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An Introduction to Infinite Ergodic Theory

Jon Aaronson



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An Introduction to Infinite Ergodic Theory

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Volume 50

An Introduction to Infinite Ergodic Theory

Jon Aaronson



American Mathematical Society

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ABSTRACT. The book is about measure preserving transformations of infinite measure spaces. It could be of interest to mathematicians working in ergodic theory, probability and/or dynamical systems and should be accessible to graduate students.

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Preface

Infinite ergodic theory is the study of measure preserving transformations of infinite measure spaces (early references being [Hop1] and [St]). It is part of "non-singular ergodic theory", the more general study of non-singular transformations (since a measure preserving transformation is also a non-singular transformation).

Non-singular ergodic theory arose as an attempt to generalise the classical ergodic theory of probability preserving transformations. Its major success was the ratio ergodic theorem. Another side to the theory also developed concentrating on facts which are valid "in the absence of invariant probabilities".

This book is more concerned with properties specific to infinite measure preserving transformations.

It should be readable by anyone initiated to metric space topology and measure theoretic probability.

Some readers may like to begin by following an example and perhaps one of the simplest in the book is Boole's transformation $T : \mathbb{R} \rightarrow \mathbb{R}$ defined by $Tx = x - \frac{1}{x}$.

This is a conservative, exact measure preserving transformation of \mathbb{R} equipped with Lebesgue measure; and for each absolutely continuous probability P on \mathbb{R} and non-negative, integrable function $f : \mathbb{R} \rightarrow \mathbb{R}$ with unit integral,

$$P\left(\left[\sum_{k=0}^{n-1} f \circ T^k \leq \frac{\sqrt{2n}}{\pi} t\right]\right) \rightarrow \frac{2}{\pi} \int_0^t e^{-\frac{s^2}{\pi}} ds$$

as $n \rightarrow \infty$.

The book begins with an introduction to basic non-singular ergodic theory (chapters 1 and 2), including recurrence behaviour, existence of invariant measures, ergodic theorems and spectral theory. One of the results in §2.4 is the collapse of absolutely normalised pointwise ergodic convergence for ergodic measure preserving transformations of infinite measure spaces.

This leaves a wide range of possible "ergodic behaviour" which is catalogued in chapter 3 mainly according to the yardsticks of intrinsic normalising constants, laws of large numbers and return sequences (the return sequence of Boole's transformation is $\frac{\sqrt{2n}}{\pi}$).

The rest of the book (excepting chapter 5) consists of illustrations of these phenomena by examples.

Markov maps which arise both in probability theory and in smooth dynamics are treated in chapter 4. They illustrate distributional convergence phenomena (mentioned above) as do the inner functions of chapter 6. Geodesic flows on hyperbolic surfaces were one of the first examples considered ([Hop1]), and these are treated in chapter 7. Some of the extremely pathological examples in the subject

can be found in the chapter on cocycles and skew products (chapter 8). In chapter 5, there is a modest beginning to the classification theory.

There is a small (but insufficient) amount of probability preserving ergodic theory in the book, and I recommend the uninitiated reader to take advantage of the excellent books available on this subject, including [**Cor-Sin-Fom**], [**De-Gr-Sig**], [**Fu**], [**Mañ**], [**Parr2**], [**Pet**], [**Rudo**], [**Wa**].

The reader will no doubt find that many (but hopefully not the reader's favourite) topics are conspicuous by their absence. By way of excuse I can only say that some of these are better covered elsewhere, while others are deemed too advanced for an introduction and yet others are too "fresh" for a book (there being no time to write about them).

Lastly I come to the thanks. I would like to thank the people who worked with me on the topics described in the book (see bibliography). Without them, none of this would have been possible. Also I would like to thank my colleagues Gilat and Lemańczyk; and my student Omri Sarig who found mistakes in early versions (any remaining errors being my sole responsibility having been introduced subsequently while correcting mistakes).

