Mathematical Surveys and Monographs

Volume 61

Boundary Value Problems and Symplectic Algebra for Ordinary Differential and Quasi-Differential Operators

W. Norrie Everitt
Lawrence Markus

American Mathematical Society

http://dx.doi.org/10.1090/surv/061

Selected Titles in This Series

61 **W. Norrie Everitt and Lawrence Markus,** Boundary value problems and symplectic algebra for ordinary differential and quasi-differential operators, 1999

60 **Iain Raeburn and Dana P. Williams,** Morita equivalence and continuous-trace C^*-algebras, 1998

59 **Paul Howard and Jean E. Rubin,** Consequences of the axiom of choice, 1998

58 **Pavel I. Etingof, Igor B. Frenkel, and Alexander A. Kirillov, Jr.,** Lectures on representation theory and Knizhnik-Zamolodchikov equations, 1998

57 **Marc Levine,** Mixed motives, 1998

56 **Leonid I. Korogodski and Yan S. Soibelman,** Algebras of functions on quantum groups: Part I, 1998

55 **J. Scott Carter and Masahico Saito,** Knotted surfaces and their diagrams, 1998

54 **Casper Goffman, Togo Nishiura, and Daniel Waterman,** Homeomorphisms in analysis, 1997

53 **Andreas Kriegl and Peter W. Michor,** The convenient setting of global analysis, 1997

52 **V. A. Kozlov, V. G. Maz′ya, and J. Rossmann,** Elliptic boundary value problems in domains with point singularities, 1997

51 **Jan Malý and William P. Ziemer,** Fine regularity of solutions of elliptic partial differential equations, 1997

50 **Jon Aaronson,** An introduction to infinite ergodic theory, 1997

49 **R. E. Showalter,** Monotone operators in Banach space and nonlinear partial differential equations, 1997

48 **Paul-Jean Cahen and Jean-Luc Chabert,** Integer-valued polynomials, 1997

47 **A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May (with an appendix by M. Cole),** Rings, modules, and algebras in stable homotopy theory, 1997

46 **Stephen Lipscomb,** Symmetric inverse semigroups, 1996

45 **George M. Bergman and Adam O. Hausknecht,** Cogroups and co-rings in categories of associative rings, 1996

44 **J. Amorós, M. Burger, K. Corlette, D. Kotschick, and D. Toledo,** Fundamental groups of compact Kähler manifolds, 1996

43 **James E. Humphreys,** Conjugacy classes in semisimple algebraic groups, 1995

42 **Ralph Freese, Jaroslav Ježek, and J. B. Nation,** Free lattices, 1995

41 **Hal L. Smith,** Monotone dynamical systems: an introduction to the theory of competitive and cooperative systems, 1995

40.3 **Daniel Gorenstein, Richard Lyons, and Ronald Solomon,** The classification of the finite simple groups, number 3, 1998

40.2 **Daniel Gorenstein, Richard Lyons, and Ronald Solomon,** The classification of the finite simple groups, number 2, 1995

40.1 **Daniel Gorenstein, Richard Lyons, and Ronald Solomon,** The classification of the finite simple groups, number 1, 1994

39 **Sigurdur Helgason,** Geometric analysis on symmetric spaces, 1994

38 **Guy David and Stephen Semmes,** Analysis of and on uniformly rectifiable sets, 1993

37 **Leonard Lewin, Editor,** Structural properties of polylogarithms, 1991

36 **John B. Conway,** The theory of subnormal operators, 1991

35 **Shreeram S. Abhyankar,** Algebraic geometry for scientists and engineers, 1990

34 **Victor Isakov,** Inverse source problems, 1990

33 **Vladimir G. Berkovich,** Spectral theory and analytic geometry over non-Archimedean fields, 1990

32 **Howard Jacobowitz,** An introduction to CR structures, 1990

Mathematical
Surveys
and
Monographs
Volume 61

Boundary Value Problems and Symplectic Algebra for Ordinary Differential and Quasi-Differential Operators

W. Norrie Everitt
Lawrence Markus

American Mathematical Society

1991 *Mathematics Subject Classification.* Primary 34B05, 34L05, 58F05;
Secondary 11E39, 47B25, 47E05.

ABSTRACT. In the classical theory of self-adjoint boundary value problems for linear ordinary differential operators there is a fundamental, but rather mysterious, interplay between the (conjugate) symmetric scalar product of the basic Hilbert function space and the skew-symmetric boundary form of the associated differential expression. Our monograph explores and exploits this interplay, for both regular and singular problems, by an innovative new conceptual framework which relates the Hilbert space to complex symplectic spaces (non-trivial generalizations of the real symplectic spaces of Lagrangian analytical dynamics), and by important simplifications introduced through quasi-differential operators. These new approaches, which incorporate and generalize well-known standard theories and results, present an effective structured method for analyzing and classifying all such self-adjoint boundary conditions.

Library of Congress Cataloging-in-Publication Data

Everitt, W. N. (William Norrie), 1924–

Boundary value problems and symplectic algebra for ordinary differential and quasi-differential operators / W. Norrie Everitt, Lawrence Markus.

p. cm. — (Mathematical surveys and monographs, ISSN 0076-5376 ; v. 61)

Includes bibliographical references (p. –) and index.

ISBN 0-8218-1080-4 (alk. paper)

1. Boundary value problems. 2. Differential operators. 3. Symplectic manifolds. I. Markus, L. (Lawrence), 1922– . II. Title. III. Series: Mathematical surveys and monographs ; no. 61.

QA379.E94 1998

515′.35—dc21 98-25674

CIP

∞ The paper used in this book is acid-free and falls within the guidelines
established to ensure permanence and durability.
Visit the AMS home page at URL: http://www.ams.org/

10 9 8 7 6 5 4 3 2 1 04 03 02 01 00 99

Dedicated to the Memory of
ANDREW LAWRENCE MARKUS
1954–1995

Contents

Preface

The origins of this monograph lie, firstly, in the pioneering contributions from H. Weyl, J. von Neumann, M.H. Stone, E.C. Titchmarsh, K. Kodaira to the theory of linear differential operators in Hilbert function spaces and, secondly, in the significant contributions made by the Ukrainian (former Soviet Union) mathematicians M.G. Krein, M.A. Naimark and I.M. Glazman to the study of boundary value problems for linear, ordinary quasi-differential equations on any real interval.

The results of Glazman in his seminal memoir of 1950, influenced by both Krein and Naimark, led to the now-named GKN theorem within the general theory of quasi-differential operators (which include and generalize the classical linear ordinary differential operators) in Hilbert function spaces. The significant contribution from Glazman, mirrored in part by the work (also in 1950) of Kodaira, led to the then new formulation of boundary conditions required to construct self-adjoint differential operators, representing the boundary value problem. The original GKN theorem is stated for real-valued, thereby necessarily of even order, quasi-differential expressions; the theorem gives an elegant, necessary and sufficient condition for Lagrange symmetric differential expressions to generate self-adjoint operators in the appropriate Hilbert space of functions on the prescribed real interval.

The Glazman idea is to represent the homogeneous boundary conditions in terms of the skew-symmetric, sesquilinear form associated with the quasi-differential expression and the corresponding Green's formula; the quasi-differential expressions are now known to define a real symplectic space, and the boundary conditions to correspond to Lagrangian subspaces of this symplectic space, as recently recognized and realized by the current authors. The properties of these real symplectic spaces, and their geometry and symplectic linear algebra, have long been advanced by mathematicians and physicists in a number of different applications; in particular in Lagrangian analytical dynamics and quantum theory.

The original GKN theory was confined to the real-valued, quasi-differential expressions of arbitrary even order. However complex-valued quasi-differential expressions, of arbitrary (positive) integer order, had been studied earlier by Halperin and Shin, and later by Everitt and Zettl. In the years following the untimely death of Glazman in 1968 these complex expressions have been extensively studied, with particular reference to the Lagrange symmetric (formally self-adjoint) expressions. This formulation of the GKN Theorem has consequently been extended to these complex quasi-differential expressions of arbitrary integer order; however this extension has required the introduction and study of linear complex symplectic geometries and the algebra of their Lagrangian subspaces, as defined and described in these pages. The consequences of this study are to be seen in the contents of this monograph.

Two special comments are called for in respect of these complex symplectic spaces:

1. The complex spaces have a much richer structure and range of properties in comparison with the real spaces. Real symplectic spaces exist in even dimensions only, and moreover there is a unique real space (up to symplectic isomorphism) in each such even dimension. Every real symplectic space can be complexified to a complex symplectic space of the same even (complex) dimension. However there exist even-order complex symplectic spaces that are not the complexification of any real space, and there exist different complex symplectic spaces of each odd integer order.

2. The complex, Lagrange symmetric, quasi-differential expressions, of arbitrary positive integer order n, also have additional structures in comparison with the corresponding real expressions. The most significant property, in this respect, is that the complex expressions lead to minimal, closed symmetric operators, defined in the appropriate Hilbert function space, which can have unequal deficiency indices; these indices are now re-interpreted as algebraic invariants of the corresponding complex symplectic space [see Section III, Theorem 1]. Of course, such a minimal symmetric operator has self-adjoint extensions if and only if the two deficiency indices are of the same value, say a non-negative integer d, which is less than or equal to the order n. In fact, an informal paraphrase of our new version of the GKN Theorem asserts (for a precise statement see Section II, Theorem 1):

Each such self-adjoint operator is specified explicitly by a Lagrangian d-space within the corresponding boundary complex symplectic 2d-space, and conversely each Lagrangian d-space corresponds to exactly one such self-adjoint operator.

However in our detailed analysis of the kinds of boundary conditions that can occur we partition the basic interval into left and right sub-intervals, on each of which the restricted differential operator may have unequal deficiency indices. Thus the full range for the deficiency indices, equal or not, plays an important role in the theory (compare Section V, Proposition 1 with the Weyl-Kodaira formulas and the Deficiency Index Conjecture 2).

It can be argued that complex symplectic spaces have richer structures in order to support the extensive properties of complex quasi-differential expressions; vice versa there is a case to state that these complex differential expressions force the structure of the complex symplectic spaces to exist in order to support their properties.

The two main and significant consequences of writing this research monograph are:

1. There is now a complete and connected account of the geometric and algebraic structure of real and complex symplectic spaces and their Lagrangian subspaces, for all integer orders, with special attention to the algebraic properties of direct sum decompositions such as are relevant for the study of boundary conditions, especially with regard to properties of separation or coupling at the boundary endpoints.

2. There is a complete account of the canonical form of all possible symmetric boundary conditions (with respect to separation or coupling at the endpoints) for the extended GKN theory of Lagrange symmetric, linear, quasi- differential expressions (real or complex) of all integer orders on arbitrary real intervals.

In addition to this main text there are two substantial appendices. The first deals with the canonical form of classical ordinary differential expressions when these are considered as quasi-differential expressions and then settles certain technical questions concerning adjoint operators; the second treats the problems of the complexification of real symplectic spaces, and the analysis of self-adjoint operators which are non-real yet arise from real differential expressions.

In all these areas the authors have made significant and extensive new contributions, in addition to re-organizing established theories into a satisfying synthesis with the results within this monograph. As an illustration of our approach and of some of the new results, we offer here two very specific and explicit findings:

(i) The balanced intersection principle (Section III, Theorem 3 for precise details) provides an algebraic criterion for describing and classifying the divers kinds of self-adjoint boundary conditions for quasi-differential expressions of arbitrary integer order, and for all boundary value problems whether regular on compact intervals or singular on general intervals. In particular the coupling grade is defined for each Lagrangian d-space (and hence for the corresponding self-adjoint operator) and from this we deduce the minimal number of coupled boundary conditions necessary in the specification of the operator domain. For a regular problem of arbitrary positive order, on a compact interval, there is always the same number of separated boundary conditions at the left endpoint as at the right endpoint of the interval (assuming that minimal coupling is employed).

For singular problems this is not necessarily the case; however there is an arithmetic formula relating the number of separated boundary conditions at each of the two endpoints with the invariants of the left and right endpoint complex sympletic spaces. For example, consider a Lagrange symmetric real quasi-differential expression of order four on the closed half-line $[0, \infty)$. We find [see Table 3 and Example 3 of Section V] that the common deficiency index d can take the values 2,3 or 4, which generalizes the limit-point and limit-circle classifications that Weyl defined for second-order differential expressions. As an indication of the explicit nature of our calculations and tabulations, we mention that it is then possible to have three (independent) boundary conditions, when $d = 3$, to define a self-adjoint operator, with one separated at the left end and two coupling the ends, but it is impossible to define a self-adjoint operator by one separated condition at the left end and two separated at the right end.

(ii) Lagrange symmetric quasi-differential expressions that are real can determine self-adjoint operators that are real (definable by real boundary conditions) or else complex operators that are non-real. An investigation of this phenomenon is conducted in Appendix B, where the complexification of real symplectic spaces, and the associated concept of self-conjugate Lagrangian subspace, are described in great detail.

We provide an affirmative answer [Appendix B, Theorem 3] to a long-standing open question concerning the existence of real differential expressions of even order ≥ 4, for which there are non-real self-adjoint differential operators specified by strictly separated boundary conditions, *i.e.* complex Lagrangian subspaces which are not self-conjugate and which have coupling grade zero; in fact, we prove the existence of such Lagrangian subspaces of every possible prescribed coupling grade. This is somewhat surprising because it is well known that for order $n = 2$ strictly separated boundary conditions can produce only real operators (that is, any such given complex boundary conditions can always be replaced by real boundary conditions). Our analyses and examples are entirely explicit for regular problems on compact intervals; moreover corresponding explicit results also hold in the general singular case.

In undertaking this project we have reviewed the theory of both differential and quasi-differential expressions and have assembled all relevant information in the opening sections of this monograph to provide a convenient source of reference. In particular we have put together details of the connections between classical differential expressions and the extended class of quasi-differential expressions.

In respect of symmetric boundary problems for these differential expressions, and the associated self-adjoint differential operators, there is complete generality; regular problems on compact intervals and the more general singular problems are all treated in full detail. From the algebraic data all the classification results for boundary conditions then follow; thereby the infinite dimensional functional analysis is reduced to finite dimensional linear symplectic algebra.

The introduction of these algebraic and geometric methods has led to the discovery of new kinds of qualitative insight into the topology of the boundary value problem in terms of the Lagrange-Grassmannian manifold.

The axiomatic formulation of these mathematical structures leads immediately to applications for other types of boundary value problems such as the multi-interval or interfacial conditions of the multi-particle systems of quantum mechanics, or the general theory of linear elliptic partial differential equations; these applications depend on the extension of the ideas considered in this monograph to infinite dimensional, complex sympletic spaces.

In concluding this work we have to survive a disappointment. It had been our hope at the start of these labors that the algebra of complex sympletic spaces would throw new light on the "deficiency index conjecture" for complex quasi-differential operators. This has not been the case and the so-called range conjecture, formulated precisely in this work, remains unsolved. We have little doubt that the conjecture is true. While our analysis and classification of symmetric boundary conditions do not rest on the validity of this conjecture, we have occasionally used it (with appropriate warnings) to give insight and guidance into the search for new properties and interrelations among classes of such boundary value problems.

Acknowledgments. The authors express their indebtedness to David Race and Tony Zettl for their contributions to the study of linear ordinary differential expressions and boundary value problems; their results have significantly contributed to the content of this monograph.

W. Norrie Everitt

Department of Mathematics and Statistics, University of Birmingham, Birmingham, B15 2TT, England

Lawrence Markus

School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455 USA and

Mathematics Institute, University of Warwick, Coventry, CV4 7AL, England

August 1998

APPENDIX A

Constructions for quasi-differential operators

In this Appendix we shall construct and exhibit explicit formulas for quasi-differential expressions M_A, and the corresponding Shin-Zettl matrices $A \in Z_n(\mathfrak{I})$, needed to represent various prescribed classical differential expressions M, following the concepts and notations introduced in Section I.

We recall, from Section I:

EXAMPLE 1. Classical differential expression of order $n \geq 2$,

$$M[y] = p_n y^{(n)} + p_{n-1} y^{(n-1)} + \cdots + p_1 y' + p_0 y, \tag{A.1}$$

with complex coefficients $p_j \in \mathcal{L}^1_{\mathrm{loc}}(\mathfrak{I})$ for $j = 0, 1, \ldots, n-1$, and furthermore $p_n \in AC_{\mathrm{loc}}(\mathfrak{I})$ with $p_n(x) \neq 0$ for all x on the real interval $\mathfrak{I}$.

The domain of M (as a linear transformation $\mathcal{D}(M) \to \mathcal{L}^1_{\mathrm{loc}}(\mathfrak{I})$) is:

$$\mathcal{D}(M) = \{y : \mathfrak{I} \to \mathbb{C} \mid y^{(r)} \in AC_{\mathrm{loc}}(\mathfrak{I}) \quad \text{for } r = 0, 1, \cdots, n-1\}. \tag{A.2}$$

EXAMPLE 2. Quasi-differential expression for $A \in Z_n(\mathfrak{I})$ of order $n \geq 2$,

$$M_A[y] = i^n y_A^{[n]} \tag{A.3}$$

with domain $\mathcal{D}(M_A)$ (also denoted $\mathcal{D}(A)$),

$$\mathcal{D}(A) = \{y : \mathfrak{I} \to \mathbb{C} \mid y_A^{[r]} \in AC_{\mathrm{loc}}(\mathfrak{I}) \quad \text{for } r = 0, 1, \ldots, n-1\}. \tag{A.4}$$

Following a general construction for $A \in Z_n(\mathfrak{I})$, exhibiting the existence of M_A for each such M, we shall give further illustrations to show that such $A \in Z_n(\mathfrak{I})$ are not unique. Then we also construct matrices $A \in Z_n(\mathfrak{I})$ for which the corresponding quasi-differential expression M_A does not represent any classical differential expression M of the form described above in Example 1.

In all these constructions we pay special attention to the case when M_A (or correspondingly M) is formally self-adjoint on $\mathfrak{I}$, as in Section I(1.8),

$$\int_{\mathfrak{I}} \{M_A[f]\bar{g} - f\overline{M_A[g]}\} dx = 0, \tag{A.5}$$

for all $f, g \in \mathcal{D}_0(M_A)$ (also denoted $\mathcal{D}_0(A)$) where

$$\mathcal{D}_0(M_A) = \{y \in \mathcal{D}(M_A) \mid \text{compact supp}\, y \text{ lies interior to } \mathfrak{I}\}. \tag{A.6}$$

This definition Section I(1.8), following the treatment of Frentzen [**FR**], [**ER**], is not easily interpreted as an explicit condition on the entries of the matrix $A \in Z_n(\mathfrak{I})$—especially when these entries are non-differentiable on $\mathfrak{I}$. However, there is an important special case when A is Lagrange symmetric, see Section I.(2.13) with further details in [**EV**]:

(A.7) $$A = A^{+} \qquad (\text{where } A^{+} :\equiv -\Lambda_n^{-1} A^{*} \Lambda_n, \text{ as in Section I}(2.14))$$

whereupon

(A.8) $$M_A = M_A^{+} \qquad (\text{where } M_A^{+} :\equiv M_{A^{+}}),$$

and from this it follows that M_A is formally self-adjoint. A partial converse, given by Frentzen [**FR**], [**ER**], is the much more difficult result, stated here without proof:

PROPOSITION 1. *Let M_B, for $B \in Z_n(\mathfrak{I})$, be formally self-adjoint. Then there exists a matrix $A \in Z_n(\mathfrak{I})$ such that $A = A^{+}$, with*

$$M_B = M_A \qquad \text{on } \mathfrak{D}(B) = \mathfrak{D}(A) \qquad (\text{for } n \text{ even})$$

and

$$M_B = M_A \text{ or } M_B = -M_A \text{ on } \mathfrak{D}(B) = \mathfrak{D}(A) \ (\text{for } n \text{ odd}).$$

Hence, if we are interested only in the quasi-differential expression M_B, as an operator on $\mathfrak{D}(B)$, then we can replace B by $A = A^{+}$ without any loss of generality (and by dealing with $-M_B$, if necessary, when n is odd). However, we should note that the quasi-derivatives $y_B^{[r]}$ may not be equal to the corresponding $y_A^{[r]}$ for $r = 0, 1, \ldots, n-1$ in Proposition 1. Furthermore it can happen that $\mathfrak{D}(B) \neq \mathfrak{D}(B^{+})$ (although clearly $\mathfrak{D}(A) = \mathfrak{D}(A^{+})$ for $A = A^{+}$), and we shall demonstrate these phenomena later.

At the end of this Appendix we present a proof of the Density Theorem (see Section I(1.15)):

(A.9)

Let $A = A^{+} \in Z_n(\mathfrak{I})$, so M_A is formally self-adjoint on $\mathfrak{I}$.

Then the corresponding linear manifold

$\mathfrak{D}_0(T_1) = \mathfrak{D}(T_1) \cap \mathfrak{D}_0(A)$ is dense in the Hilbert space $\mathfrak{L}^2(\mathfrak{I}; w)$.

Having outlined our program, we now proceed with:

EXHIBIT 1. Construct $A \in Z_n(\mathfrak{I})$ so $M_A = M$ on $\mathfrak{D}(A) = \mathfrak{D}(M)$, when M is given as in Example 1 above:

$$M[y] = p_n y^{(n)} + p_{n-1} y^{(n-1)} + \cdots + p_1 y' + p_0 y, \quad \text{for } y \in \mathfrak{D}(M).$$

We define the required $n \times n$ matrix $A \in Z_n(\mathfrak{I})$ by

(A.10)
$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & 0 & 0 & 1 & 0 & \cdots & & \vdots \\ & & & & \ddots & & & \\ & & & & & \ddots & & \\ & & & & & 1 & 0 & 0 \\ & & & & & 0 & 1 & 0 \\ 0 & 0 & 0 & & \cdots & 0 & 0 & i^n p_n^{-1} \\ -i^{-n} p_0 & -i^{-n} p_1 & -i^{-n} p_2 & & \cdots & & -i^{-n} p_{n-2} & (p_n' - p_{n-1}) p_n^{-1} \end{bmatrix},$$

where all entries not specified or indicated are zero.

This "nearly companion" matrix A in (A.10), in which we have incorporated the factors i^n in anticipation of the desired equality

$$M_A[y] = i^n y_A^{[n]} = M[y], \tag{A.11}$$

clearly belongs to $Z_n(\mathfrak{I})$ since $p_n^{-1} \in \mathfrak{L}^1_{\mathrm{loc}}(\mathfrak{I})$ and $p_n(x)^{-1}$ vanishes for no $x \in \mathfrak{I}$. In fact, $p_n^{-1} \in AC_{\mathrm{loc}}(\mathfrak{I})$, because $p_n \in AC_{\mathrm{loc}}(\mathfrak{I})$ and $p_n(x) \neq 0$ on $\mathfrak{I}$.

We compute the quasi-derivatives $y_A^{[r]}$ to find:

$$y_A^{[r]} = y^{(r)} \text{ for } r = 0, 1, \ldots, n-2 \tag{A.12}$$

and then

$$y^{(n-1)} = y_A^{[n-2]'} = i^n p_n^{-1} y_A^{[n-1]}. \tag{A.13}$$

Hence

$$\mathfrak{D}(M) = \mathfrak{D}(M_A). \tag{A.14}$$

Moreover

$$y_A^{[n]} = y_A^{[n-1]'} + i^{-n}\{p_0 y + p_1 y' + \cdots + p_{n-2} y^{(n-2)}\} - (p_n' - p_{n-1}) p_n^{-1} i^{-n} p_n y^{(n-1)}$$

so

$$M_A[y] = i^n y_A^{[n]} = (p_n y^{(n-1)})' + \{p_0 y + p_1 y' + \cdots + p_{n-1} y^{(n-1)}\} - p_n' y^{(n-1)}$$

and then

$$M_A[y] = i^n y_A^{[n]} = M[y] \qquad \text{for } y \in \mathfrak{D}(M) = \mathfrak{D}(A), \text{ as required.}$$

However, even if M is formally self-adjoint on $\mathfrak{I}$, the "nearly companion" matrix A of Exhibit 1 (A.10) fails to satisfy the condition $A = A^+$, and a different choice of the matrix $A \in Z_n(\mathfrak{I})$ will be presented later, with $M_A = M$.

EXHIBIT 2. Constructions demonstrating the non-uniqueness of $A \in Z_n(\mathfrak{I})$, such that $M_A = M$, and the non-existence of M for certain M_A—particularly in low orders $n = 2, 3, 4$.

Consider first the classical differential operator of order $n = 2$, see [**DS**],

$$M[y] = (p_1 y')' + p_0 y + i[(q_0 y)' + q_0 y'], \tag{A.15}$$

which is the most general self-adjoint operator with suitably smooth coefficients p_1, p_0, q_0 (real functions, say, p_1 and $q_0 \in AC_{\mathrm{loc}}(\mathfrak{I})$, and $p_0 \in \mathfrak{L}^1_{\mathrm{loc}}(\mathfrak{I})$—with $p_1(x) \neq 0$ for $x \in \mathfrak{I}$). If we set $p = -p_1$, $q = p_0$ and $q_0 = 0$, we obtain the real Sturm-Liouville operator

$$M[y] = -(py')' + qy, \qquad \text{for } y \in \mathfrak{D}(M), \tag{A.16}$$

$$\mathfrak{D}(M) = \{y : \mathfrak{I} \to \mathbb{C} \mid y \text{ and } y' \in AC_{\mathrm{loc}}(\mathfrak{I})\}. \tag{A.17}$$

We shall construct various matrices $A \in Z_2(\mathfrak{I})$ to represent M, that is, $M_A[y] = M[y]$ on $\mathfrak{D}(A) = \mathfrak{D}(M)$. These constructions will accompany Propositions 2 and 3.

REMARK. Note the change of notation from Example 1, for the coefficients of

$$M[y] = p_1 y'' + [p_1' + 2iq_0]y' + [p_0 + iq_0']y, \tag{A.18}$$

in order to conform to the usual literature [**DS**] on self-adjoint differential operators with complex coefficients.

PROPOSITION 2. *Consider the most general matrix in* $Z_2(\mathfrak{I})$*:*

$$A = \begin{pmatrix} r & p^{-1} \\ q & -\bar{r} + i\delta \end{pmatrix} \tag{A.19}$$

for complex p^{-1}, q, r, δ.

Then $A = A^+$ *if and only if the two conditions hold:*

$$\begin{aligned} &\text{(i)} \quad p \text{ and } q \text{ are real} \\ &\text{(ii)} \quad \delta \equiv 0. \end{aligned} \tag{A.20}$$

Furthermore, consider the most general "smooth" matrix in $Z_2(\mathfrak{I})$*:*

$$B = \begin{pmatrix} r & p^{-1} \\ q & -\bar{r} + i\delta \end{pmatrix} \in Z_2(\mathfrak{I}) \qquad \textit{(as above)}, \tag{A.21}$$

with $p^{-1}(x) \neq 0$ *in* $C^2(\mathfrak{I})$ *and the other entries in* $C^1(\mathfrak{I})$.

Then the quasi-differential expression M_B *is formally self-adjoint if and only if the two conditions hold:*

$$\begin{aligned} &\text{(iii)} \quad p \text{ and } \delta \text{ are real} \\ &\text{(iv)} \quad \operatorname{Im} q = \frac{(p\delta)'}{2} + (p\delta)(\operatorname{Re} r). \end{aligned} \tag{A.22}$$

PROOF. It is trivial that each matrix $A \in Z_2(\mathfrak{I})$ has the given form. Then we compute

$$A^+ = -\Lambda_2^{-1} A^* \Lambda_2 = \begin{pmatrix} r + i\bar{\delta} & \bar{p}^{-1} \\ \bar{q} & -\bar{r} \end{pmatrix}.$$

Thus $A = A^+$ if and only if $p = \bar{p}$, $q = \bar{q}$, $\delta \equiv 0$ so p and q are real. But this conclusion yields (i) and (ii) above.

Next consider the most general smooth matrix $B \in Z_2(\mathfrak{I})$ with complex entries p^{-1}, q, r, δ such that $p(x) \neq 0$ for all $x \in \mathfrak{I}$, and $p \in C^2(\mathfrak{I})$ with the other entries in $C^1(\mathfrak{I})$.

Compute the corresponding quasi-derivatives $y_B^{[r]}$ and the quasi-differential expression

$$M_B[y] = i^2 y_B^{[2]} \qquad \text{for } y \in \mathfrak{D}(B).$$

Here

$$y_B^{[0]} = y, \; y' = ry + p^{-1} y_B^{[1]} \quad \text{so } y_B^{[1]} = p(y' - ry).$$

Hence $\mathfrak{D}(B) = \{y : \mathfrak{I} \to \mathbb{C} \mid y \text{ and } y' \text{ in } AC_{\mathrm{loc}}(\mathfrak{I})\}$. Furthermore

$$y_B^{[2]} = y_B^{[1]\prime} - qy - (-\bar{r} + i\delta) y_B^{[1]}$$

so

$$M_B[y] = -y_B^{[2]} = -py'' + [p(r - \bar{r}) + ip\delta - p']y' + [(pr)' - ipr\delta + q + pr\bar{r}]y,$$

and then define the classical differential operator $M = M_B$ on $\mathfrak{D}(B)$.

Now recall Lagrange's necessary and sufficient condition that $M = M^+$ be formally self-adjoint—namely, the three equalities:

$$\begin{aligned} &p = \bar{p} \\ &\operatorname{Re}[p(r-\bar{r}) + ip\delta - p'] = -p' \\ &2\operatorname{Im}[(pr)' - ipr\delta + q + pr\bar{r}] = \operatorname{Im}[p(r-\bar{r}) + ip\delta - p']'. \end{aligned}$$

The first of these equalities is equivalent to the assertion that:

$$p(x) \neq 0 \text{ is real on } \mathfrak{I}.$$

The second can be then re-written $\operatorname{Re}[ip\delta] = 0$, which is equivalent to the assertion

$$\delta \text{ is real on } \mathfrak{I}.$$

On this basis the third equality is equivalent to the assertion

$$\operatorname{Im} q = \frac{(p\delta)'}{2} + (p\delta)(\operatorname{Re} r).$$

Thus the Lagrange conditions, that $M = M_B$ be formally self-adjoint, are equivalent to the two conditions (iii) and (iv) of the second part of the proposition. □

In the case of real matrices in $Z_2(\mathfrak{I})$, the Proposition 2 yields the general forms

$$A = \begin{pmatrix} r & p^{-1} \\ q & -r \end{pmatrix} \qquad \text{so } A = A^+, \text{ (in A.19);} \tag{A.23}$$

and also the same form for B, (in (A.21)) with real smooth entries, such that M_B is formally self-adjoint. Thus a real smooth matrix $B \in Z_2(\mathfrak{I})$ yields a formally self-adjoint quasi-differential expression if and only if $B = B^+$.

As a particular instance, where $r = 0$,

$$B = \begin{pmatrix} 0 & p^{-1} \\ q & 0 \end{pmatrix}, \text{ with real smooth } p(x) \neq 0 \text{ and } q \text{ on } \mathfrak{I}, \tag{A.23*}$$

defines the classical Sturm-Liouville operator $M = M_B$,

$$M[y] = -(py')' + qy \qquad \text{on } \mathfrak{D}(M), \text{ (as in (A.16)),}$$

where $\mathfrak{D}(M) = \mathfrak{D}(B)$—since $y_B^{[1]} = py'$ with p and $p^{-1} \in AC_{\text{loc}}(\mathfrak{I})$. But we can consider more general types of Sturm-Liouville operators, upon extending the domain $\mathfrak{D}(M)$, by allowing arbitrary real $p \in AC_{\text{loc}}(\mathfrak{I})$ which vanish at certain points, yet with $p^{-1} \in \mathfrak{L}^1_{\text{loc}}(\mathfrak{I})$—say $p(x) = \sqrt{|x|}$ on $\mathfrak{I} = \mathbb{R}$. More spectacularly, take $p(x) > 0$ to be everywhere continuous but nowhere differentiable on $\mathfrak{I}$. In this last case M_B *is a well defined quasi-differential expression that does not correspond to any classical differential expression* M—because

$$y_B^{[1]} = py' \notin AC_{\text{loc}}(\mathfrak{I}) \text{ for some choices of } y' \in AC_{\text{loc}}(\mathfrak{I}).$$

To illustrate the possibilities of the non-uniqueness of $A \in Z_2(\mathfrak{I})$ for which the corresponding quasi-differential expressions realize a given classical differential expression M, we start with the complex self-adjoint operator

$$M[y] = (p_1 y')' + p_0 y + i[(q_0 y)' + q_0 y'] \quad \text{(as in (A.15)),}$$

where the real coefficients p_1, p_0, q_0 are suitably smooth and $p_1(x) \neq 0$ for $x \in \mathfrak{I}$. Namely, for each real number c define

$$A_c = \begin{bmatrix} (-iq_0 - c)p_1^{-1} & -p_1^{-1} \\ p_0 + (q_0^2 + c^2)p_1^{-1} & (-iq_0 + c)p_1^{-1} \end{bmatrix} \in Z_2(\mathfrak{I}). \tag{A.24}$$

Then a direct computation yields $A_c = A_c^+$ and

$$M_{A_c} = M \qquad \text{on } \mathcal{D}(M_{A_c}) = \mathcal{D}(M), \text{ as in (A.15)}. \tag{A.25}$$

The case $c = 0$ is often written as

$$\begin{pmatrix} r & p^{-1} \\ q & -\bar{r} \end{pmatrix}, \tag{A.26}$$

where $p = -p_1$, $q = p_0 + q_0^2 p_1^{-1}$, and $ir = q_0 p_1^{-1}$ are all real. The restriction to the corresponding real case (A.23), where $q_0 = 0$, also produces *illustrations of the non- uniqueness (for* A.16), *for real values of* c *in* M_{A_c}, *and* $c = 0$ *yields* (A.23$*$).

As before, upon weakening the smoothness hypotheses in (A.25)—say take q_0 to be continuous but nowhere differentiable on $\mathfrak{I}$ (even allowing p_0 and p_1 in $C^\infty(\mathfrak{I})$), we obtain quasi-differential expressions M_{A_c} which do not equal any classical differential expression of the type prescribed in Example 1 above.

The significance of this family A_c of Shin-Zettl matrices in $Z_2(\mathfrak{I})$ is explained in the following proposition.

PROPOSITION 3. *Let* $\varphi(p_1, p_0, q_0)$, $\psi(p_1, p_0, q_0)$, $\chi(p_1, p_0, q_0)$ *and* $\omega(p_1, p_0, q_0)$ *be complex rational functions of the three variables* (p_1, p_0, q_0). *Assume that for each triple of real smooth functions* $p_1(x) \neq 0$, p_0, $q_0 \in C^\infty(\mathfrak{I})$, *the matrix*

$$A(x) = \begin{pmatrix} \varphi(x) & \psi(x)^{-1} \\ \chi(x) & \omega(x) \end{pmatrix} = A^+(x) \in Z_2(\mathfrak{I}), \tag{A.27}$$

where $\varphi(x) = \varphi(p_1(x), p_0(x), q_0(x))$ *and similarly for* $\psi(x)$, $\chi(x)$, *and* $\omega(x)$.

Assume also that for each such matrix $A = A(x)$

$$M_A[y] = i^2 y_A^{[2]} = (p_1 y')' + p_0 y + i[(q_o y)' + q_0 y'] \tag{A.28}$$

for all $y \in C^\infty(\mathfrak{I})$.

