Mathematical
Surveys
and
Monographs

Volume 49

Monotone Operators in Banach Space and Nonlinear Partial Differential Equations

R. E. Showalter

American Mathematical Society

http://dx.doi.org/10.1090/surv/049

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Monotone Operators in Banach Space and Nonlinear Partial Differential Equations

Mathematical Surveys and Monographs

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1991 *Mathematics Subject Classification.* Primary 47H15, 35A05;
Secondary 34G20, 35A15, 35J65, 35J85, 35K60, 35K65, 35K85, 47H05, 47H06, 47H20.

ABSTRACT. This book is about monotone operators and nonlinear semigroup theory and their application to partial differential equations. The primary readership is advanced graduate students in mathematics or engineering science and researchers in partial differential equations or related analysis.

Library of Congress Cataloging-in-Publication Data

Showalter, R. E. (Ralph Edwin), 1942–
Monotone operators in Banach space and nonlinear partial differential equations / R. E. Showalter.
p. cm. — (Mathematical surveys and monographs, ISSN 0076-5376 ; v. 49)
Includes bibliographical references and index.
ISBN 0-8218-0500-2 (alk. paper)
1. Monotone operators. 2. Semigroups. 3. Banach spaces. 4. Differential equations, Nonlinear. I. Title. II. Series: Mathematical surveys and monographs ; no. 49.
QA329.8.S56 1996
515′.353—dc20   96-43343
CIP

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established to ensure permanence and durability.

10 9 8 7 6 5 4 3 2 1   02 01 00 99 98 97

Contents

Preface

The objectives in this monograph are to present some topics from the theory of monotone operators and nonlinear semigroup theory which are directly applicable to the existence and uniqueness theory of initial-boundary-value problems for partial differential equations and to construct such operators as realizations of those problems in appropriate function spaces. A highlight of the presentation will be the large number and variety of examples which are introduced to illustrate this connection between the theory of nonlinear operators and partial differential equations. These include primarily semilinear or quasilinear equations of elliptic or of parabolic type, degenerate cases with change of type, related systems and variational inequalities, and spatial boundary conditions of the usual Dirichlet, Neumann, Robin or of dynamic type. The discussions of evolution equations include the usual initial-value problems as well as periodic or more general nonlocal constraints, history-value problems, those which may change type due to a possibly vanishing coefficient of the time derivative, and other implicit evolution equations or systems including hysteresis models. The scalar conservation law and semilinear wave equations are briefly mentioned, and hyperbolic systems arising from vibrations of elastic-plastic rods are developed. The origins of a representative sample of such problems is given in the Appendix.

This is the place to *begin* study of a particular problem. Once a proper setting has been established for a given problem to be well-posed, one can then proceed to investigate those properties of solutions which distinguish the problem, such as regularity, asymptotic behavior, numerical analysis, stability, special properties, controllability,.... None of these topics will be discussed here. The objective is rather to develop for the reader an instinct for the *right* place (or places) to look for a solution and the *right* techniques to use to establish existence-uniqueness in appropriate function spaces for a broad class of problems. Much attention has been devoted to develop the connection between the abstract theory and the specific problems for partial differential equations.

The work is arranged in four chapters and an appendix, and each of these is divided into numbered sections. All results are formally stated as Theorems, Propositions, Lemmas, or Corollaries which are independently numbered in the respective category by their section number and order within that section. Thus, in Section 4 of Chapter I the second Proposition is called Proposition 4.2, and it is referenced that way within Chapter I. From any other chapter it will be recalled as Proposition I.4.2. Most examples are named alphabetically within their section, so the third example of Section I.4 is Example 4.C, and from outside Chapter I it is called Example I.4.C.

The beginning chapter gives a casual but technically precise overview of the subject in the case of linear problems in one spatial dimension. Most of the major notions appear here in this simple setting, including the Lax-Milgram-Lions Theorem and the Hille-Yosida Theorem, and they are motivated by the classical Dirichlet and Neumann boundary-value problems and by various initial-boundary-value problems, respectively. The second and third chapters begin with the theory of monotone nonlinear operators from a reflexive Banach space to its dual and the solution of corresponding stationary or time-dependent problems with such operators. The remainder of each consists of the development of the applications to appropriate classes of problems. The fourth chapter is concerned with the theory of accretive operators in a single Hilbert or Banach space. Some very useful topics of convex analysis are developed independently in both the second and fourth chapters. Each chapter contains a wealth of examples of initial-boundary-value problems for partial differential equations to which the abstract results apply. The Appendix contains descriptions of the derivation of the partial differential equations and initial-boundary-value problems for heat transport, for flow through porous media, and for simple vibration problems of mechanics. These model problems begin from the most elementary considerations of the physics and proceed to many types of boundary conditions, the Stefan free-boundary problem, the porous medium equation, systems to describe diffusion in composite (fissured) media, and models of plasticity and hysteresis. These are intended to illustrate the variety of nonlinearities that are covered by the monotone theory.

There is much independence and intentional repetition between the four chapters, so one can frequently find a short path to a later section. Chapter I is not logically necessary for anything that follows, but every reader should at least look through it. For the less mathematically prepared it provides a quick but self-contained introduction to some linear functional analysis topics, and to the more seasoned reader it can serve as an orientation to the developments to follow. For one who will be satisfied with problems in one spatial dimension, such as two-point boundary-value problems and parabolic or wave equations in the plane, it contains the construction of the elementary function spaces necessary to bypass the technical Sections II.3 and II.4. A rather complete exposition of the linear version of this material is contained in Chapter I and the first three sections of Chapter III. Chapter III depends heavily on Chapter II. Chapter IV is essentially independent of all other chapters. It needs only the motivation from Section I.4, and Section II.9 is used in Section IV.9.

This work was written primarily for advanced graduate students in mathematics or engineering science, researchers in partial differential equations and related numerical analysis, applied mathematics, control theory or dynamical systems for whom a precise existence-uniqueness theory for initial-boundary-value problems is of interest. The completeness and level of the presentation should make the book much easier to read than most research papers on the subject. The primary prerequisite for the reader is a previous acquaintaince with L^p spaces and the notions of *continuous* and *linear*. The author has taught variations of this material to such audiences for the past 20 years, with classroom presentation and style modified accordingly for the specific group. This is the material that has proven to be most universally applicable and of interest to the students and seminar participants, many of whom had research interests outside of mathematics. It is not written as a traditional text, but rather as a guidebook, to lead the reader along the more accessible valley trails, to provide a view of the steep terrain of the higher peaks and an appreciation of those who travelled them before us.

No attempt has been made to include references to all the research papers in which this material was originally developed or to recount the history of these developments. The Bibliography contains references only to books which contain portions of this material, and one can consult these for further discussion of any one topic or for references to the various sources and versions of the history. If the reader develops here a sufficient appreciation of the mathematical structure and power of these methods to pursue one of these references, then this work will have served its purpose.

It is a pleasure to acknowledge a debt of personal gratitude to the people who contributed to the formulation and preparation of this book at its various stages. The text benefited substantially from the constructive comments of various students who read parts of the manuscript at a formative stage for both, and I take great pride in the success that many of them have experienced in their research. Betty Banner was responsible for the arduous metamorphosis of my scribbled notes into an early form of the manuscript. Margaret Combs graceously shared her immense TEXpertise and gave invaluable professional assistance at every stage of the preparation. Her enthusiasm in the final stages of the project contributed to its timely completion. Małgorzata Peszyńska carefully and patiently read the final version of the manuscript. The reader will benefit from her many helpful remarks and suggestions for improvement of the exposition. For myself, I claim sole responsibility for any remaining errors or offending omissions.

Finally, the author is grateful to the very professional and seemingly tireless staff of the National Science Foundation for the effective and productive support of the research reported here.

R.E. Showalter

PDE Examples by Type

First Order Equations

initial value problem	I.5.A, II.9.A, III.2.A, IV.2.D, IV.3, IV.9.A(SCL)
boundary value problem	I.4.B, II.9.A
systems, inequalities	A.3

Elliptic Equations

linear	I.3.A,B,E, I.4.C,D, II.9
semilinear	II.8.D,E,F, II.9.B, IV.2.E,F
quasilinear	II.5.A,B,C,D,E, II.6
inequalities	I.3.C,D, II.6.A,B,C,D
systems	II.5.F,G,H,I

Parabolic Equations

linear	I.5.B, II.2.B, A.1, A.2
semilinear	III.6.C (PME), IV.2.G, IV.6.B (PME), IV.7, IV.9.B,C,D (PME), A.1, A.2
quasilinear	III.4.A,B, III.6.A, IV.5.A,B, A.2
inequalities	III.7.A,B,C,D, IV.5.B, IV.9.C, A.1
degenerate	III.3.A,B, III.6.B, IV.6.A (FME), IV.6, A.1
systems	III.5, III.7.D, IV.9.C,D, A.1, A.2, A.3

Wave Equations

linear	I.6
semilinear	IV.6, A.3
systems	A.3

Appendix

A.1 Heat Conduction

Consider the diffusion of heat energy through a medium G in $\mathbb{R}^m$. For simplicity we shall assume that the medium is isotropic: its properties are independent of direction. The first experimental observation is that the heat energy in any subregion $G_0 \subset G$ is proportional to the mass of G_0 and its temperature. If we denote by $\rho(x)$ the density and by $u(x,t)$ the temperature at $x \in G$ and the time $t > 0$, then this assumption means that the heat energy in G_0 due to its temperature is given by $\int_{G_0} \rho(x)c(x)u(x,t)\,dx$, and this linear relationship defines the *heat capacity* or *specific heat* $c(x)$ of this material. Thus, $c(x)$ is the ratio of an increment in the heat energy of a unit mass to a corresponding increment in temperature at the point x. This assumption is good if the variations of temperature are not excessively large.

Fourier's law for heat conduction states that the rate at which heat energy flows through a surface element S is proportional to the surface area and to the temperature gradient in the direction normal to that surface. Thus, if we let $\vec{n}$ denote the unit normal on the surface S we have the total heat flow rate

$$-\int_S k(s)\vec{\nabla}u(x,t)\cdot\vec{n}\,ds$$

across S in the direction of $\vec{n}$. This defines the *conductivity*, $k(x)$; the *heat flux* is just $\vec{J}(x,t) = -k(x)\vec{\nabla}u(x,t)$, the vector surface density rate of heat flow. If the material is non-isotropic, then a temperature gradient will induce a heat flux which is not necessarily parallel ; in that case conductivity is given as an $n \times n$ matrix K by $\vec{J}(x,t) = -K(x)\vec{\nabla}u(x,t)$. Thus, a unit temperature gradient $\vec{e}_j$ in the j direction induces the heat flow $-(K_{1,j}, K_{2,j}, \ldots, K_{j,j})$. The matrix K is positive-definite and symmetric.

The *conservation of energy* shows that we have

$$\frac{\partial}{\partial t}\int_{G_0} \rho(x)c(x)u(x,t)\,dx = \int_{\partial G_0} k(x)\vec{\nabla}u(x,t)\cdot\vec{n} + \int_{G_0} f(x,t)\,dx$$

for every $G_0 \subset G$ and $t > 0$, where $f(x,t)$ denotes the volume-distributed heat sources in G and $\vec{n}$ is the unit outward normal. From Gauss' theorem it follows that

$$\int_{G_0} \frac{\partial}{\partial t}(\rho(x)c(x)u(x,t))\,dx = \int_{G_0} \vec{\nabla}\cdot k(x)\vec{\nabla}u(x,t)\,dx + \int_{G_0} f(x,t)\,dx$$

for every such $G_0 \subset G$, so we obtain the linear *heat equation*

$$\text{(1.1.a)} \qquad \frac{\partial}{\partial t}\rho(x)c(x)u(x,t) - \vec{\nabla}\cdot k(x)\vec{\nabla}u(x,t) = f(x,t) \text{ in } G\ .$$

Surface Discontinuity. The above accounts for energy balance when first-order derivatives of the temperature are continuous in the interior of G. Now let S be an (n-1)-dimensional surface in G , and suppose we have a solution which is smooth except along S. Note that since temperature is obtained as an average over some small generic neighborhood, it is always expected to be continuous. However, $\vec{\nabla} u$ is not necessarily continuous. In fact, it is the *flux* $k(x)\vec{\nabla} u(x,t) \cdot \vec{n} = k(x)\frac{\partial u}{\partial n}$ across the surface S with normal $\vec{n}$ which we expect to be continuous. These two observations yield the *interface conditions*

$$u(x_+) = u(x_-), \quad k(x_+)\frac{\partial u}{\partial n}(x_+) = k(x_-)\frac{\partial u}{\partial n}(x_-)$$

where x_+ and x_- denote the limiting values from the two sides at $x \in S$. It follows that the directional derivative $\frac{\partial u}{\partial n}$ is continuous exactly when $k(x)$ is continuous. In particular, if we have a non-homogeneous medium in which the material coefficient $k(x)$ is discontinuous across a surface, S, then the temperature $u(x,t)$ is continuous across S but the derivative $\frac{\partial u}{\partial n}$ will necessarily have a jump.

