An Introduction to Infinite Ergodic Theory

Jon Aaronson
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An Introduction to Infinite Ergodic Theory
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Jon Aaronson
ABSTRACT. The book is about measure preserving transformations of infinite measure spaces. It could be of interest to mathematicians working in ergodic theory, probability and/or dynamical systems and should be accessible to graduate students.
For Nilli
Contents

Contents                                     ix
Preface                                      xi

Chapter 1.  Non-singular transformations  1
§1.0  Standard measure spaces  1
§1.1  Recurrence and conservativity  14
§1.2  Ergodicity  21
§1.3  The dual operator  33
§1.4  Invariant measures  36
§1.5  Induced transformations and applications  41
§1.6  Group actions and flows  48

Chapter 2.  General ergodic and spectral theorems  53
§2.1  von Neumann’s mean ergodic theorem  54
§2.2  Pointwise ergodic theorems  56
§2.3  Converse to Birkhoff’s theorem  64
§2.4  Transformations with infinite invariant measures  70
§2.5  Spectral properties  73
§2.6  Eigenvalues  76
§2.7  Ergodicity of Cartesian products  81

Chapter 3.  Transformations with infinite invariant measures  85
§3.1  Isomorphism, factors, and similarity  86
§3.2  Intrinsic normalising constants and laws of large numbers  93
§3.3  Rational ergodicity  98
§3.4  Maharam transformations  102
§3.5  Category theorems  108
§3.6  Asymptotic distributional behaviour  112
§3.7  Pointwise dual ergodicity  118
§3.8  Wandering rates  130

Chapter 4.  Markov maps  139
§4.1  Markov partitions  139
§4.2  Graph shifts  140
§4.3  Distortion properties  143
§4.4  Ergodic properties of Markov maps with distortion properties  149
§4.5 Markov shifts 156
§4.6 Schweiger's jump transformation 161
§4.7 Smooth Frobenius-Perron operators and the Gibbs property 164
§4.8 Non-expanding interval maps 172
§4.9 Additional reading 180

Chapter 5. Recurrent events and similarity of Markov shifts 181
§5.1 Renewal sequences 181
§5.2 Markov towers and recurrent events 183
§5.3 Kaluza sequences 188
§5.4 Similarity of Markov towers 191
§5.5 Random walks 194

Chapter 6. Inner functions 201
§6.1 Inner functions on the unit disc 201
§6.2 Inner functions on the upper half plane 208
§6.3 The dichotomy 212
§6.4 Examples 214

Chapter 7. Hyperbolic geodesic flows 223
§7.1 Hyperbolic space models 224
§7.2 The geodesic flow of $H$ 227
§7.3 Asymptotic geodesics 229
§7.4 Surfaces 231
§7.5 The Poincaré series 235
§7.6 Further results 246

Chapter 8. Cocycles and skew products 247
§8.1 Skew Products 247
§8.2 Persistencies and essential values 250
§8.3 Coboundaries 253
§8.4 Skew products over Kronecker transformations 256
§8.5 Joinings of skew products 264
§8.6 Squashable skew products over odometers 267

Bibliography 275

Index 281
Preface

Infinite ergodic theory is the study of measure preserving transformations of infinite measure spaces (early references being [Hop1] and [St]). It is part of "non-singular ergodic theory", the more general study of non-singular transformations (since a measure preserving transformation is also a non-singular transformation). Non-singular ergodic theory arose as an attempt to generalise the classical ergodic theory of probability preserving transformations. Its major success was the ratio ergodic theorem. Another side to the theory also developed concentrating on facts which are valid "in the absence of invariant probabilities".

This book is more concerned with properties specific to infinite measure preserving transformations.

It should be readable by anyone initiated to metric space topology and measure theoretic probability.

Some readers may like to begin by following an example and perhaps one of the simplest in the book is Boole’s transformation $T : \mathbb{R} \to \mathbb{R}$ defined by $Tx = x - \frac{1}{x}$.

This is a conservative, exact measure preserving transformation of $\mathbb{R}$ equipped with Lebesgue measure; and for each absolutely continuous probability $P$ on $\mathbb{R}$ and non-negative, integrable function $f : \mathbb{R} \to \mathbb{R}$ with unit integral,

$$ P \left( \left\lfloor \sum_{k=0}^{n-1} f \circ T^k \right\rfloor \leq \frac{\sqrt{2n}}{\pi} \right) \to \frac{2}{\pi} \int_0^t e^{-\frac{s^2}{\pi}} \, ds $$

as $n \to \infty$.

The book begins with an introduction to basic non-singular ergodic theory (chapters 1 and 2), including recurrence behaviour, existence of invariant measures, ergodic theorems and spectral theory. One of the results in §2.4 is the collapse of absolutely normalised pointwise ergodic convergence for ergodic measure preserving transformations of infinite measure spaces.