Then there exists a real constant c *so*

(A.29)

$$\begin{aligned} &\varphi(p_1, p_0, q_0) = (-i\, q_0 - c)p_1^{-1}, \quad \psi(p_1, p_0, q_0) = -p_1 \\ &\chi(p_1, p_0, q_0) = p_0 + (q_0^2 + c^2)p_1^{-1}, \ \omega(p_1, p_0, q_0) = (-iq_0 + c)p_1^{-1}, \end{aligned}$$

and (A.27) reduces to the matrix in (A.24).

PROOF. For each triple $(p_1(x), p_0(x), q_0(x))$ construct the matrix $A = A(x) \in Z_2(\mathfrak{I})$ with the corresponding quasi-derivatives

$$\begin{aligned} y_A^{[1]} &= \psi[y' - \varphi y] \\ y_A^{[2]} &= \psi'[y' - \varphi y] + \psi[y'' - \varphi' y - \varphi y'] - \chi y - \omega\psi[y' - \varphi y] \end{aligned}$$

and then set

$$M_A[y] = -y_A^{[2]} = p_1 y'' + p_1' y' + p_0 y + i[q_0' y + 2q_0 y'], \tag{A.30}$$

for all smooth $y \in C^\infty(\mathfrak{I})$.

For each such fixed triple (p_1, p_0, q_0), the values y, y', y'' can be treated as three real independent variables, at each $x \in \mathfrak{I}$. Thus the coefficient of y'' must define an identity for $x \in \mathfrak{I}$, namely:

$$-\psi(p_1(x), p_0(x), q_0(x)) \equiv p_1(x) \quad \text{so } \psi(p_1, p_0, q_0) \equiv -p_1. \tag{A.31}$$

Next the coefficient of y' yields the identity for $x \in \mathfrak{I}$

$$-\psi' + \psi\varphi + \omega\psi \equiv p_1' + 2iq_0$$

where

$$\psi'(x) = \frac{\partial \psi}{\partial p_1} p_1'(x) + \frac{\partial \psi}{\partial p_0} p_0'(x) + \frac{\partial \psi}{\partial q_0} q_0'(x) = -p_1'(x).$$

Then

$$\varphi + \omega = -2iq_0 p_1^{-1} \qquad \text{so} \quad \omega(p_1, p_0, q_0) \equiv -\varphi(p_1, p_0, q_0) - 2iq_0 p_1^{-1}. \tag{A.32}$$

Further the coefficient of y yields the identity

$$\psi'\varphi + \psi\varphi' + \chi - \omega\psi\varphi \equiv p_0 + iq_0'. \tag{A.33}$$

But the six variables $p_1, p_0, q_0, p_1', p_0', q_0'$ can be assigned independently at each $x \in \mathfrak{I}$, so (A.33) then implies that

$$\frac{\partial \varphi}{\partial p_0} \equiv 0, \ \frac{\partial \varphi}{\partial p_1} = -\varphi p_1^{-1}, \ \frac{\partial \varphi}{\partial q_0} = -p_1^{-1}$$

so

$$\varphi(p_1, p_0, q_0) \equiv -iq_0 p_1^{-1} - cp_1^{-1} \qquad \text{for a complex constant } c. \tag{A.34}$$

Finally the hypotheses $A = A^+ \in Z_2(\mathfrak{I})$, in accord with Proposition 2, imply that

$$\omega = -\bar{\varphi} \quad \text{and } \chi \text{ is real.}$$

Elementary calculations then verify that

$$\operatorname{Im} \varphi = -q_0 p_1^{-1}, \text{ so } c \text{ is necessarily a real number.}$$

Also

$$(-p_1\varphi)' + \chi - p_1\varphi\bar{\varphi} = p_0 + iq_0'.$$

Therefore

$$\chi = p_0 + p_1\varphi\bar{\varphi} = p_0 + [q_0^2 + c^2]p_1^{-1},$$

and the proposition is proved. □

The matrix (A.27) with (A.29) reduces to the familiar format A_c in (A.24), and in the real case when $q_0 = 0$,

$$A_c = \begin{bmatrix} -cp_1^{-1} & -p_1^{-1} \\ p_0 + c^2 p_1^{-1} & cp_1^{-1} \end{bmatrix} \quad \text{(real constant } c\text{)}. \tag{A.35}$$

Each of these real matrices $A_c \in Z_2(\mathfrak{I})$ determines a quasi-differential expression M_{A_c} which is equal to the classical Sturm-Liouville differential operator (A.16).

We next turn to a few examples of order $n = 3$ and $n = 4$ to illustrate these properties of nonuniqueness, and other unusual phenomena, for self-adjoint differential and quasi-differential expressions. Moreover the case $n = 3$ will serve as a prototype for all higher odd order cases, and $n = 4$ for higher even order cases—as exhibited later—although certain peculiarities arise in these very low orders.

Consider the general classical differential expression, formally self-adjoint of order $n = 3$, [**DS**]:

$$M[y] = (p_1 y')' + (p_0 y) + i[(q_1 y')^{(2)} + (q_1 y^{(2)})'] + i[(q_0 y)' + (q_0 y')], \tag{A.36}$$

where the real functions p_1, p_0, q_1, q_0 are suitably smooth on $\mathfrak{J}$ (say, $p_1 \in C^1$, $p_0 \in C^0, q_1 \in C^2$, $q_0 \in C^1$) with $q_1(x) \neq 0$ for all $x \in \mathfrak{J}$. We can re-write (A.36) in the customary form for a classical differential expression

$$M[y] = P_3 y^{(3)} + P_2 y^{(2)} + P_1 y' + P_0 y, \tag{A.37}$$

where the complex coefficients are

$$P_3 = 2iq_1,\ P_2 = p_1 + 3iq_1',\ P_1 = p_1' + iq_1^{(2)} + 2iq_0,\ P_0 = p_0 + iq_0'. \tag{A.38}$$

Then we use the nearly companion format (A.10) to compute the appropriate matrix $B \in Z_3(\mathfrak{J})$ so $M_B = M$ on $\mathfrak{D}(B) = \mathfrak{D}(M)$, namely we find

(A.39)

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -iP_3^{-1} \\ -iP_0 & -iP_1 & (P_3' - P_2)P_3^{-1} \end{bmatrix} \quad \text{and } B^+ = \begin{bmatrix} -(\bar{P}_3' - \bar{P}_2)\bar{P}_3^{-1} & i\bar{P}_3^{-1} & 0 \\ i\bar{P}_1 & 0 & 1 \\ -i\bar{P}_0 & 0 & 0 \end{bmatrix}.$$

It is clear that $B \neq B^+$ (except for trivial cases, $q_1 = -1/2$, $p_1 = 0$, $q_0 = 0$), and also $M_B \neq M_{B^+}$ (even though $\mathfrak{D}(B) = \mathfrak{D}(B^+)$, as can be seen by a direct compution and examination of the corresponding quasi-derivatives relative to B and B^+).

Since M is formally self-adjoint, so is $M_B = M$, even though $B \neq B^+$. However, by Proposition 1, there must exist a matrix $A \in Z_3(\mathfrak{J})$ with $A = A^+$, and either $M = M_A$ or $-M = M_A$ on $\mathfrak{D}(M) = \mathfrak{D}(A)$. This required choice of the sign for $\pm M$ is apparent in the next construction for a convenient description of such a matrix A, which we denote by A_3.

For the given self-adjoint classical differential expression $\pm M$ of (A.36), we define the corresponding matrix $A_3 \in Z_3(\mathfrak{J})$:

$$A_3 = \begin{bmatrix} 0 & \beta_1 & 0 \\ q_0\beta_1 & ip_1/2q_1 & \beta_1 \\ -ip_0 & q_0\beta_1 & 0 \end{bmatrix} \tag{A.40}$$

where $\beta_1 = (-2q_1)^{-1/2} > 0$. That is, we choose the one of M or $-M$ so that $q_1(x) < 0$ for all $x \in \mathfrak{J}$, and then β_1 is a real positive function of class $C^2(\mathfrak{J})$.

It is easy to compute that $A_3 = A_3^+$ and $\mathfrak{D}(A_3) = \mathfrak{D}(M)$. Indeed the quasi-derivatives, relative to A_3, are (omitting the subscript A_3 for simplicity):

$$y^{[1]} = \beta_1^{-1} y'$$
$$y^{[2]} = \beta_1^{-1}[(\beta_1^{-1}y')' - q_0\beta_1 y - \frac{ip_1}{2q_1}(\beta_1^{-1}y')]$$

so

$$y^{[2]} = \beta_1^{-1}(\beta_1^{-1}y')' - q_0\beta_1 y + ip_1 y'.$$

Now note the identity, which will be used again in higher order cases:

$$\beta_1^{-1}(\beta_1^{-1}y')' = -q_1'y' - 2q_1 y'' = -(q_1 y')' - q_1 y''. \tag{A.41}$$

Then continue to compute

$$y^{[2]} = -(q_1 y')' - q_1 y'' - q_0 y + i p_1 y'$$

and

$$y^{[3]} = y^{[2]\prime} + i p_0 y - q_0 \beta_1 y^{[1]}.$$

It is easy to see that if y, y', y'' are all $AC_{\mathrm{loc}}(\mathfrak{J})$, then so are $y^{[0]}$, $y^{[1]}$, $y^{[2]} \in AC_{\mathrm{loc}}(\mathfrak{J})$—and vice versa.

Hence we conclude that

(A.42)
$$M_{A_3}[y] = i^3 y^{[3]} = (p_1 y')' + (p_0 y) + i[(q_1 y')^{(2)} + (q_1 y^{(2)})'] + i[(q_0 y)' + (q_0 y')]$$

and

(A.43)
$$M[y] = M_{A_3}[y] \quad \text{for all } y \in \mathfrak{D}(M) = \mathfrak{D}(A_3),$$

as required.

Of course, we can use the matrix $A_3 \in Z_3(\mathfrak{J})$ to extend the definition of the classical differential expression (A.36) to allow weaker conditions of regularity on the real functions p_1, p_0, q_1, q_0 —for instance:

(A.44)
$$\begin{aligned} &\text{(i)} \quad q_1(x) \le 0 \text{ on } \mathfrak{J} \text{ with } |q_1|^{-1/2} \in \mathfrak{L}^1_{\mathrm{loc}}(\mathfrak{J}) \\ &\text{(ii)} \quad q_0 |q_1|^{-1/2} \in \mathfrak{L}^1_{\mathrm{loc}}(\mathfrak{J}) \\ &\text{(iii)} \quad p_1 q_1^{-1} \text{ and } p_0 \in \mathfrak{L}^1_{\mathrm{loc}}(\mathfrak{J}). \end{aligned}$$

As a final example of these low order classical differential operators, we consider the general self-adjoint classical differential expression of order $n = 4$. Again we take the familiar [**DS**] self-adjoint expression with complex coefficients:

(A.45) $$M[y] = (p_2 y'')'' + (p_1 y')' + (p_0 y) + i[(q_1 y')^{(2)} + (q_1 y^{(2)})'] + i[(q_0 y)' + (q_0 y')],$$

involving the real smooth functions $p_2 \in C^2$, $p_1 \in C^1$, $p_0 \in C^1$ and also real $q_1 \in C^2$, $q_0 \in C^1$ on $\mathfrak{J}$. Then the usual format for M, as a classical differential expression, becomes:

(A.46)
$$M[y] = P_4 y^{(4)} + P_3 y^{(3)} + P_2 y^{(2)} + P_1 y' + P_0 y$$

where the complex coefficients are

(A.47)
$$\begin{aligned} &P_4 = p_2, && P_3 = 2p_2' + 2iq_1, && P_2 = p_2'' + p_1 + 3iq_1' \\ &P_1 = p_1' + iq_1'' + 2iq_0, && P_0 = p_0 + iq_0. \end{aligned}$$

Then the nearly companion matrix $B \in Z_4(\mathfrak{J})$, of the format (A.10), satisfying $M_B = M$ on $\mathfrak{D}(B) = \mathfrak{D}(M)$, is

(A.48)
$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & P_4^{-1} \\ -P_0 & -P_1 & -P_2 & (P_4' - P_3)P_4^{-1} \end{bmatrix}.$$

Of course, $M_B[y] = M[y]$ for all $y \in \mathcal{D}(B) = \mathcal{D}(M)$, so M_B is formally self-adjoint—despite the fact that $B \neq B^+$, and moreover it can be shown that $\mathcal{D}(B) \neq \mathcal{D}(B^+)$. However, by Proposition 1, there must exist a matrix $A \in Z_4(\mathfrak{I})$ with $M_A = M$ on $\mathcal{D}(A) = \mathcal{D}(M)$ and such that $A = A^+$. We next present a convenient choice for such a matrix $A \in Z_4(\mathfrak{I})$, which we denote as A_4:

$$A_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -iq_1 p_2^{-1} & p_2^{-1} & 0 \\ -iq_0 & -[p_1 + q_1^2 p_2^{-1}] & -iq_1 p_2^{-1} & 1 \\ -p_0 & -iq_0 & 0 & 0 \end{bmatrix}. \tag{A.49}$$

This matrix A_4, for the order $n = 4$, will serve as the prototype for the corresponding matrices $A = A^+ = Z_n(\mathfrak{I})$, for even orders $n \geq 4$, as described in Exhibit 3 later. Accordingly, we illustrate the techniques for the computations of the appropriate quasi-derivatives relative to A_4, in order to demonstrate that the classical differential expression M of (A.45) can be written as the quasi-differential expression M_{A_4}:

$$M[y] = M_{A_4}[y] \qquad \text{on } \mathcal{D}(M) = \mathcal{D}(A_4), \tag{A.50}$$

with $A_4 = A_4^+$.

The usual computations, show that $A_4 = A_4^+ \in Z_4(\mathfrak{I})$, and further that the corresponding quasi-derivatives are (again surpressing the subscript A_4 for simplicity):

$$\begin{aligned} y^{[1]} &= y' \\ y^{[2]} &= (p_2 y^{(2)}) + i(q_1 y') \\ y^{[3]} &= (p_2 y^{(2)})' + i(q_1 y')' + iq_0 y + [p_1 + q_1^2 p_2^{-1}]y' + iq_1 p_2^{-1}[p_2 y^{(2)} + iq_1 y'] \end{aligned}$$

so

$$y^{[3]} = (p_2 y^{(2)})' + (p_1 y') + i[(q_1 y')' + q_1 y^{(2)}] + iq_0 y.$$

Hence

$$y^{[4]} = y^{[3]'} + p_0 y + iq_0 y'$$

and

$$\begin{aligned} M_{A_4}[y] = i^4 y^{[4]}_{A_4} = &(p_2 y^{(2)})^{(2)} + (p_1 y')' + (p_0 y) \\ &+ i[(q_1 y')^{(2)} + (q_1 y^{(2)})'] + i[(q_0 y)' + (q_0 y')]. \end{aligned}$$

Also $y, y', y'', y^{(3)}$ are all in $AC_{\text{loc}}(\mathfrak{I})$, if and only if $y^{[0]}$, $y^{[1]}$, $y^{[2]}$, $y^{[3]}$ are all in $AC_{\text{loc}}(\mathfrak{I})$, so $\mathcal{D}(M) = \mathcal{D}(A_4)$. Thus (A.50) is valid.

EXHIBIT 3. Construct $A = A^+ \in Z_n(\mathfrak{I})$, so $M_A = M$ (or, possibly, $M_A = -M$ when n is odd), for any given general classical differential expression M, of order $n \geq 2$, formally self-adjoint, and with suitably smooth complex coefficients on the real interval $\mathfrak{I}$.

As indicated above, the cases for n even and odd are somewhat different, and we shall treat these separately. Also certain peculiarities arise for low order n, but we have already covered these cases $n = 2, 3, 4$—and so we examine only the general format valid for larger n.

Thus consider first the general formally self-adjoint classical differential expression M of even order $n = 2m \geq 6$, with suitably smooth complex coefficients on the real interval $\mathfrak{I}$. As is well-known [**DS**], we can write M in the special format.

$$M[y] = \sum_{r=0}^{m} (p_r y^{(r)})^{(r)} + i \sum_{r=0}^{m-1} [(q_r y^{(r)})^{(r+1)} + (q_r y^{(r+1)})^{(r)}], \tag{A.51}$$

in terms of the real smooth functions $p_m, p_{m-1}, \dots, p_1, p_0$ and $q_{m-1}, \dots, q_1, q_0$. For definiteness we assume

(A.52)

$$p_r \in C^r(\mathfrak{I}) \text{ for } r = 0, 1, \dots, m;$$

and

$$q_r \in C^{r+1}(\mathfrak{I}) \text{ for } r = 0, 1, \dots, m-1.$$

The initial leading terms for M are

$$(p_m y^{(m)})^{(m)} + i[(q_{m-1} y^{(m-1)})^{(m)} + (q_{m-1} y^{(m)})^{(m-1)}],$$

and, as usual we assume that

$$p_m(x) \neq 0 \text{ for all } x \in \mathfrak{I}. \tag{A.53}$$

Next construct the required even ordered matrix $A_e = A_e^+ \in Z_n(\mathfrak{I})$ (where $n = 2m$ is even) such that $M_{A_e} = M$ on $\mathcal{D}(A_e) = \mathcal{D}(M)$, and for this we introduce the notation in [**ER**]. For even order $n = 2m \geq 6$ the matrix for M in (A.51) has the format A_e where

(A.54)

$$A_e = \begin{bmatrix}
0 & 1 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & & & 0 & 0 \\
0 & 0 & 1 & 0 & & & & & & & 0 & 0 \\
0 & 0 & 0 & 1 & & & & & & \cdot & 0 & 0 \\
 & & & \cdot & \cdot & & & & \cdot & & & \\
\cdot & & & & 0 & 1 & 0 & 0 & & & & \cdot \\
\cdot & & & & 0 & i\beta_{m-1} & \alpha_m & 0 & & & & \cdot \\
\cdot & & & & i\beta_{m-2} & \alpha_{m-1} & i\beta_{m-1} & 1 & & & & \cdot \\
\cdot & & & \cdot & \alpha_{m-2} & i\beta_{m-2} & 0 & 0 & \cdot & & & \cdot \\
 & & \cdot & \cdot & \cdot & & & & \cdot & \cdot & & \\
0 & i\beta_1 & \alpha_2 & \cdot & & & & & & \cdot & 1 & 0 \\
i\beta_0 & \alpha_1 & i\beta_1 & & & & & & & & 0 & 1 \\
\alpha_0 & i\beta_0 & 0 & & \cdot & \cdot & \cdot & \cdot & & & 0 & 0
\end{bmatrix}.$$

The terms $i\beta_{m-1}$ are on the main diagonal, which otherwise consists of zeros; and the term α_m is on the first superdiagonal, which otherwise consists of 1's. All other entries not specified or indicated are zero.
Here

$$\text{(A.55)} \qquad \begin{aligned} &\alpha_m = (-1)^m p_m^{-1} \qquad (\text{so } \alpha_m \in C^m \text{ and } \alpha_m(x) \neq 0 \text{ on } \mathfrak{I}) \\ &\alpha_{m-1} = (-1)^{m+1}(p_{m-1} + p_m^{-1} q_{m-1}^2) \\ &\alpha_k = (-1)^{m+1} p_k \qquad (0 \leq k \leq m-2), \end{aligned}$$

and

$$\begin{aligned} &\beta_{m-1} = -q_{m-1} p_m^{-1} \\ &\beta_k = (-1)^{m+1} q_k \qquad (0 \leq k \leq m-2). \end{aligned}$$

Since all the entries $\alpha_m, \ldots, \alpha_0$, $\beta_{m-1}, \ldots, \beta_0$ are real continuous functions on $\mathfrak{I}$, and $\alpha_m(x) \neq 0$, it follows that $A_e \in Z_n(\mathfrak{I})$—also A_e is real if and only if $q_{m-1} = q_{m-2} = \cdots = q_0 = 0$, that is, M is real.

It is easy to verify that

$$\text{(A.56)} \qquad A_e = A_e^+ = -\Lambda_n^{-1} A_e^* \Lambda_n, \text{ or } \Lambda_n A_e = (\Lambda_n A_e)^*,$$

since $\Lambda_n A_e$ is obtained from A_e by reversing the order of the rows, and changing signs in alternate rows.

Next we compute the appropriate quasi-derivatives, relative to A_e of (A.54), in order to verify that

$$M_{A_e}[y] = M[y] \quad \text{on } \mathfrak{D}(A_e) = \mathfrak{D}(M).$$

Here (again surpressing the subscript A_e):

$$\text{(A.57)} \qquad \begin{aligned} y^{[0]'} &= y^{[1]} & &\text{so } y^{[1]} = y' \\ y^{[1]'} &= y^{[2]} & &\text{so } y^{[2]} = y^{(2)} \\ &\vdots & &\vdots \\ y^{[m-2]'} &= y^{[m-1]} & &\text{so } y^{[m-1]} = y^{(m-1)}. \end{aligned}$$

Further

$$\begin{aligned}
y^{[m-1]'} &= i\beta_{m-1}y^{[m-1]} + \alpha_m y^{[m]},\\
y^{[m]'} &= i\beta_{m-2}y^{[m-2]} + \alpha_{m-1}y^{[m-1]} + i\beta_{m-1}y^{[m]} + y^{[m+1]}\\
y^{[m+1]'} &= i\beta_{m-3}y^{[m-3]} + \alpha_{m-2}y^{[m-2]} + i\beta_{m-2}y^{[m-1]} + y^{[m+2]},
\end{aligned}$$

and continue

$$\begin{aligned}
y^{[m+2]'} &= i\beta_{m-4}y^{[m-4]} + \alpha_{m-3}y^{[m-3]} + i\beta_{m-3}y^{[m-2]} + y^{[m+3]}\\
y^{[m+3]'} &= i\beta_{m-5}y^{[m-5]} + \alpha_{m-4}y^{[m-4]} + i\beta_{m-4}y^{[m-3]} + y^{[m+4]}
\end{aligned}$$

so finally the next to last row determines $y^{[n-1]}$,

$$y^{[n-2]'} = i\beta_0 y + \alpha_1 y' + i\beta_1 y^{(2)} + y^{[n-1]}$$

and then

$$y^{[n]} = y^{[n-1]'} + (-1)^m p_0 y + (-1)^m i q_0 y'.$$

Next re-write and examine the critical rows:

$$y^{[m-1]'} = -i\frac{q_{m-1}}{p_m}y^{(m-1)} + (-1)^m\frac{1}{p_m}y^{[m]}$$

so

$$\begin{aligned}
(-1)^m y^{[m]} &= p_m y^{(m)} + i q_{m-1}y^{(m-1)},\\
y^{[m]'} &= i(-1)^{m+1}q_{m-2}y^{(m-2)} + (-1)^{m+1}(p_{m-1} + q_{m-1}^2/p_m)y^{(m-2)}\\
&\quad + i(-1)^{m+1}\frac{q_{m-1}}{p_m}y^{[m]} + y^{[m+1]}
\end{aligned}$$

so

$$\begin{aligned}
y^{[m+1]} &= (-1)^m(p_m y^{(m)})' + (-1)^m i(q_{m-1}y^{(m-1)})'\\
&\quad + (-1)^m i q_{m-2}y^{(m-2)} + (-1)^m\left(p_{m-1} + \frac{q_{m-1}^2}{p_m}\right)y^{(m-1)}\\
&\quad + (-1)^m i\frac{q_{m-1}}{p_m}[(-1)^m p_m y^{(m)} + (-1)^m i q_{m-1}y^{(m-1)}]
\end{aligned}$$

and

$$\begin{aligned}
(-1)^m y^{[m+1]} &= (p_m y^{(m)})' + (p_{m-1}y^{(m-1)})\\
&\quad + i\{[(q_{m-1}y^{(m-1)})' + (q_{m-1}y^{(m)})] + [(q_{m-2}y^{(m-2)})]\}.
\end{aligned}$$

The next step of computing $y^{[m+2]}$ sets the future pattern, since the exceptional elements α_m, α_{m-1}, β_{m-1} are not involved—and a similar pattern holds to $y^{[n-1]}$.

That is, we note

$$\begin{aligned}
y^{[m+2]} &= y^{[m+1]'} - i\beta_{m-3}y^{(m-3)} - \alpha_{m-2}y^{(m-2)} - i\beta_{m-2}y^{(m-1)}\\
y^{[m+2]} &= (-1)^m\{(p_m y^{(m)})^{(2)} + (p_{m-1}y^{(m-1)})'\}\\
&\quad + (-1)^m i\{[(q_{m-1}y^{(m-1)})^{(2)} + (q_{m-1}y^{(m)})'] + [(q_{m-2}y^{(m-2)})']\}\\
&\quad + (-1)^m i q_{m-3}y^{(m-3)} + (-1)^m i q_{m-2}y^{(m-1)} + (-1)^m p_{m-2}y^{(m-2)}.
\end{aligned}$$

Thus

$$\begin{aligned}(-1)^m y^{[m+2]} &= \{(p_m y^{(m)})^{(2)} + (p_{m-1}y^{(m-1)})' + p_{m-2}y^{(m-2)}\} \\ &\quad + i\{[(q_{m-1}y^{(m-1)})^{(2)} + (q_{m-1}y^{(m)})'] \\ &\qquad + [(q_{m-2}y^{(m-2)})' + (q_{m-2}y^{(m-1)})] + [(q_{m-3}y^{(m-3)}]\}.\end{aligned}$$

Thus, in passing from $y^{[m+1]}$ to $y^{[m+2]}$ we impose one additional order of differentiation on all the p-terms and q-terms of $y^{[m+1]}$, then add one additional p-term and two additional q-terms to extend the pattern. Hence we proceed step-by-step until $y^{[n-1]} = y^{[m+m-1]}$:

$$\begin{aligned}(-1)^m y^{[n-1]} &= \{(p_m y^{(m)})^{(m-1)} + (p_{m-1}y^{(m-1)})^{(m-2)} + \cdots + (p_1 y')\} \\ &\quad + i\{[(q_{m-1}y^{(m-1)})^{(m-1)} + (q_{m-1}y^{(m)})^{(m-2)}] \\ &\quad + [(q_{m-2}y^{(m-2)})^{(m-2)} + (q_{m-2}y^{(m-1)})^{(m-3)}] + \dots \\ &\quad + [(q_1 y')' + (q_1 y^{(2)})] + [(q_0 y)]\}.\end{aligned}$$

Then

$$\begin{aligned} y^{[n]} &= y^{[n-1]'} - \alpha_0 y - i\beta_0 y' = y^{[n-1]'} + (-1)^m p_0 y + (-1)^m i q_0 y' \\ (-1)^m y^{[n]} &= \{(p_m y^{(m)})^{(m)} + (p_{m-1}y^{(m-1)})^{(m-1)} + \cdots + (p_1 y')' + (p_0 y)\} \\ &\quad + i\{[(q_{m-1}y^{(m-1)})^{(m)} + (q_{m-1}y^{(m)})^{(m-1)}] \\ &\quad + [(q_{m-2}y^{(m-2)})^{(m-1)} + (q_{m-2}y^{(m-1)})^{(m-2)}] + \cdots \\ &\quad + [(q_1 y')^{(2)} + (q_1 y^{(2)})'] + [(q_0 y)' + (q_0 y')]\}.\end{aligned}$$

Hence, since $(-1)^m = (-1)^{-m} = i^{2m} = i^n$, $M_{A_e}[y] = i^n y^{[n]}_{A_e}$, as required for all suitably smooth y. But we still must verify that $\mathfrak{D}(A_e) = \mathfrak{D}(M)$.

However a careful study of the prior formulas shows that:

(A.58) if $y, y', y'', \dots, y^{(n-1)}$ are all in $AC_{\mathrm{loc}}(\mathfrak{I})$, then also the quasi-derivatives, relative to A_e, $y^{[0]}, y^{[1]}, \dots, y^{[n-1]}$ are all in $AC_{\mathrm{loc}}(\mathfrak{I})$.

This conclusion follows directly from the assumed smoothness properties of the functions p_r and q_r. It is trivial for $y^{[r]}$ on $r = 0, 1, \dots, m-1$, and thereafter the formulas for $y^{[r]}$, for $r = m, m+1, \dots, n-1$, give the required smoothness at each step.

The converse also holds by similar arguments using the inductive nature of the successive steps—say relating $y^{(r)}$ to $y, y', y'', \dots, y^{(r-1)}$ in the definition of $y^{[r]}$.

Hence we conclude that for the case of even order n, we have verified that

$$M_{A_e}[y] = M[y] \quad \text{on } \mathfrak{D}(A_e) = \mathfrak{D}(M). \tag{A.59}$$

The second construction in this Exhibit 3 treats the general formally self-adjoint classical differential expression M of odd order $n = 2m+1 \geq 5$, with suitably smooth complex coefficients on the real interval $\mathfrak{I}$. In this case we can write M in the special format [**DS**]:

$$M[y] = \sum_{r=0}^{m} (p_r y^{(r)})^{(r)} + i \sum_{r=0}^{m} [(q_r y^{(r)})^{(r+1)} + (q_r y^{(r+1)})^{(r)}], \tag{A.60}$$

involving the real smooth functions $p_m, p_{m-1}, \dots, p_1, p_0$ and real $q_m, \dots, q_1, q_0$. For definiteness we assume

$$p_r \in C^r,\ q_r \in C^{r+1} \text{ on } \mathfrak{I},\ \text{for } r = 0, \dots, m, \text{ and also} \tag{A.61}$$

$$(-1)^m q_m(x) > 0 \quad \text{for all } x \in \mathfrak{I}, \tag{A.62}$$

(otherwise replace M by $-M$ to attain this sign convention). We note that for n odd, M must have some non-real coefficients.

Then, for M in (A.60) with odd order $n = 2m+1 \geq 5$, we define the odd order matrix $A_0 = A_0^+ \in Z_n(\mathfrak{I})$ (see [**ER**]) and then verify that

$$M_{A_0}[y] = M[y] \quad \text{on } \mathcal{D}(A_0) = \mathcal{D}(M). \tag{A.63}$$

For odd order $n = 2m+1 \geq 5$ the matrix for M in (A.60) has the format A_0 where

(A.64)

$$A_0 = \begin{bmatrix}
0 & 1 & 0 & 0 & \cdot & \cdot & \cdot & & & & 0 & 0 \\
0 & 0 & 1 & 0 & & & & & & & 0 & 0 \\
0 & 0 & 0 & 1 & & & & & & \cdot & 0 & 0 \\
 & & & \cdot & \cdot & & & & \cdot & & & \\
\cdot & & & & 0 & \beta_m & 0 & & & & & \cdot \\
\cdot & & & & \beta_{m-1} & i\alpha_m & \beta_m & & & & & \cdot \\
\cdot & & & \cdot & i\alpha_{m-1} & \beta_{m-1} & 0 & \cdot & & & & \cdot \\
 & & \cdot & \cdot & & & & \cdot & & & & \\
 & & & & \cdot & & & & & \cdot & & \\
0 & \beta_1 & i\alpha_2 & \cdot & & & & & & \cdot & 1 & 0 \\
\beta_0 & i\alpha_1 & \beta_1 & & & & & & & & 0 & 1 \\
i\alpha_0 & \beta_0 & 0 & & \cdot & \cdot & \cdot & & & & 0 & 0
\end{bmatrix}.$$

The term $i\alpha_m$ is on the main diagonal, which otherwise consists of zeros; and the terms β_m are on the first superdiagonal, which otherwise consists of 1's. All other entries not specified or indicated are zero.

Here

(A.65)

$$\begin{aligned}
&\beta_m = [(-1)^m 2q_m]^{-1/2} > 0 \qquad (\text{so } \beta_m \in C^{m+1} \text{ and } \beta_m(x) > 0 \text{ on } \mathfrak{I}) \\
&\beta_{m-1} = (-1)^{m+1} q_{m-1} \beta_m \\
&\beta_k = (-1)^{m+1} q_k \qquad \text{for } (0 \leq k \leq m-2)
\end{aligned}$$

and

$$\begin{aligned}
&\alpha_m = p_m (2q_m)^{-1} \\
&\alpha_k = (-1)^m p_k \qquad \text{for } (0 \leq k \leq m-1).
\end{aligned}$$

Since all the entries $\alpha_m, \dots, \alpha_0$ and $\beta_m, \dots, \beta_0$ are real continuous functions on $\mathfrak{I}$, and $\beta_m(x) \neq 0$, it follows that $A_0 \in Z_n(\mathfrak{I})$.

It is easy to verify that

$$A_0 = A_0^+ = -\Lambda_n^{-1} A_0^* \Lambda_n, \text{ or } (\Lambda_n A_0) = -(\Lambda_n A_0)^*.$$

We now proceed to verify (A.63) by computing the appropriate quasi-derivatives, relative to the matrix A_0 of (A.64).

Here we compute (again surpressing the subscript A_0):

$$\begin{aligned} y^{[0]'} &= y^{[1]}, &&\text{so } y^{[1]} = y' \\ y^{[1]'} &= y^{[2]}, &&\text{so } y^{[2]} = y'' \\ &\vdots && \quad\vdots \\ y^{[m-2]'} &= y^{[m-1]}, &&\text{so } y^{[m-1]} = y^{(m-1)}. \end{aligned} \tag{A.66}$$

Further

$$\begin{aligned} y^{[m-1]'} &= \beta_m y^{[m]}, \qquad \text{so } y^{[m]} = \beta_m^{-1} y^{(m)} \\ y^{[m]'} &= \beta_{m-1} y^{[m-1]} + i\alpha_m y^{[m]} + \beta_m y^{[m+1]} \end{aligned}$$

so

$$y^{[m+1]} = \beta_m^{-1}\{(\beta_m^{-1} y^{(m)})' - \beta_{m-1} y^{[m-1]} - i\alpha_m y^{[m]}\}.$$

Recall that $\beta_m = [(-1)^m 2q_m]^{-1/2}$, $\beta_{m-1} = (-1)^{m+1} q_{m-1}\beta_m$ and the identity (see (A.41)):

$$\begin{aligned} \beta_m^{-1}(\beta_m^{-1} y^{(m)})' &= (-1)^m\{q_m' y^{(m)} + 2q_m y^{(m+1)}\} \\ &= (-1)^m\{(q_m y^{(m)})' + q_m y^{(m+1)}\}. \end{aligned} \tag{A.67}$$

Then we can write

$$(-1)^m y^{[m+1]} = [(q_m y^{(m)})' + (q_m y^{(m+1)})] + [q_{m-1} y^{(m-1)}] - ip_m y^{(m)},$$

and we recognize this expression as the beginning of the construction anticipated for $y^{[n]}$.