Boundary Conditions. Next we consider the conditions on the boundary of the region. One could measure the temperature on a portion Γ_0 of the boundary, $\Gamma \equiv \partial G$. Then we have the *Dirichlet boundary condition*

$$u(s,t) = h(s,t), \quad s \in \Gamma_0 \ ,$$

where $h(s,t)$ is known. One could likewise measure the heat flow across a part Γ_1 of Γ, so we obtain the *Neumann boundary condition*

$$k(s)\vec{\nabla} u(s,t) \cdot \vec{n} = g(s,t), \quad s \in \Gamma_1 \ .$$

This condition also describes the situation in which there is a heat source of density $g(x,t)$ distributed along Γ_1. More generally, one can assume the heat flux on the boundary is also driven by the difference between inside and outside temperatures,

$$\text{(1.1.b)} \qquad k(s)\vec{\nabla} u(s,t) \cdot \vec{n} + k_0(s)(u(s,t) - h(s,t)) = g(s,t), \quad s \in \Gamma_1 \ .$$

This is called a *Robin boundary condition*, and it corresponds to *Newton's law* of cooling on the surface: the heat loss rate is proportional to the temperature difference across the surface. It is a *discrete* Fourier law in which $k_0(s)$ is the surface-distributed conductivity. For example, if the boundary is insulated along Γ_1, then we have $k_0(s) = 0$. Note that the first two types of boundary condition are obtained formally as the special cases $k_0 \to +\infty$ and $k_0 \to 0$, respectively, and the third type of boundary condition results from intermediate values for $k_0(s)$.

A problem with a similar formal structure arises when we permit the boundary to consist of a material with the specific heat given by $c_0(\cdot)$. If we assume that the heat energy is supplied to the boundary by the flux from the interior, in addition to that from a source distribution on the boundary, F_0, we are led to the *dynamic boundary condition*

$$\text{(1.1.b}'\text{)} \qquad \frac{\partial}{\partial t}\rho_0(s)c_0(s)u(s,t) + k(s)\vec{\nabla} u(s,t) \cdot \vec{n} \ni F_0(s,t) \ , \qquad s \in \Gamma \ ,$$

for $t > 0$. If we also consider the local diffusion in the tangential coordinates of the boundary, then we obtain the corresponding elliptic Laplace-Beltrami operator in (1.1.b') for the manifold Γ. The appropriate initial-boundary-value problem in each case is to seek a solution of the heat equation (1.1.a) subject to a boundary condition (1.1.b) or (1.1.b') and an initial condition

$$\rho(x)c(x)u(x,0) = \rho(x)c(x)u_0(x), \quad x \in G\ , \tag{1.1.c}$$

and, additionally, in the case of (1.1.b'),

$$\rho_0(s)c_0(s)u(s,0) = \rho_0(s)c_0(s)u_0(s), \quad s \in \Gamma\ , \tag{1.1.c$'$}$$

hence, the initial energy is specified.

Nonlinear Models.

Here we assume that the density is constant, and we normalize it to unity by an appropriate change of variable. If large variations in the temperature are considered, then it may be appropriate to permit the material properties to be temperature dependent. That is, we could consider *semilinear* equations of the form

$$\frac{\partial}{\partial t}c(u) - \Delta k(u) = f(x,t) \tag{1.2}$$

for which we have $\frac{\partial}{\partial t}c(u) = c'(u)\frac{\partial u}{\partial t}$, where $c'(u)$ denotes the specific heat at the temperature u, and $-\Delta k(u) = -\vec{\nabla}\cdot k'(u)\vec{\nabla}u$, so $k'(u)$ is the corresponding conductivity. Since the specific heat and conductivity are positive quantities, it follows that the corresponding functions, $c(\cdot)$ and $k(\cdot)$, are both *monotone.* By means of a change of variable, we can eliminate either one of $c(\cdot)$ or $k(\cdot)$ with no loss of generality. The resulting equation with either of $c(\cdot)$ or $k(\cdot)$ is called the *porous medium equation.* Nonlinear boundary conditions could also be considered. Relevant examples include those of the form

$$\frac{\partial}{\partial t}c_0(u) + \vec{\nabla}k(u(s,t))\cdot\vec{n} + k_0(u(s,t) - h(s,t)) = g(x,t)\ , \qquad s \in \Gamma \tag{1.3}$$

in which $k_0(\cdot)$ and $c_0(\cdot)$ are also monotone functions. Equations of the form (1.2) arise in a variety of applications, and it is common to have *singular* or *degenerate* situations in which $c'(u) = 0$ or $k'(u) = 0$ for values of u in some interval. Next we discuss an example in which monotone but multi-valued *graphs* or *relations* appear. These are dual to the preceding situation, since portions of the graphs are vertical rather than horizontal.

The Stefan Problem. We consider a model of heat diffusion in the situation where there is a *phase change* arising from the freezing or thawing of the medium at a fixed temperature. We begin with the description of the phase relation between energy and temperature. Consider a unit volume of ice at temperature $u < 0$ and apply a uniform heat source of intensity F, so $e \equiv Ft$ is the accumulated *internal energy* after t units of time. The temperature increases according to the monotone function $e = c(u)$ until it reaches the value $u = 0$; the positive function $c'(u)$ is the *specific heat.* Then the temperature remains at $u = 0$ until L units of additional heat have been added; $L > 0$ is the *latent heat.* During this period there is a fraction w of water coexisting with the ice, and w increases at the constant rate F/L. The *water fraction* w, $0 \le w \le 1$, is the phase variable. After all the ice has melted, $w = 1$ and the temperature u begins to rise again according to $e = c(u) + L$. If the process is reversed by drawing heat out of the unit volume, the temperature falls according to $e = c(u) + L$ until it reaches $u = 0$, then w decreases until it reaches

$w = 0$, and thereafter the temperature u falls with $e = c(u)$. Note that the freezing and the melting took place at $u = 0$. This is the situation which leads to the traditional *Stefan problem*. The relation between energy and temperature is given in terms of the Heaviside graph by $e \in c(u) + LH(u)$. (Recall that the *Heaviside graph* $H(\cdot)$ is defined by $H(u) = \{0\}$ if $u < 0$, $H(0) = [0,1]$, and $H(u) = \{1\}$ if $u > 0$.) The energy is then given in the form $e = c(u) + Lw$, with $w \in H(u)$.

We shall formulate a free-boundary problem which describes heat conduction through a domain G in Euclidean space R^m subject to the constitutive assumptions above on the phase relation between energy and temperature. This is called the *Stefan problem*. Denote the boundary of G by ∂G and set $\Omega = G \times (0, \infty)$. The temperature at the point $x \in G$ and the time $t > 0$ is $u(x,t)$ and the smooth monotone functions $c(u)$, $k(u)$ are given with $c(0) = k(0) = 0$; their derivatives $c'(u)$, $k'(u)$ denote the *specific heat* and *conductivity*, respectively, of ice-water at temperature u. The phase change between water and ice occurs at $u = 0$. The space-time region Ω is then separated into an ice region Ω_- where $u < 0$, a water region Ω_+ where $u > 0$, and a region Ω_0 where $u = 0$ and in which the phase depends on it's preceding history.

Let $w(x,t)$ be the fraction of water at $(x,t) \in \Omega$, and note that according to our constitutive assumptions above we have $w \in H(u)$, the Heaviside graph. The energy is given by $e = c(u) + Lw$. Let S_- be the boundary of Ω_- in Ω and S_+ the boundary of Ω_+ in Ω. The *unit normal* $N = (N_1, \dots, N_m, N_t)$ on $S_- \cup S_+$ is oriented out of Ω_- and into Ω_+, and hence it is consistent on $S_- \cap S_+$. We shall denote by $[g]$ the *saltus* or jump in values of the function g across the boundaries, S_- and S_+, in the direction of N : for $(x,t) \in S_- \cup S_+$

$$\big[g(x,t)\big] = \lim_{h \to 0^+} \Big\{ g\big((x,t) + hN\big) - g\big((x,t) - hN\big) \Big\} .$$

The strong form of the *Stefan problem* is to find a pair of functions u and w on Ω for which

$$\frac{\partial}{\partial t} c(u) - \Delta k(u) = F(x,t) \text{ in } \Omega_- \cup \Omega_+ , \tag{1.4.a}$$

$$\frac{\partial}{\partial t} Lw = F(x,t) \text{ in } \Omega_0 , \quad w \in H(u) \text{ in } \Omega , \tag{1.4.b}$$

$$\big[\nabla k(u)\big] \cdot (N_1, \dots, N_m) = LN_t[w] \text{ on } S_- \cup S_+ , \tag{1.4.c}$$

$$u(x,0) = u_0 , \quad x \in G , \tag{1.5.a}$$

$$w(x,0) = w_0(x) \in [0,1] , \text{ where } u_0(x) = 0 , \tag{1.5.b}$$

$$u(s,t) = 0 , \quad s \in \partial G , \quad t > 0 , \tag{1.6}$$

where a distributed source $F \in L^1(\Omega)$ is given in addition to $c(\cdot)$, $k(\cdot)$, L, u_0, w_0. The classical (possibly nonlinear) heat equation (1.4.a) determines the temperature where $u \neq 0$. The water fraction is given by the relation (1.4.b) which was described above, so we have $w = 0$ in the ice, Ω_-, $w = 1$ in the water, Ω_+, and w is between 0 or 1 in the *mushy region*, Ω_0. Let n be the unit vector in the direction $(N_1, \dots, N_m)$, and let V be the *velocity* of S_- or S_+ at time t in the direction of n. By dividing (1.4.c) by $(N_1^2 + \cdots + N_m^2)^{1/2}$, we obtain

$$\left[\frac{\partial}{\partial n} k(u)\right] + LV[w] = 0 \text{ on } S_- \cup S_+,$$

and this is equivalent to (1.4.c). It means the difference in heat flux across the *free boundary* S_+ determines the velocity V of that boundary by melting the fraction of ice $1 - w = [w]$ with latent heat L, and similarly the velocity of S_- is determined by the freezing of the fraction of water $w = [w]$. The Dirichlet boundary condition (1.6) is used here for simplicity, but any of the usual types can just as easily be attained.

