$. If $E \in \mathscr{S}U$ is tame, then the map $\phi \ltimes E : \alpha \ltimes E \longrightarrow \beta \ltimes E$ is a homotopy equivalence. For any $E' \in \mathscr{S}U'$, the map $F[\phi, E') : F[\beta, E') \longrightarrow F[\alpha, E')$ is a spacewise homotopy equivalence.*

PROOF. It follows from the untwisting theorem that for any spectrum $E' \in \mathscr{S}U'$, the maps $F[\phi, E')(V)$ are homotopy equivalences. Thus for any tame spectrum E and any spectrum E' the map

$$F[\phi, E')_* : h\mathscr{S}U(E, F[\beta, E')) \longrightarrow h\mathscr{S}U(E, F[\alpha, E'))$$

is a bijection. By adjunction, this says that the map

$$(\phi \ltimes E)^* : h\mathscr{S}U'(\beta \ltimes E, E') \longrightarrow h\mathscr{S}U'(\alpha \ltimes E, E')$$

is a bijection. It is now formal that $\phi \ltimes E$ is a homotopy equivalence. $\square$

COROLLARY 7.5. *Let $\alpha_1, \alpha_2 : A \longrightarrow \mathscr{I}(U, U')$ be maps. If $E \in \mathscr{S}U$ is tame, then $\alpha_1 \ltimes E$ and $\alpha_2 \ltimes E$ are homotopy equivalent. If $E' \in \mathscr{S}U'$ is any spectrum, then $F[\alpha_1, E)$ and $F[\alpha_2, E)$ are weakly equivalent.*

PROOF. We note that since $\mathscr{I}(U, U')$ is contractible, α_1 and α_2 are homotopic. Let $H : A \times I \longrightarrow \mathscr{I}(U, U')$ be a homotopy and apply Theorem 6.4 with $B = A \times I$ and $\phi = i_t : A \longrightarrow A \times I$ for $t = 0, 1$. □

Since CW spectra are tame, this implies the following result.

COROLLARY 7.6. *If A has CW homotopy type, the functors $\alpha \ltimes -$ obtained by varying the map α are canonically and coherently equivalent upon passage to the stable category. Similarly the functors $F[\alpha, -)$ are canonically and coherently equivalent as functors on the stable category.*

8. The cofibration theorem

In this section we prove the following analog for cofibrations of Theorem 7.4. The original proof of Elmendorf, Kriz, May, and Mandell relied heavily on the properties of Elmendorf's category of spectra [20], using bundle theoretic arguments about the morphism sets in that category. We shall show that it actually follows formally from the untwisting theorem and some elementary homotopy theory.

THEOREM 8.1. *Let $\phi : A \longrightarrow B$ be a cofibration, let $\beta : B \longrightarrow \mathscr{I}(U, U')$ be a map, and let $\alpha : A \longrightarrow \mathscr{I}(U, U')$ be the composite $\beta \circ \phi$. If $E \in \mathscr{S}U$ is Σ-cofibrant or is a CW spectrum, then $\phi \ltimes E : \alpha \ltimes E \longrightarrow \beta \ltimes E$ is a cofibration.*

PROOF. Consider a test diagram

$$\begin{tikzcd}
\alpha \ltimes E \arrow[rr, "i_0"] \arrow[dd] & & (\alpha \ltimes E) \wedge I_+ \arrow[dl] \arrow[dd] \\
& E' & \\
\beta \ltimes E \arrow[ur] \arrow[rr, "i_0"'] & & (\beta \ltimes E) \wedge I_+ \arrow[ul, dashed]
\end{tikzcd}$$

for which we must prove that the dotted arrow exists making the diagram commute. By adjunction, we may consider instead the test diagram

$$\begin{tikzcd}
E \arrow[r, "f"] \arrow[d, "i_0"'] & F[\beta, E') \arrow[d, "{F[\phi, E')}"] \\
E \wedge I_+ \arrow[r, "g"'] \arrow[ur, dashed, "h"] & F[\alpha, E')
\end{tikzcd}$$

and try to prove that h exists. We are not claiming that $F[\phi, E') : F[\beta, E') \longrightarrow F[\alpha, E')$ is a fibration of spectra, but we will show that it does have the covering homotopy property (CHP) with respect to Σ-cofibrant spectra and CW spectra.