Continue to compute directly from $y^{[m+1]'}$, namely

$$y^{[m+2]} = y^{[m+1]'} - \beta_{m-2} y^{(m-2)} - i\alpha_{m-1} y^{(m-1)} - \beta_{m-1} y^{[m]}$$

so

$$\begin{aligned} (-1)^m y^{[m+2]} = {} & [(q_m y^{(m)})^{(2)} + (q_m y^{(m+1)})'] + [(q_{m-1} y^{(m-1)})' + (q_{m-1} y^{(m)})] \\ & + [q_{m-2} y^{(m-2)}] - i\{(p_m y^{(m)})' + p_{m-1} y^{(m-1)}\}. \end{aligned}$$

Now all the similar successive calculations, from $y^{[m+2]}$ to the "last full row" labeled by $y^{[n-2]'}$ and involving the term $y^{[n-1]}$, follow the same pattern, since the exceptional terms β_m, β_{m-1}, α_m no longer enter these formulas. Hence we continue to find

$$\begin{aligned} (-1)^m y^{[n-1]} = {} & [(q_m y^{(m)})^{(m)} + (q_m y^{(m+1)})^{(m-1)}] \\ & + [(q_{m-1} y^{(m-1)})^{(m-1)} + (q_{m-1} y^{(m)})^{(m-2)}] \\ & + \cdots + [(q_1 y')' + (q_1 y^{(2)})] + [q_0 y] \\ & - i\{(p_m y^{(m)})^{(m-1)} + \cdots + p_1 y'\}. \end{aligned}$$

Then

$$y^{[n]} = y^{[n-1]'} - i\alpha_0 y - \beta_0 y' = y^{[n-1]'} + (-1)^m\{q_0 y' - ip_0 y\},$$

and thus, for suitably smooth y,

$$M_{A_0}[y] = i^n y^{[n]}_{A_0} = \sum_{r=0}^{m}(p_r y^{(r)})^{(r)} + i\sum_{r=0}^{m}[(q_r y^{(r)})^{(r+1)} + (q_r y^{(r+1)})^{(r)}].$$

But just as in the case of even orders n, we can show that (see (A.58)):

(A.68) $y, y', y'', \ldots, y^{(n-1)}$ are all in $AC_{\text{loc}}(\mathcal{I})$
if and only if the quasi-derivatives relative to A_0
$y^{[0]}, y^{[1]}, \ldots, y^{[n-1]}$ are all in $AC_{\text{loc}}(\mathcal{I})$.

Hence the required result (A.63) holds:

(A.69) $$M_{A_0}[y] = M[y] \text{ on } \mathcal{D}(A_0) = \mathcal{D}(M).$$

Thus, in this Exhibit 3, we have displayed for each formally self-adjoint classical (smooth) differential operator M of order $n \geq 2$, a suitable Shin-Zettl matrix $A = A^+ \in Z_n(\mathcal{I})$ such that the corresponding quasi-differential expression satisfies

$$M_A[y] = M[y] \qquad \text{for } y \in \mathcal{D}(A) = \mathcal{D}(M),$$

(or $M_A = -M$ for some cases when n is odd).

EXHIBIT 4. Another symmetric differential expression, also a general format for the formally self-adjoint classical differential operator of order $n \geq 2$, with smooth complex coefficients, is given by (after E. Coddington, with minor notational modifications [**CL**]):

(A.70)
$$\begin{aligned} M[y] = {} & i^n q_n(\cdots(q_n(q_n y)')'\cdots)' + i^{n-1}q_{n-1}(\cdots(q_{n-1}(q_{n-1}y)')'\cdots)' \\ & + \cdots + i^2 q_2(q_2(q_2 y)')' + i q_1(q_1 y)' + q_0 y. \end{aligned}$$

The complex $q_r \in C^r$ for $r = 0, 1, \ldots, n$ on the real interval $\mathcal{I}$, must further satisfy the conditions

$$[q_r(x)]^{r+1} \in \mathbb{R} \qquad \text{with } q_n(x) \neq 0 \text{ for all } x \in \mathcal{I}.$$

Thus the $(n+1)$-st power of q_n, the n-th power of $q_{n-1}, \ldots$, the first power of q_0, are all real for $x \in \mathcal{I}$. The domain of M is

$$\mathcal{D}(M) = \{y : \mathcal{I} \to \mathbb{C} \mid y, y', y'', \cdots, y^{(n-1)} \text{ in } AC_{\text{loc}}(\mathcal{I})\}.$$

Then M describes the most general classical differential operator of order $n \geq 2$, with smooth complex coefficients, which is formally self-adjoint in the sense of Section I (1.8); and accordingly there exist representations of M by M_A, for $A = A^+ \in Z_n(\mathcal{I})$ (or of $-M$ in some cases with n odd). There does not seem to be any simple direct method of constructing such a matrix A in terms of the functions $q_n, q_{n-1}, \cdots, q_0$ of (A.70). However we can attempt to resolve this difficulty by "unwrapping" the differential expression (A.70) for M to recover the usual classical expression

$$M[y] = \sum_{r=0}^{n} b_r y^{(r)},$$

with $b_n \in AC_{\rm loc}(\mathfrak{I})$ and $b_n(x) \neq 0$ for all $x \in \mathfrak{I}$, and otherwise $b_r \in \mathfrak{L}^1_{\rm loc}(\mathfrak{I})$ for $r = 0, 1, \ldots, n-1$. This procedure is too awkward and cumbersome for large values of n, but we shall illustrate the concept for $n = 2$.

Thus consider the formally self-adjoint expression (A.70) for $n = 2$,

$$M[y] = -q_2(q_2(q_2y)')' + iq_1(q_1y)' + q_0y \tag{A.71}$$

where

$$(q_2(x))^3 \neq 0 \text{ is real, and also } (q_1)^2 \text{ and } q_0 \text{ are real on } \mathfrak{I}.$$

Then compute

$$\begin{aligned} M[y] = -(q_2)^3y'' &+ [-(q_2)^2q_2' - 2q_2'(q_2)^2 + i(q_1)^2]y' \\ &+ [-q_2(q_2')^2 - (q_2)^2q_2'' + iq_1q_1' + q_0]y. \end{aligned} \tag{A.72}$$

Now we seek $B = B^+ \in Z_2(\mathfrak{I})$ in the familiar form (A.26) for the Sturm-Liouville operator, namely take

$$B = \begin{bmatrix} r & p^{-1} \\ q & -\bar{r} \end{bmatrix}, \tag{A.73}$$

where p, q, and ir are real on $\mathfrak{I}$. In this case

$$M_B[y] = -py'' + [-p' + p(r - \bar{r})]y' + [(pr)' + q + r\bar{r}p]y. \tag{A.74}$$

Upon equating like coefficients in (A.72) and (A.74) we obtain the desired formulas:

$$\begin{aligned} &p = (q_2)^3, \; r = i(q_1)^2[2(q_2)^3]^{-1} \\ &q = q_0 - q_2(q_2')^2 - (q_2)^2q_2'' + (q_1)^4[4(q_2)^3]^{-1}, \end{aligned}$$

at each $x \in \mathfrak{I}$. This last result allows us to relax the original differentiability conditions on q_2, q_1, q_0 to the weaker demands:

$$\begin{aligned} &q_2, q_2' \text{ and } q_1 \in AC_{\rm loc}(\mathfrak{I}) \qquad (\text{with } q_2(x) \neq 0 \text{ on } \mathfrak{I}) \\ &q_0 \in \mathfrak{L}^1_{\rm loc}(\mathfrak{I}), \end{aligned}$$

which guarantee that $B = B^+ \in Z_2(\mathfrak{I})$.

The last topic in this Appendix A is an analysis and proof of the Density Theorem for quasi-differential expressions M_A, for any given Shin-Zettl matrix $A \in Z_n(\mathfrak{I})$, as described in Sections I(1.15), II(1.3 (vii)), and above in (A.3), (A.4), (A.5), (A.6), and (A.7); and applications to basic results on adjoint operators.

In order to re-state the setting for this theorem, consider the complex Hilbert space $\mathfrak{L}^2(\mathfrak{I}; w)$, relative to a real weight function $w \in \mathfrak{L}^1_{\rm loc}(\mathfrak{I})$ with $w(x) > 0$ a.e. on $\mathfrak{I}$, where $\mathfrak{I}$ is any prescribed nondegenerate real interval, say with endpoints $-\infty \leq a < b \leq +\infty$—as in Section I above. Then for any given Shin-Zettl matrix $A \in Z_n(\mathfrak{I})$ (satisfying conditions (α_1) and (α_2) of Section I (2.3)—but not necessarily (α_3) of Section I (2.13)), we construct the quasi-differential expression (as in Example 2 in Section I, and as we recalled earlier in this appendix):

$$M_A[y] = i^n y_A^{[n]}, \tag{A.3}$$

on the domain $\mathfrak{D}(M_A)$ or $\mathfrak{D}(A)$ specified by

$$\mathfrak{D}(A) = \{y : \mathfrak{I} \to \mathbb{C} \mid y_A^{[r]} \in AC_{\rm loc}(\mathfrak{I}) \text{ for } r = 0, 1, \ldots, n-1\}. \tag{A.4}$$

As in Section I, $w^{-1}M_A$ generates a maximal linear operator T_1 on the domain $\mathcal{D}(T_1) \subset \mathcal{L}^2(\mathcal{I}; w)$:

$$\mathcal{D}(T_1) = \{y \in \mathcal{D}(A) \mid y \text{ and } w^{-1}M_A[y] \text{ in } \mathcal{L}^2(\mathcal{I}; w)\}. \tag{A.75}$$

In this Appendix A we shall show that the set (compare Section I (1.11) and Section I (1.15)):

$$\mathcal{D}_0(T_1) = \{y \in \mathcal{D}(T_1) \mid (\operatorname{supp} y) \text{ lies in a compact set interior to } \mathcal{I}\} \tag{A.76}$$

is dense in the Hilbert space $\mathcal{L}^2(\mathcal{I}; w)$, under appropriate hypotheses—as asserted later in The Density Theorem 1.

LEMMA 1. *Let $A \in Z_n(\mathcal{I})$ and $w \in \mathcal{L}^1_{\mathrm{loc}}(\mathcal{I})$ be a positive weight on the interval $\mathcal{I}$, so the quasi-differential expression $w^{-1}M_A$ generates a maximal linear operator T_1 on $\mathcal{D}(T_1) \subset \mathcal{L}^2(\mathcal{I}; w)$, as above. Then $\mathcal{D}_0(T_1)$, as in* (A.76)*, is an infinite dimensional linear manifold in $\mathcal{L}^2(\mathcal{I}; w)$.*

PROOF. It is clear that $\mathcal{D}_0(T_1) \subset \mathcal{D}(T_1) \subset \mathcal{L}^2(\mathcal{I}; w)$ is a linear manifold in the complex Hilbert space $\mathcal{L}^2(\mathcal{I}; w)$; however it is not obvious that it is non-trivial (not the zero only).

Fix any non-degenerate compact interval $[\alpha, \beta]$ lying interior to $\mathcal{I}$, and we shall prove the stronger result that the linear manifold $\mathcal{D}_0([\alpha, \beta]) \subset \mathcal{D}_0(T_1)$, where

$$\mathcal{D}_0([\alpha, \beta]) :\equiv \{y \in \mathcal{D}(T_1) \mid (\operatorname{supp} y) \subset [\alpha, \beta]\}, \tag{A.77}$$

is an infinite dimensional linear manifold in $\mathcal{L}^2(\mathcal{I}; w)$.

To construct functions in $\mathcal{D}(A)$ consider the quasi-differential control system

$$y_A^{[n]} = w\varphi, \qquad \text{for controllers } \varphi \in \mathcal{L}^\infty_{\mathrm{loc}}(\mathcal{I}), \tag{A.78}$$

(so $w\varphi \in \mathcal{L}^1_{\mathrm{loc}}(\mathcal{I})$, which is sufficient for the usual existence and uniqueness theorems) or equally well, the linear differential control system

$$\underset{\sim}{y}'_A = A\underset{\sim}{y}_A + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ w \end{pmatrix} \varphi, \tag{A.79}$$

with responses $\underset{\sim}{y}_A(x) = \begin{pmatrix} y_A^{[0]}(x) \\ \vdots \\ y_A^{[n-1]}(x) \end{pmatrix} \in \mathbb{C}^n$ for all $x \in \mathcal{I}$. Take the initial data for (A.79) to be $\underset{\sim}{y}_A(\alpha) = 0$, and select an arbitrary vector $\underset{\sim}{c} \neq 0$ in $\mathbb{C}^n$ and an intermediate point γ in (α, β). Then it is known [**EM**] that there exists a controller $\varphi \in \mathcal{L}^\infty([\alpha, \gamma])$ so that the corresponding solution of (A.79) on $\alpha \leq x \leq \gamma$ is the response $\underset{\sim}{y}_A(x; \underset{\sim}{c})$ with $\underset{\sim}{y}_A(\gamma; \underset{\sim}{c}) = \underset{\sim}{c}$. Then further continue the controller φ on $[\gamma, \beta]$, with $\varphi \in \mathcal{L}^\infty([\gamma, \beta])$, so that $\underset{\sim}{y}_A(x; \underset{\sim}{c})$ is defined on $\alpha \leq x \leq \beta$ with

$$\underset{\sim}{y}_A(\alpha; \underset{\sim}{c}) = 0, \ \underset{\sim}{y}_A(\gamma; \underset{\sim}{c}) = \underset{\sim}{c}, \ \underset{\sim}{y}_A(\beta; \underset{\sim}{c}) = 0.$$

Finally continue to define $\varphi(x) \equiv 0$ for x outside $[\alpha, \beta]$ on $\mathfrak{I}$, and thus $\varphi \in \mathcal{L}^\infty(\mathfrak{I})$, and $\underset{\sim}{y}_A(x; \underset{\sim}{c})$ is defined for all $x \in \mathfrak{I}$.

Clearly $\underset{\sim}{y}_A(x; \underset{\sim}{c}) \in AC_{\mathrm{loc}}(\mathfrak{I})$ so $y(x; \underset{\sim}{c}) = y_A^{[0]}(x; \underset{\sim}{c}) \in \mathcal{D}(A)$, and also $(\operatorname{supp} y) \subset [\alpha, \beta]$. Moreover

$$|w^{-1} M_A[y]|^2 w = |\varphi^2 w| \in \mathcal{L}^1([\alpha, \beta])$$

so

$$y(x; \underset{\sim}{c}) \in \mathcal{D}(T_1), \text{ and furthermore } y(x, \underset{\sim}{c}) \in \mathcal{D}_0([\alpha, \beta]).$$

Finally replace the single intermediate point γ by an arbitrary finite number of points, say γ_k for $k = 1, 2, \ldots, \ell$,

$$\alpha < \gamma_1 < \gamma_2 < \cdots < \gamma_\ell < \beta,$$

and also take an arbitrary assignment of complex values $c_1, c_2, \ldots, c_\ell \in \mathbb{C}$ and write $c = (c_1, c_2, \ldots, c_\ell) \in \mathbb{C}^\ell$. Following [**EM**], choose the required controller $\varphi \in \mathcal{L}^\infty(\mathfrak{I})$ so that the corresponding response $\underset{\sim}{y}_A(x; c_1, \ldots, c_\ell) :\equiv \underset{\sim}{y}_A(x; c)$ lies in $AC_{\mathrm{loc}}(\mathfrak{I})$ and satisfies the conditions

$$\underset{\sim}{y}_A(x; c) \equiv 0 \text{ for } x \leq \alpha \text{ and for } x \geq \beta \text{ on } \mathfrak{I},$$

$$y(x; c) = y_A^{[0]}(x; c) \text{ satisfies } y(\gamma_k, c) = c_k \text{ for } k = 1, \ldots, \ell.$$

Just as before $y(x; c) \in \mathcal{D}(A)$ and also $y(x; c) \in \mathcal{D}(T_1)$ with $\operatorname{supp} y \subset [\alpha, \beta]$. Therefore $y(x; c) \in \mathcal{D}_0([\alpha, \beta])$.

Since the finite set of values $c_1, c_2, \ldots, c_\ell \in \mathbb{C}$ is arbitrary,

$$\dim \mathcal{D}_0([\alpha, \beta]) \geq \ell, \qquad \text{for arbitrary } \ell \geq 1,$$

as required. □

As indicated by the procedures in Lemma 1, we shall consider several spaces of complex functions defined on the compact interval $[\alpha, \beta]$ that lies interior to $\mathfrak{I}$— for instance $AC([\alpha, \beta])$, $\mathcal{L}^1([\alpha, \beta])$, $\mathcal{L}^2([\alpha, \beta]; w)$, $\mathcal{L}^\infty([\alpha, \beta])$, etc., in the familiar notations.

We note that the controllability methods, see [**EM**], applied in the proof of Lemma 1 have been employed in special cases by Naimark, leading to his result [**NA,** Ch. V 17.3, Lemma 2] which we refer to as the "patching lemma". For easy reference we reformulate this particular result as a corollary, which is an immediate consequence of the calculations of our Lemma 1 above.

COROLLARY 1 *(Patching Lemma). In the terminology of Lemma 1 above, prescribe data:*

$$\underset{\sim}{\xi} \in \mathbb{C}^n, \ \underset{\sim}{\eta} \in \mathbb{C}^n \quad \textit{at points } \gamma_1 < \gamma_2 \textit{ interior to } \mathfrak{I}.$$

Then there exists an n-vector function $\underset{\sim}{y}_A(x)$ for $x \in [\gamma_1, \gamma_2]$, with components $y_A^{[r-1]} \in AC([\gamma_1, \gamma_2])$ such that $y_A^{[r-1]}(\gamma_1) = \xi_r$ and $y_A^{[r-1]}(\gamma_2) = \eta_r$ for $r = 1, 2, \ldots, n$. Furthermore $y = y_A^{[0]}$ satisfies

$$w^{-1} M_A[y] \in \mathcal{L}^2([\gamma_1, \gamma_2]; w).$$

Using this formulation of the "patching lemma", it is clear that we can "patch together" appropriate functions on disjoint subintervals of $\mathfrak{I}$ to construct functions in $\mathcal{D}(T_1)$. We next use these techniques to produce extensions of complex-valued functions, defined on a compact interval $[\alpha, \beta] \subset \mathfrak{I}$, to functions defined on the full interval $\mathfrak{I}$—while preserving important global properties, as indicated in the following remarks.

REMARK 1. Let $z(x)$ be a complex function defined for $x \in [\alpha, \beta]$, interior to $\mathfrak{I}$, where the quasi-derivatives (relative to $A \in Z_n(\mathfrak{I})$):

$$z_A^{[r]} \in AC([\alpha, \beta]) \qquad \text{for } r = 0, 1, \ldots, n-1.$$

Also assume that

$$\int_\alpha^\beta |z|^2 w\, dx < \infty \qquad \text{and} \int_\alpha^\beta |w^{-1} M_A[z]|^2 w\, dx < \infty.$$

Then there exists an extension (non-unique) of z to a function defined on $\mathfrak{I}$ and there belonging to $\mathcal{D}_0(T_1)$.

To construct the required extension, say $\hat{z}(x)$ on $\mathfrak{I}$, choose a compact interval $[\alpha - \varepsilon, \beta + \varepsilon]$, for some small $\varepsilon > 0$, which lies interior to $\mathfrak{I}$. We then define $\hat{z}(x)$ as the solution of the quasi-differential equation (A.78)

$$y_A^{[n]} = w\varphi, \text{ with initial data } \hat{z}_A^{[r]}(\alpha) = z_A^{[r]}(\alpha) \text{ for } r = 0, 1, \ldots, n-1.$$

Here we shall select the controller $\varphi(x)$ on $\mathfrak{I}$ so that $w\varphi \in \mathcal{L}^1_{\mathrm{loc}}(\mathfrak{I})$, according to the following scheme:

(i) $\varphi(x) = w^{-1} z_A^{[n]}(x)$ for $\alpha \leq x \leq \beta$, and note that

$$\left(\int_\alpha^\beta |w\varphi| dx\right) = \int_\alpha^\beta w^{1/2} |\varphi w^{1/2}| dx \leq \left(\int_\alpha^\beta w\, dx\right)^{1/2} \left(\int_\alpha^\beta |w^{-1} z_A^{[n]}|^2 w\, dx\right)^{1/2} < \infty.$$

Then $\hat{z}(x) = z(x)$ on $[\alpha, \beta]$ by the uniqueness theorem for (A.78).

(ii) $\varphi \in \mathcal{L}^\infty$ on $[\alpha - \varepsilon, \alpha]$ and on $[\beta, \beta + \varepsilon]$, as in Lemma 1, so as to control $\hat{z}(x)$ from

$$\hat{z}_A^{[r]}(\alpha) = z^{[r]}(\alpha) \qquad \text{to} \quad \hat{z}_A^{[r]}(\alpha - \varepsilon) = 0$$

and from

$$\hat{z}_A^{[r]}(\beta) = z_A^{[r]}(\beta) \qquad \text{to} \quad \hat{z}_A^{[r]}(\beta + \varepsilon) = 0, \qquad \text{for } r = 0, 1, \ldots, n-1.$$

(iii) $\varphi(x) \equiv 0$ outside $[\alpha - \varepsilon, \beta + \varepsilon]$ on $\mathfrak{I}$, so $\hat{z}(x) \equiv 0$ for $x < \alpha - \varepsilon$ and for $x > \beta + \varepsilon$ on $\mathfrak{I}$.

Then it is easy to observe that $\hat{z}_A^{[r]} \in AC(\mathfrak{I})$, and moreover

$$\int_{\mathfrak{I}} |\hat{z}|^2 w\, dx < \infty, \qquad \int_{\mathfrak{I}} |w^{-1} M_A[\hat{z}]|^2 w\, dx < \infty,$$

so $\hat{z} \in \mathcal{D}_0(T_1)$, as required.

We shall denote $\hat{z}$ by the same symbol z, and refer to the extension of z in $\mathcal{D}_0(T_1)$.

REMARK 2. Define the complex Hilbert space $\mathcal{L}^2([\alpha,\beta];w) \subset \mathcal{L}^2(\mathfrak{I};w)$ by

$$\mathcal{L}^2([\alpha,\beta];w] :\equiv \{z \in \mathcal{L}^2(\mathfrak{I};w) \mid z = 0 \text{ a.e. outside } [\alpha,\beta]\}. \tag{A.80}$$

It is then trivial that each complex function z on $[\alpha,\beta]$ with

$$\int_\alpha^\beta |z|^2 w\, dx < \infty,$$

can be extended (say, identically zero outside $[\alpha,\beta]$) to determine a function (still called z on $\mathfrak{I}$) with $z \in \mathcal{L}^2([\alpha,\beta];w)$.

DEFINITION 1. Let $\mathcal{H}_A([\alpha,\beta]) \subset \mathcal{L}^2([\alpha,\beta];w)$ consist of all solutions $z(x)$, for $x \in [\alpha,\beta]$, of the homogeneous quasi-differential equation

$$M_A[z] = 0 \qquad \text{on } \alpha \le x \le \beta,$$

with the convention that we set

$$z(x) \equiv 0 \qquad \text{for } x \text{ outside } [\alpha,\beta],$$

so $z \in \mathcal{L}^2([\alpha,\beta];w)$.

Also define the linear manifold $\mathcal{R}_0([\alpha,\beta]) \subset \mathcal{L}^2([\alpha,\beta];w)$ as the image of the maximal linear operator T_1, as generated by $w^{-1}M_A$ (as in Lemma 1), when restricted to the domain $\mathcal{D}_0([\alpha,\beta]) \subset \mathcal{D}(T_1)$, see (A.77); that is,

$$T_1 : \mathcal{D}_0([\alpha,\beta]) \to \mathcal{R}_0([\alpha,\beta]) : y \to w^{-1}M_A[y]. \tag{A.81}$$

For the remainder of this discussion we assume that $A = A^+ \in Z_n(\mathfrak{I})$, so that M_A is formally self-adjoint. In this case $w^{-1}M_A$ generates a symmetric minimal operator T_0 on $\mathcal{D}(T_0) \subset \mathcal{L}^2(\mathfrak{I};w)$, as defined earlier in Section I(1.17), as well as the maximal operator T_1 (these facts are not needed here).

LEMMA 2. *Let $A = A^+ \in Z_n(\mathfrak{I})$ so the quasi-differential expression M_A is formally self-adjoint. Then for each compact interval $[\alpha,\beta]$ interior to $\mathfrak{I}$, using the notations of Definition 1,*

$$\mathcal{L}^2([\alpha,\beta];w) = \mathcal{R}_0([\alpha,\beta]) \oplus \mathcal{H}_A([\alpha,\beta]) \tag{A.82}$$

is an orthogonal direct sum decomposition of the Hilbert space $\mathcal{L}^2([\alpha,\beta];w)$ into two closed subspaces. Thus

$$\mathcal{R}_0([\alpha,\beta)]) = \mathcal{H}_A([\alpha,\beta])^\perp$$

and

$$\mathcal{R}_0([\alpha,\beta])^\perp = \mathcal{H}_A([\alpha,\beta])$$

in $\mathcal{L}^2([\alpha,\beta];w)$.

PROOF. (See [**NA,** page 62] for a special case, with the general case treated below.)

The Lagrange-Green identity Section II(1.3 (iii)), under the assumption $A = A^+$, becomes

$$\int_\alpha^\beta w^{-1} M_A[h]\bar{g} w \, dx - \int_\alpha^\beta h \, w^{-1}\overline{M_A[g]} w \, dx = [\![h, g]\!]_A(\beta) - [\![h, g]\!]_A(\alpha) \tag{A.83}$$

for each pair of functions $h, g \in \mathcal{D}(T_1)$. Furthermore if $g \in \mathcal{D}_0([\alpha, \beta)]$, then the boundary form $[\![h, g]\!]_A(x) \equiv 0$ for $x \leq \alpha$ and for $x \geq \beta$ in $\mathcal{I}$—because the quasi-derivatives $g_A^{[r]}(x) \equiv 0$, for $r = 0, 1, \ldots, n-1$, outside $[\alpha, \beta]$. In this case, where $g \in \mathcal{D}_0([\alpha, \beta])$,

$$\int_\alpha^\beta M[h]\bar{g} \, dx = \int_\alpha^\beta h\overline{M_A[g]} dx. \tag{A.84}$$

Now take any function $f \in \mathcal{R}_0([\alpha, \beta])$, that is,

$$w^{-1} M_A[y] = f \qquad \text{or } M_A[y] = wf,$$

for some $y \in \mathcal{D}_0([\alpha, \beta])$. We must show that f is orthogonal to $\mathcal{H}_A([\alpha, \beta])$ in $\mathcal{L}^2([\alpha, \beta]; w)$.

Let $z(x)$ be any solution of the homogeneous quasi-differential equation $M_A[z] = 0$ on $\alpha \leq x \leq \beta$. We can extend $z(x)$ for $x \in \mathcal{I}$, with either $z \in \mathcal{D}_0(T_1)$ or with $z \in \mathcal{H}_A([\alpha, \beta])$—whichever is convenient—and both choices agree for calculations restricted to $[\alpha, \beta]$. Hence we can compute (from (A.83))

$$\begin{aligned} \int_\alpha^\beta f \bar{z} \, w \, dx &= \int_\alpha^\beta w^{-1} M_A[y] \bar{z} \, w \, dx \\ &= \int_\alpha^\beta y\overline{M_A[z]} dx + [\![y, z]\!]_A(\beta) - [\![y, z]\!]_A(\alpha). \end{aligned}$$

But since $M_A[z] = 0$ on $[\alpha, \beta]$, and since $y \in \mathcal{D}_0([\alpha, \beta])$ so $[\![y, z]\!]_A(\beta) = [\![y, z]\!]_A(\alpha) = 0$, we obtain

$$\int_\alpha^\beta f \bar{z} \, w \, dx = 0 \tag{A.85}$$

However z stands for any function in $\mathcal{H}_A([\alpha, \beta])$, so (A.85) asserts that f is orthogonal to $\mathcal{H}_A[\alpha, \beta])$ in $\mathcal{L}^2([\alpha, \beta]; w)$. Hence

$$\mathcal{R}_0([\alpha, \beta]) \subset \mathcal{H}_A([\alpha, \beta])^\perp. \tag{A.86}$$

For the converse, take $f \in \mathcal{H}_A([\alpha, \beta])^\perp \subset \mathcal{L}^2([\alpha, \beta]; w)$. We must now show that $f \in \mathcal{R}_0([\alpha, \beta])$, that is, there exists a function $\hat{y} \in \mathcal{D}_0([\alpha, \beta])$ such that $w^{-1} M_A[\hat{y}] = f$.

For this purpose fix a basis $z_1, z_2, \ldots, z_n$ of solutions of the homogeneous quasi-differential equation:

$$M_A[z] = 0 \qquad \text{on } \alpha \leq x \leq \beta,$$

with the prescribed initial conditions at $x = \beta$:

$$z_\nu^{[k-1]}(\beta) = \delta_\nu^k \qquad (\text{Kronecker-}\delta) \tag{A.87}$$

for $k = 1, 2, \ldots, n$ and $\nu = 1, 2, \ldots, n$. Consider each z_ν extended as zero outside $[\alpha, \beta]$, so $z_\nu \in \mathcal{H}_A([\alpha, \beta])$ for $\nu = 1, 2, \ldots, n$. Then our assumption on $f \in \mathcal{H}_A([\alpha, \beta])^\perp$ guarantees that

$$\int_\alpha^\beta f \bar{z}_\nu \, w \, dx = 0 \qquad \text{for each } \nu = 1, 2, \ldots, n.$$

We next define $\hat{y}(x)$ on $\alpha \leq x \leq \beta$ as the unique solution of the quasi-differential equation

$$M_A[y] = wf, \qquad \text{for } \alpha \leq x \leq \beta, \tag{A.88}$$

with the initial data at $x = \alpha$

$$\hat{y}_A^{[k-1]}(\alpha) = 0 \qquad \text{for } k = 1, 2, \ldots, n.$$

Note that the existence of $\hat{y}(x)$ follows because $wf \in \mathcal{L}^1([\alpha, \beta])$; namely, $|wf| = |w^{1/2}|\,|w^{1/2} f|$ and so $\int_\alpha^\beta |wf| dx \leq (\int_\alpha^\beta w\, dx)^{1/2} (\int_\alpha^\beta |f|^2 w \, dx)^{1/2} < \infty$. The extension of $\hat{y}(x) \equiv 0$, for x outside $[\alpha, \beta]$, must be still be verified to be absolutely continuous on $\mathfrak{I}$, in order that $\hat{y} \in \mathcal{D}_0([\alpha, \beta])$.

We note that for any extension of $\hat{y}(x)$ such that $\hat{y} \in \mathcal{D}_0(T_1)$, and any extension of $z_\nu \in \mathcal{D}_0(T_1)$,

$$\begin{aligned} 0 = \int_\alpha^\beta f \bar{z}_\nu w \, dx &= \int_\alpha^\beta w^{-1} M_A[\hat{y}] \bar{z}_\nu w \, dx \\ &= \int_\alpha^\beta \hat{y} w^{-1} \overline{M_A[z_\nu]} w \, dx + [\![\hat{y}, z_\nu]\!]_A(\beta) - [\![\hat{y}, z_\nu]\!]_A(\alpha). \end{aligned} \tag{A.89}$$

Because $M_A[z_\nu] = 0$ on $[\alpha, \beta]$, and $[\![\hat{y}, z_\nu]\!]_A(\alpha) = 0$, (since $\hat{y}_A^{[k-1]}(\alpha) = 0$), we conclude that

$$[\![\hat{y}, z_\nu]\!]_A(\beta) = 0 \qquad \text{for } \nu = 1, 2, \ldots, n. \tag{A.90}$$

But we treat (A.90) as a system of n homogeneous linear (algebraic) equations in the n unknowns $\hat{y}_A^{[k-1]}(\beta)$. Therefore, using (A.87), we find that

$$\hat{y}_A^{[k-1]}(\beta) = 0 \qquad \text{for } k = 1, 2, \ldots, n. \tag{A.91}$$

This implies that every extension of $\hat{y}(x)$ from $[\alpha, \beta]$ to $\mathfrak{I}$, with $\hat{y} \in \mathcal{D}_0(T_1)$, necessarily satisfies

$$\hat{y}_A^{[k-1]}(\alpha) = \hat{y}_A^{[k-1]}(\beta) = 0 \qquad \text{for } k = 1, 2, \ldots, n. \tag{A.92}$$

Hence we conclude that the global solution of (A.88) (with $f \equiv 0$ outside $[\alpha, \beta]$) is $\hat{y}(x)$ on $\mathfrak{I}$, with $\hat{y}(x) \equiv 0$ for x outside $[\alpha, \beta]$. Therefore $\hat{y}(x) \in \mathcal{D}_0([\alpha, \beta])$, as required, and furthermore $f = w^{-1} M_A[\hat{y}] \in \mathcal{R}_0([\alpha, \beta])$.

Thus $\mathcal{H}_A([\alpha, \beta])^\perp \subset \mathcal{R}_0([\alpha, \beta])$ and so, with (A.86),

$$\mathcal{R}_0([\alpha, \beta]) = \mathcal{H}_A([\alpha, \beta])^\perp \qquad \text{in } \mathcal{L}^2([\alpha, \beta]; w). \tag{A.93}$$

But $\mathcal{H}_A([\alpha, \beta])$ is finite dimensional and hence closed, and therefore $\mathcal{R}_0([\alpha, \beta])$ is also a closed subspace of $\mathcal{L}^2([\alpha, \beta]; w)$, and they are orthogonally complementary subspaces. □

DENSITY THEOREM 1. *Let $A = A^+ \in Z_n(\mathfrak{I})$ so the quasi-differential expression M_A is formally self-adjoint, and let the positive weight $w \in \mathcal{L}^1_{\mathrm{loc}}(\mathfrak{I})$ on the interval $\mathfrak{I}$, so $w^{-1}M_A$ generates a maximal linear operator T_1 on $\mathcal{D}(T_1) \subset \mathcal{L}^2(\mathfrak{I}; w)$, as above. Then*

$$\mathcal{D}_0(T_1) \qquad \text{is dense in } \mathcal{L}^2(\mathfrak{I}; w).$$

PROOF. It is sufficient to prove that:

for each compact interval $[\alpha, \beta]$ interior to $\mathfrak{I}$, $\mathcal{D}_0([\alpha,\beta])$ is dense in $\mathcal{L}^2([\alpha,\beta]; w)$.

From this assertion the conclusion of the theorem follows immediately from the classical approximation of functions in $\mathcal{L}^2(\mathfrak{I}; w)$ by functions in various $\mathcal{L}^2([\alpha,\beta]; w)$—say, for a sequence of compact intervals $[\alpha, \beta]$ expanding to exhaust $\mathfrak{I}$.

Accordingly fix any one such compact interval $[\alpha,\beta]$ interior to $\mathfrak{I}$. Then let $h \in \mathcal{L}^2([\alpha,\beta]; w)$ satisfy

$$\text{(A.94)} \qquad (h, y) = \int_\alpha^\beta h\bar{y}\, w\, dx = 0 \qquad \text{for all } y \in \mathcal{D}_0([\alpha,\beta]).$$

We shall show that this implies $h = 0$ (a.e), and consequently that $\mathcal{D}_0([\alpha,\beta])$ must be dense in $\mathcal{L}^2([\alpha,\beta]; w)$, which will prove the Density Theorem.

Let $\hat{z}$ be any solution of the quasi-differential equation

$$\text{(A.95)} \qquad w^{-1}M_A[z] \equiv h \qquad \text{on } \alpha \le x \le \beta,$$

and extend $\hat{z}$ over $\mathfrak{I}$ so $\hat{z} \in \mathcal{D}_0(T_1)$. Then for each $y \in \mathcal{D}_0([\alpha,\beta])$

$$\text{(A.96)} \qquad (h, y) = \int_\alpha^\beta w^{-1}M_A[\hat{z}]\bar{y}w\, dx = \int_\alpha^\beta \hat{z}\, w^{-1}\overline{M_A[y]}w\, dx \qquad \text{(see A.84).}$$

Of course, the same result (A.96) holds if we use the extension $\hat{z}(x) \equiv 0$ for x outside $[\alpha,\beta]$, with $\hat{z} \in \mathcal{L}^2([\alpha,\beta]; w)$. Then (A.96) implies that $\hat{z}$ is orthogonal to $\mathcal{R}_0([\alpha,\beta])$ in $\mathcal{L}^2([\alpha,\beta]; w)$ and, by Lemma 2, we have $\hat{z} \in \mathcal{H}_A([\alpha,\beta])$.