Jon. Aaronson
Tel Aviv, October 1996

Bibliography

- [A1] J. Aaronson, *Rational ergodicity and a metric invariant for Markov shifts.*, Israel Journal of Math. **27** (1977), 93-123.
- [A2] ———, *On the ergodic theory of non-integrable functions and infinite measure spaces*, Israel Journal of Math. **27** (1977), 163-173.
- [A3] ———, *On the pointwise ergodic theory of transformations preserving infinite measures*, Israel Journal of Math. **32** (1978), 67-82.
- [A4] ———, *On the categories of ergodicity when the measure is infinite*, Ergodic Theory Proceedings, Oberwolfach 1978, Ed. M. Denker, SLN, vol. 729, 1978, pp. 1-9.
- [A5] ———, *Ergodic theory for inner functions of the upper half plane*, Ann. Inst. H. Poincaré (B) **XIV** (1978), 233-253.
- [A6] ———, *A remark on the exactness of inner functions*, Jour. L.M.S. **23** (1981), 469-474.
- [A7] ———, *The asymptotic distributional behavior of transformations preserving infinite measures*, J. D'Analyse Math. **39** (1981), 203-234.
- [A8] ———, *An ergodic theorem with large normalising constants*, Israel Journal of Math. **38** (1981), 182-188.
- [A9] ———, *The eigenvalues of non-singular transformations*, Israel Journal of Math. **45** (1983), 297-312.
- [A10] ———, *Random f -expansions*, Ann. Probab. **14** (1986), 1037-1057.
- [A11] ———, *The intrinsic normalising constants of transformations preserving infinite measures*, J. D'Analyse Math. **49** (1987), 239-270.
- [A12] ———, *Category theorems for some ergodic multiplier properties*, Israel Journal of Math. **51** (1985), 151-162.
- [A-De1] J. Aaronson, M. Denker, *Upper bounds for ergodic sums of infinite measure preserving transformations*, Trans. Amer. Math. Soc. **319** (1990), 101-138.
- [A-De2] ———, *Local limit theorems for Gibbs-Markov maps*, internet: <http://www.math.tau.ac.il/~aaro>.
- [A-De3] ———, *The Poincaré series of $\mathbb{C} \setminus \mathbb{Z}$* , internet: <http://www.math.tau.ac.il/~aaro>.
- [A-De4] ———, *Distributional limits for hyperbolic, infinite volume geodesic flows*, internet: <http://www.math.tau.ac.il/~aaro>.
- [A-De-Fi] J. Aaronson, M. Denker, A. Fisher, *Second order ergodic theorems for ergodic transformations of infinite measure spaces*, Proc. Amer. Math. Soc. **114** (1992), 115-127.
- [A-De-Ur] J. Aaronson, M. Denker, M. Urbański, *Ergodic theory for Markov fibred systems and parabolic rational maps*, Trans. Amer. Math. Soc. **337** (1993), 495-548.
- [A-Ham-Schm] J. Aaronson, T. Hamachi, K. Schmidt, *Associated actions and uniqueness of cocycles*, Algorithms fractals and dynamics, Proceedings of the Hayashibara Forum '92, Okayama, Japan; and the Kyoto symposium. Ed.: Y. Takahashi, Plenum Publishing Company, New York, 1995, pp. 1-25.
- [A-Kea1] J. Aaronson, M. Keane, *The visits to zero of some deterministic random walks*, Proc. London Math. Soc., Ser III **44** (1982), 535-553.
- [A-Kea2] ———, *Isomorphism of random walks*, Israel J. of Maths. **87** (1994), 37-63.
- [A-Le-Ma-Nak] J. Aaronson, M. Lemańczyk, C. Mauduit, H. Nakada, *Koksma's inequality and group extensions of Kronecker transformations*, Algorithms fractals and dynamics, Proceedings of the Hayashibara Forum '92, Okayama, Japan; and the Kyoto