In order to obtain a *weak formulation* of the Stefan problem, we consider a solution u, w of (1.4) for which u is smooth in each of Ω_-, Ω_0, Ω_+, and the discontinuities in flux supply the energy to drive the surfaces S_- and S_+ according to (4.c). We begin by computing $\frac{\partial e}{\partial t} - \Delta k(u)$ in the sense of distributions on Ω. Thus, for $\varphi \in C_0^\infty(\Omega)$ we have

$$\left\langle \frac{\partial e}{\partial t} - \Delta k(u), \varphi \right\rangle \equiv \int_\Omega \Big\{ -\big(c(u) + Lw\big)\varphi_t - k(u)\Delta\varphi \Big\} .$$

Since $k(u)$ is smooth in each region, we have

$$\begin{aligned} &\int_\Omega \Big\{ -\big(c(u) + Lw\big)\varphi_t + \nabla k(u) \cdot \nabla\varphi \Big\} \\ &= \int_{\Omega_-} \{ -c(u)\varphi_t + \nabla k(u) \cdot \nabla\varphi \} + \int_{\Omega_0} \Big\{ -(Lw)\varphi_t \Big\} \\ &\quad + \int_{\Omega_+} \Big\{ -\big(c(u) + L\big)\varphi_t + \nabla k(u) \cdot \nabla\varphi \Big\} \end{aligned}$$

From Gauss' theorem we can write these three successive integrals in the respective forms

$$\begin{gathered} \int_{\Omega_-} (c(u)_t - \Delta k(u))\,\varphi + \int_{S_-} \{ \nabla k(u) \cdot (N_1, \ldots, N_m) \}\,\varphi , \\ \int_{\Omega_0} (Lw)_t\,\varphi + \int_{S_-} \{ (Lw)\,N_t \}\,\varphi - \int_{S_+} \{ (Lw)\,N_t \}\,\varphi , \text{ and} \\ \int_{\Omega_+} (c(u)_t - \Delta k(u))\,\varphi - \int_{S_+} \{ \nabla k(u) \cdot (N_1, \ldots, N_m) - LN_t \}\,\varphi . \end{gathered}$$

By adding these we obtain

$$\begin{aligned} \left\langle \frac{\partial e}{\partial t} - \Delta k(u), \varphi \right\rangle &= \int_{\Omega_- \cup \Omega_0 \cup \Omega_+} \big((c(u) + Lw)_t - \Delta k(u) \big)\,\varphi \\ &\quad + \int_{S_- \cup S_+} \big(-[\nabla k(u)] \cdot (N_1, \ldots, N_m) + L[w]N_t \big)\,\varphi . \end{aligned}$$

Hence, it follows that (1.4) is equivalent to

$$\frac{\partial}{\partial t}\,(c(u) + Lw) - \Delta k(u) = F(x,t) , \quad w \in H(u) , \tag{1.7}$$

where w, u are appropriately smooth. We have shown that the weak form of the Stefan problem is to find a pair $u \in L^1(\Omega)$, $w \in L^\infty(\Omega)$ for which (1.7) and (1.5) hold, and $k(u) \in L^2(0,T; H_0^1(G))$. This last condition implies (1.6) if $k'(0) > 0$.

We close with some remarks on the enthalpy functional (1.4.b). The simple relation $H(u)$ described above is an idealization. Specifically, during the phase change the temperature u does not necessarily remain exactly constant but may increase at a very small rate. Thus, it is reasonable to replace the Heaviside relation by a (single-valued) monotone function which closely approximates it. Also, if one could manage to force temperatures past the phase-change temperature, the phase w would not be expected to respond instantly, but only at a very high rate. We can easily accommodate both of these modifications, and the latter lead to various *dynamic models* of phase change.

Hysteresis Models. The use of multi-valued graphs permits a very elegant treatment of a class of parabolic problems with *hysteresis*, and we shall describe an example of the form

$$\frac{\partial}{\partial t}(a(u) + \mathcal{H}(u)) - \Delta u = f \tag{1.8}$$

in which $\mathcal{H}$ denotes a *hysteresis* functional, that is, its value depends not only on the current value of the input, u, but also on the *history* of the input in a very nonlinear way.

We first consider an elementary but fundamental example of hysteresis. It depends on three parameters, α, β, and ϵ, with $0 < \epsilon$, $\alpha < \beta$. Denote by $[x]_+$ and $[x]_-$, respectively, the positive and negative parts of the real number x. Let's look at the response $w(\cdot)$ that arises from a given input, $u(\cdot)$. The output $w = \mathcal{H}_{\alpha,\beta,\epsilon}(u)$ varies according to the following: if $u > \beta + \epsilon$, then $w = 1$; if $u < \alpha$, then $w = 0$; if $\alpha < u < \beta + \epsilon$, then $0 \leq w \leq 1$ and

$$w'(t) = \begin{cases} \left[\dfrac{u'(t)}{\epsilon}\right]_+ & \text{if } w = \dfrac{u-\beta}{\epsilon}\,, \\ 0 \text{ if } \dfrac{u-\beta}{\epsilon} < w < \dfrac{u-\alpha}{\epsilon}\,, \\ \left[\dfrac{u'(t)}{\epsilon}\right]_- & \text{if } w = \dfrac{u-\alpha}{\epsilon}\,. \end{cases}$$

For example, suppose we start with $u(0) = 0$ and $w(0) = 0$. As long as $u(t)$ remains between α and β, the output w stays constant at the value $w(t) = 0$. If $u(t)$ increases to β and then beyond $\beta + \epsilon$, then $w(t)$ increases to $+1$ where it remains until $u(t)$ gets down to $\alpha + \epsilon$. If $u(t)$ decreases below α, then $w(t)$ will drop to 0 and remain there until $u(t)$ again reaches β, and so on. It is clear that the output $w(t)$ depends not only on the present value but also on the history of the input $u(s)$ for previous times, $0 < s < t$. The limiting case obtained from $\epsilon \to 0$ is the *relay* hysteresis functional $\mathcal{H}_{\alpha,\beta}$. When $\alpha = \beta = 0$ this reduces further to the *Heaviside graph*.

In order to see how an ordinary differential equation with constraint can produce such a hysteresis functional, we consider a somewhat more general situation. Let a maximal monotone graph $b(\cdot)$ be given; this hysteresis model will be of the type *generalized play* described by horizontal translates of $w \in b(u)$. The hysteresis functional $\mathcal{H}_{\alpha,\beta,\epsilon}$ described above is given by the choice $b = H_\epsilon$, where

$$H_\epsilon(r) = \begin{cases} 1 & \text{if } r \geq \epsilon \\ \frac{r}{\epsilon} & \text{if } o < r < \epsilon \\ 0 & \text{if } r \leq 0\,. \end{cases}$$

Introduce a new variable, v, to represent the *phase* constraints:

$$w \in b(v), \quad u - \beta \leq v \leq u - \alpha \, .$$

We use the *sign function* (or graph) sgn to realize these constraints. Recall that it is defined by $\text{sgn}(x) = 1$ if $x > 0$, $\text{sgn}(x) = -1$ if $x < 0$, and $\text{sgn}(0) = [-1, 1]$. We shall use its scaled version, $\text{sgn}_{\alpha,\beta}$, defined by $\text{sgn}_{\alpha,\beta}(x) = \beta$ if $x > 0$, $\text{sgn}_{\alpha,\beta}(x) = \alpha$ if $x < 0$, and $\text{sgn}_{\alpha,\beta}(0) = [\alpha, \beta]$. Thus, let $u(t)$ be a time-dependent input to this generalized play model, and let $w(t)$ be the corresponding output or response. There is at each time t a corresponding phase variable $v(t)$ which is related to $w(t)$ and $u(t)$ as above, and so it is required that $w(t)$ be non-decreasing when $v(t) = u(t) - \beta$, non-increasing when $v(t) = u(t) - \alpha$, and stationary ($w'(t) = 0$) in the interior region where $u - \beta < v < u - \alpha$. This is equivalent to requiring that $w(t), v(t)$ satisfy

$$w(t) \in b(v(t)) \quad , \quad w' + \text{sgn}^{-1}_{-\beta,-\alpha}(v(t) - u(t)) \ni 0 \, .$$

Thus, we are led to ordinary differential equations of the form

$$w(t) \in b(v(t)) \quad , \quad w'(t) + c(v(t) - u(t)) \ni 0,$$

with maximal monotone graphs $b(\cdot)$ and $c(\cdot)$, as models of hysteresis in which the output is the solution $w(t)$ with input $u(t)$.

We have described above the *relay hysteresis* functional which is determined by the sign graph and the two parameters $\alpha < \beta$. These translation parameters are the switching positions of the relay, and we denote the output of this relay by $w = \mathcal{H}_{\alpha,\beta}(u)$. This operator can be represented by a rectangular loop which is the input-output graph of the relay which switches up to the value $w = 1$ at $u = \beta$ and down to the value $w = -1$ at $u = \alpha$. When this hysteresis functional is combined with the nonlinear diffusion equation as above, we have a system of the form

$$(1.9.a) \quad \frac{\partial}{\partial t} a(u(x,t)) - \Delta u(x,t) - c(v(x,t) - u(x,t)) \ni f(x,t) \; , \quad x \in G \, , \; t \in (0,T] \; ,$$

$$(1.9.b) \quad \frac{\partial}{\partial t} b(v(x,t)) + c(v(x,t) - u(x,t)) \ni g(x,t) \; ,$$

$$(1.9.c) \quad -\frac{\partial}{\partial \nu} u(s,t) \in d(u(s,t)) \; , \quad s \in \partial\Omega \; ,$$

with $b(\cdot) = H(\cdot)$, $c(\cdot) = \text{sgn}^{-1}_{-\beta,-\alpha}(\cdot)$ and $g(x,t) = 0$. This is a degenerate parabolic system of coupled equations with Neumann type boundary conditions for which the initial conditions $a(u(x,0))$ and $b(v(x,y,0))$ are to be specified, and it realizes the parabolic equation (1.8) with the indicated relay hysteresis. We can show that the dynamics of problem (1.9) is determined by a nonlinear semigroup of contractions on the Banach space $L^1(G) \times L^1(G)$.

The Super-Stefan Problem. We extend the Stefan problem to include hysteresis effects that result from the assumption that the melting and the freezing temperatures are *different*. The ice melts at a slightly positive temperature and freezing occurs after a slightly negative temperature is reached.

We begin with the description of the phase relation between energy and temperature. Let α and β be given numbers with $\alpha < 0 < \beta$. Begin with a unit volume of ice at temperature $u < \alpha$ and apply the heat source F. The temperature increases according to the relation $e = c(u)$ until it reaches the value $u = \beta > 0$

where it remains until L units of additional heat have been added. During this period the fraction w of water increases at the rate F/L. After all the ice has melted, $w = 1$ and the temperature u begins to rise again according to $e = c(u) + L$. If the process is reversed, the temperature falls according to $e = c(u) + L$ until it reaches $u = \alpha < 0$, then w decreases until it reaches $w = 0$, and thereafter the temperature u falls with $e = c(u)$. Note that the freezing took place at $u = \alpha$ and the melting at $u = \beta$. If $\alpha = \beta$ this is just the traditional Stefan problem. However this model permits superheated ice and supercooled water, and it is here that hysteresis occurs.

We formulate the corresponding free-boundary problem, the *Super-Stefan problem*. The phase change from water to ice occurs at $u = \alpha < 0$ and from ice to water at $u = \beta > 0$. The space-time region Ω is then separated into an always-ice region Ω_- where $u < \alpha$, an always-water region Ω_+ where $u > \beta$, and a region Ω_0 where $\alpha \le u \le \beta$ and in which the phase depends on it's preceding history. According to our constitutive assumptions, the water fraction satisfies $w \in \mathcal{H}_{\alpha,\beta}(u)$, and the energy is given by $e = c(u) + Lw$. Let S_- be the boundary of Ω_- in Ω and S_+ the boundary of Ω_+ in Ω. The *unit normal* $N = (N_1, \dots, N_m, N_t)$ on $S_- \cup S_+$ is oriented out of Ω_- and Ω_+, and hence into Ω_0. The strong form of the problem is to find a pair of functions u and w on Ω for which

$$\frac{\partial}{\partial t} c(u) - \Delta k(u) = 0 \text{ in } \Omega_- \cup \Omega_0 \cup \Omega_+ \ , \tag{1.10.a}$$

$$w \in \mathcal{H}_{\alpha,\beta}(u) \text{ in } \Omega \ , \tag{1.10.b}$$

$$\big[\nabla k(u)\big] \cdot (N_1, \dots, N_m) = L N_t [w] \text{ on } S_- \cup S_+ \ , \tag{1.10.c}$$

$$u(x,0) = u_0 \ , \quad x \in G \ , \tag{1.11.a}$$

$$w(x,0) = w_0(x) \in [0,1] \ , \text{ where } \alpha \le u_0(x) \le \beta \ , \tag{1.11.b}$$

$$u(s,t) = 0 \ , \quad s \in \partial G \ , \quad t > 0 \ . \tag{1.12}$$

One can show as before that the *weak form* of the Super-Stefan problem is to find a pair $u \in L^1(\Omega)$, $w \in L^\infty(\Omega)$ for which

$$\frac{\partial}{\partial t}\left(c(u) + Lw\right) - \Delta k(u) = 0 \ , \quad w \in \mathcal{H}_{\alpha,\beta}(u) \ , \tag{1.13}$$

and $k(u) \in L^2(0,T; H_0^1(G))$. With the appropriate change of variable, this is just the equation (1.8) with the simple relay hysteresis.