It is easily seen that $F[\phi, E')$ is a spacewise fibration. For if $V \subset U$ and $V' \subset U'$ are such that $V \cong V'$, then $F[\phi, E')$ is equivalent to a map

$$F(\phi_+, E'V') : F(B_+, E'V') \longrightarrow F(A_+, E'V'),$$

which is clearly a fibration. It follows now that $F[\phi, E')$ has the CHP with respect to spectra of the form $\Sigma_V^\infty X$ for any $V \subset U$. In particular, $F[\phi, E')$ has the CHP with respect to sphere spectra and hence $F[\phi, E')$ may be thought of as a spectrum level Serre fibration. The standard arguments now show that $F[\phi, E')$ has the CHP with respect to CW spectra.

The argument for Σ-cofibrant spectra is a bit more delicate. Let $E = LD$ where D is a Σ-cofibrant prespectrum. We want to use the cofibration condition on the structure maps $\Sigma^{W-V} DV \longrightarrow DW$ to construct the lift h for the diagram of prespectra

$$\begin{tikzcd} D \arrow[r, "f"] \arrow[d, "i_0"'] & F[\beta, E') \arrow[d, "{F[\phi,E')}"] \\ D \wedge I_+ \arrow[r, "g"'] \arrow[ur, dashed, "h"] & F[\alpha, E'). \end{tikzcd}$$

We do this by choosing a cofinal sequence $V_0 \subset V_1 \subset \ldots$ of representations in U and then arguing that after having constructed the map $hV_i : DV_i \wedge I_+ \longrightarrow F[\beta, E')(V_i)$ we can construct the map hV_{i+1} in a compatible way so as to obtain a map of prespectra.

The problem reduces to the following: Define

$$k : DV_{i+1} \wedge \{0\}_+ \cup \Sigma^{V_{i+1}-V_i} DV_i \wedge I_+ \longrightarrow F[\beta, E')(V_{i+1})$$

to be the pushout of the maps

$$DV_{i+1} \xrightarrow{fV_{i+1}} F[\beta, E')(V_{i+1})$$

and

$$\Sigma^{V_{i+1}-V_i} DV_i \wedge I_+ \xrightarrow{\Sigma^{V_{i+1}-V_i} hV_i} \Sigma^{V_{i+1}-V_i} F[\beta, E')(V_i) \longrightarrow F[\beta, E')(V_{i+1}).$$

Then $hV_{i+1} : DV_{i+1} \wedge I_+ \longrightarrow F[\beta, E')(V_{i+1})$ must be a solution to the lifting problem

$$\begin{tikzcd} DV_{i+1} \wedge \{0\}_+ \cup \Sigma^{V_{i+1}-V_i} DV_i \wedge I_+ \arrow[r, "k"] \arrow[d] & F[\beta, E')(V_{i+1}) \arrow[d, "{F[\phi,E')(V_{i+1})}"] \\ DV_{i+1} \wedge I_+ \arrow[r, "gV_{i+1}"'] \arrow[ur, dashed, "hV_{i+1}"] & F[\alpha, E')(V_{i+1}). \end{tikzcd}$$

By the untwisting theorem, if $V' \subset U'$ is such that $V_{i+1} \cong V'$ then we can replace $F[\phi, E')(V_{i+1})$ with the map

$$F(\phi_+, E'V') : F(B_+, E'V') \longrightarrow F(A_+, E'V').$$

The existence of the lift hV_{i+1} follows from our next lemma. $\square$

LEMMA 8.2. *If $X \longrightarrow Y$ is a cofibration of based spaces, $A \longrightarrow B$ is a cofibration of unbased spaces, and Z is a based space, then any diagram of the form*

$$\begin{tikzcd} Y \wedge \{0\}_+ \cup X \wedge I_+ \arrow[r] \arrow[d] & F(B_+, Z) \arrow[d] \\ Y \wedge I_+ \arrow[r] \arrow[ur, dashed] & F(A_+, Z) \end{tikzcd}$$

can be completed by the dotted arrow.