But $\hat{z} \in \mathcal{H}_A([\alpha,\beta])$ means that $M_A[\hat{z}] = 0$ on $\alpha \le x \le \beta$, and consequently $w(x)h(x) \equiv 0$ on $[\alpha,\beta]$. Since $w(x) > 0$ a.e. on $[\alpha,\beta]$, we conclude that $h = 0$ a.e. on $[\alpha,\beta]$, as required. The final result is that $\mathcal{D}_0([\alpha,\beta])$ is dense in $\mathcal{L}^2([\alpha,\beta]; w)$, and hence the Density Theorem is proved. □

As in the Density Theorem 1, we again consider $A = A^+ \in Z_n(\mathfrak{I})$ determining the formally self-adjoint quasi-differential expression M_A on the real interval $\mathfrak{I}$, which bears the given positive weight function w. Then $w^{-1}M_A$ generates the corresponding minimal and maximal operators T_0 on $\mathcal{D}(T_0)$ and T_1 on $\mathcal{D}(T_1)$, as defined in Section I (1.17) and (1.11), respectively, in the complex Hilbert space $\mathcal{L}^2(\mathfrak{I}; w)$—according to the notations listed in Section II (1.1) (1.2) (1.3). In particular, for convenience we recall that

$$\begin{aligned}
\mathcal{D}(A) &= \{f : \mathfrak{I} \to \mathbb{C} \mid f_A^{[r]} \in AC_{\mathrm{loc}}(\mathfrak{I}) \text{ for } r = 0, 1, \dots, n-1\} \\
\mathcal{D}(T_1) &= \{f \in \mathcal{D}(A) \mid f \text{ and } T_1 f \text{ in } \mathcal{L}^2(\mathfrak{I}; w)\} \\
\mathcal{D}(T_0) &= \{f \in \mathcal{D}(T_1) \mid [f : \mathcal{D}(T_1)]_A = 0\}.
\end{aligned}$$

In the Density Theorem 1, we proved that the linear manifold (see Section I(1.15))

$$\text{(A.97)} \qquad \mathcal{D}_0(T_1) :\equiv \{f \in \mathcal{D}(T_1) \mid \text{supp } f \text{ lies in a compact set interior to } \mathfrak{I}\}$$

is dense in $\mathcal{L}^2(\mathcal{I}; w)$. We now denote the restriction of T_1 to $\mathcal{D}_0(T_1)$ by T_{00}, that is, $\mathcal{D}(T_{00}) :\equiv \mathcal{D}_0(T_1)$ and we observe that

$$T_{00} \subseteq T_0 \subseteq T_1 \text{ on } \mathcal{D}_0(T_1) \subseteq \mathcal{D}(T_0) \subseteq \mathcal{D}(T_1), \tag{A.98}$$

where all these operators are restrictions of T_1. In the following Theorem 2 we shall prove that the adjoint operators satisfy

$$T_0^* = T_1, \qquad T_1^* = T_0 \tag{A.99}$$

and that T_{00} is a symmetric operator whose closure is

$$\bar{T}_{00} = T_0. \tag{A.100}$$

LEMMA 1. *Let $A = A^+ \in Z_n(\mathcal{I})$ and consider the minimal and maximal operators T_0 on $\mathcal{D}(T_0)$ and T_1 on $\mathcal{D}(T_1)$, respectively, as generated by $w^{-1}M_A$ in the complex Hilbert space $\mathcal{L}^2(\mathcal{I}; w)$, as before. Then, for each function $F \in \mathcal{L}^2(\mathcal{I}; w)$ there exists a function $y \in \mathcal{D}(A)$ on $\mathcal{I}$ such that*

$$T_1 y = F \qquad \textit{on } \mathcal{I}.$$

PROOF. Consider the quasi-differential equation for y,

$$T_1 y = F \text{ or } y_A^{[n]} = i^{-n} w F \quad \text{on } \mathcal{I}. \tag{A.101}$$

Note that for each compact interval $[\alpha, \beta]$ interior to $\mathcal{I}$

$$\int_\alpha^\beta |wF| dx = \int_\alpha^\beta w^{1/2} |w^{1/2} F| dx \le \left(\int_\alpha^\beta w dx \right)^{1/2} \left(\int_\alpha^\beta |F|^2 w dx \right)^{1/2} < \infty,$$

so the coefficient $i^{-n} wF \in \mathcal{L}^1_{\mathrm{loc}}(\mathcal{I})$. Therefore, upon regarding this quasi-differential equation (A.101) as a first-order matrix ordinary differential system (see Section I.2), we conclude that there exist solutions, each defined on all $\mathcal{I}$ and in class $\mathcal{D}(A)$. That is, for each such solution y, the quasi-derivatives

$$y_A^{[0]} = y, \quad y_A^{[1]}, \dots, y_A^{[n-1]}$$

all exist on $\mathcal{I}$ and belong to $AC_{\mathrm{loc}}(\mathcal{I})$. □

THEOREM 2. *Let $A = A^+ \in Z_n(\mathcal{I})$ and consider the minimal and maximal operators T_0 on $D(T_0)$ and T_1 on $\mathcal{D}(T_1)$, respectively, as generated by $w^{-1}M_A$ in the complex Hilbert space $\mathcal{L}^2(\mathcal{I}; w)$, as before. Also consider the restriction T_{00} of T_1 to the domain $\mathcal{D}_0(T_1)$, so*

$$T_{00} \subseteq T_0 \subseteq T_1 \textit{ on } \mathcal{D}_0(T_1) \subseteq \mathcal{D}(T_0) \subseteq \mathcal{D}(T_1), \tag{A.102}$$

respectively, where each operator is a restriction of T_1.

Then the adjoint operators exist in $\mathcal{L}^2(\mathcal{I}; w)$ and satisfy

$$T_{00}^* = T_0^* = T_1 \textit{ and } T_1^* = T_0. \tag{A.103}$$

Hence, both T_0 and T_1 are closed operators with dense domains in $\mathcal{L}^2(\mathcal{I}; w)$. Also, the minimal closure of the symmetric operator T_{00} is $\bar{T}_{00} = T_0$.

PROOF. Since $\mathcal{D}_0(T_1) \equiv \mathcal{D}(T_{00})$ is dense in $\mathcal{L}^2(\mathcal{I}; w)$, the adjoints T_{00}^*, T_0^*, T_1^* all exist as closed operators in $\mathcal{L}^2(\mathcal{I}; w)$. Moreover, by elementary arguments [DS]

$$D(T_1^*) \subseteq \mathcal{D}(T_0^*) \subseteq \mathcal{D}(T_{00}^*), \tag{A.104}$$

and each of T_1^*, T_0^*, T_{00}^* is a restriction of T_{00}^*.

We first prove that $T_{00}^* = T_1$. Recall that the condition for a function $f \in \mathcal{L}^2(\mathcal{I}; w)$ to belong to $\mathcal{D}(T_{00}^*)$ is that there exists some (unique) $F \in \mathcal{L}^2(\mathcal{I}; w)$ for which the scalar products satisfy

$$(f, T_1 g) = (F, g), \qquad \text{for all } g \in \mathcal{D}_0(T_1) \equiv \mathcal{D}(T_{00}).$$

According to Lemma 1 above, there exists $y \in \mathcal{D}(A)$ so

$$T_1 y = F \text{ and } y_A^{[r]} \in AC_{\mathrm{loc}}(\mathcal{I}), \qquad \text{for } r = 0, 1, \ldots, n-1.$$

Then, on each compact interval $[\alpha, \beta]$ interior to $\mathcal{I}$,

$$\int_\alpha^\beta y\overline{T_1 g} w dx = \int_\alpha^\beta T_1 y \bar{g} w dx = \int_\alpha^\beta F \bar{g} w dx,$$

for every $g \in \mathcal{D}_0([\alpha, \beta])$, that is, provided $\operatorname{supp} g$ lies in $[\alpha, \beta]$ (see notation in Lemma 2 before Theorem 1 above). Hence

$$\int_\alpha^\beta f\overline{T_1 g} w dx = \int_\alpha^\beta y\overline{T_1 g} w dx,$$

for all $g \in \mathcal{D}_0([\alpha, \beta])$. We note that this implies that:

$$f - y \text{ is orthogonal to } \mathcal{R}_0([\alpha, \beta]) \text{ in } \mathcal{L}^2([\alpha, \beta]; w)$$

where $\mathcal{R}_0([\alpha, \beta])$ is the T_1-image of the domain $\mathcal{D}_0([\alpha, \beta])$.

By Lemma 2 of Theorem 1

$$f - y = z \qquad \text{on } [\alpha, \beta],$$

where z is some solution of the homogeneous equation $M_A[z] = 0$ or $z_A^{[n]} = 0$ on $\mathcal{I}$. Since $z \in \mathcal{D}(A)$ on $\mathcal{I}$, we conclude that

$$f_A^{[r]} \in AC([\alpha, \beta]) \qquad \text{for } r = 0, 1, \ldots, n-1.$$

But $[\alpha, \beta]$ is an arbitrary compact interval interior to $\mathcal{I}$, so we conclude that

$$f \in \mathcal{D}(A) \qquad \text{on } \mathcal{I}.$$

Moreover,

$$T_1 f = T_1 y \qquad \text{on } \mathcal{I}.$$

We summarize the argument thus far. For each $f \in \mathcal{D}(T_{00}^*)$ we find that $f \in \mathcal{D}(A)$ and $T_1 f = T_1 y = F$ on $\mathcal{I}$. Therefore both f and $T_1 f$ belong to $\mathcal{L}^2(\mathcal{I}; w)$ which means that $f \in \mathcal{D}(T_1)$. Also we note that

$$(f, T_1 g) = (T_1 f, g), \qquad \text{for all } g \in \mathcal{D}_0(T_1) \equiv \mathcal{D}(T_{00}).$$

Thus, we have shown that

$$\mathcal{D}(T_{00}^*) \subseteq \mathcal{D}(T_1) \qquad \text{and} \qquad T_{00}^* f = T_1 f.$$

But the reverse inclusion is trivial. Because for any $h \in \mathcal{D}(T_1)$ we always have

$$(h, T_1 g) = (T_1 h, g), \qquad \text{for all } g \in \mathcal{D}_0(T_1).$$

That is, because $A = A^+$ we know that

$$[h : g]_A = (T_1 h, g) - (h, T_1 g) = 0$$

for all $h \in \mathcal{D}(T_1)$ and $g \in \mathcal{D}_0(T_1)$. Therefore

$$\mathcal{D}(T_{00}^*) = \mathcal{D}(T_1) \quad \text{and } T_{00}^* f = T_1 f, \quad \text{for } f \in \mathcal{D}(T_1).$$

Accordingly, T_1 is a closed operator in $\mathcal{L}^2(\mathcal{I}; w)$.

Next, consider T_0^* on the domain $\mathcal{D}(T_0^*) \subseteq \mathcal{D}(T_1)$. But for each $f \in \mathcal{D}(T_1)$ we have

$$[f : g]_A = (T_1 f, g) - (f, T_1 g) = 0, \qquad \text{for all } g \in \mathcal{D}(T_0).$$

Therefore, $\mathcal{D}(T_1) \subseteq \mathcal{D}(T_0^*)$ so

$$\mathcal{D}(T_0^*) = \mathcal{D}(T_1) \quad \text{and } T_0^* f = T_1 f, \quad \text{for } f \in \mathcal{D}(T_1).$$

Now the definition of T_0 on $\mathcal{D}(T_0)$, see Section I (1.17), leads directly to the conclusion that T_0 is a closed operator in the Hilbert space $\mathcal{L}^2(\mathcal{I}; w)$. This is an elementary argument now that it is known that T_1 is closed.

Furthermore, the general theory of adjoint operators in Hilbert space, [**DS**] and [**WE**], then guarantees that

$$T_0^{**} = T_0 \quad \text{or } T_1^* = T_0 \quad \text{on } \mathcal{D}(T_1^*) = \mathcal{D}(T_0). \tag{A.105}$$

Also $T_{00}^{**} = \bar{T}_{00}$, so

$$\bar{T}_{00} = T_1^* = T_0, \tag{A.106}$$

as required. □

Note that the proof of The Density Theorem 1 does not depend on the properties of the minimal and maximal operators, T_0 and T_1, but consists essentially of an analysis of quasi-differential equations and their solutions. In Theorem 2 the basic properties of T_0 and T_1 are established, with respect to the closure and the adjoint relationships. Thus, through Theorems 1 and 2 above, the foundations of the von Neumann theory [**DS**] for self-adjoint extensions of the symmetric operator T_0 have been established, in accord with the assertions made in Section I.1.

APPENDIX B

Complexification of real symplectic spaces, and the real GKN-Theorem for real operators

In classical mechanics and geometry [**AM**] [**MH**] a real symplectic space S_R is defined as a real vector space, together with a real bilinear form $[:]_R$ that is skew-symmetric and nondegenerate. As a generalization of this concept we have defined a complex symplectic space S (see Sections I (2.32) and III (1.1)) as a complex vector space, together with a complex semibilinear form $[:]$ that is skew-Hermitian and nondegenerate. In each case we define a Lagrangian subspace as one on which the symplectic form vanishes identically. In this Appendix B we formulate results on the existence and uniqueness of the complexification of any real symplectic space S_R to a complex symplectic space S, and we then apply these ideas to assert and demonstrate a "real version" of the Glazman-Krein-Naimark (GKN)-Theorem (see Section II, Theorem 1) holding for self-adjoint real operators generated by real quasi-differential expressions. Finally we apply this real GKN-Theorem to the boundary value problems for real quasi-differential operators, and relate these considerations to the global differential geometry of real Lagrangian Grassmannians.

Definition 1. Let S be a complex symplectic space, with complex symplectic form $[:]$. A complex conjugation in S is an involutory bijective map on S:

$$Z_1 \to \bar{Z}_1 \quad \text{with} \quad \bar{\bar{Z}}_1 = Z_1, \tag{B.1}$$

such that

$$\overline{\mu_1 Z_1 + \mu_2 Z_2} = \bar{\mu}_1 \bar{Z}_1 + \bar{\mu}_2 \bar{Z}_2, \quad \overline{[Z_1 : Z_2]} = [\bar{Z}_1 : \bar{Z}_2], \tag{B.2}$$

for all vectors Z_1, $Z_2 \in S$ and all complex scalars μ_1, $\mu_2 \in \mathbb{C}$.

Remark. One should not confuse the conjugation of complex numbers $\mu = \alpha + i\beta \to \bar{\mu} = \alpha - i\beta$ with the involutory conjugation map of vectors $Z \to \bar{Z}$. In particular, it is easy to show that the 1-dimensional complex symplectic space $\mathbb{C}$, with $[1 : 1] = i$, does not admit any involutory conjugation—since each real vector r would then have to satisfy $[r : r] = 0$, see Proposition 1 below.

The concept of an isomorphism between two complex symplectic spaces $S$and $\hat{S}$ has been introduced previously; namely, a map F of S onto $\hat{S}$,

$$F : S \to \hat{S}$$

which is a vector-space isomorphism (over $\mathbb{C}$) preserving the corresponding symplectic products. We further assert that F is an isomorphism between two such complex symplectic spaces S and $\hat{S}$, *each with a distinguished complex conjugation*, in case F also commutes with the respective conjugation maps,

$$\overline{F(Z)} = F(\bar{Z}) \qquad \text{for all } Z \in S. \tag{B.3}$$

DEFINITION 2. Let S, with the symplectic form $[:]$, be a complex symplectic space having a prescribed complex conjugation.

A vector $Z_1 \in S$ is called *real* in case Z_1 is invariant (or fixed) under the conjugation map, that is,

$$Z_1 = \bar{Z}_1,$$

and we denote the set of all such real vectors in S by

$$S'_R = \{Z \in S \mid Z = \bar{Z}\}. \tag{B.4}$$

Furthermore the set S'_R, together with the algebraic operations inherited from S—for instance, see (B.2),

$$\alpha_1 Z_1 + \alpha_2 Z_2 \in S'_R, \quad [Z_1 : Z_2]_R :\equiv [Z_1, Z_2] \in \mathbb{R} \tag{B.5}$$

for all $Z_1, Z_2 \in S'_R$ and real scalars $\alpha_1, \alpha_2 \in \mathbb{R}$—specifies the real symplectic space S'_R of all real vectors of S.

REMARK. It is trivial that S'_R is, in fact, a real symplectic space, and we can further describe it as

$$S'_R = \{Z \in S \mid \operatorname{Re} Z = Z\} = \{Z \in S \mid \operatorname{Im} Z = 0\}.$$

Here we introduce the familiar notation

$$\operatorname{Re} Z = \tfrac{1}{2}(Z + \bar{Z}), \ \operatorname{Im} Z = \tfrac{1}{2i}(Z - \bar{Z}) = \operatorname{Re}(-iZ), \tag{B.6}$$

so $\operatorname{Re} Z$ and $\operatorname{Im} Z$ are real vectors in S'_R for each $Z \in S$. Furthermore

$$Z = \operatorname{Re} Z + i \operatorname{Im} Z, \ \bar{Z} = \operatorname{Re} Z - i \operatorname{Im} Z, \tag{B.7}$$

so

$$S'_R = \{\operatorname{Re} Z \mid Z \in S\}. \tag{B.8}$$

We shall verify that the complex symplectic space S, with its prescribed complex conjugation, is determined uniquely as the complexification of the real symplectic space S'_R, in the sense of the following definition and proposition.

DEFINITION 3. The complexification of a real symplectic space S_R is a complex symplectic space S, together with a distinguished complex conjugation, such that the real vectors S'_R of S constitute a real symplectic space that is isomorphic with S_R.

Often we omit the explicit specification of the complex conjugation in S, when this is apparent from the context.

PROPOSITION 1. *Let S_R, together with a real symplectic form $[:]_R$, be a real symplectic space. Then there exists a complex symplectic space S, together with a complex symplectic form $[:]$ and a specified complex conjugation, such that S is a complexification of S_R; that is, there exists an isomorphism (see (B.11) below) of the real symplectic space S'_R, of all real vectors of S, onto S_R. Furthermore, the complexification S of S_R is unique in the following sense:*

Let $\hat{S}$ be any complexification of S_R. Then $\hat{S}$ is isomorphic to S, as complex symplectic spaces with complex conjugations.

Also

$$\text{(B.9)} \qquad \textit{(real)}\ \dim S_R = \textit{(complex)}\ \dim S,$$

and, if either is infinite then so is the other.

PROOF. A complexification S of S_R has already been constructed in Section III.1. Recall that a vector Z of S is an ordered pair (X, Y) with $X, Y \in S_R$; with vector addition in S componentwise, and the multiplication by complex scalars and the computation of the complex symplectic form $[:]$ as in Section III (1.2) and (1.3). As before, we introduce the convenient notation $Z = X + iY \in S$ for X, Y in S_R. The complex involutory conjugation in S, see Section III (1.4), (1.5), is given by

$$Z = X + iY \to \bar{Z} = X - iY.$$

In this notation

$$\text{(B.10)} \qquad X = \operatorname{Re} Z = \tfrac{1}{2}(Z + \bar{Z}) \text{ and } Y = \operatorname{Im} Z = \tfrac{1}{2i}(Z - \bar{Z}) = \operatorname{Re}(-iZ),$$

are real vectors in S, and the real symplectic space S'_R of all such real vectors in S is given by

$$S'_R = \{Z = X + iY \in S \mid Y = 0\}.$$

It is then obvious that the map

$$\text{(B.11)} \qquad S'_R \to S_R : Z = X + i0 \to X$$

is a natural isomorphism between these two real symplectic spaces. In addition, (real) $\dim S_R =$ (complex) $\dim S$, see Section III (1.7).

It remains to demonstrate the required uniqueness of this complexification S of S_R. Let $\hat{S}$ be any other complex symplectic space with a specified complex conjugation, such that the corresponding set $\hat{S}'_R$ of real vectors constitutes a real symplectic space isomorphic with S_R. Since $\hat{S}'_R$ is isomorphic with S_R, it is also isomorphic with S'_R; and we fix this isomorphism and indicate it by

$$\text{(B.12)} \qquad S'_R \to \hat{S}'_R, \quad X \to \hat{X}.$$

We must extend this map to a surjective isomorphism of S onto $\hat{S}$, as complex symplectic spaces with conjugation.

Each vector $Z \in S$ can be written as

$$Z = X + iY \quad \text{for } X, Y \in S'_R,$$

and X, Y are unique; namely $X = \operatorname{Re} Z, Y = \operatorname{Im} Z$, as in (B.10). Similarly each vector of $\hat{S}$ can be expressed uniquely in terms of its real and imaginary parts, relative to the conjugation involution in $\hat{S}$. Then consider the map

$$\text{(B.13)} \qquad S \to \hat{S}, \quad Z = X + iY \to \hat{Z} = \hat{X} + i\hat{Y},$$

with $X, Y \in S'_R$ and $\hat{X}, \hat{Y} \in \hat{S}'_R$, as in (B.12). Since the map in (B.12) is surjective onto $\hat{S}'_R$, the map in (B.13) is surjective onto $\hat{S}$.

Clearly (B.13) defines an isomorphism of S onto $\hat{S}$, since the algebraic and conjugation operations in S, and also in $\hat{S}$, are defined explicitly in terms of the corresponding operations in the isomorphic spaces S'_R and $\hat{S}'_R$, respectively, i.e. compare the formulas in Section III (1.2), (1.3), and (1.4). □

REMARK 1. Recall that a real symplectic space of finite dimension D must have $D = 2m$ even, and is then unique (up to isomorphism). We denote this real symplectic space by $\mathbb{R}^{2m}$. The complexification of $\mathbb{R}^{2m}$ is the complex symplectic space $\mathbb{C}^{2m}$ with $Ex = 0$, see Theorem 1 in Section III.1 above.

REMARK 2. The complexification of a real vector space V_R to a complex vector space V_C, with a complex conjugation involution, can be constructed in an analogous manner—and this complexification is unique in the sense of Proposition 1.

For example, the real vector space $\mathbb{R}^D$ has the complexification $\mathbb{C}^D$, with the usual conjugation of complex vectors of $\mathbb{C}^D$, for each integer $D \geq 1$.

As another example, the complex Hilbert space $\mathcal{L}^2(\mathcal{I}; w)$ of Section I.1 is the complexification of the real Hilbert space

$$\mathcal{L}^2_R(\mathcal{I}; w) = \{f \in \mathcal{L}^2(\mathcal{I}; w) \mid f = \bar{f}\}. \tag{B.14}$$

As indicated, the relevant complex conjugation involution in $\mathcal{L}^2(\mathcal{I}; w)$ is the familiar conjugation of complex-valued functions. Hence the real vectors in $\mathcal{L}^2(\mathcal{I}; w)$ are just the real-valued functions (or equivalence classes of such functions agreeing a.e. on $\mathcal{I}$). As a warning, we mention the possibility of introducing some different conjugation involution in $\mathcal{L}^2(\mathcal{I}; w)$—say,

$$f + ig \to -f + ig, \quad \text{for } f, g \in \mathcal{L}^2_R(\mathcal{I}; w),$$

so then the "real vectors" would have the form ig for $g \in \mathcal{L}^2_R(\mathcal{I}; w)$. Needless to say, we shall not do this, but we shall always use the standard complex conjugation in $\mathcal{L}^2(\mathcal{I}; w)$, and also in other complex vector spaces of complex-valued functions on $\mathcal{I}$.

The next two corollaries are obvious consequences of the constructions in Proposition 1, and are presented only as convenient summaries, useful for the subsequent examples (B.17) through (B.28).

COROLLARY 1. *A complex symplectic space S, with a complex conjugation, is necessarily the complexification of some real symplectic space; namely S'_R, the real vectors in S.*

Further, a complex symplectic space S with finite dimension D, admits a complex conjugation if and only if S has invariants $\dim S = D$ even, and excess $Ex = 0$. In this case S is the complexification of the real symplectic space $\mathbb{R}^D$.

COROLLARY 2. *Let S be a complex symplectic space with a complex symplectic form $[:]$ and with a prescribed complex conjugation. Let S'_R, with the real symplectic form $[:]_R$ as in (B.4) and (B.5), be the corresponding real symplectic space of all real vectors in S. Assume*

$$\textit{(real)} \ \dim S'_R = \textit{(complex)} \dim S = D < \infty$$

for an even integer D.

Fix any basis of S'_R to define an isomorphism of S'_R onto the real symplectic space $\mathbb{R}^D$, and then the real symplectic form in S'_R is specified via a real skew-symmetric, nonsingular $D \times D$ matrix H. This same "real basis" serves also as a basis for S to define an isomorphism of S onto $\mathbb{C}^D$, with the corresponding symplectic form specified by the same real matrix H.

Furthermore, using such a real basis, the complex conjugation in S maps each vector $u = (u_1, u_2, \ldots, u_D) \in S$ by

$$u \to \bar{u} = (\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_D),$$

and hence $u \in S'_R$ if and only if all the components of u are real.

In this way we observe that the symplectic invariants of a real symplectic space S_R, with $\dim S_R = D < \infty$, are the same as the corresponding symplectic invariants of its complexification S.

Now let us return briefly to the real symplectic space $\mathbb{R}^{2m}$ and its complexification $\mathbb{C}^{2m}$. In $\mathbb{R}^{2m}$ the real symplectic form is given by

$$\text{(B.15)} \qquad [u:v]_R = uHv^t = (u_1, \ldots, u_{2m})H(v_1, \ldots, v_{2m})^t,$$

for the real row $2m$-vectors $u, v \in \mathbb{R}^{2m}$, and an appropriate real $2m \times 2m$ matrix H (depending on the chosen basis and coordinates in $\mathbb{R}^{2m}$) such that

$$\text{(B.16)} \qquad H = -H^t, \quad \det H \neq 0.$$

It is a classical result [**AM**] of elementary matrix theory that there always exists a canonical basis $\{e^1, e^2, \ldots, e^m; e^{m+1}, \ldots, e^{2m}\}$ in $\mathbb{R}^{2m}$ for which H has the format K, (see Section III(1.12)),

$$\text{(B.17)} \qquad K = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}, \quad \text{for each } m \geq 1.$$

From the existence of such a canonical basis it follows that a real symplectic space cannot have an odd dimension, and moreover any two real symplectic spaces of finite dimension $2m \geq 2$ are necessarily isomorphic.

In the real symplectic space $\mathbb{R}^{2m}$, the symplectic product (B.15) has a special expression in the corresponding canonical coordinates (using the corresponding matrix (B.17)), namely

$$\text{(B.18)} \qquad [u:v]_R = \sum_{r=1}^{m} (u_r v_{m+r} - u_{m+r} v_r).$$

In $\mathbb{C}^{2m}$ with the complex symplectic structure generated by the complexification of the real symplectic space $\mathbb{R}^{2m}$, the symplectic product of two complex row $2m$-vectors $u, v \in \mathbb{C}^{2m}$ is given by

$$\text{(B.19)} \qquad [u:v] = uHv^* = (u_1, \ldots, u_{2m})H(v_1, \ldots, v_{2m})^*,$$

where the complex $2m \times 2m$ matrix H (depending on the chosen basis and complex coordinates in $\mathbb{C}^{2m}$) is such that

$$\text{(B.20)} \qquad H = -H^*, \quad \det H \neq 0.$$

But the same canonical basis for $\mathbb{R}^{2m}$ (considered as "real vectors" in $\mathbb{C}^{2m}$) yields the formula for the symplectic products in $\mathbb{C}^{2m}$

$$[u:v] = \sum_{r=1}^{m} (u_r \bar{v}_{m+r} - u_{m+r}\bar{v}_r). \tag{B.21}$$

Other convenient bases in $\mathbb{R}^{2m}$ are available so as to choose H in the format (B.22)

$$H = \operatorname{diag}\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}, \quad (m\text{-copies, for all } m \geq 1),$$

or also

(B.23)

$$H = (-1)^s \left(\begin{array}{c|c} \begin{matrix} & & (-1)^m \\ & \cdot^{\cdot^{\cdot}} & \\ (-1)(+1) & & \end{matrix} & 0 \\ \hline 0 & \begin{matrix} & & (-1)^{m-1} \\ & \cdot^{\cdot^{\cdot}} & \\ (+1)(-1) & & \end{matrix} \end{array} \right), \quad \text{for } m = 2s \text{ even}.$$

These special bases,and corresponding expressions for H, can also serve equally well in the complexification of $\mathbb{R}^{2m}$ to the corresponding complex symplectic space $\mathbb{C}^{2m}$.

However, in $\mathbb{C}^{2m}$ (as the complexification of the real symplectic space $\mathbb{R}^{2m}$) there are other useful complex bases in terms of which H becomes other formats than K (see Section III.1, especially Theorem 1), for instance H can be replaced by the format:

$$\hat{K} = \begin{pmatrix} iI_m & 0 \\ 0 & -iI_m \end{pmatrix}, \quad \text{for } m \geq 1, \tag{B.24}$$

or else

$$J = i^m \begin{pmatrix} J_m & 0 \\ 0 & -J_m \end{pmatrix}, \quad \text{for } m \geq 1, \tag{B.25}$$

where

$$J_m = \begin{pmatrix} 0 & 0 & \dots & 0 & (-1)^m \\ \vdots & & & (-1)^{m-1} & 0 \\ & & \cdot^{\cdot^{\cdot}} & & \\ 0 & (+1) & & & \vdots \\ (-1) & 0 & \dots & & 0 \end{pmatrix} \tag{B.26}$$

(J can be examined separately in the cases where m is even, or m is odd—as in Section III.1).

NOTE 1. In the complex symplectic space $\mathbb{C}^{2m}$, with the symplectic form prescribed by the $2m \times 2m$ skew-Hermitian matrix J given by (B.25) and (B.26), the symplectic product of vectors $u = (u_1, u_2, \dots, u_m, u_{m+1}, \dots, u_{2m})$, and $v = (v_1, v_2, \dots, v_m, v_{m+1}, \dots, v_{2m})$ is computed by

$$[u : v] = uJv^* = i^m \sum_{r=0}^{m-1} (-1)^r \{u_{2m-r}\bar{v}_{m+1+r} - u_{m-r}\bar{v}_{1+r}\}. \tag{B.27}$$

When $m = 2s$ is even, then J in (B.25) reduces to the real skew-symmetric matrix H in (B.23) and then

$$[u : v] = uHv^* = (-1)^s \sum_{r=0}^{m-1} (-1)^r \{u_{2m-r}\bar{v}_{m+1+r} - u_{m-r}\bar{v}_{1+r}\}, \tag{B.28}$$

with the real case $u = \bar{u}$, $v = \bar{v}$ of special interest.

The motivation for the symplectic form (B.27) is the *regular* boundary value problem, $M_A[y] = \lambda wy$, as in Section I(1.1), for given $A = A^+ \in Z_n(\mathcal{I})$ and $w(x) > 0$ in $\mathcal{L}^1_{\mathrm{loc}}(\mathcal{I})$—where we assume that $\mathcal{I} = [a, b]$ is compact so the deficiency index $d = n$. Then the endpoint space $\mathcal{S} = \mathcal{D}(T_1)/\mathcal{D}(T_0)$ (see Section II (1.3), (1.4) and Section IV) is a complex symplectic $2n$-space, isomorphic with $\mathbb{C}^{2n}$ (complexification of $\mathbb{R}^{2n}$) under the map

$$\hat{f} = \{f + \mathcal{D}(T_0)\} \to (f_A^{[0]}(a),\ f_A^{[1]}(a), \dots, f_A^{[n-1]}(a),\ f_A^{[0]}(b), \dots, f_A^{[n-1]}(b)).$$

Further, the symplectic product of two such vectors $\hat{f}, \hat{g} \in \mathcal{S}$ can be expressed in the coordinates of $\mathbb{C}^{2n}$ according to

$$[\hat{f} : \hat{g}]_A = \hat{f}J\hat{g}^* = i^n \sum_{r=0}^{n-1} (-1)^r \{f_A^{[n-1-r]}(b)\overline{g_A^{[r]}(b)} - f_A^{[n-1-r]}(a)\overline{g_A^{[r]}(a)}\}. \tag{B.29}$$

Hence the symplectic product in $\mathcal{S}$ can be computed in terms of the skew-Hermitian matrix J of (B.25).

However there is a confusion of notation (e.g. $f_A^{[0]}(a)$ corresponds to u_1, $f_A^{[0]}(b)$ corresponds to u_{m+1}, n corresponds to m, etc.). We have used the dimension $2m$ for the general algebraic treatment of symplectic spaces, and $2n$ for the space $\mathcal{S}$ arising from quasi-differential expressions of order n. Further we have here written $m = 2s$ when m is even, yet we later shall often write $n = 2m$ when n is even (in accord with current literature), and care must be taken appropriately - although the use of a consistent notation within each discussion and topic should minimize any confusions.

Finally we remark that $\mathbb{C}^{2m}$, and also $\mathbb{C}^D$ for each dimension $D \geq 1$, bears many non-isomorphic complex symplectic structures—see Examples at the end of Section I.2, and also Theorem 1 of Section III. For instance, consider the complex symplectic space $\mathbb{C}^D$ with some arbitrarily prescribed complex symplectic structure, say specified by some skew-Hermitian, nonsingular $D \times D$ matrix H. Then H can always be chosen (in appropriate complex coordinates in $\mathbb{C}^D$) to have the format $\hat{K}$, where

$$\hat{K} = \begin{pmatrix} iI_p & 0 \\ 0 & -iI_q \end{pmatrix}, \tag{B.30}$$

and $p = q$ if and only if $\mathbb{C}^{2p}$ is the complexification of $\mathbb{R}^{2p}$, see Section III, Theorem 1. We shall often use this notation $\hat{K}$ for all such diagonal matrices with $p \geq 0$

terms of $(+i)$ and $q = D - p \geq 0$ terms of $(-i)$, in some arrangement along the diagonal.

We next turn to the study of Lagrangian subspaces of an arbitrary complex symplectic space S which is the complexification of some given real symplectic space S_R.

PROPOSITION 2. *Consider a real symplectic space S_R with real symplectic form $[:]_R$, and its complexification S with complex symplectic form $[:]$, as before.*

(1) *Then for each Lagrangian subspace $L \subset S$ the conjugate (with respect to the complex conjugation in S)*

$$\bar{L} = \{\bar{Z} \mid Z \in L\}$$

is also a Lagrangian subspace of S.

(2) *For each real Lagrangian subspace $L_R \subset S_R$ the complexification $L_C \subset S$,*

$$L_C = \{Z = X + iY \mid X, Y \in L_R\}$$

is a Lagrangian subspace of S. Furthermore

$$L_C = \bar{L}_C = \{\bar{Z} = X - iY \mid X, Y \in L_R\},$$

so L_C is self-conjugate. Different real Lagrangian subspaces of S_R have different complexifications in S.

