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An Introduction to Infinite Ergodic Theory

Jon Aaronson

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ABSTRACT. The book is about measure preserving transformations of infinite measure spaces. It could be of interest to mathematicians working in ergodic theory, probability and/or dynamical systems and should be accessible to graduate students.

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Contents

Preface

Infinite ergodic theory is the study of measure preserving transformations of infinite measure spaces (early references being [**Hop1**] and [**St**]). It is part of "non-singular ergodic theory", the more general study of non-singular transformations (since a measure preserving transformation is also a non-singular transformation).

Non-singular ergodic theory arose as an attempt to generalise the classical ergodic theory of probability preserving transformations. Its major success was the ratio ergodic theorem. Another side to the theory also developed concentrating on facts which are valid "in the absence of invariant probabilities".

This book is more concerned with properties specific to infinite measure preserving transformations.

It should be readable by anyone initiated to metric space topology and measure theoretic probability.

Some readers may like to begin by following an example and perhaps one of the simplest in the book is Boole's transformation $T : \mathbb{R} \to \mathbb{R}$ defined by $Tx = x - \frac{1}{x}$.

This is a conservative, exact measure preserving transformation of $\mathbb{R}$ equipped with Lebesgue measure; and for each absolutely continuous probability P on $\mathbb{R}$ and non-negative, integrable function $f : \mathbb{R} \to \mathbb{R}$ with unit integral,

$$P\Big(\Big[\sum_{k=0}^{n-1} f \circ T^k \le \frac{\sqrt{2n}}{\pi} t\Big]\Big) \to \frac{2}{\pi} \int_0^t e^{-\frac{s^2}{\pi}} ds$$

as $n \to \infty$.

The book begins with an introduction to basic non-singular ergodic theory (chapters 1 and 2), including recurrence behaviour, existence of invariant measures, ergodic theorems and spectral theory. One of the results in §2.4 is the collapse of absolutely normalised pointwise ergodic convergence for ergodic measure preserving transformations of infinite measure spaces.

This leaves a wide range of possible "ergodic behaviour" which is catalogued in chapter 3 mainly according to the yardsticks of intrinsic normalising constants, laws of large numbers and return sequences (the return sequence of Boole's transformation is $\frac{\sqrt{2n}}{\pi}$).

The rest of the book (excepting chapter 5) consists of illustrations of these phenomena by examples.

Markov maps which arise both in probability theory and in smooth dynamics are treated in chapter 4. They illustrate distributional convergence phenomena (mentioned above) as do the inner functions of chapter 6. Geodesic flows on hyperbolic surfaces were one of the first examples considered ([**Hop1**]), and these are treated in chapter 7. Some of the extremely pathological examples in the subject can be found in the chapter on cocycles and skew products (chapter 8). In chapter 5, there is a modest beginning to the classification theory.

There is a small (but insufficient) amount of probability preserving ergodic theory in the book, and I recommend the uninitiated reader to take advantage of the excellent books available on this subject, including [**Cor-Sin-Fom**], [**De-Gr-Sig**], [**Fu**], [**Mañ**], [**Parr2**], [**Pet**], [**Rudo**], [**Wa**].

The reader will no doubt find that many (but hopefully not the reader's favourite) topics are conspicuous by their absence. By way of excuse I can only say that some of these are better covered elsewhere, while others are deemed too advanced for an introduction and yet others are too "fresh" for a book (there being no time to write about them).

Lastly I come to the thanks. I would like to thank the people who worked with me on the topics described in the book (see bibliography). Without them, none of this would have been possible. Also I would like to thank my colleagues Gilat and Lemańczyk; and my student Omri Sarig who found mistakes in early versions (any remaining errors being my sole responsibility having been introduced subsequently while correcting mistakes).

Jon. Aaronson
Tel Aviv, October 1996

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Index

numbers refer to sections (i.e. a.b=§a.b)

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