PROOF. By playing with adjunctions we can replace the above diagram with an equivalent diagram

$$\begin{tikzcd} B_+ \wedge \{0\}_+ \cup A_+ \wedge I_+ \arrow[r] \arrow[d] & F(Y, Z) \arrow[d] \\ B_+ \wedge I_+ \arrow[r] \arrow[ur, dashed] & F(X, Z). \end{tikzcd}$$

But this diagram of based spaces is equivalent to the diagram of unbased spaces

$$\begin{tikzcd} B \times \{0\} \cup A \times I \arrow[r] \arrow[d] & F(Y, Z) \arrow[d] \\ B \times I \arrow[r] \arrow[ur, dashed] & F(X, Z). \end{tikzcd}$$

The based fibration $F(Y, Z) \longrightarrow F(X, Z)$ is also an unbased fibration and the solution now follows from the fact that $(B \times I, B \times \{0\} \cup A \times I)$ is a DR pair. □

9. Equivariant twisted half-smash products

As we have stated previously, all of our definitions and results on twisted half-smash products and function spectra generalize to the equivariant context with little change. In this section we discuss briefly the few exceptions to this assertion. More details will appear in [16].

The basic source material on categories of equivariant spectra is found in [38] and we briefly summarize that setup. For a compact Lie group G, a G-universe U is defined to be a countably infinite dimensional real inner product space on which G acts smoothly through linear isometries. We require that U contain a trivial representation and we require that whenever $V \subset U$ is a finite dimensional representation then U must contain an isomorphic copy of $V^{\oplus\infty}$. At the one extreme, a G-universe is called complete if it contains copies of all the irreducible G-representations; at the other extreme, a G-universe is called G-trivial if it contains only the trivial representations.

By a G-spectrum indexed on a G-universe U we mean a spectrum E indexed on the finite dimensional subrepresentations of U such that each component space EV is a G-space and the structure maps $\Sigma^{W-V}EV \longrightarrow EW$ are G-maps. The resulting category is denoted $G\mathscr{S}U$. Thus for non-isomorphic G-universes U, U' we have associated non-equivalent categories of G-spectra that pass to non-equivalent equivariant stable categories.

For G-universes U, U', we let G act on $\mathscr{I}(U, U')$ by conjugation. For a G-map $\alpha : A \longrightarrow \mathscr{I}(U, U')$ and G-spectra $E \in G\mathscr{S}U$, $E' \in G\mathscr{S}U'$ there is a twisted half-smash product $\alpha \ltimes E \in G\mathscr{S}U'$ and a twisted function spectrum $F[\alpha, E') \in G\mathscr{S}U$ whose definitions and properties are wholly analogous to the nonequivariant constructions. Sections 2 to 6 generalize in a straightforward way that requires little comment. The category $G\mathscr{S}(U'; U)$ is defined in the evident way. In the equivariant version of Definition 3.2, for an object $\mathscr{E} \in G\mathscr{S}(U'; U)$ and G-spectrum $E' \in G\mathscr{S}U'$ the G-space $F(\mathscr{E}, E')(V) = \mathscr{S}U'(\mathscr{E}_V, E')$ must be understood to mean the space of all nonequivariant maps of spectra $\mathscr{E}_V \longrightarrow E'$ with G acting by conjugation. The equivariant versions of the untwisting results, Lemma 4.5 and Theorem 5.5, are true, except that the existence of a $V' \subset U'$ isomorphic to the given G-indexing space $V \subset U$ is a non-trivial hypothesis that may not be satisfied.

Fix a copy of $\mathbb{R}^\infty$ in U. It is a crucial fact [38, I.4.6] that for a map $f : E_1 \longrightarrow E_2$ of G-spectra indexed on U to be a spacewise weak equivalence of G-spaces, it suffices that $f\mathbb{R}^n : E_1\mathbb{R}^n \longrightarrow E_1\mathbb{R}^n$ be a weak equivalence for all n. Since all G-universes contain the trivial representations, the untwisting theorem applies to the trivial representations, and we see that the equivariant versions of Proposition 7.1 and Corollary 7.2 hold. The equivariant version of Proposition 7.3 only holds when G is finite. In analogy with [38, I.4.6], it can be shown that in order for a map of G-spectra to be a spacewise homotopy equivalence, it is sufficient to consider only the trivial representations. Hence the equivariant version of Theorem 7.4 holds. While $\mathscr{I}(U, U')$ is not a G-contractible G-space, it does have the property that if A is a G-CW complex then any two maps $\alpha_1, \alpha_2 : A \longrightarrow \mathscr{I}(U, U')$ are G-homotopic. Hence the equivariant versions of Corollaries 7.5 and 7.6 hold.