- symposium. Ed.: Y. Takahashi, Plenum Publishing Company, New York, 1995, pp. 27-50.
- [A-Le-V] J. Aaronson, M. Lemańczyk, D. Volný, *A salad of cocycles*, preprint, internet: <http://www.math.tau.ac.il/~aaro>.
- [A-Lig-P] J. Aaronson, T. Liggett, P. Picco, *Equivalence of renewal sequences and isomorphism of random walks*, Israel J. of Maths. **87** (1994), 65-76.
- [A-Lin-W] J. Aaronson, M. Lin and B. Weiss, *Mixing properties of Markov operators and ergodic transformations*, Israel Journal of Math. **33** (1979), 198-224.
- [A-Nad] J. Aaronson, M. Nadkarni, *L^∞ eigenvalues and L^2 spectra of non-singular transformations*, Proc. London Math. Soc. **55** (1987), 538-570.
- [A-Su] J. Aaronson, D. Sullivan, *Rational ergodicity of geodesic flows*, Ergod. Theory & Dynam. Syst. **4** (1984), 165-178.
- [A-W1] J. Aaronson and B. Weiss, *Generic distributional limits for measure preserving transformations*, Israel Journal of Math. **47** (1984), 251-259.
- [A-W2] ———, *A \mathbb{Z}^d ergodic theorem with large normalising constants*, Convergence in ergodic theory and probability, Eds. V. Bergelson, P. March, J. Rosenblatt, O.S.U. Math. Res. Inst. Publ., vol. 5, de Gruyter, Berlin, 1996.
- [A-W3] ———, *On the asymptotics of a 1-parameter family of infinite measure preserving transformations*, internet: <http://www.math.tau.ac.il/~aaro>.
- [Ad] R. Adler, *F-expansions revisited*, Recent advances in topological dynamics, L.N.Math. 318, Springer, Berlin, Heidelberg, New York, 1973, pp. 1-5.
- [Ad-W] R. Adler, B. Weiss, *The ergodic, infinite measure preserving transformation of Boole*, Israel Journal of Math. **16** (1973), 263-278.
- [Ahl] L.V. Ahlfors, *Conformal invariants topics in geometric function theory*, McGraw-Hill, New York, 1973.
- [Arn] L. K. Arnold, *On σ -finite invariant measures*, Z. Wahrsch. u.v. Geb. **9** (1968), 85-97.
- [At] G. Atkinson, *Recurrence of co-cycles and random walks*, J. London math. Soc., II. Ser. **13** (1976), 486-488.
- [Ban] S. Banach, *Theorie des operations lineaires*, Chelsea, New York, 1932.
- [Bar-Nin] M. N. Barber, B. W. Ninham, *Random and restricted walks*, Gordon and Breach, New York, 1970.
- [Bir] G. D. Birkhoff, *Proof of the ergodic theorem*, Proc. Nat. Acad. Sci. USA **17** (1931), 656-660.
- [Boo] G. Boole, *On the comparison of transcendents with certain applications to the theory of definite integrals*, Philos. Transac. R. Soc. London **147** (1857), 745-803.
- [Bow] R. Bowen, *Invariant measures for Markov maps of the interval*, Comm. Math. Phys. **69** (1979), 1-17.
- [Bra] R. Bradley, *On the ψ -mixing condition for stationary random sequences*, Transac. Amer. Math. Soc. **276** (1983), 55-66.
- [Bre] L. Breiman, *Probability*, Addison-Wesley, Reading, Mass., U.S., 1968.
- [Bru] A. Brunel, *New conditions for the existence of invariant measures in ergodic theory*, Contributions to ergodic theory and probability, S.L.N. Math., vol. 160, 1970, pp. 7-17.
- [Cha-Orn] R. Chacon, D. Ornstein, *A general ergodic theorem*, Illinois J. Math. **4** (1960), 153-160.
- [Cho-Ro] Y.S. Chow, H. Robbins, *On sums of independent random variables with ∞ moments*, Proc. Nat. Acad. Sci. U.S.A. **47** (1961), 330-335.
- [Chu] K.L. Chung, *Markov chains with stationary transition probabilities*, Springer, Heidelberg, 1960.
- [Coh] D. Cohn, *Measure theory*, Birkhauser, Boston, 1980.
- [Con] J.P. Conze, *Ergodicite d'un flot cylindrique*, Bull. Soc. Mat. de France **108** (1980), 441-456.
- [Cor-Sin-Fom] I. Cornfeld, S.V. Fomin, Ya. G. Sinai, *Ergodic theory*, Springer, New York, 1982.
- [Cra] M. Craizer, *The Bernoulli property of inner functions*, Ergodic Theory Dyn. Syst. **12** (1992), 209-215.
- [Da-Kac] D.A. Darling, M. Kac, *On occupation times for Markov processes*, Trans. Amer. Math. Soc. **84** (1957), 444-458.

