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Jacobi Operators and Completely Integrable Nonlinear Lattices

Gerald Teschl

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Jacobi Operators and Completely Integrable Nonlinear Lattices

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Jacobi Operators and Completely Integrable Nonlinear Lattices

Gerald Teschl

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ABSTRACT. This book is intended to serve both as an introduction and a reference to spectral and inverse spectral theory of Jacobi operators (i.e., second order symmetric difference operators) and applications of these theories to the Toda and Kac-van Moerbeke hierarchy.

Starting from second order difference equations we move on to self-adjoint operators and develop discrete Weyl-Titchmarsh-Kodaira theory, covering all classical aspects like Weyl m-functions, spectral functions, the moment problem, inverse spectral theory, and uniqueness results. Next, we investigate some more advanced topics like locating the essential, absolutely continuous, and discrete spectrum, subordinacy, oscillation theory, trace formulas, random operators, almost periodic operators, (quasi-)periodic operators, scattering theory, and spectral deformations.

Then, the Lax approach is used to introduce the Toda hierarchy and its modified counterpart, the Kac-van Moerbeke hierarchy. Uniqueness and existence theorems for the initial value problem, solutions in terms of Riemann theta functions, the inverse scattering transform, Bäcklund transformations, and soliton solutions are discussed.

Corrections and complements to this book are available from:

`http://www.mat.univie.ac.at/~gerald/ftp/book-jac/`

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To Susanne

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Preface

The name of the game

Jacobi operators appear in a variety of applications. They can be viewed as the discrete analogue of Sturm-Liouville operators and their investigation has many similarities with Sturm-Liouville theory. Spectral and inverse spectral theory for Jacobi operators plays a fundamental role in the investigation of completely integrable nonlinear lattices, in particular the Toda lattice and its modified counterpart, the Kac-van Moerbeke lattice.

Why I have written this book

Whereas numerous books about Sturm-Liouville operators have been written, only few on Jacobi operators exist. In particular, there is currently no monograph available which covers all basic topics (like spectral and inverse spectral theory, scattering theory, oscillation theory and positive solutions, (quasi-)periodic operators, spectral deformations, etc.) typically found in textbooks on Sturm-Liouville operators.

In the case of the Toda lattice a textbook by M. Toda [**230**] exists, but none of the recent advances in the theory of nonlinear lattices are covered there.

Audience and prerequisites

As audience I had researchers in mind. This book can be used to get acquainted with selected topics as well as to look up specific results. Nevertheless, no previous knowledge on difference equations is assumed and all results are derived in a self-contained manner. Hence the present book is accessible to graduate students as well. Previous experience with Sturm-Liouville operators might be helpful but is not necessary. Still, a solid working knowledge from other branches of mathematics is needed. In particular, I have assumed that the reader is familiar with the theory of (linear) self-adjoint operators in Hilbert spaces which can be found in (e.g.)

[**192**] or [**241**]. This theory is heavily used in the first part. In addition, the reader might have to review material from complex analysis (see Appendix A and B) and differential equations on Banach manifolds (second part only) if (s)he feels (s)he does not have the necessary background. However, this knowledge is mainly needed for understanding proofs rather than the results themselves.

The style of this book

The style of this monograph is strongly influenced by my personal bias. I have striven to present an intuitive approach to each subject and to include the simplest possible proof for every result. Most proofs are rather sketchy in character, so that the main idea becomes clear instead of being drowned by technicalities. Nevertheless, I have always tried to include enough information for the reader to fill in the remaining details (her)himself if desired. To help researchers, using this monograph as a reference, to quickly spot the result they are looking for, most information is found in display style formulas.

The entire treatment is supposed to be mathematically rigorous. I have tried to prove *every* statement I make and, in particular, these little obvious things, which turn out less obvious once one tries to prove them. In this respect I had Marchenko's monograph on Sturm-Liouville operators [**167**] and the one by Weidmann [**241**] on functional analysis in mind.

Literature

The first two chapters are of an introductory nature and collect some well-known "folklore", the successive more advanced chapters are a synthesis of results from research papers. In most cases I have rearranged the material, streamlined proofs, and added further facts which are not published elsewhere. All results appear without special attribution to who first obtained them but there is a section entitled "Notes on literature" in each part which contains references to the literature plus hints for additional reading. The bibliography is selective and far from being complete. It contains mainly references I (am aware of and which I) have actually used in the process of writing this book.