Boundary Evolution System. Next we describe a problem with the same formal structure as (1.9), a degenerate-parabolic initial boundary value problem for $t > 0$

$$\frac{\partial}{\partial t} a(u) - \Delta u \ni f \ , \qquad x \in G \ , \tag{1.14.a}$$

$$\frac{\partial}{\partial t} b(v) + \frac{\partial u}{\partial \nu} \ni g \text{ and} \tag{1.14.b}$$

$$\frac{\partial u}{\partial \nu} \in c(v - u) \ , \qquad s \in \Gamma \tag{1.14.c}$$

with initial values specified at $t = 0$ for $a(u)$ and $b(v)$. At each $t > 0$, $u(t)$ is a function on the bounded domain G in $\mathbb{R}^m$ with smooth boundary Γ, and $v(t)$ is a function on Γ. Each of $a(\cdot), b(\cdot), c(\cdot)$ is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$. Thus, the system (1.14) consists of a generalized porous medium equation in the interior of G subject to a nonlinear dynamic constraint on the boundary.

A variety of boundary conditions is obtained in (1.14). For example, if $b \equiv 0$ we have an explicit *Neumann boundary condition*, and if $c \equiv 0$ it is homogeneous. If $b(0) = \mathbb{R}$ (i.e., $b^{-1} = 0$), then $v \equiv 0$ and we have a nonlinear Robin constraint, and if $c(0) = \mathbb{R}$ we get $v = u$ on Γ, and this satisfies the nonlinear *dynamic boundary condition*

$$\frac{\partial}{\partial t} b(u) + \frac{\partial u}{\partial \nu} \ni g. \tag{1.14.b$'$}$$

If $b(0) = c(0) = \mathbb{R}$ we have the homogeneous *Dirichlet boundary condition*. Note that (1.14.b) together with (1.14.c) can represent *boundary hysteresis*. The dynamics of the problem (14) is given by a nonlinear semigroup of contractions on the Banach space $L^1(G) \times L^1(\Gamma)$.

Composite Media.

We shall develop some models for the flow of heat in a *composite material* composed of two finely interwoven and (possibly) connected components. Note that it is impossible to satisfy both of these geometric constraints in $\mathbb{R}^2$. We treat the two components of the system as independent media with different physical parameters. The discontinuities in these parameter values across the common interface can be severe, with the ratios of their values in the two media being of some orders of magnitude. Consequently, the exact microscopic model, written as a classical interface problem, is numerically and analytically intractable. Thus one constructs models by averaging methods. We shall first construct the simplest such type of model for conduction, a *parallel flow* model. At each point of the region, we center a generic neighborhood, and then we compute two quantities at that point, the average temperature of the first material within that neighborhood and then the average temperature of the second material within that same neighborhood. Thus there are two temperatures identified with each point of the region, and each of these two temperature fields satisfies the classical heat conduction equation together with an appropriate coupling term that combines them into a system.

Consider now small temperature variations in a system consisting of two components as above. We let $u(x,t)$ denote the temperature of the first material (averaged over the neighborhood centered at x), and let $v(x,t)$ denote the corresponding temperature in the second material at x. In order to account for the exchange of heat between the two components, we assume the simplest (linear) coupling, namely, Newton's law: the heat flux between the components is proportional to the difference between their temperatures. This leads to the linear system

$$\begin{aligned} &\frac{\partial}{\partial t}\rho_1(x)c_1(x)u(x,t) - \vec{\nabla}\cdot k_1(x)\vec{\nabla}u(x,t) + k_0(u(x,t) - v(x,t) = f_1(x,t)\ , \\ &\frac{\partial}{\partial t}\rho_2(x)c_2(x)v(x,t) - \vec{\nabla}\cdot k_2(x)\vec{\nabla}v(x,t) + k_0(v(x,t) - u(x,t)) = f_2(x,t)\ , \end{aligned} \tag{1.15}$$

the simplest example of a *parallel flow model* which describes heat conduction in a composition of two media. The subscripts refer to the characteristic properties of the respective media in the two components, and k_0 is a conductivity for the volume distributed heat exchange between the components. The name *parallel flow* is used because this is the same system that arises from the description of heat flow in two parallel planes that are in contact.

Fissured Structures. We consider the modifications of the system (1.15) that are appropriate to model a fissured medium. The characteristics of a fissured structure are that the first component is a matrix which occupies a much larger volume than the fissure system which comprises the second component and that the matrix is much more resistant to heat conduction than is the conducting material which is imbedded in the pores of the matrix and more highly concentrated in the fissure system. As a consequence, most of the heat is conducted through the system of fissures, while storage of heat takes place primarily inside the matrix. Some part of the heat flow is through the matrix, but the primary flow is that from the matrix into the fissures followed by flow within the fissures. There will be some flow in the matrix: the temperatures at nearby parts of the matrix can influence each other directly, not just indirectly via the system of fissures. This effect can be substantially less prominent than the bulk flow of heat through the fractures, but it can have a noticeable effect when the connectivity of the matrix is sufficiently well developed or the flow through the fissure system is not so overwhelming. Suppose the composition is such that the first material occurs as a system of particles suspended within the second is such a way that they are isolated from one another. Then there is no direct diffusion from one of these particles to the neighboring particles, but only an exchange into the surrounding second material. We realize this situation in the system (1.15) by setting $k_1 = 0$. The resulting system consists of an ordinary differential equation coupled to a parabolic partial differential equation,

$$\frac{\partial}{\partial t}\rho_1(x)c_1(x)u(x,t) + k_0(u(x,t) - v(x,t)) = f_1(x,t) \ , \tag{1.16.a}$$

$$\frac{\partial}{\partial t}\rho_2(x)c_2(x)v(x,t) - \vec{\nabla}\cdot k_2(x)\vec{\nabla}v(x,t) + k_0(v(x,t) - u(x,t)) = f_2(x,t) \ , \tag{1.16.b}$$

and this is called a *first-order kinetic model.* The equation (16.a) serves as a memory term. In the model of heat flow in two parallel planes, the system (1.16) results from the assumption that the top layer consists of a fine distribution of isolated particles.

If we additionally assume that the second material is a connected component so thinly distributed throughout the composite that its volume is negligible, then this second component cannot store any substantial amount of heat energy, and we realize this assumption by setting $\rho_2 = 0$ to obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}\rho_1(x)c_1(x)u(x,t) + k_0(u(x,t) - v(x,t)) = f_1(x,t) \ , \\
&-\vec{\nabla}\cdot k_2(x)\vec{\nabla}v(x,t) + k_0(v(x,t) - u(x,t)) = f_2(x,t) \ .
\end{aligned}$$

This is a coupled ordinary-elliptic system in which all storage is in the first component and all diffusion is in the second component. We can eliminate v to obtain the *fissured medium equation*

$$\begin{aligned}
\frac{\partial}{\partial t}\rho_1(x)c_1(x)u(x,t) &- \vec{\nabla}\cdot k_2(x)\vec{\nabla}\big(u(x,t) + \frac{1}{k_0}\frac{\partial}{\partial t}\rho_1(x)c_1(x)u(x,t)\big) \\
&= \big(I - \frac{1}{k_0}\vec{\nabla}\cdot k_2(x)\vec{\nabla}\big)f_1(x,t) + f_2(x,t) \ .
\end{aligned}$$

This can also be written in the form

$$\left(I - \frac{1}{k_0}\vec{\nabla} \cdot k_2(x)\vec{\nabla}\right)\frac{\partial}{\partial t}\rho_1(x)c_1(x)u(x,t) - \vec{\nabla} \cdot k_2(x)\vec{\nabla}u(x,t)$$
$$= \left(I - \frac{1}{k_0}\vec{\nabla} \cdot k_2(x)\vec{\nabla}\right)f_1(x,t) + f_2(x,t)$$

which we recognize as the *Yosida approximation* with parameter $\alpha = \frac{1}{k_0}$ of the heat equation.

For the construction of parallel flow models of heat flow when the temperature variations are larger and consequently the components are described by the semilinear equations (1.2), we obtain the nonlinear system

$$\begin{aligned} &\frac{\partial}{\partial t}a_1(u) - \Delta k_1(u) + \tau(u - v) = f_1(x,t) \\ &\frac{\partial}{\partial t}a_2(v) - \Delta k_2(v) - \tau(u - v) = f_2(x,t) \end{aligned} \tag{1.17}$$

in which the coupling is described by the monotone function $\tau(\cdot)$, and the monotone functions $a_1(\cdot)$ and $a_2(\cdot)$ include the density. As a naturally occuring example of a *Stefan system*, one finds fissured media in which, for example, immobile water is the conductor and it can undergo a phase change. Then the first component is the water temperature in the matrix of porous material and the second component is water temperature in the system of fissures. Both will undergo a phase change, so one gets a Stefan problem in each component of the system (1.17). Furthermore, since each component acts on the other as a distributed source, there will always be mushy regions present. This system generalizes (1.9), and its dynamics is determined by a nonlinear semigroup of contractions on $L^1(G) \times L^1(G)$.

Distributed Microstructure Models. We describe an elementary example in which a *distributed microstructure* model arises very naturally. The problem concerns the following situation. A system of pipes is used to absorb heat from circulating hot water and then later to return this heat to colder water, thereby serving as an energy storage device. This system has two sources of singularity, a geometric one due to the high ratio between the pipe length and the cross section, and a material one due to the very different conduction properties of the two materials involved. We seek a model in which these two singularities are properly balanced in order to obtain a good description of the exchange process.

We begin by examining the heat exchange in a single pipe in such a system. The heat transport in the water is primarily convective, due to the unit velocity of the water. The transport in the walls of the pipe is purely conductive and relatively very slow. The cross-sections of the pipe are very small, and these must be properly scaled to accurately model the exchange of heat between them and the water. We introduce a small parameter $\varepsilon > 0$ to quantify this. A reference cell $Y = \{y = (y_1, y_2) \in \mathbb{R}^2 : |y_1|^2 + |y_2|^2 < 2^2\}$ defines the structure of the cross-sections, and we write the parts of this double component domain as $\overline{Y} = \overline{Y_1} \cup \overline{Y}_2$, with $Y_1 = \{y \in \mathbb{R}^2 : \|y\| < 1\}$ representing the internal flow region and $Y_2 = \{y \in \mathbb{R}^2 : 1 < \|y\| < 2\}$ the walls of the pipe. The boundary of Y_1 is the unit circle, Γ_{11}, and that of Y is a larger circle, Γ_{22}. The boundary of Y_2 is given by $\Gamma_2 = \Gamma_{11} \cup \Gamma_{22}$. By ν we denote the unit vector normal to Γ_2

which points in the direction *out* of Y_2. We shall represent the very small cross-sections by $Y_1^\varepsilon = \varepsilon Y_1 = \{z = (z_1, z_2) = \varepsilon(y_1, y_2) \in \mathbb{R}^2 : (y_1, y_2) \in Y_1\}$ and $Y_2^\varepsilon = \varepsilon Y_2 = \{z = \varepsilon y \in \mathbb{R}^2 : y \in Y_2\}$. Similarly, we denote the corresponding boundaries by $\Gamma_{11}^\varepsilon, \Gamma_{22}^\varepsilon$, and Γ_2^ε.