The equivariant version of Theorem 8.1 holds in full generality for the case that E is a G-CW spectrum. The reason is that the sphere spectra relevant to a G-CW structure are of the form $S^n \wedge (G/H)_+$ where n is an integer. Hence the problem of lifting cells over the map $F[\phi, E')$ only requires consideration of the maps $F[\phi, E')(\mathbb{R}^n)$ for which the untwisting theorem applies. For general universes U, U', if E is tame, but not G-CW, I do not know whether the map $\phi \ltimes E : \alpha \ltimes E \longrightarrow \beta \ltimes E$ must always be a cofibration. Our proof of Theorem 8.1, like the original proof given by Elmendorf, Kriz, May, and Mandell, works equivariantly only if we add the additional assumption that the universe U' is "as big or bigger" than U in the sense that there exists a G-linear isometry $U \longrightarrow U'$.

Bibliography

[1] J. F. Adams. Lectures on generalised cohomology. Springer Lecture Notes Vol 99, 1969, 1–138.

[2] J. F. Adams. Stable homotopy and generalized homology. The University of Chicago Press. 1974. (Reprinted 1995.)

[3] J. F. Adams. A variant of E. H. Brown's representability theorem. Topology, 10(1971), 185–198.

[4] M. Barr and C. Wells. Toposes, triples, and theories. Springer-Verlag. 1985.

[5] D. Blanc. New model categories from old. Preprint. 1994.

[6] J. M. Boardman. Stable homotopy theory. Thesis, Warwick 1964; mimeographed notes from Warwick and Johns Hopkins Universities, 1965-1970.

[7] J. M. Boardman. Conditionally convergent spectral sequences. Preprint. Johns Hopkins University, 1981.

[8] M. Bökstedt. Topological Hochschild homology. Preprint, 1990.

[9] M. Bökstedt. The topological Hochschild homology of $\mathbb{Z}$ and $\mathbb{Z}/p$. Preprint, 1990.

[10] M. Bökstedt, W. C. Hsiang, and I. Madsen. The cyclotomic trace and algebraic K-theory of spaces. Invent. Math. 111(1993), 465-539.

[11] A. K. Bousfield. The localization of spaces with respect to homology. Topology 14(1975), 133–150.

[12] A. K. Bousfield. The localization of spectra with respect to homology. Topology 18(1979), 257–281.

[13] E. H. Brown, Jr. Abstract homotopy theory. Trans. Amer. Math. Soc. 119 (1965), 79–85.

[14] R. R. Bruner, J. P. May, J. E. McClure, and M. Steinberger. H_∞ ring spectra and their applications. Springer Lecture Notes Vol 1176. 1986.

[15] H. Cartan and S. Eilenberg. Homological algebra. Princeton University Press. 1956.

[16] M. Cole. Twisted half-smash products and function spectra. Chapter XXII of "Equivariant homotopy and cohomology theory", by J. P. May, et al. NSF-CBMS Regional Conference Monograph. To appear in 1996.

[17] W. G. Dwyer, M. J. Hopkins, and D. M. Kan. The homotopy theory of cyclic sets. Trans. Amer. Math. Soc. 291(1985), 281-289.

[18] W. G. Dwyer and J. Spalinski. Homotopy theories and model categories. A handbook of algebraic topology, edited by I. M. James. Elsevier Science. 1995.

[19] S. Eilenberg and J. C. Moore. Homology and fibrations, I. Comm. Math. Helv. 40(1966), 199–236.

[20] A. D. Elmendorf. The Grassmannian geometry of spectra. J. Pure and Applied Algebra, 54(1988), 37–94.

[21] A. D. Elmendorf. Geometric realization of simplicial spectra. Manuscripta Mathematica. To appear.

[22] A. D. Elmendorf, J. P. C. Greenlees, I. Kriz, and J. P. May. Commutative algebra in stable

homotopy theory and a completion theorem. Mathematical Research Letters 1(1994), 225-239.