- [De] A. Denjoy, *Fonctions contractent le cercle* $|Z| < 1$, C.R.Acad. Sci. Paris **182** (1926), 255-257.
- [De-Gr-Sig] M. Denker, C. Grillenberger, K. Sigmund, *Ergodic theory on compact spaces*, Lecture notes in Mathematics vol. 527, Springer, Berlin, 1976.
- [Do-Mañ] C. Doering, R. Mañé, *The dynamics of inner functions*, Ensaios Matemáticos **3** (1991), Soc. Brasileira de Mat., 1-79.
- [Doo] J. Doob, *Stochastic processes*, Wiley, New York, 1953.
- [E-Fr] M. Ellis, N. Friedman, *On eventually weakly wandering sequences*, Studies in probability and ergodic theory, Ed. G. Rota, Adv. Math., Suppl. Stud., vol. 2, 1978, pp. 185-194.
- [Fe1] W. Feller, *A limit theorem for random variables with infinite moments*, Amer. J. Math. **68** (1946), 257-262.
- [Fe2] ———, *An introduction to probability theory and its applications, volume I*, John Wiley, New York, 1968.
- [Fe3] ———, *An introduction to probability theory and its applications, volume II*, Wiley, New York, 1966.
- [Fo] S. Foguel, *The ergodic theory of Markov processes*, van Nostrand, New York, 1969.
- [Fo-Lin] S. Foguel, M. Lin, *Some ratio limit theorems for Markov operators*, Z. Wahrsch. u.v. gebiete **23**, 55-66.
- [Fr] N. Friedman, *Introduction to ergodic theory*, van Nostrand, New York, 1970.
- [Fu-W] H. Furstenberg, B. Weiss, *The finite ergodic multipliers of infinite ergodic transformations*, Lecture notes in math. **668** (1978), Springer, Berlin, 127-132.
- [Fu] H. Furstenberg, *Recurrence in ergodic theory and combinatorial number theory*, Princeton university press, Princeton, N.J., 1981.
- [Ga] A. Garsia, *Topics in almost everywhere convergence*, Lectures in Advanced Mathematics : NO. 4, Markham, Chicago, 1970.
- [Go-Sin] V.I. Golodets, S.D. Sinel'shchikov, *Locally compact groups appearing as ranges of cocycles of ergodic Z -actions*, Ergod. Th. and Dynam. Sys. **5** (1985), 45-57.
- [G] Y. Guivarc'h, *Propriétés ergodiques, en mesure infinie, de certains systèmes dynamiques fibrés*, Ergod. Th. and Dynam. Sys. **9** (1989), 433-453.