Let the interval $G = (0,1)$ denote the axis of the very narrow pipe in which water is flowing at unit velocity through the cross section Y_1^ε and for which Y_2^ε is the pipe wall, a material of relatively *very low* conductivity. This situation is described by the initial-boundary-value problem

$$\begin{aligned}
&\frac{\partial U^\varepsilon}{\partial t} + \frac{\partial U^\varepsilon}{\partial x} = \frac{\partial^2 U^\varepsilon}{\partial x^2} + \nabla_z \cdot \nabla_z U^\varepsilon(x,z,t) \ , && z \in Y_1^\varepsilon \ , \ x \in G \ , \\
&\frac{\partial U^\varepsilon}{\partial t} = \varepsilon^2 \big(\nabla_z \cdot \nabla_z U^\varepsilon(x,z,t) + \frac{\partial^2 U^\varepsilon}{\partial x^2}\big) \ , && z \in Y_2^\varepsilon, \ \ x \in G \ , \\
&\nabla_z U^\varepsilon \cdot \nu|_{Y_1^\varepsilon} = \varepsilon^2 \nabla_z U^\varepsilon \cdot \nu|_{Y_2^\varepsilon} \ , && z \in \Gamma_{11}^\varepsilon \ , x \in G \ , \\
&\nabla_z U^\varepsilon \cdot \nu = 0 \ , && z \in \Gamma_{22}^\varepsilon \ , x \in G \ ,
\end{aligned}$$

which is the *exact* ε*-model*. The ε^2-scaled conductivity permits *very high* gradients in $G \times Y_2^\varepsilon$. However, only those components in the cross-section direction are responsible for the exchange flux, and in order to see that it is balanced with the other transport terms we rescale with $z = \varepsilon y$ and $\nabla_z = \frac{1}{\varepsilon}\nabla_y$ to get

$$\begin{aligned}
&(1.18.\text{a}) \quad \frac{\partial U^\varepsilon}{\partial t} + \frac{\partial U^\varepsilon}{\partial x} = \frac{\partial^2 U^\varepsilon}{\partial x^2} + \frac{1}{\varepsilon^2}\nabla_y \cdot \nabla_y U^\varepsilon(x,y,t) \ , && y \in Y_1 \ , \ x \in G \ , \\
&(1.18.\text{b}) \quad \frac{\partial U^\varepsilon}{\partial t} = \nabla_y \cdot \nabla_y U^\varepsilon(x,y,t) + \varepsilon^2 \frac{\partial^2 U^\varepsilon}{\partial x^2} \ , && y \in Y_2 \ , \ x \in G \ , \\
&(1.18.\text{c}) \quad \nabla_y U^\varepsilon \cdot \nu|_{Y_1} = \varepsilon^2 \nabla_y U^\varepsilon \cdot \nu|_{Y_2} \ , && y \in \Gamma_{11} \ , x \in G \ , \\
&(1.18.\text{d}) \quad \nabla_y U^\varepsilon \cdot \nu = 0 \ , && y \in \Gamma_{22} \ , x \in G \ .
\end{aligned}$$

Now we can recognize the limit as $\varepsilon \to 0$. In Y_1 we get $U(x,y,t) = u(x,t)$ in the limit, i.e., it is independent of $y \in Y_1$. Average (1.18.a) over Y_1 and use (1.18.c) with Gauss' theorem to get an integral over Γ_{11}. This gives the following limiting form of the problem:

$$\begin{aligned}
&(1.19.\text{a}) \quad \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} - \frac{1}{|Y_1|}\int_{\Gamma_{11}} \nabla_y U \cdot \nu \ ds \ , && && x \in G \ , \\
&(1.19.\text{b}) \quad \frac{\partial U}{\partial t} = \nabla_y \cdot \nabla_y U(x,y,t) \ , && y \in Y_2 \ , && x \in G \ , \\
&(1.19.\text{c}) \quad U(x,s,t) = u(x,t) \ , && s \in \Gamma_{11} \ , && x \in G \ , \\
&(1.19.\text{d}) \quad \nabla_y U^\varepsilon \cdot \nu = 0 \ , && s \in \Gamma_{22} \ , && x \in G \ .
\end{aligned}$$

The integral term in (1.19.a) is the total heat lost to the pipe wall at the cross-section $x \in G^\varepsilon$, computed from the normal component of the heat gradient in the wall, and it is comparable to the transport terms representing convection and conduction along the pipe. This balances the heat gained in the water with its moderate gradients against that exchanged with the pipe where the gradients are very high.

This problem approximates the heat transport and the exchange between the water and the pipe walls. It is essentially the classical *heat recuperator* problem, and it is the earliest example of a *distributed microstructure model*. The *global* domain is the interval G. For each point $x \in G$ there is identified a *local cell*, Y_2, which is located at or identified with that point. In the exchange of heat with the global medium, this family of cells acts as a distributed source, and the global medium is coupled to the cell at that point only on its boundary, Γ_{11}. It provides a well-posed problem which is a good approximation to the apparently singular ε-problem. We see the apparently singular term in (1.18.a) arose in the rescaling from the singular geometry in the original problem, but it is balanced by the corresponding term in (1.18.b) without the small coefficient because of the flux condition (1.18.c). In particular, the choice of ε^2 as coefficient in (1.18.b) exactly balanced the competing singularities introduced by the geometry and the materials.

A.2 Flow in Porous Media

A porous medium G in $\mathbb{R}^m$ is filled with a fluid, either liquid or gas, and this fluid diffuses from locations of higher to those of lower pressure. We begin by modeling this situation. Let $p(x,t)$ denote the *pressure* of fluid at the point $x \in G$ and time $t > 0$, and denote the corresponding *density* by $\rho(x,t)$. The quantity of fluid in an element of volume V is $\int_V c(x)\rho(x,t)\,dx$, and this defines the *porosity* $c(x)$ of the medium at the point x. This is the volume fraction of the medium that is accessible to the fluid. The *flux* is the vector flow rate $\vec{J}(x,t)$, so the rate at which fluid moves across a surface element S with normal $\vec{\nu}$ is given by $\int_S \vec{J}(x,t)\cdot\vec{\nu}\,dS$. Then the *conservation of fluid* takes the integral form

$$\frac{\partial}{\partial t}\int_{G_0} c(x)\rho(x,t)\,dx + \int_{\partial G_0} \vec{J}\cdot\vec{\nu}\,dS = \int_{G_0} f(x,t)\,dx\ , \qquad G_0 \subset G\ ,$$

in which $f(x,t)$ denotes any volume distributed *source density*. When the flux and density are differentiable, we can write this in the differential form

$$\frac{\partial}{\partial t}c(x)\rho(x,t) + \vec{\nabla}\cdot\vec{J}(x,t) = f(x,t)\ , \qquad x \in G\ .$$

The statement that the flux depends on the pressure gradient is *Darcy's law*, and it takes the form

$$\vec{J}(x,t) = -\frac{k(x)}{\mu}\rho\vec{\nabla}p(x,t)\ .$$

This defines the *permeability* $k(x)$ of the porous medium. The value of k is a measure of the ease with which the fluid flows through the medium, and μ is the *viscosity* of the fluid. That is, $\frac{1}{k}$ is the *resistance* of the medium to flow, and μ is a corresponding property of the fluid. Finally, the type of fluid considered is described by the *equation of state*,

$$\rho = s(p)\ .$$

The function $s(\cdot)$ which relates the pressure and density is monotone, in fact, it is usually chosen to be strictly increasing. However in problems with *partial saturation*, it is necessary to let $s(\cdot)$ be a *graph*, since at $p = 0$ fluid may only partially fill the pores and thereby give an effective density within the interval $[0, \rho(0)]$ related to the fraction of fluid present. By substituting the appropriate quantities above we obtain

$$\frac{\partial}{\partial t}c(x)s(p(x,t)) - \vec{\nabla}\cdot\frac{k(x)}{\mu}\vec{\nabla}S(p(x,t)) = f(x,t)\ , \qquad x \in G\ ,\ t > 0\ ,$$

where we denote by $S(\cdot)$ the antiderivative of $s(\cdot)$. That is, we set $S(p) = \int_0^p s(r)\,dr$. If we introduce the variable $u = S(p)$ we obtain the *generalized porous medium equation*

$$(2.1) \qquad \frac{\partial}{\partial t} c(x)a(u(x,t)) - \vec{\nabla} \cdot \frac{k(x)}{\mu} \vec{\nabla} u(x,t) = f(x,t) \ , \qquad x \in G \ , \ t > 0 \ ,$$

with $a(\cdot) \equiv s(\cdot) \circ S^{-1}(\cdot)$.

The classical case is obtained by choosing an equation of state that is specific to the flow of gas. This corresponds to the choice of the function

$$s(p) = \rho_0 p^\alpha$$

in the equation of state where ρ_0 and α are positive constants with $\alpha \leq 1$. If the medium is homogeneous, so the porosity and permeability are independent of $x \in G$, then this leads to the *classical porous medium equation*

$$(2.2) \qquad \frac{\partial}{\partial t} c\rho - \frac{k}{\mu(\alpha+1)\rho_0^{m-1}} \Delta \rho^m = f \ ,$$

where $m = 1 + \frac{1}{\alpha}$. Other situations in which (.) arises include boundary layer theory, where $m = 2$, population models, interstellar diffusion of galactic civilizations, and certain problems of plasma physics. The nonlinearity satisfies $m > 1$ in all but the last of these examples, and there $0 < m < 1$. The situation with $m > 1$ is called the *slow diffusion* case, and $0 < m < 1$ is the *fast diffusion* case. In the former case disturbances have a finite speed of propagation, unlike the linear case $m = 1$, and any solution which initially has compact support will continue to have compact support for all later times.

The simplest situation is that of a *slightly compressible* fluid. Here the equation of state has the form $s(p) = \exp c_0\, p$ where $c_0 > 0$ is the *compressibility* of the fluid. Then (2.1) simplifies to the linear parabolic equation

$$(2.3) \qquad \frac{\partial}{\partial t} c\rho - \frac{k c_0}{\mu} \Delta \rho = f \ .$$

This is formally the same as the heat equation, i.e., it is (2.2) with $m = 1$.

If we assume that the Darcy Law has the nonlinear form

$$\vec{J} = -\frac{k(\rho \vec{\nabla} p)}{\mu} \rho \vec{\nabla} p$$

in which the permeability depends on the flux, then in terms of the variable $u = S(p)$ we obtain the *quasilinear porous medium equation*

$$(2.4) \quad \frac{\partial}{\partial t} c(x)a(u(x,t)) - \vec{\nabla} \cdot \frac{k(\vec{\nabla} u(x,t))}{\mu} \vec{\nabla} u(x,t) = f(x,t) \ , \qquad x \in G \ , \ t > 0 \ ,$$

with $a(\cdot) \equiv s(\cdot) \circ S^{-1}(\cdot)$ as before. Finally, if we specialize this to the case of a slightly compressible fluid, we obtain

$$(2.5) \quad \frac{\partial}{\partial t} c(x) c_0 u(x,t) - \vec{\nabla} \cdot \frac{k(\vec{\nabla} u(x,t))}{\mu} \vec{\nabla} u(x,t) = f(x,t) \ , \qquad x \in G \ , \ t > 0 \ .$$

Fissured Media. The flow of fluid through a fissured porous medium leads to some related systems. A fissured medium consists of a matrix of porous and permeable blocks of pores or *cells* which are isolated from each other by a highly developed system of *fissures* through which the bulk of the fluid transport occurs. Due to this separation of the cells by the fissure system, there is a negligible amount of transport directly from cell to cell. Another feature of fissured media is that the volume occupied by the fissures is considerably smaller than that occupied by the matrix of cells. Thus, most transport occurs in the fissures and the bulk of the storage takes place in the cells. The system is by nature very much unsymmetric in the structure and function of its components. Specifically, the fissure system provides the primary transport component and the cell system accounts for all storage. Thus, the exchange between the fissure system and the matrix of cells is an important component of the model.