This leaves a wide range of possible "ergodic behaviour" which is catalogued in chapter 3 mainly according to the yardsticks of intrinsic normalising constants, laws of large numbers and return sequences (the return sequence of Boole’s transformation is $\frac{\sqrt{2n}}{\pi}$).

The rest of the book (excepting chapter 5) consists of illustrations of these phenomena by examples.

Markov maps which arise both in probability theory and in smooth dynamics are treated in chapter 4. They illustrate distributional convergence phenomena (mentioned above) as do the inner functions of chapter 6. Geodesic flows on hyperbolic surfaces were one of the first examples considered ([Hop1]), and these are treated in chapter 7. Some of the extremely pathological examples in the subject
can be found in the chapter on cocycles and skew products (chapter 8). In chapter 5, there is a modest beginning to the classification theory.

There is a small (but insufficient) amount of probability preserving ergodic theory in the book, and I recommend the uninitiated reader to take advantage of the excellent books available on this subject, including [Cor-Sin-Fom], [De-Gr-Sig], [Fu], [Mañ], [Parr2], [Pet], [Rudo], [Wa].

The reader will no doubt find that many (but hopefully not the reader’s favourite) topics are conspicuous by their absence. By way of excuse I can only say that some of these are better covered elsewhere, while others are deemed too advanced for an introduction and yet others are too ”fresh” for a book (there being no time to write about them).

Lastly I come to the thanks. I would like to thank the people who worked with me on the topics described in the book (see bibliography). Without them, none of this would have been possible. Also I would like to thank my colleagues Gilat and Lemańczyk; and my student Omri Sarig who found mistakes in early versions (any remaining errors being my sole responsibility having been introduced subsequently while correcting mistakes).

Jon. Aaronson
Tel Aviv, October 1996
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Index

numbers refer to sections (i.e. a.b=§a.b)

adapted pair 4.7
adding machine 1.2
algebraic joinings 8.5
analytic section theorem 1.0
analytic set 1.0
aperiodic 4.1
approximately generate 8.4
Arzela-Ascoli theorem 4.7
asymptotic renewal equation 3.8
asymptotic type 3.3
basic partition 4.2
Bernoulli shift,
    one sided, two sided 1.2
Birkhoff’s ergodic theorem 2.2,
    converses to 2.3
Blaschke,
    series 6.1,
    theorem 6.1
Boole’s transformation
    1.1 1.3 2.2 3.7 3.8 6.4
Borel sets 1.0
Cartesian product space 1.0
Cauchy measure 6.2
Chacon-Ornstein lemma 2.2
Chung taboo distribution 4.5
coboundary 8.1
cocycles 8.1
cohomologous 8.1
completely squashable 8.6
completion of a measure space 1.0
conditional expectations 1.0
conjugacy lemma 3.5
conservative, part 1.1
continued fraction 8.4,
    -mixing 3.7
    -mixing, exponential 4.7
convergent 8.4
cylinder set 4.1
Darling-Kac,
    sets 3.7,
    theorem 3.6
Denjoy-Wolff,
    point 6.1 6.2,
    theorem 6.1
denominator 8.4
dichotomy 6.3
Dirichlet,
    fundamental domain 7.4,
    set, weak 2.6
disintegration theorem 1.0
dissipative, part 1.1 1.6
distortion 4.3,
    property 4.3, strong -, weak - 4.3,
    proposition 4.3
dyadic integers 1.2
eigenfunction 2.6
eigenvalue 2.6,
    -theorem 2.6
equidistribution corollary 7.5
ergodic decomposition 2.2
ergodic multiplier theorem 2.7
ergodic 1.0 1.6
essential values 8.2
exact 1.2
exhaustion lemma 1.0
existence of invariant probability 1.4
extension 3.1,
  \( G^- \), 8.1,
  Maharam 3.4 8.4,
  natural 3.1
factor 1.0 3.1,
  map 1.0 3.1,
  measure space 1.0,
  proposition 1.0 3.1,
  transformation 1.0 3.1
feeble convergence 3.3
fibre,
  expectation 3.1,
  measures 1.0, dilation of 1.0
fixed points,
  attractive, repulsive,
  indifferent 4.3
flow 1.6,
  \( G^- \), 7.0, 7.2, 7.4
  geodesic 7.1
genesis 7.1,
  directed 7.2,
  flow 7.2
Gibbs property 4.6
graph 4.1,
  shift 4.1
group,
  Fuchsian 7.0,
  norm 1.6,
  topological 1.0 1.6
group joinings,
  admissible, dilation of 8.5
Halmos’ recurrence theorem 1.1
harmonic measure 6.1 6.2
hereditary collection 1.0,
  measurable union of 1.0
homogeneous, weakly - 3.3
Hopf’s ergodic theorem 2.2
Hopf’s lemma 7.3
Hopf-Tsuji theorem 7.5
horocycle 7.3
Hurewicz’s ergodic theorem 2.2
hyperbolic distance 7.1
incidence graph 4.1
inner function 6.1, 6.2
invariant factor 2.2
invariant measure 1.5,
  maximally supported 1.5
inverse theorem 3.6
invertible map 1.0
invertible, locally 1.0
irreducible 4.1 4.5
isomorphism 3.1
isomorphism of measure spaces 1.0
isomorphism 1.0
joining,
  algebraic 8.5,
  group- 8.5,
  self- 8.5
jump,
  distribution 5.5,
  transformation 4.1
Kac’s formula 1.5
Kaluza sequence 5.3
Karamata’s Tauberian theorem 3.6
Kolmogorov’s zero-one law 1.2
Komlos’ theorem 1.4
Krengel’s theorem 1.4
Kronecker transformation 1.2
Kuratowski’s isomorphism theorem 1.0
law of large numbers 3.2
Lebesgue space 1.0
lifetime distribution 5.1
line elements 7.2
Lipschitz continuity, local 4.6
local,
  invertibility lemma 1.0,
  limit theorem 5.5
Lusin’s theorem 1.0
Maharam,
  extension 3.4,
  ’s recurrence theorem 1.1
Markov,
  interval map 4.3,
  map 4.2,
partition 4.2,  period of a state 4.1
property 3.7,  periods of invariant functions of a skew product 8.2
shift 4.1 4.3 4.5,  persistencies 8.2
tower, simple- 5.2  persistent state 4.1