(3) *In fact, a complex Lagrangian subspace $L_1 \subset S$ is self-conjugate (that is, $L_1 = \bar{L}_1$) if and only if L_1 is the complexification of a real Lagrangian subspace $L_R \subset S_R$, namely $L_R = L_1 \cap S'_R = L_1 \cap S_R$ (identifying $S_R = S'_R \subset S$, as usual).*

PROOF. Conclusions (1) and (2) are trivial. In order to demonstrate (3) let $L_1 = \bar{L}_1$ be a self-conjugate Lagrangian subspace of the complex symplectic space S. Then $X = \operatorname{Re} Z = \frac{1}{2}(Z + \bar{Z})$ and $Y = \operatorname{Im} Z = \frac{1}{2i}(Z - \bar{Z})$ both belong to L_1, for each $Z = X + iY \in L_1$.

Now consider the Lagrangian subspace $L_R = L_1 \cap S_R$, (identifying $S_R = S'_R \subset S$), consisting of all the real vectors in L_1. But X and Y are each real vectors in L_R, so L_1 is the complexification of L_R, as required. □

COROLLARY 1. *Under the circumstances of Proposition 2,*

$$\begin{aligned}
(\text{real}) \dim S_R &= (\text{complex}) \dim S \\
(\text{real}) \dim L_R &= (\text{complex}) \dim L_C \\
(\text{complex}) \dim L &= (\text{complex}) \dim \bar{L}
\end{aligned}$$

(and in each case, if one is infinite so is the other).

It should be noted that, under the conditions of Proposition 2, $[Z : \bar{Z}] = 0$ for each vector $Z \in S$. This follows easily because we can write $Z = X + iY$, with X, Y real vectors in S'_R, and then

$$[Z : \bar{Z}] = [X + iY : X - iY] = [X : X]_R - [Y : Y]_R + i\{[Y : X]_R + [X : Y]_R\} = 0.$$

Yet it does not follow that $Z \in L$ implies $\bar{Z} \in L$. It may happen that $L \neq \bar{L}$, so that L is not the complexification of any real Lagrangian subspace of S'_R, and we give illustrations of this phenomenon later in the study of GKN-Theory for real quasi-differential expressions.

We shall next apply these algebraic results of Proposition 2 to formulate and demonstrate a real version of the complex GKN-Theorem which was previously propounded in Section II Theorem 1. That is, we prove a real version of the GKN-Theorem corresponding to real operators and real boundary conditions (as expressed by real Lagrangian spaces)—see [**NA**]. For this purpose we shall require that the Shin-Zettl matrix $A = A^+ \in Z_n(\mathfrak{I})$ shall be real-valued and of even order $n = 2m$, so that the quasi-differential expression $M_A[y] = i^n y_A^{[n]}$, and also $w^{-1}M_A$ for real $w(x) > 0$ in $\mathcal{L}^1_{\text{loc}}(\mathfrak{I})$), are formal differential operators with real coefficients on the interval $\mathfrak{I}$. In this situation M_A is also formally self-adjoint, and hence $w^{-1}M_A$ generates maximal and minimal operators T_1 on $\mathcal{D}(T_1)$ and T_0 on $\mathcal{D}(T_0)$, respectively, in addition to self-adjoint operators T on $\mathcal{D}(T)$ in the complex Hilbert space $\mathcal{L}^2(\mathfrak{I}; w)$, (note: the deficiency index $d = d^\pm \geq 0$ automatically in this real case)—see Section II (1.1), (1.2), (1.3).

Under these conditions the complex GKN-Theorem is certainly valid and asserts the existence of a natural one-to-one correspondence

$$\{T\} \leftrightarrow \{L\}, \tag{B.31}$$

between the set $\{T\}$ of all self-adjoint operators T on $\mathcal{D}(T)$, as generated by $w^{-1}M_A$ in $\mathcal{L}^2(\mathfrak{I}; w)$, and the set $\{L\}$ of all Lagrangian d-spaces L in the complex symplectic $2d$-space $\mathcal{S} = \mathcal{D}(T_1)/\mathcal{D}(T_0)$. Namely, take the natural correspondence $T \leftrightarrow L$ as defined by

$$\Psi\mathcal{D}(T) = L \quad \text{and } \mathcal{D}(T) = \Psi^{-1}L, \tag{B.32}$$

where

$$\Psi\mathcal{D}(T_1) \to \mathcal{S} = \mathcal{D}(T_1)/\mathcal{D}(T_0), \; f \to \Psi f = \hat{f} = \{f + \mathcal{D}(T_0)\} \tag{B.33}$$

is the natural projection (set) map—see Section II (1.5) and Theorem 1.

We seek to refine this GKN-correspondence (B.31) to "real operators" $T_R \in \{T\}$ and "real Lagrangian d-spaces" L_R corresponding to self-conjugate Lagrangian d-spaces $L_C = \bar{L}_C \in \{L\}$, under the hypothesis that $A = A^+ \in Z_n(\mathfrak{I})$ is a real $n \times n$ matrix with even order $n = 2m$—and, with this goal in mind, we next present the required definitions and special constructs. However, we remark that even under this "reality hypothesis" there can exist self-adjoint non-real operators, as generated by $w^{-1}M_A$ in $\mathcal{L}^2(\mathfrak{I}; w)$, corresponding to complex boundary conditions (specified by complex Lagrangian d-spaces in $\mathcal{S}$). This phenomenon will be illustrated later in this Appendix B by an elementary example of a real Sturm-Liouville operator with $n = 2$—and then investigated for higher order quasi-differential expressions with various kinds of boundary conditions.

Definition 4. A linear operator τ in the complex Hilbert space $\mathcal{L}^2(\mathfrak{I}; w)$ is a *real operator* in case:

(1) The domain $\mathcal{D}_\tau$ of τ is self-conjugate, that is,

$$f \in \mathcal{D}_\tau \quad \text{if and only if } \bar{f} \in \mathcal{D}_\tau,$$

and

(2) τ commutes with the (usual) complex conjugation in $\mathcal{L}^2(\mathcal{I}; w)$, that is,

$$\tau f = g \quad \text{implies } \tau \bar{f} = \bar{g}, \quad \text{for all } f \in \mathcal{D}_\tau.$$

In particular, a real operator τ maps real functions into real functions.

PROPOSITION 3. *Consider the quasi-differential expression $w^{-1}M_A$, where $A = \bar{A} = A^+ \in Z_n(\mathcal{I})$ is a real Shin-Zettl matrix of even order $n = 2m$, and $w \in \mathcal{L}^1_{\mathrm{loc}}(\mathcal{I})$ is a positive weight function on the real interval $\mathcal{I}$—as before. Then M_A is formally self-adjoint, and $w^{-1}M_A$ has real coefficients (as a formal differential expression).*

Let T_1 on $\mathcal{D}(T_1)$ and T_0 on $\mathcal{D}(T_0)$ be the maximal and minimal operators, respectively, with deficiency indices $0 \le d^-, d^+ \le n$, as generated by $w^{-1}M_A$ on the complex Hilbert space $\mathcal{L}^2(\mathcal{I}; w)$. Also let $\mathcal{S} = \mathcal{D}(T_1)/\mathcal{D}(T_0)$ be the corresponding complex symplectic space with endpoint decomposition $\mathcal{S} = \mathcal{S}_- \oplus \mathcal{S}_+$, so $[\mathcal{S}_- : \mathcal{S}_+]_A = 0$.

Then, under these circumstances, T_1 and T_0 are real operators on $\mathcal{L}^2(\mathcal{I}; w)$ with

$$\mathcal{D}(T_1) = \overline{\mathcal{D}(T_1)}, \quad \mathcal{D}(T_0) = \overline{\mathcal{D}(T_0)}, \tag{B.34}$$

and

$$d^- = d^+.$$

In addition

$$\mathcal{S} = \bar{\mathcal{S}}, \quad \mathcal{S}_\pm = \bar{\mathcal{S}}_\pm$$

are all complex symplectic spaces with complex conjugation (defined, as usual, by the complex conjugation $f \to \bar{f}$ for $f \in \mathcal{D}(T_1)$). Hence $\mathcal{S}$ has finite dimension $2d = 2d^\pm \le 2n$ and excess $Ex = 0$. Similarly the excess for $\mathcal{S}_\pm$ is $Ex_\pm = 0$, and the corresponding symplectic invariants satisfy

$$\Delta_- + \Delta_+ = \Delta = d.$$

PROOF. Since $w^{-1}M_A$ is a real linear quasi-differential operator on $\mathcal{D}(T_1) \subset \mathcal{L}^2(\mathcal{I}; w)$, it is clear that $\mathcal{D}(T_1) = \overline{\mathcal{D}(T_1)}$ and thereon $\overline{T_1 f} = T_1 \bar{f}$, so T_1 is a real operator on $\mathcal{L}^2(\mathcal{I}; w)$. From the formula, for $f = f_R + if_I \in \mathcal{D}(T_1)$ (with real $f_R, f_I \in \mathcal{D}(T_1)$),

$$[f : \mathcal{D}(T_1)]_A = [f_R + if_I : \mathcal{D}(T_1)]_A = [f_R : \mathcal{D}(T_1)]_A + i[f_R : \mathcal{D}(T_1)]_A,$$

it follows that $\bar{f} = f_R + if_I \in \mathcal{D}(T_0)$ if and only if its conjugate function $\bar{f} = f_R - if_I \in \mathcal{D}(T_0)$. Hence $\mathcal{D}(T_0) = \overline{\mathcal{D}(T_0)}$, and T_0 is a real operator on $\mathcal{L}^2(\mathcal{I}; w)$.

Now take $\hat{f} = \{f + \mathcal{D}(T_0)\} \in \mathcal{S}$ and define the complex conjugation involution on $\mathcal{S}$ by using the map $\hat{f} \to \{\bar{f} + \overline{\mathcal{D}(T_0)}\} = \{\bar{f} + \mathcal{D}(T_0)\} = \bar{\hat{f}} \in \mathcal{S}$. Then it is easy to verify that $\mathcal{S}$ is a complex symplectic space with a complex conjugation (see Lemma 1 below for further details), and moreover each of the symplectic subspaces $\mathcal{S}_-$ and $\mathcal{S}_+$ is individually invariant under this conjugation and hence also inherits the complex conjugation involution.

But the deficiency spaces specified by $w^{-1}M_A$ in $\mathcal{L}^2(\mathcal{I}; w)$ satisfy $\mathcal{D}^- = \overline{\mathcal{D}^+}$, from which it is obvious that $d^- = d^+ = d$. Thus $\dim \mathcal{S} = 2d \le 2n$ and its excess is $Ex = 0$ (see Corollary 1 to Proposition 1 above). It also follows that each of the complex symplectic subspaces $\mathcal{S}_\pm$ has finite dimension $\dim \mathcal{S}\pm = 2\Delta_\pm$, so then $\Delta_- + \Delta_+ = \Delta = d$, as required. □

REMARKS. Let $B \in Z_n(\mathcal{I})$ be a real Shin-Zettl matrix. Then the quasi-differential expression $M_B[y] = i^n y_B^{[n]}$ has real coefficients (so it is real for all real $y \in \mathcal{D}(B)$) if and only if $n = 2m$ is even. As an aside note the interesting example $B_1 \in Z_3(\mathcal{I})$:

$$B_1 = \bar{B}_1 = B_1^+ = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \in Z_3(\mathcal{I}) \quad \text{with } M_{B_1}[y] = i^3 y_{B_1}^{[3]},$$

but M_{B_1} is not a real operator on $\mathcal{L}^2(\mathcal{I})$, in the sense of Definition 4.

Now return to a general real Shin-Zettl matrix $B = \bar{B} \in Z_{2m}(\mathcal{I})$ and assume that M_B is formally self-adjoint. By Proposition 1 of Appendix A, there exists a matrix $A = A^+ \in Z_{2m}(\mathcal{I})$ for which $M_A = M_B$ on $\mathcal{D}(A) = \mathcal{D}(B)$. We would wish to take A to be real (as guaranteed for smooth B by the formulas of Appendix A), but the general case is unclear and still awaits a re-interpretation of the Theorem of Frentzen [**FR**] in Appendix A.

With this caution we henceforth follow the pattern of the previous complex cases, but study quasi-differential expressions M_A with the assumption that $A = \bar{A} = A^+ \in Z_{2m}(\mathcal{I})$, hence by-passing any need for extensions of [**FR**].

DEFINITION 5. Consider a real matrix $A = A^+ \in Z_n(\mathcal{I})$ of even order $n \geq 2$, so M_A is formally self-adjoint and has real coefficients. Let the maximal and minimal operators T_1 on $\mathcal{D}(T_1)$and T_0 on $\mathcal{D}(T_0)$, respectively, be generated by $w^{-1}M_A$ in the complex Hilbert space $\mathcal{L}^2(\mathcal{I}; w)$, as in Proposition 3 above.

Then define the real Hilbert space (see (B.14))

$$\mathcal{L}^2_R(\mathcal{I}; w) = \{f \in \mathcal{L}^2(\mathcal{I}; w) \mid f = \bar{f}\}$$

and the corresponding real linear submanifolds

(B.35)
$$\begin{aligned} \mathcal{D}_R(T_1) &= \mathcal{D}(T_1) \cap \mathcal{L}^2_R(\mathcal{I}; w) \\ \mathcal{D}_R(T_0) &= \mathcal{D}(T_0) \cap \mathcal{L}^2_R(\mathcal{I}; w), \end{aligned}$$

and the quotient or identification space

(B.36)
$$\mathcal{S}_R = \mathcal{D}_R(T_1)/\mathcal{D}_R(T_0).$$

We note that $\mathcal{D}_R(T_1)$ and $\mathcal{D}_R(T_0)$ consist of all the real functions in $\mathcal{D}(T_1)$ and $\mathcal{D}(T_0)$, respectively. Moreover, under these circumstances we can alternatively define

(B.37)
$$\begin{aligned} \mathcal{D}_R(T_1) = \{f : \mathcal{I} \to \mathbb{R} \mid f_A^{[r]} \in AC_{\mathrm{loc}}(\mathcal{I}) \quad &\text{for } r = 0, 1, \ldots, n-1, \\ \text{and both } f \text{ and } w^{-1}M_A[f] &\in \mathcal{L}^2_R(\mathcal{I}; w)\}, \end{aligned}$$

and, noting (B.34),

(B.38)
$$\mathcal{D}_R(T_0) = \{f \in \mathcal{D}_R(T_1) \mid [f : g]_A = 0 \quad \text{for all } g \in \mathcal{D}_R(T_1)\}.$$

Also we recognize that $\mathcal{S}_R$ is a real vector space, with the natural algebraic operations performed on the cosets

(B.39)
$$\hat{h}_R = \{h + \mathcal{D}_R(T_0)\} \qquad \text{for each } h \in \mathcal{D}_R(T_1),$$

so that the projection map

$$\begin{aligned}(\text{B.40}) \qquad \Psi_R : \mathcal{D}_R(T_1) &\to \mathcal{S}_R = \mathcal{D}_R(T_1)/\mathcal{D}_R(T_0) \\ h &\to \Psi_R h = \hat{h}_R = \{h + \mathcal{D}_R(T_0)\}\end{aligned}$$

is a linear map of $\mathcal{D}_R(T_1)$ onto S_R. As before, we use the notations Ψ_R and Ψ_R^{-1} for the corresponding induced maps on subsets of $\mathcal{D}_R(T_1)$ or $\mathcal{S}_R$, respectively.

In the next lemma we show that $\mathcal{S}_R$ is a real symplectic $2d$-space. Then in the subsequent real GKN-Theorem we shall demonstrate the existence of a natural one-to-one correspondence between the self-adjoint real operators, generated by $w^{-1}M_A$ in $\mathcal{L}^2(\mathcal{I}; w)$, and the real Lagrangian d-spaces in $\mathcal{S}_R$. But first we must clarify the relation between $\mathcal{S}_R$ and $\mathcal{S}$; compare (B.33) and (B.40).

LEMMA 1. *Consider a real matrix $A = A^+ \in Z_n(\mathcal{I})$, of even order $n \geq 2$, and the corresponding formally self-adjoint quasi-differential expression M_A, so $w^{-1}M_A$ generates the maximal and minimal operators T_1 on $\mathcal{D}(T_1)$ and T_0 on $\mathcal{D}(T_0)$, respectively, with deficiency index $0 \leq d \leq 0$ in $\mathcal{L}^2(\mathcal{I}; w)$ - as in Proposition 3 above.*

Then $\mathcal{S} = \mathcal{D}(T_1)/\mathcal{D}(T_0)$ is a complex symplectic $2d$-space with complex conjugation (with algebraic operations inherited from $\mathcal{D}(T_1)$). An element $\hat{f} = \{f + \mathcal{D}(T_0)\}$ is real in $\mathcal{S}$ if and only if $f - \bar{f} \in \mathcal{D}(T_0)$, or, equally well, $\hat{f}$ contains a real function, say $h = \bar{h} \in \mathcal{D}_R(T_1)$, so $\hat{f} = \hat{h} = \{h + \mathcal{D}(T_0)\}$. As in (B.6) and (B.7) the real vectors of $\mathcal{S}$ constitute a real symplectic $2d$-space $\mathcal{S}'_R$, whose complexification is $\mathcal{S}$.

Further, $\mathcal{S}_R = \mathcal{D}_R(T_1)/\mathcal{D}_R(T_0)$ is a real symplectic $2d$-space (with algebraic operations inherited from $\mathcal{D}(T_1)$). If a coset $\{h + \mathcal{D}_R(T_0)\}$ of $\mathcal{S}_R$ intersects a coset $\{g + \mathcal{D}(T_0)\}$ of $\mathcal{S}$ (as subsets of $\mathcal{D}(T_1)$), then

$$\{h + \mathcal{D}_R(T_0\} \subset \{g + \mathcal{D}(T_0)\} = \{h + \mathcal{D}(T_0)\},$$

and in this way $\{h + \mathcal{D}_R(T_0)\}$ specifies the unique element $\{h + \mathcal{D}(T_0)\} \in \mathcal{S}$. Therefore the natural map of $\mathcal{S}_R$ into $\mathcal{S}$,

$$(\text{B.41}) \qquad \mathcal{S}_R \to \mathcal{S}'_R,\ \hat{h}_R = \{h + \mathcal{D}_R(T_0)\} \to \hat{h} = \{h + \mathcal{D}(T_0)\}$$

is a symplectic isomorphism of $\mathcal{S}_R$ onto $\mathcal{S}'_R$, (and we use (B.41) to identify $\mathcal{S}_R$ with $\mathcal{S}'_R$).

PROOF. According to the GKN-Theorem 1 and Lemma 1 in Section II, $\mathcal{S} = \mathcal{D}(T_1)/\mathcal{D}(T_0)$ is a complex symplectic $2d$-space, with the complex symplectic product of vectors $\hat{f} = \{f + \mathcal{D}(T_0)\}$ and $\hat{g} = \{g + \mathcal{D}(T_0)\}$,

$$(\text{B.42}) \qquad [\hat{f} : \hat{g}]_A = [f + \mathcal{D}(T_0) : g + \mathcal{D}(T_0)]_A = [f : g]_A,$$

for all $f, g \in \mathcal{D}(T_1)$. Now define the complex conjugation in $\mathcal{S}$ by

$$(\text{B.43}) \qquad \hat{f} \to \{\overline{f + \mathcal{D}(T_0)}\} = \{\bar{f} + \mathcal{D}(T_0)\},$$

noting that $\overline{\mathcal{D}(T_0)} = \mathcal{D}(T_0)$.

A vector $\hat{f} = \{f + \mathcal{D}(T_0)\} \in \mathcal{S}$ is real just in case $\{f + \mathcal{D}(T_0)\} = \{\bar{f} + \mathcal{D}(T_0)\}$, which holds if and only if $f - \bar{f} \in \mathcal{D}(T_0)$, or equally well, $f - \frac{1}{2}(f + \bar{f}) = \frac{1}{2}(f - \bar{f}) \in \mathcal{D}(T_0)$. But the function $h = \frac{1}{2}(f + \bar{f})$ is real; hence the vector $\hat{f}$ is real in $\mathcal{S}$ if and only if $\{f + \mathcal{D}(T_0\} = \{h + \mathcal{D}(T_0)\}$, with $h \in \mathcal{D}_R(T_1)$. As previously discussed, see (B.6) and (B.7) and Proposition 1 above, the real vectors of $\mathcal{S}$ constitute a real symplectic $2d$-space $\mathcal{S}'_R$ whose complexification is $\mathcal{S}$.

Next consider the real vector space $\mathcal{S}_R = \mathcal{D}_R(T_1)/\mathcal{D}_R(T_0)$, with vectors $\hat{k}_R = \{k + \mathcal{D}_R(T_0)\}$, $\hat{\ell}_R = \{\ell + \mathcal{D}_R(T_0)\}$ for real functions $k, \ell \in \mathcal{D}_R(T_1)$, and define the real bilinear form

$$[\hat{k}_R : \hat{\ell}_R]_A = [k + \mathcal{D}_R(T_0) : \ell + \mathcal{D}_R(T_0)]_A = [k : \ell]_A. \tag{B.44}$$

Since $[k : \ell]_A$ is defined already on $\mathcal{D}(T_1)$, it is clear that this bilinear product is well-defined on $\mathcal{S}_R$, and furthermore it is skew-symmetric, $[\hat{k}_R : \hat{\ell}_R]_A = -[\hat{\ell}_R : \hat{k}_R]_A$.

We now verify that this real bilinear form is nondegenerate on the real vector space $\mathcal{S}_R$. Accordingly, suppose $[\hat{k}_R : \hat{\ell}_R]_A = 0$ for all $\hat{\ell}_R = \{\ell + \mathcal{D}_R(T_0)\} \in \mathcal{S}_R$. But using the reality of the matrix A and the function k, we compute

$$[k : \mathcal{D}(T_1)]_A = [k : \mathcal{D}_R(T_1) + i\mathcal{D}_R(T_1)]_A = [k : \mathcal{D}_R(T_1)]_A - i[k : \mathcal{D}_R(T_1)]_A = 0,$$

and so $[k : \mathcal{D}(T_1)]_A = 0$. This implies that $k \in \mathcal{D}(T_0)$, so $k \in \mathcal{D}_R(T_0)$ and $\hat{k}_R = 0$ in $\mathcal{S}_R$. This proves that the skew-symmetric bilinear form $[:]_A$ is nondegenerate on $\mathcal{S}_R$, and we conclude that $\mathcal{S}_R$ is a real symplectic space.

We still wish to relate the real symplectic space $\mathcal{S}_R$ to the real vectors $\mathcal{S}'_R \subset \mathcal{S}$. For this purpose assume that a coset

$$\hat{h}_R = \{h + \mathcal{D}_R(T_0)\} \in \mathcal{S}_R \quad \text{intersects a coset } \hat{g} = \{g + \mathcal{D}(T_0)\} \in \mathcal{S}$$

(with $h \in \mathcal{D}_R(T_1)$ and $g \in \mathcal{D}(T_1)$—and both these cosets interpreted as subsets of $\mathcal{D}(T_1)$). Then $\{h + \mathcal{D}(T_0)\}$ meets $\{g + \mathcal{D}(T_0)\}$ within $\mathcal{D}(T_1)$, and hence these two cosets are the same element in $\mathcal{S} = \mathcal{D}(T_1)/\mathcal{D}(T_0)$. In this case

$$\{h + \mathcal{D}_R(T_0)\} \subset \{h + \mathcal{D}(T_0)\} = \{g + \mathcal{D}(T_0)\}.$$

Since h is real, the coset $\{h + \mathcal{D}(T_0)\}$ is necessarily a real vector in $\mathcal{S}'_R \subset \mathcal{S}$.

In this way we define a linear map of $\mathcal{S}_R$ onto $\mathcal{S}'_R \subset \mathcal{S}$,

$$\hat{h}_R = \{h + \mathcal{D}_R(T_0)\} \to \hat{h} = \{h + \mathcal{D}(T_0)\},$$

as in (B.41). If $\hat{h} = 0$ in $\mathcal{S}$, then $h \in \mathcal{D}(T_0)$ so $h \in \mathcal{D}_R(T_0)$ and hence $\hat{h}_R = 0$ in $\mathcal{S}_R$. Thus the map (B.41) is injective, and so it defines a (natural) isomorphism of $\mathcal{S}_R$ onto $\mathcal{S}'_R$, as real vector spaces and also as real symplectic spaces. □

The next lemma re-casts the results of Proposition 2 so as to apply to the real and complex symplectic spaces $\mathcal{S}_R$ and $\mathcal{S}$, respectively.

LEMMA 2. *Consider a real matrix $A = A^+ \in Z_n(\mathcal{I})$ of even order $n \geq 2$, and the corresponding formally self-adjoint quasi-differential expression M_A, so $w^{-1}M_A$ generates the maximal and minimal operators T_1 on $\mathcal{D}(T_1)$ and T_0 on $\mathcal{D}(T_0)$, respectively, with deficiency index $0 \leq d \leq n$ in $\mathcal{L}^2(\mathcal{I}; w)$—as before. Consider also the complex symplectic $2d$-space $\mathcal{S} = \mathcal{D}(T_1)/\mathcal{D}(T_0)$, with complex conjugation involution and real vectors $\mathcal{S}'_R \subset \mathcal{S}$, so $\mathcal{S}'_R$ is a real symplectic $2d$-space. Identify $\mathcal{S}_R = \mathcal{D}_R(T_1)/\mathcal{D}_R(T_0)$ with $\mathcal{S}'_R$ as in Lemma 1.*

Then each real Lagrangian d-space $L_R \subset \mathcal{S}_R$ has a complexification L_C which is a self-conjugate complex Lagrangian d-space in $\mathcal{S}$. Moreover, each such self-conjugate complex Lagrangian d-space in $\mathcal{S}$ is the complexification of exactly one real Lagrangian d-space in $\mathcal{S}_R$.

In this way there is established a one-to-one correspondence

$$\{L_R\} \leftrightarrow \{L_C\} \tag{B.45}$$

between the set $\{L_R\}$ of all real Lagrangian d-space in $\mathcal{S}_R$, and the set $\{L_C\}$ of all self-conjugate complex Lagrangian d-spaces in $\mathcal{S}$.

PROOF. These conclusions follow immediately from Proposition 2, upon the identification of the isomorphic real symplectic spaces $\mathcal{S}_R$ and $\mathcal{S}'_R \subset \mathcal{S}$, as prescribed in Lemma 1 above. □

The next theorem is the principal goal of this Appendix B, and it gives the real version of the complex GKN-Theorem 1 of Section II—but under additional reality assumptions: namely, assuming a real matrix $A = A^+ \in Z_n(\mathcal{I})$ of even order $n \geq 2$ so that the corresponding quasi-differential expression $M_A[y] = i^n y_A^{[n]}$ is a formally self-adjoint quasi-differential operator with real coefficients on the interval $\mathcal{I}$. Then, with a given real weight function $w(x) > 0$ in $\mathcal{L}^1_{\text{loc}}(\mathcal{I})$, $w^{-1}M_A$ generates self-adjoint operators T on $\mathcal{D}(T)$ in the complex Hilbert space $\mathcal{L}^2(\mathcal{I}; w)$—since necessarily the deficiency index $d = d^+ = d^-$—see Section II (1.3).

The complex GKN-Theorem then applies, and asserts, as we recall from Section II, the existence of the one-to-one correspondence $\{T\} \leftrightarrow \{L\}$, see (B.31), (B.32), (B.33), between all such self-adjoint operators $T \in \{T\}$, and all complex Lagrangian d-spaces $L \in \{L\}$ of the complex symplectic $2d$-space $\mathcal{S}$.

THEOREM 1. (Real GKN). *Consider the quasi-differential expression*

$$w^{-1}M_A[y] = i^n w^{-1} y_A^{[n]},$$

on the real interval $\mathcal{I}$, as defined by a real matrix $A = A^+ \in Z_n(\mathcal{I})$ of even order $n \geq 2$, and with the given real weight function $w(x) > 0$ in $\mathcal{L}^1_{\text{loc}}(\mathcal{I})$. Let T_1 on $\mathcal{D}(T_1)$ and T_0 on $\mathcal{D}(T_0)$ be the maximal and minimal operators, respectively, as generated by $w^{-1}M_A$ in the complex Hilbert space $\mathcal{L}^2(\mathcal{I}; w)$ (recall that the deficiency index $0 \leq d \leq n$—as in Section II (1.3)).

Then for this real case in the GKN-Theorem (where we refer to the general GKN-correspondence $\{T\} \leftrightarrow \{L\}$ of (B.31)):

(1) $T_R \in \{T\}$ is a self-adjoint real operator on $\mathcal{D}(T_R) \subset \mathcal{L}^2(\mathcal{I}; w)$ if and only if the corresponding complex Lagrangian d-space $L_C \in \{L\}$ is self-conjugate, that is, $L_C = \bar{L}_C$ in the complex symplectic $2d$-space $\mathcal{S} = \mathcal{D}(T_1)/\mathcal{D}(T_0)$.

Furthermore, the real operator T_R can always be determined by real boundary conditions, as next follows:

(2) There exists a natural one-to-one correspondence between the set $\{T_R\}$ of all self-adjoint real operators T_R on $\mathcal{D}(T_R)$ (as generated by $w^{-1}M_A$ in $\mathcal{L}^2(\mathcal{I}; w)$), and the set $\{L_R\}$ of all real Lagrangian d-spaces L_R in the real symplectic $2d$-space $\mathcal{S}_R = \mathcal{D}_R(T_1)/\mathcal{D}_R(T_0)$. Namely:

For each $T_R \in \{T_R\}$ take the corresponding $L_R \in \{L_R\}$ to be the unique real Lagrangian d-space in $\mathcal{S}_R$, whose complexification $L_C = \bar{L}_C$ in $\mathcal{S}$ is defined by

$$\Psi\mathcal{D}(T_R) = L_C.$$

Conversely, for each real Lagrangian d-space L_R *in* $\mathcal{S}_R$ *the complexification* $L_C = \bar{L}_C$ *in* $\mathcal{S}$ *determines* $T_R \in \{T_R\}$ *by*

$$\mathcal{D}(T_R) = \Psi^{-1} L_C,$$

where

$$\Psi : \mathcal{D}(T_1) \to \mathcal{S}$$

is the natural projection (set) map (B.33).

In more detail, for each set of d real functions $f^1, f^2, \dots, f^d$ in $\mathcal{D}_R(T_1)$, such that $\{\hat{f}_R^1, \dots, \hat{f}_R^d\}$ constitutes a basis for $L_R \subset \mathcal{S}_R$ (and consequently $[f^r : f^s]_A = 0$ for $1 \leq r, s \leq d$), the domain $\mathcal{D}(T_R)$ of the corresponding self-adjoint real operator T_R is

$$\mathcal{D}(T_R) = \{f \in \mathcal{D}(T_1) \mid [f : f^s]_A = 0 \quad \text{for } s = 1, \dots, d\}$$

or equally well,

$$\mathcal{D}(T_R) = c_1 f^1 + c_2 f^2 + \cdots + c_d f^d + \mathcal{D}(T_0),$$

where $c_1, c_2, \dots, c_d$ are arbitrary complex constants, as in (B.50).

PROOF. Take a self-adjoint real operator $T_R \in \{T\}$ with the domain $\mathcal{D}(T_R) \subset \mathcal{D}(T_1)$, as generated by $w^{-1} M_A$ on $\mathcal{L}^2(\mathcal{I}; w)$. Then the complex GKN-Theorem 1 of Section II asserts that there is determined a complex Lagrangian d-space $L_C \in \{L\}$ in the complex symplectic $2d$-space $\mathcal{S} = \mathcal{D}(T_1)/\mathcal{D}(T_0)$, according to (B.32)

$$\Psi \mathcal{D}(T_R) = L_C \quad \text{and} \quad \mathcal{D}(T_R) = \Psi^{-1} L_C. \tag{B.46}$$

But this natural projection (set) map

$$\Psi : \mathcal{D}(T_1) \to \mathcal{S}, \quad f \to \hat{f} = \{f + \mathcal{D}(T_0)\}, \tag{B.47}$$

carries the conjugate $\bar{f}$ to $\Psi \bar{f} = \{\bar{f} + \mathcal{D}(T_0)\}$. That is, Ψ preserves the conjugation operations that are given on the two complex vector spaces $\mathcal{D}(T_1)$ and $\mathcal{S}$. Since T_R is a real operator $\mathcal{D}(T_R) = \overline{\mathcal{D}(T_R)}$, and hence $L_C = \bar{L}_C$ is self-conjugate in $\mathcal{S}$.

On the other hand, take any such self-conjugate $L_C = \bar{L}_C \in \{L\}$ in $\mathcal{S}$. Then the corresponding self-adjoint operator $T \in \{T\}$ has a domain $\mathcal{D}(T) \subset \mathcal{D}(T_1)$ specified by $\mathcal{D}(T) = \Psi^{-1} L_C$. That is, $\mathcal{D}(T)$ is the union in $\mathcal{L}^2(\mathcal{I}; w)$ of all the cosets $\hat{f} = \{f + \mathcal{D}(T_0)\}$ for $\hat{f} \in L_C$. But $\overline{\{f + \mathcal{D}(T_0)\}} = \{\bar{f} + \mathcal{D}(T_0)\}$ also belongs to $L_C = \bar{L}_C$, and hence $\mathcal{D}(T) = \overline{\mathcal{D}(T)}$ is self-conjugate in $\mathcal{L}^2(\mathcal{I}; w)$. Further, $w^{-1} M_A$ has real coefficients, since $A = A^+$ is real of even order $n \geq 2$, and therefore T is a real operator in accord with Definition 4 above, and we denote it by T_R to match the conclusion (1) of this theorem.

Now conclusion (2) is an easy consequence of the complex GKN-Theorem, especially the existence of the bi-unique correspondence $\{T\} \leftrightarrow \{L\}$. Namely, this same GKN-correspondence, when restricted to $\{T_R\} \subset \{T\}$, establishes a one-to-one correspondence between $\{T_R\}$ and the set $\{L_C\}$ of all complex Lagrangian d-space which are self-conjugate in $\mathcal{S}$.

However, in Lemma 2 above, we have defined a natural one-to-one correspondence (B.45) $\{L_R\} \leftrightarrow \{L_C\}$ between the real Lagrangian d-spaces L_R of $\mathcal{S}_R$, and the self-conjugate complex Lagrangian d-spaces L_C in $\mathcal{S}$. In this bijective correspondence $L_C = \bar{L}_C$ is the complexification of a unique L_R, and L_R is the unique real Lagrangian d-space in $\mathcal{S}_R$ which has L_C as its complexification.

These two bijective correspondences combine to produce the required one-to-one correspondence

$$\{T_R\} \leftrightarrow \{L_R\}, \tag{B.48}$$

as asserted in the conclusion (2) in this theorem. □

REMARK. Let $w^{-1}M_A$ be a quasi-differential expression for a given complex matrix $A = A^+ \in Z_n(\mathcal{I})$ of order $n \geq 2$, with the corresponding deficiency index $0 \leq d \leq n$—as in the complex GKN-Theorem 1 of Section II. Then the boundary value problem (or eigenvalue problem for $\lambda \in \mathbb{C}$),

$$M_A[y] = \lambda w y, \quad \text{on real interval } \mathcal{I},$$

is self-adjoint on some linear domain $\mathcal{D}(T) \subset \mathcal{L}^2(\mathcal{I}; w)$; that is the operator $T = w^{-1}M_A$ is self-adjoint, in case there exist complex functions $f^1, \ldots, f^d \in \mathcal{D}(T_1)$ such that $\hat{f}^1, \ldots, \hat{f}^d$ constitute a basis for a Lagrangian d-space L of the complex symplectic space $\mathcal{S} = \mathcal{D}(T_1)/\mathcal{D}(T_0)$. This can also be stated as:

$$[f^r : f^s]_A = 0 \qquad \text{for } 1 \leq r, s \leq d,$$

and $\{f^r\}$ are linearly independent over $\mathbb{C}$, $\pmod{\mathcal{D}(T_0)}$. In terms of this basis the linear domain for T is precisely

$$\mathcal{D}(T) = c_1 f^1 + c_2 f^2 + \cdots + c_d f^d + \mathcal{D}(T_0), \tag{B.49}$$

where the complex constants $c_1, c_2, \ldots, c_d$ range over $\mathbb{C}$.