The *double porosity model* is a classical description of diffusion in heterogeneous media. The idea is to introduce at each point in space a density, pressure or concentration for each component, each being obtained by averaging in the respective medium over a generic neighborhood sufficiently large to contain a representative sample of each component. The rate of exchange between the components must be expressed in terms of these quantities, and the resulting expressions become distributed source and sink terms for the diffusion equations in the individual components. Thus, one obtains a *system* of diffusion equations, one for each component. The classical linear double porosity model for the flow of slightly compressible fluid in a general heterogeneous medium consisting of two components is

$$
\begin{aligned}
&\frac{\partial}{\partial t} c_1 u_1 - \frac{k_1 c_0}{\mu} \Delta u_1 + \frac{1}{\alpha}(u_1 - u_2) = f\ , \\
&\frac{\partial}{\partial t} c_2 u_2 - \frac{k_2 c_0}{\mu} \Delta u_2 + \frac{1}{\alpha}(u_2 - u_1) = g\ .
\end{aligned}
\tag{2.6}
$$

The first equation describes the flow in fissures — regions of small relative volume but large permeability. The second describes the flow in the cell system — regions of large porosity or volume, but largely isolated from one another. Both of these equations are to be understood macroscopically; that is, these equations are obtained by averaging over neighborhoods containing a large number of cells (called "blocks" of pores) and fissures. Although the components of (2.6) are structured symmetrically, fissured media characteristics are modeled by very small coefficients c_1 and k_2.

For the fissured medium model, the coefficient c_1 is almost zero because the relative volume of the fissures is small, and $k_2 = 0$ because there is no direct flow from one block of pores to another, i.e., each cell is isolated from the adjacent cells by the fissure system. The last term on the left of each equation represents the exchange of fluid between the cells and the fissures. The parameter α represents the resistance of the medium to this exchange. When $\alpha = \infty$, no exchange flow is possible. An alternative interpretation is that $1/\alpha$ represents the degree of fissuring in the medium. When the degree of fissuring is infinite, the exchange flow encounters no resistance and $u_0 = u_1$. The external sources of fluid represented by f and g are located in the fissures and in the cells, respectively.

Consider the analogous quasilinear situation in which the permeability is flux-dependent. Then we obtain the system

$$
\begin{aligned}
&\frac{\partial}{\partial t}c_1(x)a(u_1(x,t)) - \vec{\nabla}\cdot\frac{k_1(\vec{\nabla}u_1(x,t))}{\mu}\vec{\nabla}u_1(x,t) + \tau(u_1-u_2) = f(x,t)\ ,\\
&\frac{\partial}{\partial t}c_2(x)a(u_2(x,t)) - \vec{\nabla}\cdot\frac{k(\vec{\nabla}u_2(x,t))}{\mu}\vec{\nabla}u_2(x,t) - \tau(u_1-u_2) = g(x,t)\ ,
\end{aligned}
\tag{2.7}
$$

in terms of the variables $u_1 = S(p_1)$ and $u_2 = S(p_2)$. The fluid exchange rate between the cells and fissures is assumed to be determined by a monotone function $\tau(\cdot)$ of the pressure difference and the density, so it is given by $\tau(\tilde{\rho}(p_1-p_2))$. Here we choose $\tilde{\rho}$ to be the *average* density on the pressure interval $[p_1,p_2]$, so we have

$$\tilde{\rho}(p_2-p_1) = \int_{p_1}^{p_2} s(r)\,dr = u_2 - u_1\ .$$

In this way we have obtained an exchange term which is a function of the difference of the components instead of a difference of functions of the two components. If we specialize this to the case of a slightly compressible fluid we obtain the system

$$
\begin{aligned}
&\frac{\partial}{\partial t}c_1(x)c_0(u_1(x,t)) - \vec{\nabla}\cdot\frac{k_1(\vec{\nabla}u_1(x,t))}{\mu}\vec{\nabla}u_1(x,t) + \gamma(u_1-u_2) = f(x,t)\ ,\\
&\frac{\partial}{\partial t}c_2(x)c_0(u_2(x,t)) - \vec{\nabla}\cdot\frac{k(\vec{\nabla}u_2(x,t))}{\mu}\vec{\nabla}u_2(x,t) - \gamma(u_1-u_2) = g(x,t)\ .
\end{aligned}
\tag{2.8}
$$

These provide examples of parabolic partial differential equations and systems with multiple nonlinearities as well as a variety of *types* of nonlinearity that can arise.

A.3 Continuum Mechanics

Our objective is to describe the formulation and the well-posedness of some classical problems of vibration of a body composed of visco-elastic or plastic material. Although these can easily be developed for a general body in R^m, we restrict our attention here for simplicity to the 1−dimensional case of longitudinal vibrations of a rod. The structure of the resulting system equations is the same, and these provide some interesting and instructive examples of evolution equations in Hilbert space.

Longitudinal vibrations. Consider the longitudinal vibrations in a long narrow cylindrical rod consisting of a material which is elastic and (possibly) viscous. We position the rod along the x−axis and identify it with the interval $[0,1]$ in $\mathbb{R}$. The rod stretches or contracts in the horizontal direction, and we assume that the vertical plane cross-sections of the rod move only horizontally. Denote by $u(x,t)$ the *displacement* in the positive direction from the point $x \in [0,1]$ at the time $t > 0$. The corresponding *displacement rate* or *velocity* is denoted by $v(x,t) \equiv u_t(x,t)$.

Let $\sigma(x,t)$ denote the local *stress*, the force per unit area with which the part of the rod to the left of the point x acts on the part to the right of x. For a section of the rod, $x_1 < x < x_2$, during the time interval, $t_1 < t < t_2$, the total impulse is given by

$$\int_{t_1}^{t_2} \big(\sigma(x_2,s) - \sigma(x_1,s)\big)\,ds\ .$$

If the density of the string at x is given by $\rho > 0$, the change in momentum of this section is just

$$\int_{x_1}^{x_2} \rho \left(u_t(x,t_2) - u_t(x,t_1)\right) dx \ .$$

If we let $F(x,t)$ denote any external applied force per unit of length in the positive $x-$direction, then we obtain from Newton's second law by equating the above that

$$\int_{x_1}^{x_2} \rho \left(u_t(x,t_2) - u_t(x,t_1)\right) dx = \int_{t_1}^{t_2} \left(\sigma(x_2,s) - \sigma(x_1,s)\right) ds + \int_{t_1}^{t_2}\int_{x_1}^{x_2} F(x,s)\, dx\, ds$$

for any such $x_1 < x_2$ and $t_1 < t_2$. For a sufficiently smooth displacement $u(x,t)$, the displacement therefore satisfies the *momentum equation*

$$\text{(3.1.a)} \qquad \frac{\partial^2}{\partial t^2}\left(\rho\, u(x,t)\right) - \frac{\partial}{\partial x}\sigma(x,t) = F(x,t) \ , \qquad 0 < x < 1 \ , \ t > 0 \ .$$

The *stress* $\sigma(x,t)$ is determined by the type of material of which the rod is composed and the amount by which the neighboring region is stretched or compressed, i.e., on the *elongation* or *strain* $\varepsilon(x,t) \equiv \frac{\partial u(x,t)}{\partial x}$ at the point x. In the simplest case we assume that $\sigma(x,t)$ is proportional to $\varepsilon(x,t)$, i.e., that there is a constant E called *Young's modulus* for which

$$\sigma(x,t) = E\,\varepsilon(x,t) \ .$$

The constant E is a property of the material, and in this case we say the material is purely *elastic*. A rate-dependent component of the stress-strain relationship arises when the force generated by the elongation depends not only on the magnitude of the elongation but also on the speed at which it is changed, i.e., on the *elongation rate*

$$\frac{\partial \varepsilon(x,t)}{\partial t} = \frac{\partial v(x,t)}{\partial x} \ .$$

The simplest such case is that of a *visco-elastic* material defined by the linear *constitutive equation*

$$\text{(3.1.b)} \qquad \sigma(x,t) = E\,\varepsilon(x,t) + \mu\,\frac{\partial \varepsilon(x,t)}{\partial t} \ ,$$

in which the material constant μ is the *viscosity* of the material.

The system (3.1) consisting of the momentum and constitutive equations together with appropriate boundary and initial conditions comprises a model for the simplest problems of elasticity theory, the one-dimensional case. Note that by substitution of (3.1.b) into (3.1.a), we obtain the *viscous wave equation*

(3.2)

$$\frac{\partial^2}{\partial t^2}\left(\rho\, u(x,t)\right) - \frac{\partial}{\partial x}\left(\mu\,\frac{\partial^2 u(x,t)}{\partial t \partial x} + E\,\frac{\partial u(x,t)}{\partial x}\right) = F(x,t) \ , \qquad 0 < x < 1 \ , \ t > 0 \ .$$

See Proposition I.6.1 and Corollary IV.6.3.

For the special case of $\mu = 0$ we first consider the system (3.1) in the form

$$\text{(3.3.a)} \qquad v_t - E^{\frac{1}{2}}\sigma_x = F(x,t) \ ,$$

$$\text{(3.3.b)} \qquad \sigma_t - E^{\frac{1}{2}} v_x = 0 \ .$$

(Here we have replaced σ by $E^{\frac{1}{2}}\sigma$ in order to obtain the symmetry.) We shall show that the dynamics is governed by a semigroup of contractions in a product of L^2

spaces. That is, the spatial part of this system is the realization of an *m-accretive* operator in the appropriate Hilbert space.

DEFINITION. The operator A is determined as follows: $[f,g] = A[v,\sigma]$ if

$$[f,g] \in L^2(0,1) \times L^2(0,1),\ [v,\sigma] \in H^1(0,1) \times H^1(0,1)\ ,$$

and

$$-E^{\frac{1}{2}}\sigma_x(x) = f(x)\ ,\ 0 < x < 1\ ,\ \sigma(1) = 0$$
$$-E^{\frac{1}{2}}v_x(x) = g(x)\ ,\ v(0) = 0\ .$$

Here we have chosen for illustration boundary conditions associated with the requirements of no motion at the left end and no force at the right end of the rod. It is easy to check that A is accretive in $L^2(0,1) \times L^2(0,1)$. In fact, we have $(A[v,\sigma],[v,\sigma])_{L^2\times L^2} = 0$. Define the Hilbert spaces

$$V = \{v \in H^1(0,1) :\ v(0) = 0\}\ , \qquad W = \{\sigma \in H^1(0,1) :\ \sigma(1) = 0\}\ .$$

The corresponding resolvent equation, $(I + A)[v,\sigma] = [f,g]$, is given by

$$v \in V : \quad v - E^{\frac{1}{2}}\sigma_x = f\ ,$$
$$\sigma \in W : \quad \sigma - E^{\frac{1}{2}}v_x = g\ .$$

If we eliminate v, we can write this as a single equation or *variational equality*. To be precise, this problem has the following variational form: find a function

$$\sigma \in W : \quad \int_0^1 (\sigma w + E^{\frac{1}{2}}(E^{\frac{1}{2}}\sigma_x + f)w_x\,dx = \int_0^1 gw\,dx\ , \qquad w \in W\ .$$

In particular, this is the characterization of the solution to the problem of minimizing the convex function

$$\Phi(\sigma) = \int_0^1 (\frac{1}{2}\sigma^2 + \frac{E}{2}\sigma_x^2\,dx - \int_0^1 (g\sigma - f\sigma_x)\,dx$$

over the space W, so it is known to have a solution by Theorem I.2.1A and, hence, $\mathrm{Rg}(I + A) = L^2(0,1) \times L^2(0,1)$. This shows that A is m-accretive, so Corollary I.5.2 and Corollary I.5.3 will show directly that there is a unique solution of (3.3) with

$$v \in C^1([0,T],V)\ , \quad \sigma \in C^1([0,T],W)\ .$$

The case with $\mu > 0$ is different. Here we have the corresponding system

$$v_t - E^{\frac{1}{2}}\sigma_x = F\ , \tag{3.4.a}$$

$$\sigma_t - E^{\frac{1}{2}}v_x - \frac{\mu}{E^{\frac{1}{2}}}\sigma_{xx} = \frac{\mu}{E^{\frac{1}{2}}}F_x\ , \tag{3.4.b}$$

and we cannot deduce from (3.4.b) that $v(t) \in H^1(0,1)$. Note that if we had $E = 0$ we could substitute (3.1.b) directly into (3.1.a) to obtain the *heat equation*

$$\rho\,v_t - \mu\,v_{xx} = F(x,t)$$

for which we have not only well-posedness of the appropriate initial-boundary-value problem but also some *regularizing effects*.