[23] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. Modern foundations for stable homotopy theory. A handbook of algebraic topology, edited by I. M. James. Elsevier Science. 1995.

[24] D. Grayson. Localization for Flat Modules in Algebraic K-Theory. J. Alg. 61(1979), 463-496.

[25] J. P. C. Greenlees and J. P. May. Generalized Tate cohomology. Memoirs Amer. Math. Soc. 1995.

[26] J. P. C. Greenlees and J. P. May. Completions in algebra and topology. A handbook of algebraic topology, edited by I. M. James. Elsevier Science. 1995

[27] J. P. C. Greenlees and J. P. May. Equivariant stable homotopy theory. A handbook of algebraic topology, edited by I. M. James. Elsevier Science. 1995.

[28] J. P. C. Greenlees and J. P. May. A completion theorem for MU-module spectra. Preprint, 1995.

[29] V. K. A. M. Gugenheim and J. P. May. On the theory and applications of differential torsion products. Memoirs Amer. Math. Soc. 142, 1974.

[30] L. Hesselholt and I. Madsen. On the K-theory of finite algebras over Witt vectors of perfect fields. Aarhus Universitet Preprint Series 1995, No. 2.

[31] H. Hochschild. On the cohomology groups of an associative algebra. Annals of Math. 46(1945), 58-67.

[32] M. J. Hopkins. Notes on E_∞ ring spectra. Typed notes. 1993.

[33] G. M. Kelly. Basic concepts of enriched category theory. London Math. Soc. Lecture Note Series Vol. 64. Cambridge University Press. 1982.

[34] I. Kriz. Towers of E_∞ ring spectra with an application to BP. Preprint, 1993.

[35] I. Kriz. and J. P. May. Operads, algebras, modules, and motives. Astérisque Vol. 233. 1995.

[36] L. G. Lewis. When is the natural map $X \longrightarrow \Omega\Sigma X$ a cofibration? Trans. Amer. Math. Soc. 273(1982), 147-155.

[37] L. G. Lewis, Jr. Is there a convenient category of spectra? J. Pure and Applied Algebra 73(1991). 233-246.

[38] L. G. Lewis. Jr., J. P. May, and M. Steinberger (with contributions by J. E. McClure). Equivariant stable homotopy theory. Springer Lecture Notes in Mathematics Volume 1213. 1986.

[39] J. Lillig. A union theorem for cofibrations. Arch. Math. 24(1973), 410-415.

[40] T.-Y. Lin. Adams type spectral sequences and stable homotopy modules. Indiana Math. J. 25(1976), 135-158.

[41] F. E. J. Linton. Coequalizers in categories of algebras. Springer Lecture Notes in Mathematics 80, 1969, 75-90.

[42] J. L. Loday. Opérations sur l'homologie cyclique des algébras commutatives. Invent. Math. 96(1989), 205-230.

[43] S. Mac Lane. Categories for the Working Mathematician. Springer-Verlag. 1971.

[44] J. P. May. Simplicial objects in algebraic topology, 1967. Reprinted by the University of Chicago Press, 1982 and 1992.

[45] J. P. May. The Geometry of Iterated Loop Spaces. Springer Lecture Notes in Mathematics Volume 271. 1972.

[46] J. P. May. E_∞ spaces, group completions, and permutative categories. London Math. Soc. Lecture Notes No.11, 1974, 61-93.

[47] J. P. May. Classifying spaces and fibrations. Memoirs Amer. Math. Soc. 155. 1975.

[48] J. P. May (with contributions by F. Quinn, N. Ray, and J. Tornehave). E_∞ ring spaces and E_∞ ring spectra. Springer Lecture Notes in Mathematics Volume 577. 1977.

[49] J. P. May. A_∞ Ring Spaces and Algebraic K-Theory. Springer Lecture Notes in Mathematics Volume 658. 1978, 240-315.

[50] J. P. May. Multiplicative infinite loop space theory. J. Pure and Applied Algebra, 26(1982). 1-69.

[51] J. P. May. Derived categories in algebra and topology. In the Proceedings of the Eleventh

International Conference on Topology, Rendiconti dell Istituto Matematico dell Università di Trieste 25(1993), 363-377.