Incidentally, such a basis for L can also be specified by an appropriate set of d linearly independent functionals on $\mathcal{S}$, that is, by d linearly independent homogeneous complex boundary conditions that vanish on $\mathcal{D}(T_0)$—as explained earlier in Sections III.2 and again in Section IV.

We compare this construction with the corresponding construction for the *real version* of the GKN-Theorem 1 above, where $A = A^+ \in Z_n(\mathcal{I})$ is a real matrix of even order $n \geq 2$. Then we can determine every self-adjoint *real operator* T_R on $\mathcal{D}(T_R) \subset \mathcal{L}^2(\mathcal{I}; w)$ by a selection of d *real functions* $f^1, f^2, \ldots, f^d \in \mathcal{D}_R(T_1)$ such that $\hat{f}^1_R = \{f^1 + \mathcal{D}_R(T_0)\}, \cdots, \hat{f}^d_R = \{f^d + \mathcal{D}_R(T_0)\}$ constitute a basis for a *real Lagrangian d-space* L_R in the *real symplectic* $2d$*-space* $\mathcal{S}_R = \mathcal{D}_R(T_1)/\mathcal{D}_R(T_0)$. This can also be stated as:

$$[f^r : f^s]_A = 0 \quad \text{for } 1 \leq r,\ s \leq d,$$

and $f^1, \ldots, f^d$ are linearly independent over $\mathbb{R}$, $\pmod{\mathcal{D}_R(T_0)}$.

But the complexification $L_C = \bar{L}_C$ of L_R has the basis $\hat{f}^1 = \{f^1 + \mathcal{D}(T_0)\}, \ldots, \hat{f}^d = \{f^d + \mathcal{D}(T_0)\}$ over $\mathbb{C}$, $\pmod{\mathcal{D}(T_0)}$. Hence, here

$$\mathcal{D}(T_R) = c_1 f^1 + c_2 f^2 + \cdots + c_d f^d + \mathcal{D}(T_0), \tag{B.50}$$

where again the complex constants $c_1, c_2, \ldots, c_d$ range over $\mathbb{C}$.

We shall later note how such a basis of *real functions* $f^1, f^2, \ldots, f^d \in \mathcal{D}_R(T_1)$ can be specified by d linearly independent homogeneous *real boundary conditions* that vanish on $\mathcal{D}_R(T_0)$.

REMARKS ON REAL OPERATORS. A real operator τ on the complex Hilbert space $\mathcal{L}^2(\mathcal{I};w)$, as in Definition 4, defines a linear operator on the real Hilbert space $\mathcal{L}^2_R(\mathcal{I};w)$, by the restriction to real functions. Moreover, each linear operator on $\mathcal{L}^2_R(\mathcal{I};w)$ extends to a unique real operator on the complex Hilbert space $\mathcal{L}^2(\mathcal{I};w)$.

If we wish to consider the real operator T_R in (B.50) as a self-adjoint operator on the real Hilbert space $\mathcal{L}^2_R(\mathcal{I};w)$, then its domain would consist of the real functions in $\mathcal{D}(T_R)$, namely

$$\text{(B.50R)} \qquad \mathcal{D}_R(T_R) = c_1 f^1 + c_2 f^2 + \cdots + c_d f^d + \mathcal{D}_R(T_0),$$

where the constants $c_1, c_2, \ldots, c_d$ are now restricted to be real. However, we shall consider only operators defined on the complex Hilbert space $\mathcal{L}^2(\mathcal{I};w)$.

We have seen that the further developments in the general GKN-Theory, and the corresponding boundary value problems, have been guided by the theory of complex symplectic spaces, as in Sections III and IV. These methods also apply to the theory of real quasi-differential expressions via real symplectic spaces—but significantly, they require that we consider their complexifications. However, these results may be easier to obtain and simpler to state in the real cases, since each real symplectic space S of finite dimension D must have $D = 2m$ even, and excess $Ex = 0$, so S is thus symplectically isomorphic to the standard $\mathbb{R}^D$. For instance, the real symplectic space $\mathcal{S}_R$, in the real GKN-Theorem 1, has the even dimension $2d$ (for the corresponding deficiency index $d = d^{\pm}$) and also its endpoint decomposition $\mathcal{S}_R = \mathcal{S}_{R-} \oplus \mathcal{S}_{R+}$ leads always to the real symplectic subspaces $\mathcal{S}_{R\pm}$ of even dimensions—according to Proposition 5 below.

However, it is often possible, even advantageous, to establish interesting versions of these results for real boundary value problems without any explicit reference to their complexifications—and with the conclusions phrased entirely within real analysis, just as for the real GKN-Theorem 1. Since these concepts and arguments in both the real and complex cases are generally parallel, we merely state some of the most important of those propositions of real symplectic geometry, which correspond to the main body of theory of Sections III and IV.

PROPOSITION 4. *Let S, together with a real symplectic form $[:]_R$, be a real symplectic space of finite dimension $D = 2\Delta$. Assume given a prescribed direct sum decomposition*

$$S = S_- \oplus S_+, \quad \textit{with } [S_- : S_+]_R = 0,$$

where the real symplectic subspaces $S_\pm$ have

$$\dim S_\pm = 2\Delta_\pm, \quad \textit{so} \quad \Delta_- + \Delta_+ = \Delta.$$

Further let L be a real Lagrangian Δ-space in S.

Then

$$0 \le \Delta_- - \dim L \cap S_- = \Delta_+ - \dim L \cap S_+ = \min\{\Delta_-, \Delta_+\}.$$

Moreover, in the special case where

$$\dim S_\pm = \Delta \qquad (\textit{so } \Delta_- = \Delta_+),$$

then

$$0 \le \dim L \cap S_- = \dim L \cap S_+ \le \Delta_\pm.$$

The proof of the proposition (which may be denoted as the *real version of the balanced intersection principle*) follows closely the argument of Theorem 3 in Section III.1 since the methods of linear algebra only are involved. Moreover, other results of complex symplectic algebra in Section III.1 are also much easier, or even trivial, in the real case—for instance the analogues of Theorems 1 and 2 there. Accordingly, we do not present these arguments or statements here.

REMARK. As in Section III.1, but now in accord with the circumstances of Proposition 4 above, we define the *coupling grade* of a real Lagrangian Δ-space L in the real symplectic 2Δ-space $S = S_- \oplus S_+$, namely

$$\text{(B.51)} \qquad \operatorname{grade} L = \Delta_- - \dim L \cap S_- = \Delta_+ - \dim L \cap S_+.$$

As before, a non-zero vector $v \in S$ is separated at the left (or at the right) in case $v \in S_-$ (or $v \in S_+$); otherwise v is a coupled vector.

Then L is *strictly separated* in case grade $L = 0$, that is, $L = \operatorname{span}\{L \cap S_-, L \cap S_+\}$ so that L has a basis of vectors, each of which is separated. Also L is *totally coupled* in case grade $L = \Delta_- = \Delta_+$, that is $L \cap S_- = L \cap S_+ = 0$ so that every basis of L consists of vectors, each of which is coupled.

It is easy to see that if the real symplectic 2Δ-space $S = S_- \oplus S_+$, and the real Lagrangian Δ-space $L \subset S$, are all complexified to $S_C = S_{C-} \oplus S_{C+}$ and L_C, respectively, then grade $L = \operatorname{grade} L_C$.

Proposition 4 can be applied to the analysis of real boundary value problems, say

$$M_A[y] = \lambda w y \qquad \text{(as in Section I(1.1))}$$

where $A = \bar{A} = A^+ \in Z_n(\mathcal{J})$ is a real Shin-Zettl matrix of even order $n = 2m$, and with the real weight function $w(x) > 0$ in $\mathcal{L}^1_{\mathrm{loc}}(\mathcal{J})$, as before. Then the corresponding (real) endpoint space

$$\mathcal{S}_R = \mathcal{D}_R(T_1)/\mathcal{D}_R(T_0),$$

is a real symplectic $2d$-space, for the deficiency index $0 \le d \le n$ as in the notation of the real GKN-Theorem 1 above.

In order to construct the left and right endpoint decomposition of $\mathcal{S}_R$, in analogy to Definition 4 in Section III.2, we define

$$\text{(B.52)} \qquad \mathcal{D}_{R\pm}(T_1) = \mathcal{D}_\pm(T_1) \cap \mathcal{D}_R(T_1)$$

and the real endpoint spaces

$$\text{(B.53)} \qquad \mathcal{S}_{R\pm} = \Psi_R \mathcal{D}_{R\pm},$$

in terms of the natural projection (set) map (B.40)

$$\text{(B.54)} \qquad \begin{aligned} \Psi_R : \mathcal{D}_R(T_1) &\to \mathcal{S}_R = \mathcal{D}_R(T_1)/\mathcal{D}_R(T_0) \\ f &\to \Psi_R f = \hat{f}_R = \{f + \mathcal{D}_R(T_0)\}. \end{aligned}$$

Thus $\mathcal{S}_{R-}$ can be thought of as "real functions supported near the left endpoint a," and $\mathcal{S}_{R+}$ as "the real functions supported near the right endpoint b" of $\mathcal{J}$.

In this terminology we can now assert the analogue of Theorem 5 in Section III.2, which provides the general endpoint space decomposition for the real symplectic space $\mathcal{S}_R$.

PROPOSITION 5. *Consider the real symplectic* $2d$*-space* $\mathcal{S}_R = \mathcal{D}_R(T_1)/\mathcal{D}_R(T_0)$, *determined as the endpoint space for a real quasi-differential expression* $w^{-1}M_A$. *Here* $A = \bar{A} = A^+ \in Z_n(\mathcal{I})$ *is a real matrix of even order* $n = 2m$, $w \in \mathcal{L}^1_{loc}(\mathcal{I})$ *is the positive weight function on the real interval* $\mathcal{I}$, *and* $0 \leq d \leq n$ *is the deficiency index, as in Theorem 1.*

Then $\mathcal{S}_R$ *has a natural direct sum decomposition*

$$\mathcal{S}_R = \mathcal{S}_{R-} \oplus \mathcal{S}_{R+}, \quad \text{with } [\mathcal{S}_{R-} : \mathcal{S}_{R+}]_A = 0, \tag{B.55}$$

where the left and right endpoint spaces $\mathcal{S}_{R-}$ *and* $\mathcal{S}_{R+}$, *respectively, are each real symplectic subspaces;*

$$\dim \mathcal{S}_{R-} = 2\Delta_-, \quad \dim \mathcal{S}_{R+} = 2\Delta_+$$

and $\Delta_- + \Delta_+ = \Delta = d$ *(in terms of the usual symplectic invariants of Theorem 1 in Section III.1).*

In the special case where $\mathcal{I} = [a, b]$ *is compact, so the corresponding boundary value problem is regular, then*

$$\Delta = d = n \quad \text{and} \quad \Delta_- = \Delta_+ = m.$$

The proof of Proposition 5 follows just as in the analogous complex case, once we observe that the patching lemma of Naimark [**NA**], or the corresponding controllability results [**EM**] introduced in Appendix A above, remains valid within the framework of real analysis.

Further, classifications of real Lagrangian d-spaces $L_R \subset \mathcal{S}_R$ can be ordered by means of the coupling grade of L_R

$$\operatorname{grade} L_R = \Delta_\pm - \dim L_R \cap \mathcal{S}_{R\pm},$$

as in Proposition 4 and the subsequent discussion (B.51).

COROLLARY 1 (For real GKN-Theorem 1). *Consider the quasi-differential expression* $w^{-1}M_A$ *on the real interval* $\mathcal{I}$, *as defined by the real matrix* $A = \bar{A} = A^+ \in Z_n(\mathcal{I})$ *of even order* $n = 2m \geq 2$, *and by the given positive weight function* $w \in \mathcal{L}^1_{\mathrm{loc}}(\mathcal{I})$, *as in Theorem 1. Define the real symplectic* $2d$*-space*

$$\mathcal{S}_R = \mathcal{D}_R(T_1)/\mathcal{D}_R(T_0) = \mathcal{S}_{R_-} \oplus \mathcal{S}_{R_+},$$

where the left and right endpoint spaces $\mathcal{S}_{R_-}$ *and* $\mathcal{S}_{R_+}$, *respectively, are real symplectic subspaces with*

$$\dim \mathcal{S}_{R_-} = 2\Delta_-, \quad \dim \mathcal{S}_{R_+} = 2\Delta_+ \qquad (\text{so} \quad \Delta_- + \Delta_+ = \Delta = d),$$

as in Proposition 5 above.

Let $L_\ell \subset \mathcal{S}_R$ *be a real Lagrangian* d*-space with*

$$\operatorname{grade} L_\ell = \Delta_\pm - \dim L_R \cap \mathcal{S}_{R_\pm} = \ell,$$

for an assigned integer $0 \leq \ell \leq \min\{\Delta_-, \Delta_+\}$.

Then each basis of L_ℓ *contains:*

at most $(\Delta_- - \ell)$ *vectors in* $\mathcal{S}_-$
(each separated at left of $\mathcal{I}$*),*

at most $(\Delta_+ - \ell)$ *vectors in* $\mathcal{S}_+$
(each separated at right of $\mathcal{I}$*),*
and

at least (Nec-coupling $L_\ell = 2\ell$) vectors
(each coupled on $\mathcal{I}$).

Furthermore there exists a minimally coupled basis $\{\hat{g}_R^1, \hat{g}_R^2, \dots, \hat{g}_R^d\}$ of the d-space $L_\ell \subset \mathcal{S}_R$, in which precisely 2ℓ vectors are each coupled, and hence precisely $(\Delta_- - \ell)$ vectors are each separated at the left, and precisely $(\Delta_+ - \ell)$ vectors are each separated at the right of the interval $\mathcal{I}$. In fact, we can choose the corresponding representative functions $g^1, g^2, \dots, g^d$ in $\mathcal{D}_R(T_1)$ such that:

the first $(\Delta_- - \ell)$ of these functions lie in $\mathcal{D}_{R_-}(T_1)$
(each vanishes in a neighborhood of the right end of $\mathcal{I}$),

the next $(\Delta_+ - \ell)$ of these functions lie in $\mathcal{D}_{R+}(T_1)$
(each vanishes in a neighborhood of the left end of $\mathcal{I}$)

the last 2ℓ of these functions lie neither in $\mathcal{D}_{R_-}(T_1)$ nor $\mathcal{D}_{R+}(T_1)$
(each has a support which meets every neighborhood of the left end of $\mathcal{I}$, and also every neighborhood of the right end of $\mathcal{I}$—in non-empty intersections).

The proof of Corollary 1 follows immediately from the real GKN-Theorem 1, Proposition 5—and the conceptual developments in Theorem 7 of Section III.2. The problem of the existence of real Lagrangian d-spaces $L_\ell \subset \mathcal{S}_R$, with grade $L_\ell = \ell$ assigned on $0 \leq \ell \leq \min\{\Delta_-, \Delta_+\}$, is discussed later in Theorem 3 and the subsequent Note 2 at the end of this Appendix B.

We can also define the real Lagrangian d-space $L_\ell \subset \mathcal{S}_R$, as in Corollary 1, in terms of real linear functionals on the $2d$-space $\mathcal{S}_R$ (namely, elements of the dual space $\mathcal{S}_R^\#$). Recall that a real boundary condition for the quasi-differential expression $w^{-1}M_A$ (as in Theorem 1) is a real linear functional ω on the linear $2d$-space $\mathcal{S}_R = \mathcal{D}_R(T_1)/\mathcal{D}_R(T_0)$ (or more precisely, the null space of ω, abbreviated as "the solutions of $\omega = 0$").

Accordingly, consider any linear functional

$$\omega : \mathcal{S}_R \to \mathbb{R} \tag{B.56}$$

(or equally well, a real linear functional on $\mathcal{D}_R(T_1)$ which vanishes on $\mathcal{D}_R(T_0)$). We denote the set of all such boundary conditions as the dual space $\mathcal{S}_R^\#$.

Just as in Definition 5 and Theorem 6 of Section III.2 we define the duality map

$$\mathcal{S}_R \to \mathcal{S}_R^\# : v \to v^\# \tag{B.57}$$

where the real linear functional $v^\#$ is given by

$$v^\#[\hat{f}_R] = [v : \hat{f}_R]_A \qquad \text{for all } \hat{f}_R \in \mathcal{S}_R, \tag{B.58}$$

and we use the map to induce a real symplectic product on the real linear space $\mathcal{S}_R^\#$. Then the duality map is a real symplectic isomorphism carrying $\mathcal{S}_R = \mathcal{S}_{R-} \oplus \mathcal{S}_{R+}$ onto $\mathcal{S}_R^\# = \mathcal{S}_{R-}^\# \oplus \mathcal{S}_{R+}^\#$. Further, the real symplectic subspaces $\mathcal{S}_{R\pm}^\#$, the images of $\mathcal{S}_{R\pm}$, respectively, can also be recognized as annihilators (or nullifiers), as before:

$$\mathcal{S}_{R_-}^\# = \mathcal{N}(\mathcal{S}_{R+}) \text{ and } \mathcal{S}_{R+}^\# = \mathcal{N}(\mathcal{S}_{R-}). \tag{B.59}$$

Thus a real boundary condition can be specified by either a functional $v^\# \in \mathcal{S}_R^\#$, or equally well by the corresponding vector $v \in \mathcal{S}_R$, just as in Section III.2.

Now let us turn to the regular boundary value problem

$$M_A[y] = \lambda w y \qquad (\text{spectral parameter } \lambda \in \mathbb{C}),$$

where $A = \bar{A} = A^+ = Z_n(\mathcal{I})$ is a real Shin-Zettl matrix of even order $n = 2m$, and with the positive weight function w, both in $\mathcal{L}^1(\mathcal{I})$ on the compact interval $\mathcal{I} = [a, b]$ as in Section IV (1.1), (1.2), (1.3). Then the results of the real GKN-Theorem 1, and Propositions 4 and 5 above, remain valid—but special properties hold for the real endpoint spaces $\mathcal{S}_R = \mathcal{D}_R(T_1)/\mathcal{D}_R(T_0) = \mathcal{S}_{R-} \oplus \mathcal{S}_{R+}$.

These real endpoint spaces $\mathcal{S}_{R\pm}$ can be defined quite explicitly for *regular* boundary value problems, as in Section IV where the interval $\mathcal{I} = [a, b]$ is compact and where A and w both lie in $\mathcal{L}^1(\mathcal{I})$. In such a case

(B.60)

$$\begin{aligned} \mathcal{S}_{R-} &= \{\hat{f}_R \in \mathcal{S}_R \mid f_A^{[0]}(b) = f_A^{[1]}(b) = \cdots = f_A^{[n-1]}(b) = 0\}, \text{ and} \\ \mathcal{S}_{R+} &= \{\hat{f}_R \in \mathcal{S}_R \mid f_A^{[0]}(a) = f_A^{[1]}(a) = \cdots = f_A^{[n-1]}(a) = 0\}. \end{aligned}$$

because

$$\mathcal{D}_R(T_0) = \{f \in \mathcal{D}_R(T_1) \mid f_A^{[r]}(a) = f_A^{[r]}(b) = 0, \text{ for } r = 0, 1, \ldots, n-1\}.$$

For the case of a regular boundary value problem on the compact interval $\mathcal{I} = [a, b]$ the evaluation map, see Section IV,

$$\text{(B.61)} \quad \mathcal{V}_R : \hat{f}_R \equiv \{f + \mathcal{D}_R(T_0)\} \to (f_A^{[0]}(a), \ldots, f_A^{[n-1]}(a), f_A^{[0]}(b), \ldots, f_A^{[n-1]}(b))$$

is a symplectic isomorphism of $\mathcal{S}_R = \mathcal{S}_{R-} \oplus \mathcal{S}_{R+}$ onto the real symplectic space $\mathbb{R}^{2n} = \mathbb{R}^{2m} \oplus \mathbb{R}^{2m}$. With these coordinates in $\mathcal{S}_R$ we can write the symplectic product as follows

$$\text{(B.62)} \qquad [\hat{f}_R : \hat{g}_R]_A = \hat{f}_R H \hat{g}_R^t,$$

where the real skew-symmetric matrix (see(B.23))

$$\text{(B.63)} \quad H = (-1)^m \begin{pmatrix} H_n & 0 \\ 0 & -H_n \end{pmatrix}, \text{ and } H_n = \begin{pmatrix} 0 & \cdot & \cdot & \cdot & 0 & (+1) \\ \vdots & & & & (-1) & 0 \\ & & & \cdot^{\cdot^{\cdot}} & & \\ 0 & (+1) & & & & \vdots \\ (-1) & 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix}.$$

In this way we compute (see (B.28)),

$$\text{(B.64)} \quad [\hat{f}_R : \hat{g}_R]_A = (-1)^m \sum_{r=0}^{n-1} (-1)^r \{f_A^{[n-1-r]}(b) g_A^{[r]}(b) - f_A^{[n-1-r]}(a) g_A^{[r]}(a)\}$$

for each $\hat{f}_R = \{f + \mathcal{D}_R(T_0)\}$, $\hat{g}_R = \{g + \mathcal{D}_R(T_0)\}$ in $\mathcal{S}_R$, with $f, g \in \mathcal{D}_R(T_1)$. Furthermore the skew-symmetric matrices $(-1)^m H_n$ and $(-1)^{m+1} H_n$, immediately yield the symplectic forms on the corresponding subspaces $\mathcal{S}_{R_-}$ and S_{R_+} in $\mathcal{S}_R$.

For the regular boundary value problem, as described in Proposition 5, in the special case where $\mathcal{I} = [a, b]$ is compact so $\Delta_- = \Delta_+ = m$ and $d = n = 2m$, examples of such Lagrangian n-spaces of all possible coupling grades are shown to exist in Section III.1 Theorem 4—(while Section III.1 deals with complex Lagrangian spaces, these particular examples are also valid, where meaningful, for the real case). Furthermore, it is important to emphasize that these examples of

Section III.1, although described entirely in terms of abstract algebra, also *yield valid constructions for real operators* T_R, once $A = \bar{A} = A^+ \in Z_{2m}(\mathfrak{I})$ and $w^{-1}M_A$ have been specified on $\mathfrak{I} = [a, b]$.

We now re-formulate and summarize these conclusions in the next theorem, which is the real version of Theorem 2 in Section IV.

THEOREM 2 (Real GKN-Theorem: Regular Case).
Consider the real quasi-differential expression

$$w^{-1}M_A[y] = i^n w^{-1} y_A^{[n]} \tag{B.65}$$

defined on the real compact interval $\mathfrak{I} = [a, b]$, *in terms of the real matrix* $A = \bar{A} = A^+ \in Z_n(\mathfrak{I})$ *of even order* $n = 2m$, *and the positive weight function* w, *both in* $\mathcal{L}^1(\mathfrak{I})$, *as for a regular real boundary value problem on* $\mathfrak{I}$.

Let T_1 *on* $\mathcal{D}(T_1)$ *and* T_0 *on* $\mathcal{D}(T_0)$ *be the maximal and minimal operators, respectively, as generated by* $w^{-1}M_A$ *on the complex Hilbert space* $\mathcal{L}^2(\mathfrak{I}; w)$*—as in the real GKN-Theorem 1. Then the real symplectic* $2n$*-space* $\mathcal{S}_R = \mathcal{D}_R(T_1)/\mathcal{D}_R(T_0)$ *has the direct sum decomposition into the left and right endpoint spaces, as before:*

$$\mathcal{S}_R = \mathcal{S}_{R-} \oplus \mathcal{S}_{R+}, \ \textit{with}\ [\mathcal{S}_{R-} : \mathcal{S}_{R+}]_A = 0, \tag{B.66}$$

where the real symplectic spaces $\mathcal{S}_{R\pm}$ *have the common dimension* $2\Delta_- = 2\Delta_+ = 2m = n$.

Each real Lagrangian n*-space* $L_R \subset \mathcal{S}_R$ *has a coupling grade*

$$\operatorname{grade} L_R = m - \dim L_R \cap \mathcal{S}_{R\pm}, \tag{B.67}$$

which is an integer $\ell \in [0, m]$*; and moreover for each integer* $\ell \in [0, m]$ *there exists (at least) one real Lagrangian* n*-space, say* L_ℓ *with*

$$\operatorname{grade} L_\ell = \ell.$$

According to the real GKN-Theorem 1 above, there exists a natural one-to-one correspondence between the self-adjoint real operators, as generated by $w^{-1}M_A$ *on* $\mathcal{L}^2(\mathfrak{I}; w)$*, and the real Lagrangian* n*-spaces in* $\mathcal{S}_R$*. For each such real Lagrangian* n*-space in* $\mathcal{S}_R$*, say* L_ℓ *with grade* $L_\ell = \ell$*, the corresponding self-adjoint real operator, say* $T_{R\ell}$*, is specified by its domain*

$$\mathcal{D}(T_{R\ell}) = \{f \in \mathcal{D}(T_1) \mid [f : f^s]_A = 0 \qquad \textit{for } s = 1, 2, \ldots, n\}, \tag{B.68}$$

or equally well,

$$\mathcal{D}(T_{R\ell}) = c_1 f^1 + c_2 f^2 + \cdots + c_n f^n + \mathcal{D}(T_0),$$

where $f^1, f^2, \ldots, f^n$ *are any set of* n *real functions in* $\mathcal{D}_R(T_1)$ *such that* $\{\hat{f}_R^1, \hat{f}_R^2, \ldots, \hat{f}_R^n\}$ *constitutes a basis for* L_ℓ*, and* $c_1, c_2, \ldots, c_n$ *are arbitrary complex constants, as in (B.50).*

Furthermore,

$$\dim L_\ell \cap \mathcal{S}_{R\pm} = m - \ell, \ \textit{for}\ \ell = 0, 1, 2, \ldots, m,$$

so

$$L_\ell \textit{ is strictly separated if and only if } \ell = 0, \tag{B.69}$$

and

(B.70) $\qquad L_\ell$ *is totally coupled if and only if* $\ell = m$.

In the special case of a regular problem where $\mathcal{I} = [a, b]$ is compact, the annihilator relations (B.57), (B.58), (B.59) can be demonstrated by explicit calculations in terms of the coordinates in $\mathcal{S}_R$ induced via the evaluation isomorphism (B.61) with $\mathbb{R}^{2n}$. For instance, take any vector $v \in \mathcal{S}_{R-}$, written in the notation (motivated by the formula (B.72) below)

$$v = (-1)^m(\alpha_{n-1}, -\alpha_{n-2}, \alpha_{n-3}, \dots, \alpha_1, -\alpha_0, 0, 0, \dots, 0), \tag{B.71}$$

where the real coordinates $(-1)^m\alpha_{n-1}, \dots, (-1)^m(-\alpha_0)$ are not all zero. In this situation we can compute (B.62), (B.63), (B.64):

$$v^\#[\hat{f}_R] = [v : \hat{f}_R]_A = \alpha_0 f_A^{[0]}(a) + \alpha_1 f_A^{[1]}(a) + \dots + \alpha_{n-1} f_A^{[n-1]}(a) \tag{B.72}$$

for each vector $\hat{f}_R \equiv (f_A^{[0]}(a), \dots, f_A^{[n-1]}(a), f_A^{[0]}(b), \dots, f_A^{[n-1]}(b))$ in $\mathcal{S}_R$ (where $\hat{f}_R = \{f + \mathcal{D}_R(T_0)\}$ is represented by the real function $f \in \mathcal{D}_R(T_1)$, as usual). But since $v^\#[\hat{f}_R]$ involves the quasi-derivatives of f only at the left endpoint a of $\mathcal{I} = [a, b]$, we see that $v^\#$ annihilates all $\hat{f}_R \in \mathcal{S}_{R+}$. In this way we can easily verify directly that

$$\mathcal{S}_{R-}^\# = \mathcal{N}(\mathcal{S}_{R+}), \text{ and similarly } \mathcal{S}_{R+}^\# = \mathcal{N}(\mathcal{S}_{R-}).$$

Thus for $v \in \mathcal{S}_{R-}$ the functional $v^\# \in \mathcal{S}_{R-}^\#$, and these are each separated at the left endpoint a of $\mathcal{I} = [a, b]$. But this fact is emphasized by the real boundary condition corresponding to $v^\#$, namely,

$$v^\#[\hat{f}_R] = 0 \quad \text{or} \quad \alpha_0 f_A^{[0]}(a) + \dots + \alpha_{n-1} f_A^{[n-1]}(a) = 0. \tag{B.73}$$

Similar conclusions hold in case $v \in \mathcal{S}_{R+}$, so $v_{R+}^\# \in \mathcal{S}_{R+}^\# = \mathcal{N}(\mathcal{S}_{R-})$, and then the corresponding boundary condition involves data only at the right endpoint b of $\mathcal{I} = [a, b]$. In this case v, and $v^\#$ the corresponding boundary condition, are each separated at the right endpoint b of $\mathcal{I} = [a, b]$, that is,

$$v^\#[\hat{f}_R] = 0 \quad \text{or} \quad \beta_0 f_A^{[0]}(b) + \dots + \beta_{n-1} f_A^{[n-1]}(b) = 0, \tag{B.74}$$

where the real constants $\beta_0, \beta_1, \dots, \beta_{n-1}$ are not all zero. If v is neither separated at the left nor at the right endpoint of $\mathcal{I}$, then the boundary condition corresponding to $v^\#$ involves data at both ends a and b of $\mathcal{I}$. Then v (or $v^\#$) is called nonseparated or coupled, and

$$v^\#[\hat{f}_R] = \alpha_0 f_A^{[0]}(a) + \dots + \alpha_{n-1} f_A^{[n-1]}(a) + \beta_0 f_A^{[0]}(b) + \dots + \beta_{n-1} f_A^{n-1]}(b),$$

where at least one $\alpha_j \neq 0$ and one $\beta_k \neq 0$, for $0 \leq j,\ k \leq n-1$. When A is smooth, the boundary condition can then be expressed as a linear combination of the ordinary derivatives $f^{(0)}(a), \dots, f^{(n-1)}(a), f^{(0)}(b), \dots, f^{(n-1)}(b)$, but the real constant coefficients then depend on the matrix A, and the analogous conditions hold for separation or coupling.

From these notations and comments concerning boundary conditions the following corollary is immediate.

COROLLARY 1. *For each integer* $\ell = 0, 1, 2, \ldots, m$ *consider the real Lagrangian* n*-space* $L_\ell \subset \mathcal{S}_R$ *with coupling grade* ℓ*, as in Theorem 2. Let* $T_{R\ell}$ *on domain* $\mathcal{D}(T_{R\ell})$ *be the corresponding self-adjoint real operator, on the complex Hilbert space* $\mathcal{L}^2(\mathcal{I}; w)$*, as generated by the real quasi-differential expression* $w^{-1}M_A$*, with* $A = \bar{A} = A^+ \in Z_n(\mathcal{I})$ *of even order* $n = 2m$ *on the compact interval* $\mathcal{I} = [a, b]$*, in accord with Theorems 1 and 2 above.*

Then there exists a basis of n *linearly independent real boundary conditions (a basis for* $L_\ell^\#$*) whose solutions in* $\mathcal{S}_R$ *are precisely the vectors of* L_ℓ*, and thus whose solutions in* $\mathcal{D}_R(T_1)$ *then* span $\mathcal{D}(T_{R\ell})$ *(mod* $\mathcal{D}(T_0)$*), as in* (B.68).

Moreover there exists a minimally coupled basis of n *boundary conditions (basis for* $L_\ell^\#$*) with precisely*

(B.75)
$$\begin{aligned} &m - \ell \textit{ separated at the left endpoint of } \mathcal{I}, \\ &m - \ell \textit{ separated at the right endpoint of } \mathcal{I}, \\ &2\ell \textit{ coupled (nonseparated).} \end{aligned}$$

Furthermore, every basis of n *boundary conditions (basis of* $L_\ell^\#$*) must contain at least* 2ℓ *coupled boundary conditions.*

REMARK. Of course, a minimally coupled basis for $L_\ell^\#$ can be also interpreted as a minimally coupled basis for L_ℓ, compare Corollary 1 to Theorem 1—also see Theorem 2 of Section IV.

PROOF. According to the argument in Corollary 1 of Theorem 6 in Section III.2,

$$L_\ell^\# = \mathcal{N}(L_\ell) \text{ and } L_\ell = \mathcal{N}(L_\ell^\#).$$

Hence for each basis of $L_\ell^\#$ (or the corresponding set of n linearly independent real boundary conditions), L_ℓ is precisely the set of solutions (annihilators) of these n boundary conditions.

But grade $L_\ell = m - \dim L_\ell \cap \mathcal{S}_{R\pm} = \ell$ if and only if

$$\dim L_\ell \cap \mathcal{S}_{R-} = m - \ell \quad \text{and} \quad dim L_\ell \cap \mathcal{S}_{R+} = m - \ell.$$

Therefore there exists a basis for L_ℓ which contains $m - \ell$ vectors in $\mathcal{S}_{R-}$ and also $m - \ell$ vectors in $\mathcal{S}_{R+}$, with each of the remaining vectors lying in neither the left nor right endpoint space. But then the dual basis in $L_\ell^\#$ has the required properties:

$$\begin{aligned} &m - \ell \text{ separated at the left endpoint of } \mathcal{I}, \\ &m - \ell \text{ separated at the right endpoint of } \mathcal{I}, \\ &2\ell \text{ coupled (nonseparated).} \end{aligned}$$

Of course, a perturbation of this basis can create a basis for $L_\ell^\#$ with more basis vectors nonseparated, even all basis vectors nonseparated or coupled. However, every basis for $L_\ell^\#$ must contain at least 2ℓ nonseparated boundary conditions, as required. □

The next topic in this Appendix B deals with the self-adjoint but *non-real* operators T, as generated by the real quasi-differential expression $w^{-1}M_A$ on the complex Hilbert space $\mathcal{L}^2(\mathcal{I}; w)$—where the real matrix $A = \bar{A} = A^+ \in Z_n(\mathcal{I})$ is of even order $n = 2m$, and the positive weight function $w \in \mathcal{L}^1_{\text{loc}}(\mathcal{I})$ on the real interval $\mathfrak{I}$, as before. Such a formally self-adjoint real quasi-differential expression $w^{-1}M_A$ can generate self-adjoint operators T on $\mathfrak{D}(T) \subseteq \mathfrak{D}(T_1)$, that are not real, through the use of certain complex non-real boundary conditions.

We first clarify this phenomenon for the classical Sturm-Liouville differential operator of order $n = 2$, and then proceed to the more general self-adjoint real quasi-differential operators of order $n \geq 4$.

