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Nonlocal Bifurcations

Yu. Ilyashenko
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ABSTRACT. The book studies nonlocal bifurcations that occur on the boundary of the domain of Morse–Smale systems in the space of all dynamical systems. These bifurcations provide a series of new fascinating scenarios for the transition from simple dynamical systems to complicated ones. The main effects are: generation of hyperbolic periodic orbits, nontrivial hyperbolic invariant sets and the elements of hyperbolic theory. All the results are rigorously proved and exposed in a uniform way. The foundations of normal forms and hyperbolic theories are presented from the very first steps. The proofs are preceded by heuristic descriptions of ideas. Most of the results have never been exposed in monographs; some of them are new.

The book is addressed to graduate students in mathematics, as well as specialists in pure and applied mathematics, physics and engineering.

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Contents

Preface

The complete title of this book should be "Nonlocal bifurcations, normal forms, and elements of hyperbolic theory".

We present the modern theory of normal forms for local families of vector fields and diffeomorphisms. This presentation contains a complete list of integrable normal forms in the finitely smooth classification. This classification is the most suitable one for applications to nonlocal theory.

Hyperbolic and partial hyperbolic theory is the tool for the description of the invariant sets that occur under nonlocal bifurcations.

We restrict ourselves to the study of the bifurcations that occur on the boundary of the set of Morse–Smale systems and are generated by the loss of hyperbolicity of singular points and periodic orbits. Even this moderate goal leads to a variety of effects, which are only partly investigated up to now. A very rich domain of study is related to the homoclinic tangency of stable and unstable manifolds of singular points and periodic orbits. This subject, discussed in [**PT**], is beyond the scope of this book.

The most celebrated nonlocal bifurcations are those of the homoclinic orbit of a saddle. Bifurcations of this kind in space, both those that generate one periodic orbit and an arbitrary number of Smale horseshoes, are described in many books. The first occur at the boundary of the Morse–Smale set; the others are distant from this boundary. The general case of bifurcations of homoclinic orbits of saddles in spaces of arbitrary dimension was not described in detail in previous monographs on the same subject. All these results are presented below.

Less celebrated are bifurcations of homoclinic orbits of saddlenode singular points. One homoclinic curve generates one periodic orbit. But without increasing the rate of degeneracy, several homoclinic curves of the same point may occur. Their bifurcation gives rise to a nontrivial hyperbolic set: an Ω-explosion takes place.

The most complicated are the bifurcations of homoclinic surfaces of a saddle-node periodic orbit. It is not difficult to imagine a semistable cycle in a 2-torus filled by homoclinic orbits of this cycle. A homoclinic surface of this type may occur in a space of arbitrary dimension. But much more complicated homoclinic surfaces may occur as well. There may be a Klein bottle instead of a torus. Several homoclinic surfaces of the same periodic orbit may occur simultaneously. Their bifurcations lead to a new class of dynamical systems, whose investigation is now at the very beginning. The homoclinic surfaces may be topologically complicated. The so-called twisted homoclinic surfaces may occur in a higher-dimensional space (see Figure 1.15 below). The corresponding bifurcation gives rise to a hyperbolic attractor of solenoidal type.

Some new results related to bifurcations of homoclinic surfaces described above are presented in this book. Namely,

• smooth homoclinic tori and Klein bottles preserved under small perturbations;

• an invariant set with a random dynamical system on it occurring under the bifurcation of several homoclinic surfaces;

• a strange attractor generated under the bifurcation of the twisted homoclinic surface (Shilnikov and Turaev, 1995; the first complete proof was given by the first author during the work on this book).

The theory of normal forms drastically simplifies the study of nonlocal bifurcations. It provides integrable normal forms not only for the unperturbed equation near the equilibrium point, but for the perturbation as well. Therefore the map of the cross-sections transversal to invariant manifolds of the singular point along the orbits of the vector field may be explicitly calculated. Thus simple formulas replace the delicate estimates of the previous investigations.

There are two types of applications of these normal forms to nonlocal bifurcations.

The first is related to planar bifurcations. Here there are two directions of study as well: families with few and with many parameters.

Families with a small number of parameters may be investigated in full detail. The classical results of Andronov and his school are related to the nonlocal bifurcations in planar one-parameter families. The theory of normal forms transforms the proofs of these results into simple exercises. The general study of two- and three-parameter families in the plane was suggested by Arnold in 1985. At the beginning of the nineties Kotova collected the "zoo" of all polycycles that may occur in generic two- and three-parameter families. The cyclicity of these polycycles was investigated in a number of papers by Dumortier, Grozovskiĭ, Kotova–Stanzo, Morsalami, Mourtada, Roussarie, Rousseau, Sotomayor and others. The concluding paper was recently published by Trifonov.

The study of many-parameter families in the plane is mainly related to the so-called Hilbert–Arnold problem: to prove that the polycycle that occurs in a generic finite-parameter family generates no more than a finite number of limit cycles, and this number depends on the number of parameters. This problem is solved for polycycles with elementary singular points as vertexes, the so-called elementary polycycles. These results are included in two collections [**I2, IYa3**], where the other sources are quoted. Recently, Kaloshin obtained an explicit estimate of the number $E(k)$ of limit cycles generated by elementary polycycles in typical k-parameter families.

The second application of normal forms to nonlocal bifurcations is the study of spatial bifurcations. A systematic study is carried out for one-parameter families. This is the subject of this book. It seems that two-parameter spatial families may be studied in detail as well. The complete study of nonlocal bifurcations in three-parameter spatial and four-parameter planar families seems to be too complicated to be ever obtained.