- [Haj-It-Kak] A. Hajian, Y. Ito, S. Kakutani, *Invariant measures and orbits of dissipative transformations*, Adv. Math **9** (1972), 52-65.
- [Haj-Kak1] A. Hajian, S. Kakutani, *Weakly wandering sets and invariant measures*, Trans. Amer. Math. Soc. **110** (1964), 136-151.
- [Haj-Kak2] ———, *An example of an ergodic measure preserving transformation defined on an infinite measure space*, Contributions to ergodic theory and probability, S.L.N. Math., vol. 160, 1970, pp. 45-52.
- [Half] M. Halfant, *Analytic properties of Renyi's invariant density*, Israel Journal of Math. **27** (1977), 1-20.
- [Halm1] P. Halmos, *Measure theory*, van Nostrand, New York, 1950.
- [Halm2] ———, *Lectures on ergodic theory*, Chelsea, New York, 1956.
- [Ham] T. Hamachi, *The normaliser group of an ergodic automorphism of type III and the commutant of an ergodic flow*, Jour. Funct. Anal. **40** (1981), 387-403.
- [Hed] G. Hedlund, *Fuchsian groups and mixtures*, Ann. Math. **40** (1939), 370-383.
- [Hei] M. Heins, *On the finite angular derivatives of an analytic function mapping the disk onto itself*, J. London Math. Soc. **15** (1977), 239-254.
- [Hew-Ros] E. Hewitt, K. Ross, *Abstract harmonic analysis*, Springer Verlag, Berlin, 1979.
- [Hof-Kel] F. Hofbauer; G. Keller, *Ergodic properties of invariant measures for piecewise monotonic transformations.*, Math. Zeitschrift **180** (1982), 119-140.
- [Hop1] E. Hopf, *Ergodentheorie*, Ergeb. Mat., vol. 5, Springer, Berlin, 1937.
- [Hop2] ———, *The general temporally discrete Markov process*, J. Rat. Mech. Anal. **3** (1954), 13-45.
- [Hop3] ———, *Ergodic theory and the geodesic flow on surfaces of constant negative curvature*, Bull. Am. Math. Soc. **77** (1971), 863-877.
- [Hos-Me-Par] B. Host, J-F. Melá, F. Parreau, *Nonsingular transformations and spectral analysis of measures*, Bull. Soc. Math. de France **119** (1991), 33-90.
- [Hur] W. Hurewicz, *Ergodic theorem without invariant measure*, Ann. Math. **45** (1944), 192-206.