maximal,  Poincaré series,
  ergodic theorem 2.2,  Abelian, asymptotic 7.5
  inequality 2.2  Poincaré’s recurrence theorem 1.1
mean ergodic theorem 2.1  Poisson measure 6.1
measurable,  pointwise,
  function 1.0,  convergence 2.2,
  image theorem 1.0,  dual ergodic 3.7,
  map 1.0  ergodic theorems 2.2

measure,  Polish space 1.0
  Cauchy 6.2,  Pommerenke’s theorem 6.3
  fibre 1.0,  positive,
  harmonic 6.1,  definite 2.5,
  Poisson 6.1  -null decomposition 1.4,

measure algebra 1.0,  part 1.4
  conjugacy 1.0  product type cocycles 8.4

measure preserving,  proper 4.1
  map 1.0,  pure 1.0
  transformation 1.0  random walk 5.5
mixing 2.5,  recurrence theorem,
  continued fraction 3.7,  Halmos’ 1.1,
  mild 2.5 2.7,  Maharam’s 1.1,
  topological 4.1,  Poincaré’s 1.1,
  weak 2.5 2.7  recurrent,

Mittag-Leffler distribution 3.6 5.2  event 5.2,
moment,  null 4.5,
  sequence 5.3,  positive 4.5,
  set 3.6  topologically 4.1

Möbius transformation 6.1 7.1  regular source 4.3
non-singular,  regularly varying function 3.6
  map 1.0,  relative normalisation lemma 3.2
  transformation 1.0, action 1.6  renewal process 5.1
odd function 6.4  renewal sequence, recurrent 5.1 ,equivalent 5.4
odometer 8.4  Renyi,
Osikawa’s theorem 2.6  inequality 1.1,
partial quotient 8.4  property 4.3
partition,
  basic 4.2,  resolvent 4.7
  Markov 4.2  restriction 6.1
period of a state 4.1  Return sequence 3.3
Rokhlin’s,  structure theorem 1.0,
INDEX

tower theorem 1.5
rotations of the circle 1.2
saturation 2.6
Schwarz,
-’s lemma 6.1,
-Pick lemma 6.1 7.1
Schweiger collection 4.3
self joining, ergodic, dilation of 8.5
separable measure space 1.0
similarity 3.1
skew product 8.1
slowly varying function 3.6,
representation of 3.6
spectral,
measure 2.5,
property 2.5,
radius theorem 4.7,
theorem, scalar- 2.6,
type 2.5
squashable 8.4,
completely- 8.6
stable,
distribution 5.2,
manifold 7.3
standard measurable space 1.0
standard measure space 1.0
stochastic,
ergodic theorem 2.2,
matrix 4.3
strong disjointness 3.1
strong distortion property 4.3
strong distributional convergence 3.6
symmetric 1.0
tail 1.0 1.2
Thaler’s assumptions 4.3
topological,
Markov shift 4.1
topologically,
mixing 4.1,
recurrent 4.1,
transitive 4.1
transfer function 8.1, partial 8.4
unicity,
of invariant probability 1.4,
of invariant measure 1.5
uniform set 3.8
universal measurability theorem 1.0
wandering,
rate 3.8,
set 1.1 1.6
weak distortion property 4.3