Terminology and notation

For the most part, the terminology used agrees with generally accepted usage. Whenever possible, I have tried to preserve original notation. Unfortunately I had to break with this policy at various points, since I have given higher priority to a consistent (and self-explaining) notation throughout the entire monograph. A glossary of notation can be found towards the end.

Contents

For convenience of the reader, I have split the material into two parts; one on Jacobi operators and one on completely integrable lattices. In particular, the second part is to a wide extent independent of the first one and anybody interested

only in completely integrable lattices can move directly to the second part (after browsing Chapter 1 to get acquainted with the notation).

Part I

Chapter 1 gives an introduction to the theory of second order difference equations and bounded Jacobi operators. All basic notations and properties are presented here. In addition, this chapter provides several easy but extremely helpful gadgets. We investigate the case of constant coefficients and, as a motivation for the reader, the infinite harmonic crystal in one dimension is discussed.

Chapter 2 establishes the pillars of spectral and inverse spectral theory for Jacobi operators. Here we develop what is known as discrete Weyl-Titchmarsh-Kodaira theory. Basic things like eigenfunction expansions, connections with the moment problem, and important properties of solutions of the Jacobi equation are shown in this chapter.

Chapter 3 considers qualitative theory of spectra. It is shown how the essential, absolutely continuous, and point spectrum of specific Jacobi operators can be located in some cases. The connection between existence of α-subordinate solutions and α-continuity of spectral measures is discussed. In addition, we investigate under which conditions the number of discrete eigenvalues is finite.

Chapter 4 covers discrete Sturm-Liouville theory. Both classical oscillation and renormalized oscillation theory are developed.

Chapter 5 gives an introduction to the theory of random Jacobi operators. Since there are monographs (e.g., [**40**]) devoted entirely to this topic, only basic results on the spectra and some applications to almost periodic operators are presented.

Chapter 6 deals with trace formulas and asymptotic expansions which play a fundamental role in inverse spectral theory. In some sense this can be viewed as an application of Krein's spectral shift theory to Jacobi operators. In particular, the tools developed here will lead to a powerful reconstruction procedure from spectral data for reflectionless (e.g., periodic) operators in Chapter 8.

Chapter 7 considers the special class of operators with periodic coefficients. This class is of particular interest in the one-dimensional crystal model and several profound results are obtained using Floquet theory. In addition, the case of impurities in one-dimensional crystals (i.e., perturbation of periodic operators) is studied.

Chapter 8 again considers a special class of Jacobi operators, namely reflectionless ones, which exhibit an algebraic structure similar to periodic operators. Moreover, this class will show up again in Chapter 10 as the stationary solutions of the Toda equations.

Chapter 9 shows how reflectionless operators with no eigenvalues (which turn out to be associated with quasi-periodic coefficients) can be expressed in terms of Riemann theta functions. These results will be used in Chapter 13 to compute explicit formulas for solutions of the Toda equations in terms of Riemann theta functions.

Chapter 10 provides a comprehensive treatment of (inverse) scattering theory for Jacobi operators with constant background. All important objects like reflection/transmission coefficients, Jost solutions and Gel'fand-Levitan-Marchenko

equations are considered. Again this applies to impurities in one-dimensional crystals. Furthermore, this chapter forms the main ingredient of the inverse scattering transform for the Toda equations.

Chapter 11 tries to deform the spectra of Jacobi operators in certain ways. We compute transformations which are isospectral and such which insert a finite number of eigenvalues. The standard transformations like single, double, or Dirichlet commutation methods are developed. These transformations can be used as powerful tools in inverse spectral theory and they allow us to compute new solutions from old solutions of the Toda and Kac-van Moerbeke equations in Chapter 14.

Part II

Chapter 12 is the first chapter on integrable lattices and introduces the Toda system as hierarchy of evolution equations associated with the Jacobi operator via the standard Lax approach. Moreover, the basic (global) existence and uniqueness theorem for solutions of the initial value problem is proven. Finally, the stationary hierarchy is investigated and the Burchnall-Chaundy polynomial computed.

Chapter 13 studies various aspects of the initial value problem. Explicit formulas in case of reflectionless (e.g., (quasi-)periodic) initial conditions are given in terms of polynomials and Riemann theta functions. Moreover, the inverse scattering transform is established.

The final Chapter 14 introduces the Kac van-Moerbeke hierarchy as modified counterpart of the Toda hierarchy. Again the Lax approach is used to establish the basic (global) existence and uniqueness theorem for solutions of the initial value problem. Finally, its connection with the Toda hierarchy via a Miura-type transformation is studied and used to compute N-soliton solutions on arbitrary background.