PROPOSITION 6. *Consider the Sturm-Liouville real quasi-differential operator of order $n = 2m = 2$,*

$$M_A[y] = -y_A^{[2]} = -(py')' + qy, \tag{B.76}$$

$$\text{where } A = \bar{A} = A^+ = \begin{pmatrix} 0 & p^{-1} \\ q & 0 \end{pmatrix} \in Z_2(\mathfrak{I}) \tag{B.77}$$

for smooth real coefficients p, q with $p(x) \neq 0$ on the compact interval $\mathfrak{I}[a,b]$. Then the quasi-differential expression $w^{-1}M_A$, for given positive weight function $w \in \mathfrak{L}^1(\mathfrak{I})$, has real coefficients, so the corresponding maximal and minimal operators T_1 on $\mathfrak{D}(T_1)$ and T_0 on $\mathfrak{D}(T_0)$, respectively, are both real operators in $\mathfrak{L}^2(\mathfrak{I}; w)$.

As usual, define the endpoint real symplectic 4-space.

$$\mathcal{S}_R = \mathfrak{D}_R(T_1)/\mathfrak{D}_R(T_0) = \mathcal{S}_{R-} \oplus \mathcal{S}_{R+}, \tag{B.78}$$

and its complexification, which is isomorphic to the complex symplectic 4-space

$$\mathcal{S} = \mathcal{S}_- \oplus \mathcal{S}_+, \quad \text{with} \quad [\mathcal{S}_- : \mathcal{S}_+]_A = 0. \tag{B.79}$$

Then necessarily

$$\dim \mathcal{S}_{R\pm} = 2, \quad \dim \mathcal{S}_\pm = 2. \tag{B.80}$$

Let L_R be a real Lagrangian 2-space in $\mathcal{S}_R$, so L_R determines a self-adjoint real operator T_R on $\mathfrak{D}(T_R) \subseteq \mathfrak{D}(T_1)$, as generated by $w^{-1}M_A$ in $\mathfrak{L}^2(\mathfrak{I}; w)$—according to the real GKN-Theorem 1 above.

Then either:

(i) *grade $L_R = 0$ (L_R is strictly separated);*

In this case L_R can be defined as the annihilator of an independent pair of real boundary conditions, one of which is separated at the left endpoint a of $\mathfrak{I}$ and the other is separated at the right endpoint b of $\mathfrak{I}$,

or else:

(ii) *grade $L_R = 1$ (L_R is totally coupled);*

In this case each basis of real boundary conditions defining L_R must consist of two boundary conditions each of which is coupled.

Next let L_C be a complex Lagrangian 2-space in $\mathcal{S}$ so L_C determines a self-adjoint operator T on $\mathfrak{D}(T) \subseteq \mathfrak{D}(T_1)$, as generated by $w^{-1}M_A$, according to the general GKN-Theorem 1 of Section II.

Then either:

(iii) *grade $L_C = 0$ (L_C is strictly separated);*

In this case L_C is defined by some basis of complex boundary conditions, each of which is separated in the sense of (i) above. More surprisingly, L_C is necessarily the complexification of some real Lagrangian 2-space in $\mathcal{S}_R$, and hence $L_C = \bar{L}_C$ and is defined by a basis of real boundary conditions, *each of which is separated. Moreover, the self-adjoint operator T on $\mathcal{D}(T)$ is necessarily a real operator,*

or else:

(iv) *grade $L_C = 1$ (L_C is totally coupled).*

In this case L_C is defined by a basis of complex boundary conditions, each of which is coupled in the sense of (ii) above. Moreover, L_C may or may not be the complexification of a real Lagrangian 2-space.

If $L_C = \bar{L}_C$, then it is such a complexification, and T on $\mathcal{D}(T)$ is a real operator.

If $L_C \neq \bar{L}_C$, then it is not a complexification, and then T on $\mathcal{D}(T)$ is not a real operator.

Examples of both kinds exist.

PROOF. Conclusions (i) and (ii) are trivial consequences of Corollary 1 of Theorem 2 above. Accordingly we consider only the cases (iii) and (iv) for a complex Lagrangian 2-space L_C in the complex symplectic 4-space $\mathcal{S} = \mathcal{S}_- \oplus \mathcal{S}_+$.

Since $\mathcal{I} = [a, b]$ is compact, the evaluation isomorphisms of $\mathcal{S}$ with $\mathbb{C}^4$, as in Section IV, identifies

$$\hat{f} = \{f + \mathcal{D}(T_0)\} \text{ with } (f_A^{[0]}(a), f_A^{[1]}(a), f_A^{[0]}(b), f_A^{[1]}(b)) \in \mathbb{C}^4. \tag{B.81}$$

Then the symplectic product can be computed by

$$[\hat{f} : \hat{g}]_A = \hat{f}\, H\, \hat{g}^*, \quad \text{with } H = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \tag{B.82}$$

for the complex 4-vectors $\hat{f}$ and $\hat{g} \in \mathcal{S}$.

The real skew-symmetric matrix H also defines the symplectic product in $\mathcal{S}_R = \mathcal{S}_{R-} \oplus \mathcal{S}_{R+}$, as well as in its complexification $\mathcal{S} = \mathcal{S}_- \oplus \mathcal{S}_+$. Thus there is a complex conjugation in $\mathcal{S}$, with real vectors $\mathcal{S}_R$, and each vector $u = (u_1, u_2, u_3, u_4) \in \mathcal{S}$ has the conjugate $\bar{u} = (\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4) \in \mathcal{S}$. Then u is real just in case all its components are real numbers.

Now assume that grade $L_C = 0$, so that $\dim L_C \cap \mathcal{S}_- = 1$ and $\dim L_C \cap \mathcal{S}_+ = 1$. Take vectors $u_- = (u_1, u_2, 0, 0)$ as a basis for $L_C \cap \mathcal{S}_-$ and $v_+ = (0, 0, v_1, v_2)$ as a basis for $L_C \cap \mathcal{S}_+$. We can choose $u_- = (1, \eta, 0, 0) \in \mathcal{S}_-$ (or possibly $(0, 1, 0, 0)$) after a suitable multiplication by a complex scalar.

But u_- is a neutral vector in the Lagrangian 2-space L_C, and so

$$u_- H u_-^* = (1, \eta) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (1, \bar{\eta})^t = \eta - \bar{\eta} = 0.$$

Hence η is a real number, and so $u_- \in \mathcal{S}_{R_-}$ is a real base vector for $L_C \cap \mathcal{S}_-$.

A similar calculation shows that we can take v_+ to be a real base vector for $L_C \cap \mathcal{S}_+$, and thus $\{u_-, v_+\}$ is a real basis (lying in $\mathcal{S}_R$) for the complex Lagrangian 2-space L_C.

Now observe that the conjugation in $\mathcal{S}$ carries

$$c_1 u_- + c_2 v_+ \quad \text{to} \quad \overline{c_1 u_- + c_2 v_+} = \bar{c}_1 u_- + \bar{c}_2 v_+ \in L_C,$$

for complex constants c_1 and c_2. Hence $L_C = \bar{L}_C$ is self-conjugate in $\mathcal{S}$, as required. Clearly L_C is thus the complexification of a real Lagrangian 2-space, say $L'_R \subset \mathcal{S}_R$,

$$L'_R = \{r_1 u_- + r_2 v_+\} \qquad \text{for } r_1, r_2 \in \mathbb{R}, \tag{B.83}$$

with grade $L'_R = 0$; and conclusion (iii) is proved.

Finally we exhibit two examples for complex Lagrangian 2-spaces $L_C \subset \mathcal{S}$ with grade $L_C = 1$, say L'_C with $L'_C = \overline{L'_C}$, and also L''_C with $L''_C \neq \overline{L''_C}$. The other conclusions in (iv) then follow from Proposition 2 above.

For example take L'_C defined by the basis of vectors $(1,0,1,0)$ and $(0,1,0,1)$ in $\mathcal{S} = \mathcal{S}_- \oplus \mathcal{S}_+ \approx \mathbb{C}^4$. Because this basis consists of real vectors in $\mathcal{S}$, $L'_C = \bar{L}'_C$ is self-conjugate and so L'_C is the complexification of a real Lagrangian 2-plane in $\mathcal{S}_R$. Moreover it is obvious that $L'_C \cap \mathcal{S}_- = L'_C \cap \mathcal{S}_+ = 0$ so that grade $L'_C = 1$ and L'_C is totally coupled.

For the next example take L''_C defined by the basis of vectors $u = (1,0,i,0)$ and $v = (0,1,0,i)$—or equally well by the boundary conditions

$$f_A^{[0]}(b) = i f_A^{[0]}(a), \qquad f_A^{[1]}(b) = i f_A^{[1]}(a). \tag{B.84}$$

But $\bar{u} = (1,0,-i,0) \notin L''_C$ so $L''_C \neq \overline{L''_C}$. Again it is obvious that $L''_C \cap \mathcal{S}_- = L''_C \cap \mathcal{S}_+ = 0$, so L''_C is totally coupled, yet it determines a self-adjoint operator T on $\mathcal{D}(T)$ which is not real. □

The situation in Proposition 6 (iv)—where certain complex boundary conditions, necessarily coupled for the *real* Sturm-Liouville differential operator with $n = 2$, determine a self-adjoint operator T on $\mathcal{D}(T)$ that is *not real* in $\mathcal{L}^2(\mathcal{I}; w)$—can be illustrated more explicitly as follows. Take $p(x) = q(x) = w(x) \equiv 1$ in $\mathcal{I} = [0,1]$. Then the function

$$y(x) = \frac{1}{2}[\cos \pi x + 1] + \frac{i}{2}[\cos \pi(1-x) + 1] \tag{B.85}$$

lies in the domain $\mathcal{D}(T)$, yet $\bar{y} \notin \mathcal{D}(T)$. Hence $\mathcal{D}(T) \neq \overline{\mathcal{D}(T)}$ and T is not a real operator.

Analogous examples are easily constructed for self-adjoint real quasi-differential expressions of even orders $n \geq 4$.

On the other hand the phenomenon in Proposition 6 (iii)—where complex boundary conditions which are strictly separated force the corresponding self-adjoint operator T on $\mathcal{D}(T)$ to be a real operator in $\mathcal{L}^2(\mathcal{I}; w)$—is peculiar to the order $n = 2$, and fails for higher orders $n \geq 4$.

The next example illustrates this failure for $n = 4$, and then the subsequent theorem demonstrates the corresponding result for all even $n \geq 4$.

Example 1. Let $w^{-1} M_A$ be a real quasi-differential expression, where the real matrix $A = \bar{A} = A^+ \in Z_4(\mathcal{I})$ has even order $n = 2m = 4$, and the positive weight function $w \in \mathcal{L}^1(\mathcal{I})$ on the compact interval $\mathcal{I} = [a,b]$. Also consider the real endpoint space $\mathcal{S}_R = \mathcal{D}_R(T_1)/\mathcal{D}_R(T_0) = \mathcal{S}_{R_-} \oplus \mathcal{S}_{R_+}$, and its complexification

$$\mathcal{S} = \mathcal{D}(T_1)/\mathcal{D}(T_0) = \mathcal{S}_- \oplus \mathcal{S}_+, \quad \text{with } [\mathcal{S}_- : \mathcal{S}_+]_A = 0,$$

as usual. Then $\mathcal{S}$ is a complex symplectic 8-space, with complex conjugation that leaves the left and right endpoint spaces $\mathcal{S}_\pm$ invariant. Thus each of $\mathcal{S}_\pm$ is a complex symplectic 4-space which is the complexification of the corresponding real endpoint space $\mathcal{S}_{R\pm}$.

Under these conditions we shall construct explicitly a complex Lagrangian 4-space $L_0 \subset \mathcal{S}$ such that:

(B.86) $$\operatorname{grade} L_0 = 0 \qquad (\text{so } \dim L_0 \cap \mathcal{S}_{\pm} = 2)$$

and yet

(B.87) $$L_0 \neq \bar{L}_0 \qquad (\text{so } L_0 \text{ is not self-conjugate}).$$

Hence L_0 is strictly separated, yet still defines a self-adjoint operator T on $\mathcal{D}(T) \subseteq \mathcal{D}(T_1) \subset \mathcal{L}^2(\mathcal{I}; w)$, which is not a real operator. We shall further describe L_0 in terms of separated complex boundary conditions at the endpoints of $\mathcal{I} = [a, b]$, in the customary manner.

Let the evaluation isomorphism of $\mathcal{S}$ onto $\mathbb{C}^8$ define the complex coordinates for each vector $\hat{f} = \{f + \mathcal{D}(T_0)\} \in \mathcal{S}$ according to:

(B.88) $$\hat{f} \leftrightarrow (f_A^{[0]}(a), f_A^{[1]}(a), f_A^{[2]}(a), f_A^{[3]}(a), f_A^{[0]}(b), f_A^{[1]}(b), f_A^{[2]}(b), f_A^{[3]}(b)),$$

as in Section IV above. The corresponding skew-Hermitian matrix for evaluating the symplectic products in $\mathcal{S}$ is (see (B.63):

(B.89) $$H = \begin{pmatrix} H_4 & 0 \\ 0 & -H_4 \end{pmatrix}, \text{ where } H_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Using these evaluation coordinates in $\mathcal{S}$, we compute

(B.90) $$[\hat{f} : \hat{g}]_A = \hat{f}\, H \hat{g}^*, \qquad \text{for } \hat{f} \text{ and } \hat{g} \in \mathcal{S}.$$

More particularly, if $\hat{f} \in \mathcal{S}_-$ so that

$$f_A^{[0]}(b) = f_A^{[1]}(b) = f_A^{[2]}(b) = f_A^{[3]}(b) = 0,$$

and similarly for $\hat{g} \in \mathcal{S}_-$, then

(B.91)
$$[\hat{f} : \hat{g}]_A = (f_A^{[0]}(a), f_A^{[1]}(a), f_A^{[2]}(a), f_A^{[3]}(a)) H_4 (g_A^{[0]}(a), g_A^{[1]}(a), g_A^{[2]}(a), g_A^{[3]}(a))^*.$$

Analogous formulas hold for $\hat{f}, \hat{g} \in \mathcal{S}_+$. As usual $[\mathcal{S}_- : \mathcal{S}_+]_A = 0$.

Now we specify the required Lagrangian 4-space $L_0 \subset \mathcal{S}$ by assigning the following four vectors as a basis for L_0:

(B.92) $$\begin{aligned} u_- &= (i+1, 0, i-1, 0, 0, 0, 0, 0) \\ v_- &= (0, i+1, 0, i-1, 0, 0, 0, 0) \\ u_+ &= (0, 0, 0, 0, i+1, 0, i-1, 0) \\ v_+ &= (0, 0, 0, 0, 0, i+1, 0, i-1). \end{aligned}$$

The motivations and notations for this choice of basis for L_0 arose in Proposition 6 (iv), specifically in formula (B.84). In particular we use the special calculation,

$$Q\, H_4 Q^t = \begin{pmatrix} -H_2 & 0 \\ 0 & H_2 \end{pmatrix}, \text{ as in (B.63)},$$

for the real orthogonal matrix $Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$.

It is then straightforward to verify that L_0 is a complex Lagrangian 4-space in $\mathcal{S}$, and also that

$$\dim L_0 \cap \mathcal{S}_\pm = 2, \text{ so grade } L_0 = 0.$$

Thus L_0 is strictly separated in $\mathcal{S} = \mathcal{S}_- \oplus \mathcal{S}_+$.

Moreover the vector u_- has the conjugate

$$\bar{u}_- = (-i+1, 0, -i-1, 0, 0, 0, 0, 0),$$

because H is real; and we note that

$$u_- \in L_0 \text{ but } \bar{u}_- \notin L_0.$$

Hence $L_0 \neq \bar{L}_0$ and L_0 is not self-conjugate, so L_0 is not the complexification of any real Lagrangian subspace of $\mathcal{S}_R$.

As an alternative to the description of L_0 through basis vectors, we can specify L_0 by means of boundary conditions (linear functionals on $\mathcal{S}$). For instance, L_0 consists of all functions (mod $\mathcal{D}(T_0)$) satisfying

$$(i-1)f_A^{[0]}(a) = (i+1)f_A^{[2]}(a), \qquad (i-1)f_A^{[1]}(a) = (i+1)f_A^{[3]}(a) \tag{B.93}$$

and

$$(i-1)f_A^{[0]}(b) = (i+1)f_A^{[2]}(b), \qquad (i-1)f_A^{[1]}(b) = (i+1)f_A^{[3]}(b),$$

where the first pair of these boundary conditions are separated at the left endpoint a, and the second pair are separated at the right endpoint b of $\mathcal{I}$.

As a classical treatment of this special example, let us write these boundary conditions for L_0, as generated by a Shin-Zettl matrix, see Appendix A (A.49),

$$A = \bar{A} = A^+ = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & p_2^{-1} & 0 \\ 0 & -p_1 & 0 & 1 \\ -p_0 & 0 & 0 & 0 \end{pmatrix}, \tag{B.94}$$

for smooth real functions p_0, p_1, p_2 with $p_2(x) \neq 0$ for $x \in \mathcal{I} = [a, b]$. In this case the quasi-differential expression M_A (say, take $w(x) \equiv 1$ for $x \in \mathcal{I}$) reduces to the classical real linear differential operator

$$M[y] = (p_2 y'')'' + (p_1 y')' + p_0 y, \tag{B.95}$$

for all suitably smooth functions $y \in \mathcal{D}(A)$. The usual formulas for the quasi-derivatives $y_A^{[r]}$ are related to the ordinary derivatives $y^{(r)}$, for $r = 0, 1, 2, 3, 4$, according to

$$y_A^{[0]} = y, y_A^{[1]} = y', y_A^{[2]} = p_2 y'', y_A^{[3]} = (p_2 y'')' + p_1 y'$$

and

$$y^{[4]} = (p_2 y'')'' + (p_1 y')' + p_0 y.$$

Hence the prior boundary conditions specifying the complex Lagrangian 4-space $L_0 \subset \mathcal{S}$ are now

$$(i-1)y(a) = (i+1)p_2(a)y''(a) \tag{B.96}$$

$$(i-1)y'(a) = (i+1)[p_2'(a)y''(a) + p_2(a)y'''(a) + p_1(a)y'(a)]$$

at the left endpoint, and also

$$(i-1)y(b) = (i+1)p_2(b)y''(b) \tag{B.97}$$

$$(i-1)y'(b) = (i+1)[p_2'(b)y''(b) + p_2(b)y'''(b) + p_1(b)y'(b)]$$

at the right endpoint b of $\mathcal{I}$.

With these explicit results we conclude our analyses of this Example 1.

In the next theorem we demonstrate the existence of various kinds of Lagrangian n-spaces within the complex symplectic $2n$-space $\mathcal{S} = \mathcal{S}_- \oplus \mathcal{S}_+$, as determined by a formally self-adjoint real quasi-differential expression $w^{-1}M_A$ on a compact interval $\mathcal{I} = [a, b]$. However, we do not calculate the explicit descriptions of these Lagrangian n-spaces in terms of boundary conditions involving the quasi-derivatives, as was done in the preceding Example 1.

THEOREM 3. *Let $w^{-1}M_A$ be a real quasi-differential expression, where the real matrix $A = \bar{A} = A^+ \in Z_n(\mathcal{I})$ has even order $n = 2m \geq 4$, and the positive weight function $w \in \mathcal{L}^1(\mathcal{I})$ on the compact interval $\mathcal{I} = [a, b]$.*

Let T_1 on $\mathcal{D}(T_1)$ and T_0 on $\mathcal{D}(T_0)$ be the maximal and minimal operators, respectively, as generated by $w^{-1}M_A$ on the complex Hilbert space $\mathcal{L}^2(\mathcal{I}; w)$. Also let

$$\mathcal{S} = \mathcal{D}(T_1)/\mathcal{D}(T_0) = \mathcal{S}_- \oplus \mathcal{S}_+, \qquad \text{with } [\mathcal{S}_- : \mathcal{S}_+]_A = 0,$$

be the endpoint complex symplectic $2n$-space with complex conjugation which leaves invariant the left and right endpoint n spaces $\mathcal{S}_\pm$. Then the corresponding real symplectic $2n$-space is

$$\mathcal{S}_R = \mathcal{D}_R(T_1)/\mathcal{D}_R(T_0) = \mathcal{S}_{R_-} \oplus \mathcal{S}_{R+}, \qquad \text{with } [\mathcal{S}_{R-} : \mathcal{S}_{R+}]_A = 0,$$

(with notations as in the general GKN-Theorem 1 of Section II, and the real GKN Theorem 1 above).

Then for each integer $\ell = 0, 1, 2, \ldots, m$ there exist complex Lagrangian n-spaces L_ℓ' and L_ℓ'', each of grade ℓ in $\mathcal{S}$, that is

$$\dim L_\ell' \cap \mathcal{S}_\pm = \dim L_\ell'' \cap \mathcal{S}_\pm = m - \ell, \tag{B.98}$$

and such that

$$L_\ell' = \overline{L_\ell'} \quad \text{but} \quad L_\ell'' \neq \overline{L_\ell''}. \tag{B.99}$$

Hence L_ℓ' is the complexification of a real Lagrangian n-space in $\mathcal{S}_R$, but L_ℓ'' is not.

As before, in each case L_ℓ' or L_ℓ'' can be defined by n linearly independent complex boundary conditions at the endpoints of $\mathcal{I}$ (corresponding to a minimally coupled basis), with

(B.100)

$m - \ell$	*separated at the left endpoint of $\mathcal{I}$,*
$m - \ell$	*separated at the right endpoint of $\mathcal{I}$,*
2ℓ	*coupled,*

and moreover every other basis of n boundary conditions for L_ℓ' or L_ℓ'' must have at least 2ℓ coupled.

For each fixed $\ell = 0, 1, 2, \ldots, m$ this basis of n boundary conditions for L'_ℓ can be chosen to be real, and hence L'_ℓ determines a self-adjoint operator T_R on $\mathcal{D}(T_R)$, as generated by $w^{-1}M_A$—and T_R is a real operator in $\mathcal{L}^2(\mathcal{I}; w)$.

On the other hand, for each fixed $\ell = 0, 1, 2, \ldots, m$ these n boundary conditions for L''_ℓ cannot all be real, and L''_ℓ determines a self-adjoint operator T on $\mathcal{D}(T)$, as generated by $w^{-1}M_A$—and T is definitely not a real operator in $\mathcal{L}^2(\mathcal{I}; w)$.

In particular L''_0 is strictly separated, yet the corresponding self-adjoint operator T is not real in $\mathcal{L}^2(\mathcal{I}; w)$.

PROOF. We first review some of the constructions and notations of Section III.1 Theorem 4, referring to L'_ℓ, since these techniques will then be modified in the construction of L''_ℓ.

Take a canonical basis for $\mathcal{S}_{R-} \subset \mathcal{S}_-$,

(B.101) $$\{e^1, e^2, \ldots, e^m;\ e^{m+1}, e^{m+2}, \ldots, e^n\},$$

which are n independent real vectors in $\mathcal{S}_-$ such that:

(B.102) $$[e^j : e^k]_A = 0, \qquad [e^{m+j} : e^{m+k}]_A = 0$$

and

$$[e^j : e^{m+k}]_A = \delta^{jk}, \text{ for } 1 \le j, k \le m.$$

Then $\operatorname{span}\{e^1, e^2, \ldots, e^m\}$ and $\operatorname{span}\{e^{m+1}, e^{m+2}, \ldots, e^n\}$ are each Lagrangian m-spaces in $\mathcal{S}_-$, and they are canonical duals of one another.

Further take another such canonical basis of real vectors on $\mathcal{S}_+$, namely

(B.103) $$\{e^{n+1}, e^{n+2}, \ldots, e^{n+m};\ e^{n+m+1}, e^{n+m+2}, \ldots, e^{2n}\}$$

with

(B.104) $$[e^{n+j} : e^{n+k}]_A = 0, \qquad [e^{n+m+j} : e^{n+m+k}]_A = 0$$

and

$$[e^{n+j} : e^{n+m+k}]_A = \delta^{jk}, \text{ for } 1 \le j, k, \le m.$$

As we previously explained in Section III.1,

(B.105) $$L'_0 = \operatorname{span}\{e^1, e^2, \ldots, e^m, e^{n+1}, e^{n+2}, \ldots, e^{n+m}\}$$

is a complex Lagrangian n-space in $\mathcal{S}$, and is such that

$$\dim L'_0 \cap \mathcal{S}_- = \dim L'_0 \cap \mathcal{S}_+ = m,$$

so L'_0 is strictly separated and

$$\operatorname{grade} L'_0 = 0.$$

In addition $L'_0 = \overline{L'_0}$ since L_0 has a basis of real vectors in $\mathcal{S}$.

Next for each integer $\ell = 1, 2, \ldots, m$ we proceed to specify a real basis for the complex Lagrangian n-space L'_ℓ. Namely, we take the following four sets of real vectors to constitute the basis of L'_ℓ in $\mathcal{S}$:

(B.106) $$\begin{aligned} &[e^1, e^2, \ldots, e^{m-\ell}\}, && (m-\ell) \text{ vectors in } \mathcal{S}_- \\ &\{e^{n+1}, e^{n+2}, \ldots, e^{n+m-\ell}\}, && (m-\ell) \text{ vectors in } \mathcal{S}_+ \\ &(\text{omit these } 2(m-\ell) \text{ vectors when } \ell = m), \end{aligned}$$

and then

$$\{e^{m-\ell+1}+e^{n+m-\ell+1},\ldots,e^m+e^{n+m}\}$$

and also

$$\{e^{n-\ell+1}-e^{n+2m-\ell+1},\ldots,e^n-e^{2n}\}$$

(these last 2ℓ vectors span a subspace that meets each of $\mathcal{S}_-$ and $\mathcal{S}_+$ only at the origin). The schematic Diagram 1 illustrates the arrangement of these four group of basis vectors within $\mathcal{S}=\mathcal{S}_-\oplus\mathcal{S}_+$.

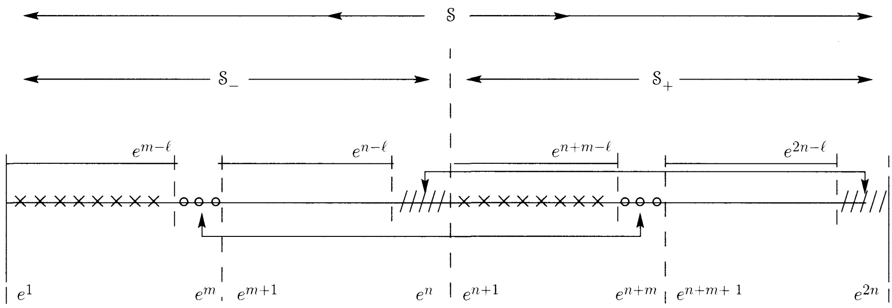

DIAGRAM 1

A straightforward calculation, using the prescribed symplectic products, shows that these $2(m-\ell)+2\ell=2m=n$ real vectors span a Lagrangian n-space $L'_\ell\subset\mathcal{S}$ with

$$\text{grade}\,L'_\ell=\ell \text{ and } L'_\ell=\overline{L'_R}. \tag{B.107}$$

Now we modify the previous construction of L'_ℓ by minor changes to produce the other Lagrangian n-spaces L''_ℓ for each fixed $\ell=1,2,3,\ldots,n$. Later we return to resolve the more difficult case of the strictly separated Lagrangian n-space L''_0, by means of the techniques illustrated in Example 1 above.

We use the same real canonical bases for $\mathcal{S}_\pm$ as given above, but we now select the basis for L''_ℓ to consist of the following four sets of vectors:

(B.108)

$$\begin{aligned}&\{e^1,e^2,\ldots,e^{m-\ell}\}, \qquad (m-\ell) \text{ vectors in } \mathcal{S}_-\\&\{e^{n+1},e^{n+2},\ldots,e^{n+m-\ell}\}, \quad (m-\ell) \text{ vectors in } \mathcal{S}_+\\&\text{(omit these } 2(m-\ell) \text{ vectors if } \ell=m),\end{aligned}$$

and then

$$\{e^{m-\ell+1}+e^{n+m-\ell+1},\ldots,e^{m-1}+e^{n+m-1},e^m+ie^{n+m}\}$$

and also

$$\{e^{n-\ell+1}-e^{n+n-\ell+1},\ldots,e^{n-1}-e^{2n-1},e^n-ie^{2n}\}.$$

As before, it is easy to verify that these $2(m-\ell)+2\ell = 2m = n$ vectors are linearly independent in $\mathcal{S}$. Furthermore the verification that L''_ℓ is a Lagrangian n-space is routine, and involves only one new calculation

$$\text{(B.109)} \qquad [e^m + ie^{n+m} : e^n - ie^{2n}]_A = [e^m : e^n]_A + i^2[e^{n+m} : e^{2n}]_A = 0.$$

From the designation of the basis for L''_ℓ we see that

$$\dim L''_\ell \cap \mathcal{S}_- = \dim L''_\ell \cap \mathcal{S}_+ = m - \ell$$

so

$$\operatorname{grade} L''_\ell = m - \dim L''_\ell \cap \mathcal{S}_\pm = \ell.$$

However the complex Lagrangian n-space L''_ℓ is not self-conjugate in $\mathcal{S}$, since the base vector

$$e^m + ie^{n+m} \in L''_\ell, \text{ but its complex conjugate } e^m - ie^{n+m} \notin L''_\ell.$$

Here the complex conjugation in $\mathcal{S}$ carries $e^m + ie^{n+m}$ to its complex conjugate $e^m - ie^{n+m}$, since the given base vectors of $\mathcal{S}_\pm$ are all real, and we also use the fact that the vectors e^m and e^{n+m} occur in L''_ℓ only in the combination $e^m + ie^{n+m}$.

There still remains the question of the existence of the complex Lagrangian n-space L''_0 with grade $L''_0 = 0$ and with $L''_\ell \neq \overline{L''_\ell}$ in $\mathcal{S}$. That is, we must construct a strictly separated Lagrangian n-space L''_0 which is not self-conjugate and hence which is not the complexification of any real Lagrangian n-space in $\mathcal{S}_R$. By Proposition 6 we know that no such L''_0 exists for $n = 2$, but in Example 1 above we did construct such a Lagrangian n-space L''_0 for $n = 4$.

We now proceed to construct the required complex Lagrangian n-space L''_0 for all even orders $n = 2m \geq 6$, so $m \geq 3$. Use the same given real canonical bases for $\mathcal{S}_-$ and $\mathcal{S}_+$ as before, but now we define certain other complex symplectic subspaces of $\mathcal{S}$, as indicated in the schematic Diagram 2.

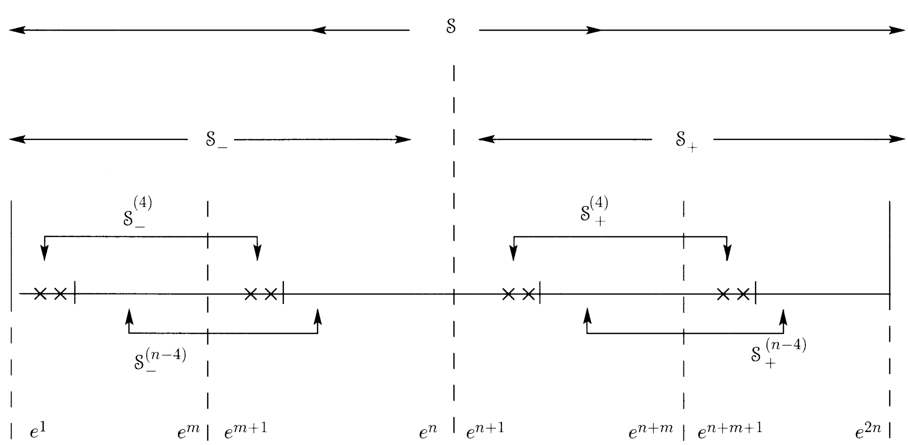

DIAGRAM 2

Namely, define

(B.110)
$$\mathcal{S}_-^{(4)} = \operatorname{span}\{e^1, e^2, e^{m+1}, e^{m+2}\}$$
$$\mathcal{S}_+^{(4)} = \operatorname{span}\{e^{n+1}, e^{n+2}, e^{n+m+1}, e^{n+m+2}\}$$
(recall $n = 2m \geq 6$ so $m+2 < n$), and again

(B.111)
$$\mathcal{S}_-^{(n-4)} = \operatorname{span}\{e^3, \dots, e^m, e^{m+3}, \dots, e^n\}$$
$$\mathcal{S}_+^{(n-4)} = \operatorname{span}\{e^{n+3}, \dots, e^{n+m}, e^{n+m+3}, \dots, e^{2n}\}.$$

Now each of these four complex symplectic subspaces of $\mathcal{S}$ is invariant under the complex conjugation of $\mathcal{S}$, and hence inherits this conjugation so as to specify the corresponding real symplectic subspace of $\mathcal{S}_R$. In this sense the bases given for $\mathcal{S}_\pm^{(4)}$ and $\mathcal{S}_\pm^{(n-4)}$ are each real, and furthermore each such basis contains all of the canonical duals of its members. Hence we can now define the complex symplectic spaces

(B.112)
$$\mathcal{S}^{(4)} = \mathcal{S}_-^{(4)} \oplus \mathcal{S}_+^{(4)}, \text{ with } [\mathcal{S}_-^{(4)} : \mathcal{S}_+^{(4)}]_A = 0$$

and

(B.113)
$$\mathcal{S}^{(n-4)} = \mathcal{S}_-^{(n-4)} \oplus \mathcal{S}_+^{(n-4)}, \text{ with } [\mathcal{S}_-^{(n-4)} : \mathcal{S}_+^{(n-4)}]_A = 0.$$

The dimension of these complex symplectic spaces are then

$$\dim \mathcal{S}^{(4)} = 8, \text{ and } \dim \mathcal{S}_\pm^{(4)} = 4,$$
$$\dim \mathcal{S}^{(n-4)} = 2(n-4), \text{ and } \dim \mathcal{S}_\pm^{(n-4)} = n-4.$$

Also they each have excess zero, since they admit a complex conjugation.

Note that $\mathcal{S}^{(4)}$ is isomorphic, as a complex symplectic space with complex conjugation, to the complex symplectic space analysed in the Example 1 above. As in Example 1, there exists a complex Lagrangian 4-space $L_0^{(4)} \subset \mathcal{S}^{(4)}$ such that $L_0^{(4)}$ is strictly separated within $\mathcal{S}^{(4)} = \mathcal{S}_-^{(4)} \oplus \mathcal{S}_+^{(4)}$, that is, grade $L_0^{(4)} = 0$, and furthermore

$$L_0^{(4)} \neq \overline{L_0^{(4)}}.$$

Next consider the complex symplectic space

$$\mathcal{S}^{(n-4)} = \mathcal{S}_-^{(n-4)} \oplus \mathcal{S}_+^{(n-4)}, \text{ with } [\mathcal{S}_-^{(n-4)} : \mathcal{S}_+^{(n-4)}]_A = 0.$$

Use the techniques employed in the first part of this theorem to construct a complex Lagrangian $(n-4)$-space $L_0^{(n-4)}$ which is strictly separated in $\mathcal{S}^{(n-4)}$ (without any regard to demands on self-conjugacy of $L_0^{(n-4)}$).