In order to take advantage of the possibility to substitute for the viscous term, we write the system (3.1) in the form

$$\frac{\partial}{\partial t}v - \sigma_x = F \ , \quad \sigma = E^{\frac{1}{2}}\sigma_1 + \mu^{\frac{1}{2}}\sigma_2 \ ,$$
$$\sigma_1 = E^{\frac{1}{2}}\frac{\partial u}{\partial x} \ , \quad \sigma_2 = \mu^{\frac{1}{2}}\frac{\partial v}{\partial x} \ .$$

This formulation displays the total stress as the sum of two components, the *parallel* addition of the elastic stress (corresponding to σ_1) with a purely viscous stress (corresponding to σ_2). By substituting the second back into the first equation and dropping the subscript on σ_1, we obtain the system

(3.5.a) $$v_t - E^{\frac{1}{2}}\sigma_x - \mu v_{xx} = F(x,t) \ ,$$

(3.5.b) $$\sigma_t - E^{\frac{1}{2}}v_x = 0 \ .$$

Note that we have moved the viscosity term from the second line of (3.4) to the first line of the system (3.5).

We shall show that the corresponding operator A is m-accretive in the space $H \equiv L^2(0,1) \times L^2(0,1)$. To this end, as well as to motivate our notation in the next section, we introduce the (unbounded) operator on $L^2(0,1)$

$$D = \frac{d}{dx} : V \to L^2(0,1)$$

and its *adjoint* operator

$$D^* = -\frac{d}{dx} : W \to L^2(0,1) \ .$$

DEFINITION. The operator A on $L^2(0,1) \times L^2(0,1)$ is determined as follows: $[f,g] \in A[v,\sigma]$ if

$$[f,g] \in L^2(0,1) \times L^2(0,1), \ [v,\sigma] \in V \times W \ ,$$

and

$$D^*\big(E^{\frac{1}{2}}\sigma + \mu Dv\big) = f \ ,$$
$$-E^{\frac{1}{2}}Dv = g \ .$$

Note that it is implicit in the definition that the total stress satisfies $E^{\frac{1}{2}}\sigma + \mu Dv \in W$. The corresponding resolvent equation, $(I + A)[v,\sigma] = [f,g]$, is given by

$$v \in V : \quad v + D^*\big(E^{\frac{1}{2}}\sigma + \mu Dv\big) = f \ ,$$
$$\sigma \in W : \quad \sigma - E^{\frac{1}{2}}Dv = g \ .$$

We compute

$$(A[v,\sigma],[v,\sigma])_{L^2\times L^2} = \mu(Dv,Dv)_{L^2} \ ,$$

so A is accretive. The resolvent equation is equivalent to the single equation

$$v + D^*\big(EDv + E^{\frac{1}{2}}g + \mu Dv\big) = f \ ,$$

that is, to the variational equation

$$v \in V : \quad (v,\varphi)_{L^2\times L^2} + \big((E+\mu)Dv + E^{\frac{1}{2}}g, D\varphi\big)_{L^2\times L^2} = (f,\varphi)_{L^2\times L^2}, \quad \varphi \in V \ ,$$

and this is guaranteed by Theorem I.2.1A to have a unique solution if $E + \mu > 0$. Thus the dynamic problem has a unique solution by Corollary I.5.2.

Nonlinear Models.

We consider briefly the modifications above in the case that the constitutive equation (3.1.b) is nonlinear. First we note that if the elasticity term were nonlinear, then we obtain a *quasilinear wave equation*

$$\frac{\partial^2}{\partial t^2}(\rho\, u(x,t)) - \frac{\partial}{\partial x}(E(\frac{\partial u(x,t)}{\partial x})) = F(x,t) \ .$$

Such equations are not handled by monotone methods, and we have not discussed them. However in the case of a *purely viscous* material, we obtain the *quasilinear diffusion equation*

$$\frac{\partial \rho\, v}{\partial t} - \frac{\partial}{\partial x}(\mu(\frac{\partial v}{\partial x})) = F(x,t)$$

with the monotone viscosity function, $\mu(\cdot)$. Such problems were resolved by Proposition IV.5.2 in Example IV.5.A. More generally, if we substitute (3.1.b) with such a viscosity term into (3.1.a), we obtain the *quasilinear-damped wave equation*

$$\frac{\partial^2}{\partial t^2}(\rho\, u(x,t)) - \frac{\partial}{\partial x}(\mu(\frac{\partial^2 u(x,t)}{\partial x \partial t}) + E\frac{\partial u(x,t)}{\partial x}) = F(x,t) \ ,$$

or equivalently

$$\frac{\partial^2}{\partial t^2}(\rho\, v(x,t)) - \frac{\partial}{\partial t}\frac{\partial}{\partial x}(\mu(\frac{\partial v(x,t)}{\partial x}) + E\frac{\partial v(x,t)}{\partial x}) = \frac{\partial}{\partial t}F(x,t) \ , \quad 0 < x < 1, \ t > 0 \ .$$

See Theorem IV.6.2 and the abstract equations IV.6.13 and IV.6.14.

Plasticity Models.

We describe two models of plasticity in very simple form. For each of these examples, we shall describe the operator in L^2 that realizes the corresponding initial-boundary value problem. The general approach is to express the constitutive relation (3.1.1.b) as a variational equation of evolution type

$$\sigma_t + \partial\varphi(\sigma) \ni Dv \tag{3.6}$$

whose input, the strain-rate $\frac{\partial \varepsilon}{\partial t} = Dv$ corresponding to the displacement-rate $v = u_t$ is in L^2. Here $\varphi(\cdot)$ denotes either the indicator function $I_K(\cdot)$ of a given closed convex set K characterizing the particular plasticity model or a smooth convex function for the viscosity models, and $\partial\varphi$ is the corresponding subgradient or derivative, respectively. The existence and uniqueness of a solution $\sigma(t)$, $t > 0$ of (3.6) with prescribed initial value $\sigma(0)$ is guaranteed by Theorem IV.4.3.

Elastic–perfectly Plastic. Consider a material whose stress-strain relationship is described as follows. When we apply an increasing load to one end of a segment of the rod material, there is initially no elongation within the segment, and the resulting stress at the other end, σ, similarly increases until the value $\sigma = 1$ is reached. Then the material *yields*, and any additional increase in the elongation ε occurs with no additional stress at the other end. If the load is incrementally decreased from $\sigma = 1$, the elongation remains fixed and the stress decreases until the value -1 is reached. Then the material again yields with decreasing elongation

ε and limited stress at the value $\sigma = -1$. This is a *perfectly plastic* material, and its constitutive relation is given by

$$\sigma \in \operatorname{sgn}(\frac{d\varepsilon}{dt}) ,$$

where the *sign graph* is given by

$$sgn(x) = \begin{cases} \{1\} , & \text{if } x > 0 , \\ [-1,1] , & \text{if } x = 0 , \\ \{-1\} , & \text{if } x < 0 . \end{cases}$$

If we add the perfectly elastic to the perfectly plastic model in *series*, then their elongations add, and we obtain the material defined by the relation

$$\sigma_t + \operatorname{sgn}^{-1}(\sigma) \ni E\,\varepsilon_t .$$

This is the *Prandtl* model of elasto-plasticity. Hereafter we take $E = 1$. The phase diagram showing the relationship between stress σ and strain ε is given in Figure 1. Since the value of σ depends on the past history of ε and is rate-independent, this relationship is a *hysteresis* functional.

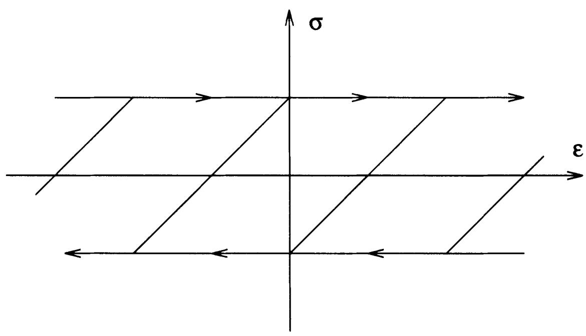

FIGURE 1

The resulting dynamical system is given by

(3.7.a) $$v_t - \sigma_x = f\,,\ 0 < x < 1\,,\ 0 < t\,, \quad v(0,t) = 0$$

(3.7.b) $$\sigma_t - v_x + \operatorname{sgn}^{-1}(\sigma) \ni 0\,, \quad \sigma(1,t) = 0$$

with appropriate initial conditions on v and σ. From the unbounded operator $D^* = -\frac{d}{dx} : W \to L^2(0,1)$ we construct its *continuous dual*

$$(D^*)' : L^2(0,1) \to W' .$$

Denote by $I_1(\cdot)$ the *indicator function* of the interval $[-1,1]$. Thus, $I_1 : \mathbb{R} \to +\mathbb{R}_\infty$ is convex, proper, and lower-semi-continuous, and its subgradient is characterized as follows: $f \in \partial I_1(x)$ means

$$|x| \le 1 \text{ and } \begin{cases} f \ge 0\,, & \text{for } x = 1, \\ f = 0\,, & \text{for } -1 < x < 1\,, \\ f \le 0\,, & \text{for } x = -1\,. \end{cases}$$

Thus, ∂I_1 is just the inverse of the *sign graph*. Let the function φ_1 be the corresponding realization on the Hilbert space W (see Example II.8.B) given by

$$\varphi_1(w) = \int_0^1 I_1(w(x))\,dx\;, \qquad w \in W\;.$$

DEFINITION. The (weak) operator A is determined as follows: $[f,g] \in A[v,\sigma]$ if
$[f,g] \in L^2(0,1) \times L^2(0,1)$, $[v,\sigma] \in L^2(0,1) \times W$, and there exists a $c \in W'$ for which

$$\sigma \in W, \quad D^*\sigma = f\;, \tag{3.8.a}$$

$$v \in L^2(0,1), \quad -(D^*)'v + c = g\;,\; c \in \partial\varphi_1(\sigma)\;. \tag{3.8.b}$$

REMARKS. The first term in (3.8.b) is defined in W' by

$$-v(D^*w) = \int_0^1 v(x)w'(x)\,dx\;, \qquad w \in W\;.$$

Formally (3.8.b) means

$$-v_x(x) + c(x) = g(x)\,,\; c(x) \in \operatorname{sgn}^{-1}(\sigma(x))\,,\; v(0) = 0\;,$$

but this holds only if c is sufficiently regular, e.g., if $c \in L^2(0,1)$, for then $v \in H^1(0,1)$ and the boundary condition is meaningful. The range, $\mathrm{Rg}(A)$, is the set of pairs $[f,g] \in L^2(0,1) \times L^2(0,1)$ for which $|\int_x^1 f\,dx| \le 1$ for each $x \in [0,1]$. Neither A nor A^{-1} is a function.