The theory of normal forms for local families is presented in this book from the very beginning. We describe the homotopy method, which is the most convenient tool for the local smooth theory. The results we present are in a sense complete: we give the list of smooth integrable normal forms for the simplest families; smooth classification of more complicated families has functional moduli.

Elements of the hyperbolic theory are also presented from the very beginning and lead up to the newest results. The presentation of the nonlinear Smale horseshoe map deals with classical material. On the other hand, we study parallel results for partially hyperbolic maps. Roughly speaking, these maps have stable, unstable, and central subbundles of the tangent bundle. In general, stable and unstable foliations for such maps do exist, while the center-stable and center-unstable do not. The maps we study are subject to special geometric assumptions that guarantee the existence of all the invariant foliations mentioned above. This gives rise to a new kind of dynamics: random dynamical systems appear to be subsystems of smooth ones. The investigation of these systems is beyond the scope of this book. Here we only prove their birth under the bifurcations of several homoclinic surfaces of a saddlenode periodic orbit.

We describe all the bifurcations from a uniform point of view. Namely, all the proofs are obtained by studying the interaction of the theory of normal forms and hyperbolic theory. The main subject is the global Poincaré map represented as a product of singular and regular maps. The singular map is a map of cross-sections along the orbits passing near a singular point. This map is not everywhere defined and produces an unbounded distortion. It contracts in some directions and expands in others. The regular map is once more a map of cross-sections, this time along the orbits distant from a singular point. It is well defined, and bounded together with its derivatives.

The theory of normal forms gives explicit formulas for singular maps. The genericity assumptions guarantee that the regular map does not mix the contracting and expanding directions of the singular one. For the product of these maps hyperbolicity wins: the unbounded distortion of the singular maps overcomes the influence of the smooth regular map. Thus the global Poincaré map becomes the subject of hyperbolic theory, which provides the description of the invariant sets of this map.

Our uniform approach is illustrated in Figures 4.9, 4.12, 5.12, 7.7, 7.12, 7.18.

The idea of this book arose when the survey "Bifurcation theory" by Afraimovich, Arnold, Ilyashenko, and Shilnikov, 1986, was written. In the process, Arnold said that the survey should reflect the development of bifurcation theory for at least the twenty five years to come. The present book is a partial response to this challenge. Bifurcations, like torches, shed light on the transition from simple dynamical systems to complicated ones. Complicated systems occurring under bifurcations of homoclinic surfaces of nonhyperbolic periodic orbits partially described in Chapters 5 and 6 are the subject for promising future study.

The present book develops the third chapter of the survey [**AAIS**] (description of the bifurcations themselves) and the end of the second chapter of the same survey, where normal forms for local families were listed for the first time. In 1988 Professor Zhang Zhifen from Beijin organized a seminar on nonlocal bifurcations, where the proofs missing in the survey were reconstructed. The second author was an active participant of this seminar. In 1991–93 he had a scholarship in Moscow. During this period the first draft of the book was written. The present version is the result of several rewritings produced by both authors.

The first chapter contains all the necessary definitions beginning with very elementary ones. Its goal is to present the main results about nonlocal bifurcations. They are summarized in the main table at the end of the chapter.

Chapter 2 contains two sections that are in fact outside the main content of the book. The first discusses "prevalence": the concept of genericity different from the traditional "category" notion. The revision of the genericity assumptions throughout this book from the "prevalent" point of view is a challenging problem. The second large section is devoted to the Hausdorff dimension of attractors. The results of this section are applied only once, in Chapter 6. Yet the subject seems to be of independent interest, and was therefore included. The third section contains an elementary description of the Smale horseshoe, understandable for high school students. The rest of the chapter contains a summary of the hyperbolic and normal form theories used throughout the book.

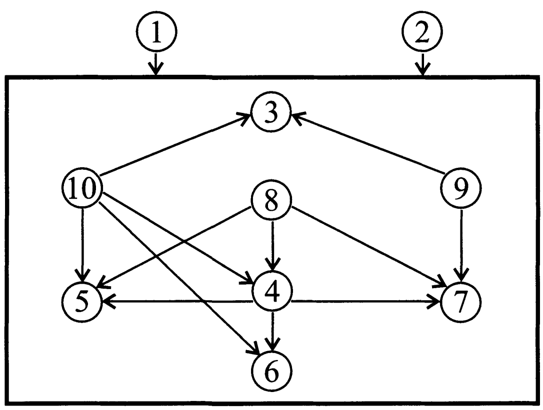
The next five chapters present nonlocal bifurcations. Chapter 3 is elementary and deals with one-parameter planar families of vector fields. The only exception is the end of the chapter, where simultaneous occurrence of several saddle connections in two-parameter families is studied.

Chapter 4 contains the main ideas of our subsequent study. It demonstrates the mechanism of the occurrence of hyperbolicity via the singular and transversality properties of the Poincaré map.

The next three chapters are the main ones. Chapter 5 studies bifurcations of smooth homoclinic surfaces of saddlenode cycles. It contains the new results mentioned above. Chapter 6 deals with nonsmooth homoclinic surfaces. It presents the first detailed exposition of the study of the strange attractor begun by Afraimovich and Shilnikov in the seventies and followed by Newhouse, Palis, Takens in the eighties. Chapter 7 describes bifurcations of the homoclinic orbit of a saddle in the space of arbitrary dimension.

The hyperbolic and partial hyperbolic theory is presented in Chapter 8. The last two chapters are devoted to normal forms for local families. These three chapters contain proofs of the results stated in the last two sections of Chapter 2.

The dependence of chapters is shown below.



The system of references is as follows. Theorems, lemmas, propositions, and formulas have double numbers $a.b$, where a is the number of the section, and b is the number of the statement. The references inside the same chapter use this numeration. The reference to item $a.b$ in Chapter A looks like $A.a.b$.

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