Finally define the required complex Lagrangian n-space $L_0'' \subset \mathcal{S}$ by

(B.114)
$$L_0'' = \operatorname{span}\{L_0^{(4)}, L_0^{(n-4)}\} = L_0^{(4)} \oplus L_0^{(n-4)}.$$

Clearly L_0'' is strictly separated in $\mathcal{S} = \mathcal{S}_- \oplus \mathcal{S}_+$, (see Diagram 2 as a guide), and so

(B.115)
$$\operatorname{grade} L_0'' = 0.$$

Also

(B.116) $$L_0'' \neq \overline{L_0''}, \text{ since } L_0^{(4)} \neq \overline{L_0^{(4)}}.$$

Therefore L_0'' is the required Lagrangian n-space which is strictly separated in $\mathcal{S} = \mathcal{S}_- \oplus \mathcal{S}_+$, yet which is not self-conjugate.

Thus the constructions for L_ℓ' and L_ℓ'' as in (B.98) and (B.99) are completed, and the remaining conclusions in (B.100) are immediate consequences. In particular, the Lagrangian n-space L_0'' of $\mathcal{S}$, with

$$\operatorname{grade} L_0'' = 0, \text{ yet } L_0'' \neq \overline{L_0''},$$

can be defined by n complex boundary conditions; which cannot be all real, but which can be chosen so that m are separated at the left endpoint a, and m are separated at the right endpoint b of $\mathcal{J} = [a,b]$. Furthermore, since $L_0'' \neq \overline{L_0''}$, the self-adjoint operator T on $\mathcal{D}(T) \subseteq \mathcal{D}(T_1)$, generated by $w^{-1}M_A$ and uniquely determined by L_0'', is definitely not a real operator in $\mathcal{L}^2(\mathcal{J}; w)$. □

NOTE 2. Return briefly to the general singular boundary value problem for the real quasi-differential expression $w^{-1}M_A$, with real matrix $A = \bar{A} = A^+ \in Z_n(\mathcal{J})$ of even order $n \geq 2$ and positive weight function $w \in \mathcal{L}^1_{\mathrm{loc}}(\mathcal{J})$—as in Theorem 3 above—but on a non-compact interval $\mathcal{J}$. Then the real sympletic $2d$-space $\mathcal{S}_R$ has a direct sum decomposition

$$\mathcal{S}_R = \mathcal{S}_{R-} \oplus \mathcal{S}_{R+}, \quad [\mathcal{S}_{R-} : \mathcal{S}_{R+}]_A = 0,$$

in which the real symplectic subspaces $\mathcal{S}_{R-}$ and $\mathcal{S}_{R+}$ can have different dimensions $2\Delta_-$ and $2\Delta_+$, respectively. Here

$$\dim \mathcal{S}_R = 2\Delta = 2\Delta_- + 2\Delta_+,$$

where the deficiency index $d = \Delta \leq n$. If $\Delta_- = \Delta_+$, then the analysis and the results are just as in Theorem 3 above, with $\Delta_\pm$ playing the role of m. Otherwise, we assume $\Delta_+ \neq \Delta_-$, and for definiteness $\Delta_+ > \Delta_- \geq 0$.

Nevertheless, the constructions for complex Lagrangian Δ-spaces L_ℓ, with grade $L_\ell = \ell$ within the complexification

$$\mathcal{S} = \mathcal{S}_- \oplus \mathcal{S}_+, \quad [\mathcal{S}_- : \mathcal{S}_+]_A = 0,$$

can be achieved for $\ell = 0, 1, 2, \ldots, \min\{\Delta_-, \Delta_+\}$, according to Theorem 4 of Section III.1.

However, we now can modify the arguments of our preceding Theorem 3 (say, with $\Delta_+ > \Delta_- \geq 2$) to obtain the corresponding Lagrangian Δ-spaces L_ℓ' and L_ℓ'', real and non-real respectively,

$$L_\ell' = \overline{L_\ell'} \quad \text{but} \quad L_\ell'' \neq \overline{L_\ell''},$$

for every grade $\ell = 0, 1, 2, \ldots, \Delta_-$. [Note that if $\Delta_+ \geq 2$ but $\Delta_- = 0$ or 1, then both real and non-real Lagrangian Δ-spaces exist with all grades $\leq \Delta_-$; and the special properties of the special cases $\Delta_+ = 1$ and $\Delta_- = 0$ or 1 have already been treated in earlier examples.]

We extend the techniques of Theorem 3 by writing

$$\mathcal{S}_+ = \mathcal{S}_+^{(2\Delta_-)} \oplus \mathcal{S}_+^{2(\Delta_+ - \Delta_-)},$$

where the complex symplectic subspaces have the indicated dimensions $(2\Delta_-)$ and $2(\Delta_+ - \Delta_-)$, and each has excess zero. Then apply the constructions in Theorem 3 to the complex symplectic space $\mathcal{S}_- \oplus \mathcal{S}_+^{(2\Delta_-)}$ to obtain complex Lagrangian spaces $L_\ell^{(2\Delta_-)}$ of dimension $(2\Delta_-)$, and with grade of $\ell = 0, 1, 2, \ldots, \Delta_-$ in $\mathcal{S}_- \oplus \mathcal{S}_+^{(2\Delta_-)}$. Afterward, take the direct sum of such a Lagrangian subspace $L_\ell^{(2\Delta_-)}$ with a Lagrangian subspace of $\mathcal{S}_+^{2(\Delta_+ - \Delta_-)}$ having the dimension $(\Delta_+ - \Delta)$. The resulting Lagrangian space $L_\ell \subset \mathcal{S}$ has dimension $\Delta = (2\Delta_-) + (\Delta_+ - \Delta_-) = \Delta_- + \Delta_+$, and moreover

$$\dim L_\ell \cap \mathcal{S}_- = \ell, \qquad \dim L_\ell \cap \mathcal{S}_+ = \ell + (\Delta_+ - \Delta_-)$$

so

$$\operatorname{grade} L_\ell = \Delta_- - \ell, \qquad \text{for } \ell = 0, 1, 2, \ldots, \Delta_-.$$

We can select $L_\ell^{(2\Delta_-)}$ so as to produce either the real L'_ℓ or the non-real L''_ℓ, as required, and the corresponding conclusions of Theorem 3 also hold in this singular boundary value problem.

Based on the treatment of the real boundary value problem for real quasi-differential operators $w^{-1}M_A$, with $A = \bar{A} = A^+ \in Z_n(\mathcal{J})$ for $n = 2m$ even, as is described in the regular case in Theorems 2 and 3 above, and in the singular case in the subsequent Note 2, we next tabulate all the kinds of boundary conditions that arise for real self-adjoint operators. In this analysis we follow the earlier Theorem 7 of Section III.2, and Theorem 2 of Section IV for the descriptions and calculations for minimally coupled bases of the appropriate (real or complex) Lagrangian spaces. In particular the table of symplectic invariants in Theorem 1 of Section V, with the subsequent remarks on the real cases, and the exhaustive results in the Examples 2 and 3 of Section V, all provide a complete guide for the following examples below.

Example 2. Consider a real quasi-differential expression $w^{-1}M_A$ of even order $n = 2m \geq 2$, with both positive weight w and $A = \bar{A} = A^+ \in Z_n(\mathcal{J})$ in $\mathcal{L}^1(\mathcal{J})$ on the compact interval $\mathcal{J} = [a, b]$, so as to specify a regular boundary value problem. Then, just as in Section IV, the endpoint real symplectic $2n$-space $\mathcal{S}_R$ has the decomposition.

$$\mathcal{S}_R = \mathcal{S}_{R-} \oplus \mathcal{S}_{R+} \quad \text{with} \quad [\mathcal{S}_{R-} : \mathcal{S}_{R+}]_A = 0,$$

and the corresponding symplectic invariants are

$$\begin{aligned} &\dim \mathcal{S}_R = 2n, \qquad Ex = 0, \qquad \Delta = d = n \\ &\dim \mathcal{S}_{R\pm} = n, \qquad Ex_\pm = 0, \qquad \Delta_\pm = m. \end{aligned}$$

Here $d = d^\pm$, and $Ex = 0$, $Ex_\pm = 0$, because $w^{-1}M_A$ is a real quasi-differential expression.

We use these invariants to describe all possible kinds of minimally coupled boundary conditions (BC) that can define real Lagrangian n-spaces L in $\mathcal{S}_R$, and hence self-adjoint real operators T_R on $\mathcal{D}(T_R)$, as generated by $w^{-1}M_A$ on $\mathcal{L}^2(\mathcal{J}; w)$. We follow the format of Example 1 in Section IV, and comment that each such case for prescribed n and grade L does actually arise for some existent real quasi-differential expression $w^{-1}M_A$ on $\mathcal{J} = [a, b]$; and we tabulate these for $n \leq 6$.

$n = 2$ so $\Delta_\pm = m = 1$, and $\dim L \cap \mathcal{S}_{R\pm} = m-$ grade L.

grade $L = 0$ so Nec-coupling $L = 2$ grade $L = 0$.

1 separated at left, 1 separated at right, 0 coupled BC.

grade $L = 1$ so Nec-coupling $L = 2$.

0 separated at left, 0 separated at right, 2 coupled BC.

$n = 4$ so $\Delta_\pm = m = 2$.

grade $L = 0$: 2 left, 2 right, 0 coupled BC.
grade $L = 1$: 1 left, 1 right, 2 coupled BC.
grade $L = 2$: 0 left, 0 right, 4 coupled BC.

$n = 6$ so $\Delta_\pm = m = 3$.

grade $L = 0$: 3 left, 3 right, 0 coupled BC.
grade $L = 1$: 2 left, 2 right, 2 coupled BC.
grade $L = 2$: 1 left, 1 right, 4 coupled BC.
grade $L = 3$: 0 left, 0 right, 6 coupled BC.

EXAMPLE 3. Consider a real quasi-differential expression $w^{-1}M_A$ of even order $n = 2m \geq 2$, with positive weight w and $A = \bar{A} = A^+ \in Z_n(\mathcal{I})$ both in $\mathcal{L}^1_{\text{loc}}(\mathcal{I})$ on the interval $\mathcal{I} = (a, b)$, so as to specify a singular boundary value problem in the sense of Section V. Then the endpoint real symplectic $2d$-space $\mathcal{S}_R$ has the decomposition

$$\mathcal{S}_R = \mathcal{S}_{R-} \oplus \mathcal{S}_{R+} \quad \text{with} \quad [\mathcal{S}_{R-} : \mathcal{S}_{R+}]_A = 0,$$

and the corresponding symplectic invariants are

$$\begin{array}{lll} \dim \mathcal{S}_R = 2d, & Ex = 0, & \Delta = d \leq n \\ \dim \mathcal{S}_{R-} = 2d_\ell - n, & Ex_- = 0, & \Delta_- = d_\ell - m \\ \dim \mathcal{S}_{R+} = 2d_r - n, & Ex_+ = 0, & \Delta_+ = d_r - m \end{array}$$

(See Proposition 5 above, and apply the real version of Theorem 7 in Section III.2). Here $d = d^\pm$, $d_\ell = d_\ell^\pm$, $d_r = d_r^\pm$, and $m \leq d_\ell, d_r \leq n$, with $Ex = Ex_- = Ex_+ = 0$, according to the reality assumptions.

Now use $d_\ell + d_r = d + n$ for $d = 1, 2, \ldots, n$ (omit the trivial case $d = 0$ where there are no boundary conditions) and we describe all possible kinds of minimally coupled boundary conditions (BC) that can define real Lagrangian d-spaces $L \subset \mathcal{S}_R$, and hence self-adjoint real operators T_R on $\mathcal{D}(T_R)$, as generated by $w^{-1}M_A$ in $\mathcal{L}^2(\mathcal{I}; w)$. We follow the format of Example 2 of Section V, and comment that each possible pair of deficiency indices $\{d_\ell, d_r\}$ (satisfying the Weyl-Kodaira conditions above) does actually arise from a real quasi-differential expression $w^{-1}M_A$ on $\mathcal{I} = (a, b)$, (see [**AG**,App. 2] and [**GZ**]); and we tabulate these cases for $n \leq 6$ in terms of n, d, $\{d_\ell, d_r\}$, $\Delta_\pm$ and $\dim L \cap \mathcal{S}_{R\pm} = \Delta_\pm -$ grade L.

$n = 2$ so $m = 1$, $d = 1$, $d_\ell + d_r = 3, \Delta_- = d_\ell - 1, \Delta_+ = d_r - 1$.

$\{d_\ell, d_r\} = \{1, 2\}$, so $\Delta_- = 0$, $\Delta_+ = 1$
grade $L = 0$: 0 left, 1 right, 0 coupled BC

$\{d_\ell, d_r\} = \{2, 1\}$, so $\Delta_- = 1$, $\Delta_+ = 0$
grade $L = 0$: 1 left, 0 right, 0 coupled BC

$n = 2$ so $m = 1$, $d = 2$, $d_\ell + d_r = 4, \Delta_- = d_\ell - 1, \Delta_+ = d_r - 1$.
$\{d_\ell, d_r\} = \{2, 2\}$, so $\Delta_- = 1$, $\Delta_+ = 1$

grade $L = 0$: 1 left, 1 right, 0 coupled BC
grade $L = 1$: 0 left, 0 right, 2 coupled BC.

$\underline{n = 4 \text{ so } m = 2,\ d = 1}$, $d_\ell + d_r = 5, \Delta_- = d_\ell - 2, \Delta_+ = d_r - 2$.
$\{d_\ell, d_r\} = \{2, 3\}$, so $\Delta_- = 0$, $\Delta_+ = 1$
grade $L = 0$: 0 left, 1 right, 0 coupled BC.

$\{d_\ell, d_r\} = \{3, 2\}$, so $\Delta_- = 1$, $\Delta_+ = 0$
grade $L = 0$: 1 left, 0 right, 0 coupled BC.

$n = 4$ so $m = 2$, $d = 2$, $d_\ell + d_r = 6, \Delta_- = d_\ell - 2, \Delta_+ = d_r - 2$.
$\{d_\ell, d_r\} = \{2, 4\}$, so $\Delta_- = 0$, $\Delta_+ = 2$
grade $L = 0$: 0 left, 2 right, 0 coupled BC.

$\{d_\ell, d_r\} = \{3, 3\}$, so $\Delta_- = 1$, $\Delta_+ = 1$
grade $L = 0$: 1 left, 1 right, 0 coupled BC.
grade $L = 1$: 0 left, 0 right, 2 coupled BC.

$\{d_\ell, d_r\} = \{4, 2\}$, so $\Delta_- = 2$, $\Delta_+ = 0$
grade $L = 0$: 2 left, 0 right, 0 coupled BC.

$n = 4$ so $m = 2$, $d = 3$, $d_\ell + d_r = 7, \Delta_- = d_\ell - 2, \Delta_+ = d_r - 2$.
$\{d_\ell, d_r\} = \{3, 4\}$, so $\Delta_- = 1$, $\Delta_+ = 2$
grade $L = 0$: 1 left, 2 right, 0 coupled BC.
grade $L = 1$: 0 left, 1 right, 2 coupled BC.

$\{d_\ell, d_r\} = \{4, 3\}$, so $\Delta_- = 2$, $\Delta_+ = 1$
grade $L = 0$: 2 left, 1 right, 0 coupled BC.
grade $L = 1$: 1 left, 0 right, 2 coupled BC.

$n = 4$ so $m = 2$, $d = 4$, $d_\ell + d_r = 8, \Delta_- = d_\ell - 2, \Delta_+ = d_r - 2$.
$\{d_\ell, d_r\} = \{4, 4\}$, so $\Delta_- = 2$, $\Delta_+ = 2$
grade $L = 0$: 2 left, 2 right, 0 coupled BC.
grade $L = 1$: 1 left, 1 right, 2 coupled BC.
grade $L = 2$: 0 left, 0 right, 4 coupled BC.

$\underline{n = 6 \text{ so } m = 3,\ d = 1}$, $d_\ell + d_r = 7, \Delta_- = d_\ell - 3, \Delta_+ = d_r - 3$.
$\{d_\ell, d_r\} = \{3, 4\}$, so $\Delta_- = 0$, $\Delta_+ = 1$
grade $L = 0$: 0 left, 1 right, 0 coupled BC.

$\{d_\ell, d_r\} = \{4, 3\}$, so $\Delta_- = 1$, $\Delta_+ = 0$
grade $L = 0$: 1 left, 0 right, 0 coupled BC.

$n = 6$ so $m = 3$, $d = 2$, $d_\ell + d_r = 8, \Delta_- = d_\ell - 3, \Delta_+ = d_r - 3$.
$\{d_\ell, d_r\} = \{3, 5\}$, so $\Delta_- = 0$, $\Delta_+ = 2$
grade $L = 0$: 0 left, 2 right, 0 coupled BC.

$\{d_\ell, d_r\} = \{4, 4\}$, so $\Delta_- = 1$, $\Delta_+ = 1$
grade $L = 0$: 1 left, 1 right, 0 coupled BC.
grade $L = 1$: 0 left, 0 right, 2 coupled BC.

$\{d_\ell, d_r\} = \{5, 3\}$, so $\Delta_- = 2$, $\Delta_+ = 0$
grade $L = 0$: 2 left, 0 right, 0 coupled BC.

$n = 6$ so $m = 3$, $d = 3$, $d_\ell + d_r = 9, \Delta_- = d_\ell - 3, \Delta_+ = d_r - 3$.
$\{d_\ell, d_r\} = \{3, 6\}$, so $\Delta_- = 0$, $\Delta_+ = 3$
grade $L = 0$: 0 left, 3 right, 0 coupled BC.

$\{d_\ell, d_r\} = \{4, 5\}$, so $\Delta_- = 1$, $\Delta_+ = 2$
grade $L = 0$: 1 left, 2 right, 0 coupled BC.
grade $L = 1$: 0 left, 1 right, 2 coupled BC.

$\{d_\ell, d_r\} = \{5, 4\}$, so $\Delta_- = 2$, $\Delta_+ = 1$
grade $L = 0$: 2 left, 1 right, 0 coupled BC.
grade $L = 1$: 1 left, 0 right, 2 coupled BC.

$\{d_\ell, d_r\} = \{6, 3\}$, so $\Delta_- = 3$, $\Delta_+ = 0$
grade $L = 0$: 3 left, 0 right, 0 coupled BC.

$n = 6$ so $m = 3$, $d = 4$, $d_\ell + d_r = 10, \Delta_- = d_\ell - 3, \Delta_+ = d_r - 3$.
$\{d_\ell, d_r\} = \{4, 6\}$, so $\Delta_- = 1$, $\Delta_+ = 3$
grade $L = 0$: 1 left, 3 right, 0 coupled BC.
grade $L = 1$: 0 left, 2 right, 2 coupled BC.

$\{d_\ell, d_r\} = \{5, 5\}$, so $\Delta_- = 2$, $\Delta_+ = 2$
grade $L = 0$: 2 left, 2 right, 0 coupled BC.
grade $L = 1$: 1 left, 1 right, 2 coupled BC.
grade $L = 2$: 0 left, 0 right, 4 coupled BC.

$\{d_\ell, d_r\} = \{6, 4\}$, so $\Delta_- = 3$, $\Delta_+ = 1$
grade $L = 0$: 3 left, 1 right, 0 coupled BC.
grade $L = 1$: 2 left, 0 right, 2 coupled BC.

$n = 6$ so $m = 3$, $d = 5$, $d_\ell + d_r = 11, \Delta_- = d_\ell - 3, \Delta_+ = d_r - 3$.
$\{d_\ell, d_r\} = \{5, 6\}$, so $\Delta_- = 2$, $\Delta_+ = 3$
grade $L = 0$: 2 left, 3 right, 0 coupled BC.
grade $L = 1$: 1 left, 2 right, 2 coupled BC.
grade $L = 2$: 0 left, 1 right, 4 coupled BC.

$\{d_\ell, d_r\} = \{6, 5\}$, so $\Delta_- = 3$, $\Delta_+ = 2$
grade $L = 0$: 3 left, 2 right, 0 coupled BC.
grade $L = 1$: 2 left, 1 right, 2 coupled BC.
grade $L = 2$: 1 left, 0 right, 4 coupled BC.

$n = 6$ so $m = 3$, $d = 6$, $d_\ell + d_r = 12, \Delta_- = d_\ell - 3, \Delta_+ = d_r - 3$.
$\{d_\ell, d_r\} = \{6, 6\}$, so $\Delta_- = 3$, $\Delta_+ = 3$
grade $L = 0$: 3 left, 3 right, 0 coupled BC.
grade $L = 1$: 2 left, 2 right, 2 coupled BC.
grade $L = 2$: 1 left, 1 right, 4 coupled BC.
grade $L = 3$: 0 left, 0 right, 6 coupled BC.

EXAMPLE 4. As a modification of the analysis and classification scheme of Example 3 above, we next consider the real quasi-differential expression $w^{-1}M_A$ of even order $n = 2m \geq 2$, with positive weight w and $A = \bar{A} = A^+ \in Z_n(\mathcal{I})$, but now in $\mathcal{L}^1_{\text{loc}}(\mathcal{I})$ where $\mathcal{I} = [a, b)$, so the boundary value problem is regular at the left endpoint a of $\mathcal{I}$. Then, in the notation of Example 3,

$$\begin{aligned}
&\dim \mathcal{S}_R = 2d, && Ex = 0, && \Delta = d \leq n \\
&\dim \mathcal{S}_{R-} = n, && Ex_- = 0, && \Delta_- = \quad m \\
&\dim \mathcal{S}_{R+} = 2d - n, && Ex_+ = 0, && \Delta_+ = d - m
\end{aligned}$$

since $d_\ell = d_\ell^\pm = n$, $d_r = d_r^\pm = d$, and $m \leq d \leq n$. Again we tabulate all cases arising (algebraically) for $n \leq 6$, and each such possibility can actually arise from some real quasi-differential expression $w^{-1}M_A$ on $\mathfrak{I} = [a, b)$.

$\underline{n = 2 \text{ so } m = 1}$, $d = 1$, $\Delta_- = 1, \Delta_+ = d - 1 = 0$.
grade $L = 0$: 1 left, 0 right, 0 coupled BC.

$n = 2$, $d = 2$, $\Delta_- = 1, \Delta_+ = 1$
grade $L = 0$: 1 left, 1 right, 0 coupled BC.
grade $L = 1$: 0 left, 0 right, 2 coupled BC.

$\underline{n = 4 \text{ so } m = 2}$, $d = 2$, $\Delta_- = 2, \Delta_+ = 0$.
grade $L = 0$: 2 left, 0 right, 0 coupled BC.

$n = 4$, $d = 3$, $\Delta_- = 2, \Delta_+ = 1$
grade $L = 0$: 2 left, 1 right, 0 coupled BC.
grade $L = 1$: 1 left, 0 right, 2 coupled BC.

$n = 4$ so $d = 4$, $\Delta_- = 2, \Delta_+ = 2$.
grade $L = 0$: 2 left, 2 right, 0 coupled BC.
grade $L = 1$: 1 left, 1 right, 2 coupled BC.
grade $L = 2$: 0 left, 0 right, 4 coupled BC.

$\underline{n = 6 \text{ so } m = 3}$ $d = 3$, $\Delta_- = 3$, $\Delta_+ = 0$
grade $L = 0$: 3 left, 0 right, 0 coupled BC

$n = 6$ so $d = 4$, $\Delta_- = 3, \Delta_+ = 1$.
grade $L = 0$: 3 left, 1 right, 0 coupled BC.
grade $L = 1$: 2 left, 0 right, 2 coupled BC.

$n = 6$ so $d = 5$, $\Delta_- = 3, \Delta_+ = 2$.
grade $L = 0$: 3 left, 2 right, 0 coupled BC.
grade $L = 1$: 2 left, 1 right, 2 coupled BC.
grade $L = 2$: 1 left, 0 right, 4 coupled BC.

$n = 6$, $d = 6$, $\Delta_- = 3$, $\Delta_+ = 3$
grade $L = 0$: 3 left, 3 right, 0 coupled BC
grade $L = 1$: 2 left, 2 right, 2 coupled BC
grade $L = 2$: 1 left, 1 right, 4 coupled BC
grade $L = 3$: 0 left, 0 right, 6 coupled BC

As the final topic in this Appendix B, we indicate briefly an approach to the real boundary value problem through the methods of global differential topology—compare the parallel treatment for the complex case at the end of Section III.2 above.

The set of all (non-oriented) real linear k-spaces through the origin in $\mathbb{R}^{2n}$ is the real Grassmannian, denoted by $\mathrm{Grass}_R(k, 2n)$. This real Grassmannian can thus be constructed as a homogeneous space whereon the real orthogonal group $O(2n)$ acts transitively so that

$$\mathrm{Grass}_R(k, 2n) = O(2n)/[O(k) \times O(2n - k)], \tag{B.117}$$

since the stability subsgroup $O(k) \times O(2n - k) \subset O(2n)$ consists of all real matrices $\mathrm{diag}\{F, G\}$ with $F \in O(k)$ and $G = O(2n-k)$. In this way we define the topology on $\mathrm{Grass}_R(k, 2n)$ as that of a homogeneous space of $O(2n)$ or, equally well, by means of convenient coordinate charts, say with Plücker coordinates. Then $\mathrm{Grass}_R(k, 2n)$ is a (connected) compact real-analytic manifold [**ST**]. In the case of greatest interest to us, when $k = n$,

$$\dim \mathrm{Grass}_R(n, 2n) = n^2, \tag{B.118}$$

and this n^2-manifold is not simply connected but has a fundamental group $\mathbb{Z}_2$, see [**ST**]. For example, when $n = 2$ we have

$$\dim \mathrm{Grass}_R(2, 4) = 4.$$

Next we impose the (unique) real symplectic structure on $\mathbb{R}^{2n}$, and define the real Lagrangian Grassmannian

$$\mathrm{Lag}_R(n, 2n) \subset \mathrm{Grass}_R(n, 2n) \tag{B.119}$$

which consists of all real Lagrangian n-spaces in $\mathbb{R}^{2n}$. It can be shown that $\mathrm{Lag}_R(n, 2n)$ is a (connected) compact real-analytic submanifold of $\mathrm{Grass}_R(n, 2n)$. In fact,

$$\mathrm{Lag}_R(n, 2n) = U(n)/O(n), \tag{B.120}$$

and so it has dimension $n(n+1)/2$, and has the fundamental group $\mathbb{Z}$, see [**MS**].

We shall not pursue further these introductory remarks on the global geometry of the real Lagrangian Grassmannians, but merely comment that this study has applications to parametrized families of self-adjoint boundary conditions for any specified formally self-adjoint real quasi-differential expression of even order $n = 2m$. We illustrate one simple application for the regular real Sturm-Liouville problem, following the line of development in Proposition 6 above.

PROPOSITION 7. *Consider the Sturm-Liouville real quasi-differential operator of order $n = 2m = 2$,*

$$M_A[y] = -y_A^{[2]} = -(py')' + qy,$$

where

$$A = \bar{A} = A^+ = \begin{pmatrix} 0 & p^{-1} \\ q & 0 \end{pmatrix} \in Z_2(\mathfrak{I}),$$

for smooth real coefficients p, q with $p(x) \neq 0$ on the compact interval $\mathfrak{I} = [a, b]$. Then the quasi-differential expression $w^{-1}M_A$, for a given positive weight function $w \in \mathcal{L}^1(\mathfrak{I})$, specifies the endpoint real symplectic 4-space (as in Proposition 6):

$$\mathcal{S}_R = \mathcal{S}_{R-} \oplus \mathcal{S}_{R+} \approx \mathbb{R}^4.$$

Consider the corresponding real Lagrangian Grassmannian

$$\mathrm{Lag}_R(2, 4) = U(2)/O(2),$$

which is a compact 3-manifold. Then the set of all strictly separated real Lagrangian 2-spaces in $\mathcal{S}_R \approx \mathbb{R}^4$ constitutes a 2-dimensional submanifold of $\mathrm{Lag}_R(2, 4)$*, which is topologically a torus surface T^2. The set of all totally coupled real Lagrangian 2-spaces in $\mathcal{S}_R$ constitutes an open-dense submanifold of* $\mathrm{Lag}_R(2, 4)$*, namely, the complement of* $T^2 \subset \mathrm{Lag}_R(2, 4)$.

PROOF. Let L_0 be a real Lagrangian 2-space with grade $L_0 = 0$ in $\mathcal{S}_R$. That is, L_0 is strictly separated so $\dim L_0 \cap \mathcal{S}_{R\pm} = 1$. Then there exist linearly independent vectors $v_- \in L_0 \cap \mathcal{S}_{R_-}$ and $v_+ \in L_0 \cap \mathcal{S}_{R+}$ which form a basis for L_0 in $\mathcal{S}_R$. That is, v_- and v_+ (or equally well, their 1-dimensional subspaces ℓ_- and ℓ_+, respectively) span L_0.

On the other hand, each such pair of lines ℓ_- and ℓ_+, through the origin in the 2-planes $\mathcal{S}_{R-}$ and $\mathcal{S}_{R+}$, respectively, determine exactly one such 2-space which is necessarily a real Lagrangian 2-space that is strictly separated in $\mathcal{S}_R$. But, trivially, the set of all lines through the origin in $\mathbb{R}^2$ is homeomorphic to a circle S^1. Hence the set of all strictly separated real Lagrangian 2-spaces in $\mathcal{S}_R$ is topologically the product of two circles $S^1 \times S^1$, which is a torus surface T^2 in $\mathrm{Lag}_R(2,4)$.

Since every real Lagrangian 2-space in $\mathcal{S}_R$ has a coupling grade of either 0 or 1, the set of all totally coupled Lagrangian 2-spaces is the complement of the set of all strictly separated Lagrangians within $\mathrm{Lag}_R(2,4)$. As a consequence, the set of all totally coupled real Lagrangian 2-spaces in $\mathcal{S}_R$ fills the complement of $T^2 \subset \mathrm{Lag}_R(2,4)$, which is an open-dense 3-manifold in $\mathrm{Lag}_R(2,4)$. □

In the sense of Proposition 7 the totally coupled real Lagrangian 2-spaces in $\mathcal{S}_R$, for the regular real Sturm-Liouville problem, describe the "generic case", and the strictly separated real Lagrangian 2-spaces are non-generic.

References

[AG] Akhiezer, N. I. and Glazman, I. M., *Theory of linear operators in Hilbert space:* volumes I and II, Pitman and Scottish Academic Press, London, 1981; translated from the third Russian edition of 1977

[AM] Abraham, R. and Marsden, J. E., *Foundations of mechanics*, Benjamin/Cummings Publ. Co., Reading, Mass., 1978.

[BM] Birkhoff, G. and Mac Lane, S., *A survey of modern algebra*, Macmillan Co., New York, 1947.

[CL] Coddington, E. A. and Levinson, N., *Theory of ordinary differential equations*, McGraw-Hill, New York, 1955.

[DS] Dunford, N. and Schwartz, J. T., *Linear operators:* Part II, Wiley, New York, 1963.

[EI] Everitt, W. N., *On the deficiency index problem for ordinary differential operators 1910–1977*, Proceedings of *The 1977 Uppsala International Conference: Differential Equations*, 62–81, Published by the University of Uppsala, Sweden, 1977, distributed by Almquist and Wiksell International, Stockholm, Sweden.

[EV] ———, *Linear ordinary quasi-differential expressions*, Lecture notes for *The Fourth International Symposium on Differential Equations and Differential Geometry*, Beijing, Peoples' Republic of China, 1–28. (Department of Mathematics, University of Peking, Peoples' Republiic of China; 1986.).

[EM] Everitt, W. N. and Markus, L., *Controllability of* $[r]$*-matrix quasi-differential equations*, Journal of Differential Equations **89** (1991), 95–109.

[EM 1] ———, *The Glazman-Krein-Naimark theorem for ordinary differential operators*, New Results in Operator Theory and its Applications, *Operator Theory Adv. Appl.* **98** (1997), Birkhäuser, Basel, 118–130.

[EM 2] ———, *Complex symplectic geometry with applications to ordinary differential operators*, to appear in TAMS.

[ER] Everitt, W. N. and Race, D., *Some remarks on linear ordinary quasi-differential expressions*, Proc. London Math. Soc. (3) **54** (1987), 300–320.

[EZ] Everitt, W. N. and Zettl, A., *Differential operators generated by a countable number of quasi-differential expressions on the real line*, Proc. London Math. Soc. (3) **64** (1992), 524–544.

[FR] Frentzen, H., *Equivalence, adjoints, and symmetry of quasi-differential expressions with matrix-valued coefficients and polynomials in them*, Proc. Royal Soc. Edinburgh (A) **92** (1982), 123–146.

[GI] Gilbert, R. C., *The deficiency index of a symmetric ordinary differential operator with complex coefficients*, J. Differential Equations **25** (1977), 425–459.

[GL] ———, *A class of symmetric ordinary differential operators whose deficiency numbers differ by an integer*, Proc. Royal Soc. Edinburgh (A) **81** (1978), 57–70.

[GZ] Glazman, I. M., *On the theory of singular differential operators*, Uspehi Math. Nauk **40** (1950), 102–135; English translation in Amer. Math. Soc. Translations (1) **4** (1962), 331–372.

[HA] Halperin, I., *Closures and adjoints of linear differential operators*, Ann. of Math. **38** (1937), 880–919.

[KB] Kogan, V.I. and Rofe-Beketov, F. S., *On square-integrable solutions of symmetric systems of differential equations of arbitrary order*, Proc. Royal Soc. Edinburgh (A) **74** (1974/75), 5–40.

[KR] ———, *On the question of the deficiency indices of differential operators with complex coefficients*, Proc. Royal Soc. Edinburgh (A) **72** (1973/74), 281–298; Translated from the Russian Mat. Fiz. Funk. Anal **2** (1971), 45–60.

[MA] Markus, L., *Hamiltonian dynamics and symplectic manifolds*, Lecture Notes, University of Minnesota, University of Minnesota Bookstores, 1973, 1–256.

[MH] Meyer, K.R. and Hall, G.R., *Introduction to Hamiltonian dynamical systems and the n-body problem*, Springer NY, 1992.

[MS] McDuff, D. and Salamon, D., *Introduction to symplectic topology*, Oxford Univ. Press, 1995.

[NA] Naimark, M. A., *Linear differential operators:* Part II, Ungar, New York, 1969; translated from the second Russian edition, 1966

[RO] Robbin, J. W., *Symplectic mechanics*, Symp. on Global Analysis and its applications, Internat. Centre Th. Physics, Trieste, Proc. Vol. III, 97–120 at Internat. Atomic Energy Agency, Vienna 1974. See M. R. Vol 57, #2033.

[SH] Shin, D., *Existence theorems for quasi-differential equations of order n*, Doklad. Akad. Nauk SSSR **18** (1938), 515–518.

[SI] ———, *On quasi-differential operators in Hilbert space*, Doklad. Akad. SSSR **18** (1938), 523–526.

[SN] ———, *On the solutions of a linear quasi-differential equation of order n*, Mat. Sb **7** (1940), 479–532.

[ST] Steenrod, N., *The topology of fibre bundles*, Princeton University Press, Princeton, 1951.

[TI] Titchmarsh, E. C., *Eigenfunction expansions:* Part I, (Oxford University Press; second edition; 1962).

[WE] Weidmann, J., *Linear operators in Hilbert spaces*, Springer NY, 1980.

[ZE] Zettl, A., *Formally self-adjoint quasi-differential operators*, Rocky Mountain J. Math. **5** (1975), 453–474.

Notation Index

Index

ISBN 0-8218-1080-4

9 780821 810804