The corresponding resolvent equation, $(I + A)[v,\sigma] \ni [f,g]$, is given by

$$v \in L^2(0,1): \quad v + D^*\sigma = f\;,$$
$$\sigma \in W: \quad \sigma - (D^*)'v + \partial\varphi_1(\sigma) \ni g\;.$$

Note that $(D^*)'v$ and $\partial\varphi_1(\sigma)$ are in W', so this is a *weak* solution in our notation below, and there is no boundary value assigned to $v(0)$. If we eliminate v, we can write this as a single equation or *variational inequality*, formally of the form

$$\sigma \in W: \quad \sigma + \partial\varphi_1(\sigma) + (D^*)'(D^*\sigma) \ni g + (D^*)'f\;.$$

To be precise, this problem has the following variational form: find a pair of functions

$$\sigma \in W\,,\; c \in W' \text{ such that } c \in \partial\varphi_1(\sigma)\,, \text{ and}$$
$$\int_0^1 (\sigma\varphi + (\sigma_x + f)\varphi_x\,dx + c(\varphi) = \int_0^1 g\varphi\,dx \text{ for all } \varphi \in W\;.$$

This is the characterization of the solution to the problem of minimizing the Dirichlet integrand

$$\Phi(\sigma) = \int_0^1 (\frac{1}{2}\sigma^2 + \frac{1}{2}\sigma_x^2 + I_1(\sigma))\,dx - \int_0^1 (g\sigma - f\sigma_x)\,dx$$

over the space W. This is contained in Example II.8.D (also see Proposition IV.1.4 or Proposition IV.1.5), so it is known to have a solution and, hence, $\mathrm{Rg}(I + A) = L^2(0,1) \times L^2(0,1)$. It follows as before that the map $[f,g] \to [v,\sigma]$ is a contraction on $L^2(0,1) \times L^2(0,1)$, so A is m-accretive, and Theorem IV.4.1 shows that there is a unique *weak* solution of (3.7) with

$$\frac{\partial v}{\partial t}, \frac{\partial \sigma}{\partial t} \in L^\infty(0,T;L^2(0,1)) , \qquad v \in L^\infty(0,T;V),\ \sigma \in L^\infty(0,T;W) .$$

REMARK. The corresponding equation for v is degenerate in the gradient, hence, not coercive.

Kinematic Hardening. Here we assume that the material *work-hardens* each time the yield stress is reached. In this case the *position* of the stress interval is moved upward or downward. Momentum and constitutive equations are, respectively,

$$\tag{3.9} \begin{gathered} \frac{\partial}{\partial t} v - \sigma_x = f \ , \quad \sigma = \beta_1\sigma_1 + \beta_2\sigma_2 \ , \\ \frac{\partial}{\partial t}\sigma_1 + \partial\varphi_1(\sigma_1) \ni \beta_1 \frac{\partial}{\partial x} v \ , \quad \frac{\partial}{\partial t}\sigma_2 = \beta_2 \frac{\partial}{\partial x} v \ . \end{gathered}$$

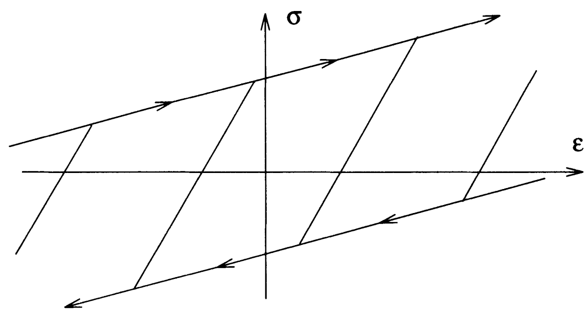

FIGURE 2

This model results from the *parallel* addition of the elastic-plastic stress from above (corresponding to σ_1) with a purely elastic stress (corresponding to σ_2) which records the position of the center of the yield stress interval. Thus the lines in Figure 2 representing the upper and lower yield surfaces are at a vertical distance apart of 2, and they have slope β_2.

We shall write the system (3.9) as an evolution equation in the appropriate product space.

DEFINITION. The (strong) operator A is determined as follows:
$[f, g_1, g_2] \in A[v, \sigma_1, \sigma_2]$ if

$$[f, g_1, g_2] \in L^2(0,1)^3,\ [v, \sigma_1, \sigma_2] \in V \times L^2(0,1)^2,\ \beta_1\sigma_1 + \beta_2\sigma_2 \in W \ ,$$

and there exists a $c \in L^2(0,1)$ for which

$$\tag{3.10} \begin{gathered} D^*(\beta_1\sigma_1 + \beta_2\sigma_2) = f \ , \\ -\beta_1 Dv + c = g_1 \ , \ c \in \partial\varphi_1(\sigma_1) \ , \\ -\beta_2 Dv = g_2 \ . \end{gathered}$$

Note that since $c \in L^2(0,1)$, the inclusion $c(x) \in \partial\varphi_1(\sigma_1(x))$ is a pointwise variational inequality in $\mathbb{R}$ for a.e. $x \in [0,1]$. Namely, it is equivalent to

$$c(x) \in \mathbb{R},\ |\sigma_1(x)| \le 1 : \quad c(x)(\rho - \sigma_1(x)) \le 0 \text{ for all } \rho \in \mathbb{R} \text{ with } |\rho| \le 1 \ ,$$

the sign graph. We shall show that the operator A is m-accretive in the space $H \equiv L^2(0,1)^3$ and, since $v_x \in L^2(0,1)$, that it leads to a *strong* solution.

We introduce the following:

$$\beta = [\beta_1 I, \beta_2 I] : L^2(0,1) \to L^2(0,1)^2 \text{ where } \beta_1, \beta_2 \in \mathbb{R} \text{ are given,}$$
$$\beta^*[\sigma] = \beta_1\sigma_1 + \beta_2\sigma_2, \quad \beta^* : L^2(0,1)^2 \to L^2(0,1)$$
$$W_0 = \{\sigma = [\sigma_1, \sigma_2] \in L^2(0,1)^2 : \beta^*\sigma \in H^1(0,1),\ \beta^*\sigma(1) = 0\}$$

Note that for any solution of (3.9), either weak or strong, we have $\beta^*\sigma \in W$ in the momentum equation. In particular, $\beta^*\sigma$ satisfies the appropriate boundary condition.

First we check by a direct estimate that A is accretive. Second, the resolvent equation $(I + A)[v, \sigma] \ni [f, g]$ is equivalent to solving the system

$$v \in V : \quad v + D^*\beta^*\sigma = f \ ,$$
$$\sigma \in W_0 : \quad \sigma - \beta D v + [\partial\varphi_1(\sigma_1), 0] = g \in L^2(0,1)^2 \ .$$

This is equivalent to solving for v the equation

$$v \in V : \quad v + D^*(\beta_1(I + \partial\varphi_1)^{-1}(\beta_1 D v + g_1) + \beta_2^2 D v + \beta_2 g_2) = f \text{ in } V' \ .$$

Since $\beta_2^2 > 0$, the form is coercive, and existence of a solution follows. (See Example II.8.D.) The components of $[\sigma_1, \sigma_2] \in W_0$ are obtained directly from the second and third terms in this equation, respectively, and then we check that $\sigma \in W_0$. In particular, the boundary condition at $x = 1$ is satisfied. These remarks show that Theorem IV.4.1 applies directly to give existence and uniqueness of a *strong* solution of (3.9) with

$$v \in L^\infty(0,T;V), \quad \sigma \in L^\infty(0,T;W_0), \quad \frac{\partial v}{\partial t}, \frac{\partial \sigma}{\partial t} \in L^\infty(0,T;L^2(0,1)) \ .$$

REMARK 1. Since v belongs to V instead of merely to $L^2(I)$, the solution here is *smoother* than that above. This is made possible here by the coercivity resulting from the β_2 term.

REMARK 2. $A(v, \sigma)$ is single valued only if $\sigma_1 \neq 0$ a.e., and $A^{-1}(f, g)$ is single valued only if $\beta_2 g_1 \neq \beta_1 g_2$ a.e..

REMARK 3. We can include a viscous element in parallel to the above by adding a third equation of the form

$$\frac{1}{k}\frac{\partial}{\partial t}\sigma_3 + \frac{1}{\mu}\sigma_3 = \beta_3 \frac{\partial}{\partial x} v \ .$$

More generally, we can include *visco-elastic* elements in the form

$$\frac{1}{k}\frac{\partial}{\partial t}\sigma_3 + J'(\sigma_3) = c\frac{\partial}{\partial x} v \ .$$

where J has a *bounded* derivative. This represents a series combination of elastic element and a purely viscous element, and one obtains *strong* solutions as above.

REMARK 4. By setting $v = u_t$, eliminating $\sigma_2 = \beta_2 \frac{\partial}{\partial x} u$, and replacing the term $\partial\varphi_1(\sigma_1)$ by $-\Delta\sigma_1$ in (3.9), we obtain the system

$$u_{tt} - \beta_1 \frac{\partial}{\partial x}\sigma_1 - \beta_2^2 u_{xx} = f \ ,$$
$$\frac{\partial}{\partial t}\sigma_1 - \Delta\sigma_1 = \beta_1 \frac{\partial}{\partial x} u_t \ .$$

This is the classical problem of *thermoelasticity*, and it is easier and can be handled similarly.

We close with a simple but important extension of the preceding example to a plasticity model built on four stress components. This will motivate the consideration of generalized sums or *integrals* of a collection or even a *continuum* of such components. The system is given by

$$\begin{aligned} &\frac{\partial}{\partial t}v - \sigma_x = f, \qquad \sigma = \sigma_1 + \frac{1}{2}\sigma_2 + \frac{1}{4}\sigma_3 + \frac{1}{4}\sigma_4 \ , \\ &\frac{\partial}{\partial t}\sigma_1 + \partial\varphi_1(\sigma_1) \ni \frac{\partial}{\partial x}v \ , \\ &\frac{\partial}{\partial t}\sigma_2 + \partial\varphi_2(\sigma_2) \ni \frac{1}{2}\frac{\partial}{\partial x}v \ , \\ &\frac{\partial}{\partial t}\sigma_3 + \partial\varphi_3(\sigma_3) \ni \frac{1}{4}\frac{\partial}{\partial x}v \ , \\ &\frac{\partial}{\partial t}\sigma_4 = \frac{1}{4}\frac{\partial}{\partial x}v \ . \end{aligned} \tag{3.11}$$

For each $j = 1, 2, 3$, φ_j is the indicator function of the interval $[-j, j]$, so the corresponding stress component σ_j is constrained to lie within that interval. The relation between total stress σ and strain ε is indicated by Figure 3. For an example of a typical output of this system, we begin with all components at 0, increase the strain, ε, from 0 to 5, decrease it to -5, then increase it to 2, and we follow the resulting stress, σ. The slope of the stress starting upward from the origin is 2, then it decreases to 1 and to $\frac{1}{2}$ on successive intervals until only σ_4 with slope $\frac{1}{4}$ is active for $\varepsilon \geq 3$. When the curve begins to decrease from $\varepsilon = 5$, the slope is initially 2, and then the slope decreases successively to 1 and to $\frac{1}{2}$ on intervals of length 2 until only σ_4 with slope $\frac{1}{4}$ is active for $\varepsilon \leq -1$. The applied strain ε reverses direction again at -5, and the resulting stress begins to rise with slope 2 again. The limiting positive slope $\frac{1}{4}$ is the *work-hardening* component, and it is this component of the stress that will lead to a *strong* solution of (3.11) as before.

Since the bounding lines in this hysteresis functional are straight lines, such models are called *multilinear*. By using a collection of such components, one can approximate a large class of convex bounding curves; with a continuum of such components, the corresponding class of convex functions can be matched. Most models of plasticity involve such multiple yield surfaces, and these provide an approximation of the observed smooth transitions between elastic and plastic regimes. Such smooth transitions are best modeled by a continuum of elastic-plastic elements with varying yield surfaces.

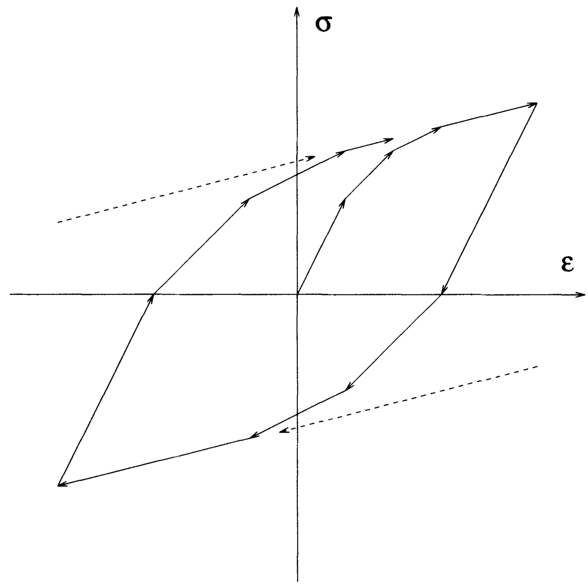

FIGURE 3

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