- [Io-Mar] Ionescu-Tulcea, G. Marinescu, *Théorie ergodique pour des classes d'opérations non complètement continues*, Ann. Math. **47** (1946), 140-147.
- [Kac] M. Kac, *On the notion of recurrence in discrete stochastic processes*, Bull. Amer. Math. Soc. **53** (1947), 1002-1010.
- [Kah-Sal] J-P. Kahane, R. Salem, *Ensembles parfaits et series trigonometriques*, Hermann, Paris, 1963.
- [Kak] S. Kakutani, *Induced measure preserving transformations*, Proc. Imp. Acad. Sci. Tokyo **19** (1943), 635-641.
- [Kar] J. Karamata, *Sur un mode de croissance reguliere. Theoreme s fondamentaux*, Bull. Soc. Math. France **61** (1933), 55-62.
- [Kal] T. Kaluza, *Über die Koeffizienten reziproker Potenzreihen*, Math. Z. **28** (1928), 161-170.
- [Kat] Y. Katznelson, *An Introduction to Harmonic Analysis*, Dover Publ. inc., New York, 1967.
- [Kec] A. Kechris, *Classical descriptive set theory*, Graduate texts in mathematics : 156, Springer-Verlag, Berlin, 1995.
- [Kel] G. Keller, *Exponents, attractors and Hopf decompositions for interval maps*, Ergodic Theory Dyn. Syst. **10** (1990), 717-744.
- [Ken] D. G. Kendall, *Delphic semigroups*, Springer Lecture Notes in Math. **31** (1967), 147-175.
- [Key-New] H. Keynes, D. Newton, *The structure of ergodic measures for compact group extensions*, Israel J. Math. **18** (1974), 363-389.
- [Kh] A. Ya. Khinchin, *Continued Fractions*, University of Chicago Press, Chicago and London, 1964.
- [Ki] J.F.C. Kingman, *Regenerative Phenomena*, John Wiley, New York, 1972.
- [Kol] A.N. Kolmogorov, *Foundations of the theory of probability*, Chelsea, New York, 1956.
- [Kom] J. Komlos, *A generalization of a problem of Steinhaus*, Acta Math. Acad. Sci. Hung. **18** (1967), 217-229.
- [Kre] U. Krengel, *Ergodic theorems*, de Gruyter, Berlin, 1985.
- [Kre-Suc] U. Krengel, L. Sucheston, *On mixing in infinite measure spaces*, Z. Wahrsch. u.v. Geb. **13** (1969), 150-164.
- [Kri] W. Krieger, *On ergodic flows and isomorphism of factors*, Math. Annalen **223** (1976), 19-70.
- [Kui-Ni] L. Kuipers, H. Niederreiter, *Uniform Distribution of Sequences*, Wiley, N.Y., 1974.
- [Kur] K. Kuratowski, *Topology vols. 1 and 2*, Academic press, New York, 1966.
- [La] S. Lang, *$SL_2(\mathbb{R})$* , Addison-Wesley, Reading, Mass. USA., 1975.
- [Lem] M. Lemańczyk, *Ergodic compact Abelian group extensions*, Habilitation thesis, Nicholas Copernicus University, Toruń, 1990.
- [Let] G. Letac, *Which functions preserve Cauchy laws?*, Proc. Amer. Math. Soc. **67** (1977), 277-286.
- [Lév] P. Lévy, *Théorie de l'addition des variables aléatoires*, 2nd. edition, Gauthier-Villars, Paris, 1954.
- [Li-Schw] T-Y. Li, F. Schweiger, *The generalized Boole's transformation is ergodic*, Manuscr. Math. **25** (1978), 161-167.
- [Lin] M. Lin, *Mixing for Markov operators*, Z. Wahrsch. u.v. Geb. **19** (1971), 231-243.
- [Mah] D. Maharam, *Incompressible transformations*, Fund. Math. **56** (1964), 35-50.
- [Mañ] R. Mañé, *Ergodic theory and differential dynamics*, Springer-Verlag, Berlin, 1987.
- [Me] J. F. Melá, *Groupes de valeurs propres des systemes dynamiques et sous groupes satures du cercle*, C.R. Acad. Sci. Paris Ser. I Math **296** (1983), 419-422.
- [Mo-Schm] C. Moore, K. Schmidt, *Coboundaries and homomorphisms for nonsingular actions and a problem of H. Helson*, Proc. L.M.S. **40** (1980), 443-475.
- [Myr] P.J. Myrberg, *Über die Existenz der Greenschen Funktionen auf einer gegebenen Riemannschen Fläche*, Acta Math. **61** (1933), 39-79.
- [Neu-v1] von Neumann, J., *Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren*, Math. Annalen **102** (1929), 49-131.
- [Neu-v2] ———, *Proof of the quasi ergodic hypothesis*, Proc. Nat. Acad. Sci. USA **18** (1932), 70-82.