[52] J. P. May, et al. Equivariant homotopy and cohomology theory. NSF-CBMS Regional Conference Monograph. To appear in 1996.

[53] R. McCarthy. On Fundamental Theorems of Algebraic K-Theory. Topology 32(1993), 325–328.

[54] J. E. McClure, R. Schwänzl, and R. Vogt. $THH(R) \cong R \otimes S^1$ for E_∞ ring spectra. J. Pure and Applied Algebra. To appear.

[55] J. E. McClure and R. E. Staffeldt. On the topological Hochschild homology of bu, I. Amer. J. Math. 115(1993), 1-45.

[56] A. Neeman. On a theorem of Brown and Adams. Preprint, 1995.

[57] D. G. Quillen. Homotopical algebra. Springer Lecture Notes in Mathematics Volume 43. 1967.

[58] D. G. Quillen. Higher Algebraic K-Theory I. Springer Lecture Notes in Mathematics Volume 341, 1973, 85–147.

[59] A. Robinson. Derived tensor products in stable homotopy theory. Topology 22(1983), 1-18.

[60] A. Robinson. Spectra of derived module homomorphisms. Math. Proc. Camb. Phil. Soc. 101(1987), 249-257.

[61] A. Robinson. The extraordinary derived category. Math. Z. 196(1987), 231-238.

[62] A. Robinson. Obstruction theory and the strict associativity of Morava K-theories. London Math. Soc. Lecture Note Series, vol. 139, 1989, 111-126.

[63] A. Robinson. Composition products in RHom, and ring spectra of derived endomorphisms. Springer Lecture Notes Vol 1370, 1989, 374-386.

[64] R. Schwänzl and R. M. Vogt. The categories of A_∞ and E_∞ monoids and ring spaces as closed simplicial and topological model categories. Arch. Math. 56(1991), 405-411.

[65] R. Schwänzl and R. M. Vogt. Basic Constructions in the K-Theory of Homotopy Ring Spaces. Trans. Amer. Mat. Soc 341(1994), 549-584.

[66] G. Segal. Categories and Cohomology Theories. Topology 13(1974), 293–312.

[67] R. E. Staffeldt. On Fundamental Theorems of Algebraic K-Theory. K-Theory 2(1989), 511–532.

[68] R. W. Thomason and T. Trobaugh. Higher Algebraic K-Theory of Schemes. The Grothendieck Festschrift, v. III, 1990, 247-435.

[69] J. L. Verdier. Catégories dérivées. Springer Lecture Notes Vol 569, 1977, 262-311.

[70] F. Waldhausen. Algebraic K-Theory of Topological Spaces II. Springer Lecture Notes in Mathematics Volume 763, 1979, 356–394.

[71] F. Waldhausen. Algebraic K-Theory of Spaces. Springer Lecture Notes in Mathematics Volume 1126, 1985, 318–419.

[72] J. Wolbert. Towards an algebraic classification of module spectra. J. Pure and Applied Algebra. To appear.

[73] N. Yoneda. On the homology theory of modules. J. Fac. Sci. Univ. Tokyo Sec. I, 7(1954), 193-227.

Index

Selected Titles in This Series

(Continued from the front of this publication)

11 **J. Cronin,** Fixed points and topological degree in nonlinear analysis, 1964

10 **R. Ayoub,** An introduction to the analytic theory of numbers, 1963

9 **Arthur Sard,** Linear approximation, 1963

8 **J. Lehner,** Discontinuous groups and automorphic functions, 1964

7.2 **A. H. Clifford and G. B. Preston,** The algebraic theory of semigroups, Volume II, 1961

7.1 **A. H. Clifford and G. B. Preston,** The algebraic theory of semigroups, Volume I, 1961

6 **C. C. Chevalley,** Introduction to the theory of algebraic functions of one variable, 1951

5 **S. Bergman,** The kernel function and conformal mapping, 1950

4 **O. F. G. Schilling,** The theory of valuations, 1950

3 **M. Marden,** Geometry of polynomials, 1949

2 **N. Jacobson,** The theory of rings, 1943

1 **J. A. Shohat and J. D. Tamarkin,** The problem of moments, 1943

ISBN 978-0-8218-4303-1

9 780821 843031

SURV/47.S