- [Neu-v3] ———, *Zur Operatorenmethode in der Klassischen Mechanik*, Ann. Math **33** (1932), 587–642.
- [Neu] J. H. Neuwirth, *Ergodicity of some mappings of the circle and the line*, Israel Journal of Math. **31** (1978), 359–367.
- [Nor] E. A. Nordgren, *Composition operators*, Canad. J. Math. **20** (1968), 442–449.
- [Nic] P. Nicholls, *The ergodic theory of discrete groups*, London Math. Soc. Lecture Notes, vol. 143, Cambridge University Press, Cambridge, 1989.
- [Ore] I. Oren, *Ergodicity of cylinder flows arising from irregularities of distribution*, Israel J. Math. **44** (1983), 127–138.
- [Os] M. Osikawa, *Point spectra of non-singular flows*, Publ. R.I.M.S. Kyoto U. **13** (1977), 167–172.
- [Parr1] W. Parry, *Ergodic and spectral analysis of certain infinite measure preserving transformations*, Proc. Am. Math. Soc. **16** (1965), 960–966.
- [Parr2] ———, *Topics in ergodic theory*, Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge, 1981.
- [Part] K. R. Parthasarathy, *Probability measures on metric spaces*, Academic press, New York, 1967.
- [Pas] D.A. Pask, *Skew products over the irrational rotation*, Israel J. Math. **69** (1990), 65–74.
- [Pet] K. Petersen, *Ergodic theory*, Cambridge University Press, Cambridge, 1983.
- [Poi] H. Poincaré, *Les méthodes nouvelles de la mécanique céleste*, vol. 3, Gauthier-Villars, Paris, 1899.
- [Pol-Sh] M. Pollicott, R. Sharp, *Orbit counting for some discrete groups acting on simply connected manifolds with negative curvature*, Invent. Math. **117** (1994), 275–302.
- [Pom] C. Pommerenke, *On the iteration of analytic functions in a half plane*, J. London Math. Soc. **19** (1979), 439–447.
- [Ren1] A. Rényi, *Representations for real numbers and their ergodic properties*, Acta. Math. Acad. Sci. Hung. **8** (1957), 477–493.
- [Ren2] ———, *Probability theory*, North-Holland, Amsterdam, 1970.
- [Ree] M. Rees, *Checking ergodicity of some geodesic flows with infinite Gibbs measure*, Ergodic Theory Dyn. Syst. **1** (1981), 107–133.
- [Rie-Sz.N] F. Riesz, B. Szokefalvi-Nagy, *Lecons d'analyse fonctionnelle*, Akademiai Kiado, Budapest, 1953.
- [Ro1] V.A. Rokhlin, *On the fundamental ideas of measure theory*, Mat. Sb. **25** (1949), 107–150; A.M.S.Transl. **71** (1952).
- [Ro2] ———, *Selected topics from the metric theory of dynamical systems*, Uspehi Mat. Nauk. **4** (1949), 57–125; A.M.S.Transl. Ser. 2 **49** (1966).
- [Rudi] W. Rudin, *Real and complex analysis*, Tata McGraw-Hill, New Delhi, India, 1974.
- [Rudo] D.J. Rudolph, *Fundamentals of measurable dynamics (ergodic theory on lebesgue spaces)*, Clarendon Press, Oxford, 1990.
- [Rudo-Sil] D. J. Rudolph, C. Silva, *Minimal self-joinings for nonsingular transformations*, Ergodic Theory Dyn. Syst. **4** (1989), 759–800.
- [Rue] D. Ruelle, *Thermodynamic formalism (the mathematical structures of classical equilibrium statistical mechanics)*, Encyclopedia of Mathematics and its applications, vol. 5, Addison-Wesley, Reading, Mass., 1978.
- [Sach] U. Sachdeva, *On category of mixing in infinite measure spaces*, Math. Syst. Theory **5** (1971), 319–330.
- [Schm1] K. Schmidt, *Cocycles of Ergodic Transformation Groups*, Lect. Notes in Math. Vol. 1, Mac Millan Co. of India, 1977.
- [Schm2] ———, *Spectra of ergodic group actions*, Israel Journal of Math. **41** (1982), 151–153.
- [Schm3] ———, *On recurrence*, Wahrscheinlichkeitstheor. Verw. Geb. **68** (1984), 75–95.
- [Schw1] F. Schweiger, *Number theoretical endomorphisms with σ -finite invariant measures*, Israel Journal of Math. **21** (1975), 308–318.
- [Schw2] ———, *$\tan x$ is ergodic*, Proc. Am. Math. Soc. **71** (1978), 54–56.
- [Schw3] ———, *Ergodic theory of fibred systems and metric number theory*, Clarendon Press, Oxford, 1995.

- [Sie] C.L Siegel, *Some remarks on discontinuous groups*, Ann. Math. **46** (1945), 708-718.
- [Sil] C. Silva, *On μ -recurrent nonsingular endomorphisms*, Israel Journal of Math. **61** (1988), 1-13.
- [So] M. Souslin, *Sur une définition des ensembles mesurable B sans nombres transfinis*, C. R. Acad. Sci. Paris **164** (1917), 88-91.
- [St] V. V. Stepanov, *Sur une extension du theoreme ergodique*, Compositio Math. **3** (1936), 239-253.
- [Su] D. Sullivan, *On the ergodic theory at infinity of an arbitrary discrete group of hyperbolic motions*, Riemann surfaces and related topics: Proc. 1978 Stony Brook Conf., Ann. Math. Stud., vol. 97, 1981, pp. 465-496.
- [Ta] D. Tanny, *A $0-1$ law for stationary sequences*, Z. Wahrsch. u.v. Geb. **30** (1974), 139-148.
- [Te] A. Tempelman, *Ergodic theorems for group actions (informational and thermodynamical aspects)*, Ser: Mathematics and its applications, vol. 78, Kluwer Academic, Dordrecht, The Netherlands, 1992.
- [Tha1] M. Thaler, *Estimates of the invariant densities of endomorphisms with indifferent fixed points*, Israel Journal of Math. **37** (1980), 303-314.
- [Tha2] ———, *Transformations on $[0, 1]$ with infinite invariant measures*, Israel Journal of Math. **46** (1983), 67-96.
- [Tha3] ———, *A limit theorem for the Perron-Frobenius operator of transformations on $[0, 1]$ with indifferent fixed points*, Israel Journal of Math. **91** (1995), 111-127.
- [Tho] J-P. Thouvenot, *Some properties and applications of joinings in ergodic theory*, Ergodic theory and its connections with harmonic analysis (Alexandria 1993 conference proceedings, Eds: K. Petersen, I. Salama), London Math. Soc. Lecture Notes, vol. 205, Cambridge University Press, Cambridge, 1995, pp. 207-235.
- [Ts] M. Tsuji, *Potential theory in modern function theory*, Maruzen Co. Ltd, Tokyo, 1959.
- [vN1] von Neumann, J., *Proof of the quasi ergodic hypothesis*, Proc. Nat. Acad. Sci. USA **18** (1932), 70-82.
- [Var] V. S. Varadarajan, *Groups of automorphisms of Borel spaces*, Trans. Amer. Math. Soc. **109** (1963), 191-220.
- [vN2] von Neumann, J., *Zur Operatorenmethode in der Klassischen Mechanik*, Ann. Math **33** (1932), 587-642.
- [Wa] P. Walters, *An introduction to ergodic theory*, Springer, New York, 1982.
- [Wi] N. Wiener, *The ergodic theorem*, Duke Math. J. **5** (1939), 1-18.
- [Wo] J. Wolff, *Sur l'iteration des fonctions dans une region*, C.R.Acad. Sci. Paris **182** (1926), 42-43.
- [Yo-Kak] K. Yosida, S. Kakutani, *Birkhoff's ergodic theorem and the maximal ergodic theorem*, Proc. Imp. Acad. Sci. Tokyo **15** (1939), 165-168.
- [Yu] M. Yuri, *Multi-dimensional maps with infinite invariant measures and countable state sofic shifts*, Indag. Math., New Ser. **6** (1995), 355-383.
- [Zi] R. J. Zimmer, *Ergodic theory and semisimple groups*, Birkhauser, Boston, 1984.

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