Appendix

Appendix A reviews the theory of Riemann surfaces as needed in this monograph. While most of us will know Riemann surfaces from a basic course on complex analysis or algebraic geometry, this will be mainly from an abstract viewpoint like in [**86**] or [**129**], respectively. Here we will need a more "computational" approach and I hope that the reader can extract this knowledge from Appendix A.

Appendix B compiles some relevant results from the theory of Herglotz functions and Borel measures. Since not everybody is familiar with them, they are included for easy reference.

Appendix C shows how a program for symbolic computation, `Mathematica`®, can be used to do some of the computations encountered during the main bulk. While I don't believe that programs for symbolic computations are an indispensable tool for doing research on Jacobi operators (or completely integrable lattices), they are at least useful for checking formulas. Further information and `Mathematica`®notebooks can be found at

`http://www.mat.univie.ac.at/~gerald/ftp/book-jac/`

respectively

`ftp://ftp.mat.univie.ac.at/pub/teschl/book-jac/`

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Finally, no book is free of errors. So if you find one, or if you have comments or suggestions, please let me know. I will make all corrections and complements available at the URL above.

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May, 1999

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Notes on literature

Chapter 12. The Toda lattice was first introduced by Toda in [**233**]. For further material see his selected papers [**232**] or his monograph [**230**].

The Toda hierarchy was first introduced in [**237**] and [**238**] (see also [**236**]). Our introduction uses a recursive approach for the standard Lax formalism ([**164**]) which was first advocated by Al'ber [**11**], Jacobi [**137**], McKean [**169**], and Mumford [**178**] (Sect. I-II a). It follows essentially the monograph [**35**] (with some extensions). An existence and uniqueness theorem for the semi infinite case seems to appear first in [**65**], the form stated here is taken from [**220**]. The Burchnall-Chaundy polynomial in the case of commuting differential expressions was first obtained in [**36**], [**37**] (cf. also [**236**]).

Further references I refer to a few standard monographs such as [**78**], [**79**] [**184**], and [**190**].

Chapter 13. Section 1 and 2 closely follow [**35**] with several additions made. The Baker-Akhiezer function, the fundamental object of our approach, goes back to the work of Baker [**23**], Burchnall and Chaundy [**36**], [**37**], and Akhiezer [**8**]. The modern approach was initiated by Its and Matveev [**136**] in connection with the Korteweg-de Vries equation and further developed into a powerful machinery by Krichever (see, e.g., [**156**]–[**158**]) and others. Gesztesy and Holden are also working on a monograph on hierarchies of soliton equations and their algebro-geometric solutions [**103**]. We refer, in particular, to the extensive treatments in [**25**], [**74**], [**75**], [**76**], [**168**], [**177**], and [**184**]. In the special context of the Toda equations we refer to [**3**], [**75**], [**76**], [**156**], [**159**], [**168**], [**170**], and [**175**].

The periodic case was first investigated by Kac and van Moerbeke [**143**], [**144**]. Further references for the periodic case are [**3**], [**33**], [**53**], [**56**], [**146**], [**168**], and [**173**]–[**175**].

In the case of the Toda lattice ($r = 0$) the formulas for the quasi periodic finite gap case in terms of Riemann theta functions seem due to [**210**]. However, some norming constants and the fact $\tilde{b} = \tilde{d}$ (see Remark 13.9) are not established there.

In case of the Toda equation on the half line a different approach using the spectral measure of H_+ is introduced and extended in [**28**] – [**32**] (see also [**15**]). It turns out that the time evolution of the spectral measure can be computed explicitly as the Freud transform of the initial measure.

The inverse scattering transform for the case $r = 0$ has been formally developed by Flaschka [**85**] (see also [**230**] and [**80**] for the case of rapidly decaying sequences) for the

Toda lattice. In addition, Flaschka also worked out the inverse procedure in the reflectionless case (i.e., $R_{\pm}(k,t) = 0$). His formulas clearly apply to the entire Toda hierarchy upon using the t dependence of the norming constants given in (13.72) and coincide with those obtained using the double commutation method (i.e., the last example in Section 14.5). All results from the fourth section are taken from [**219**] and [**224**].

For additional results where the Toda equation is driven from one end by a particle see [**63**], [**64**] and also [**67**]. For results on a continuum limit of the Toda lattice see [**59**]. For connections of the Riemann-Hilbert problem and the inverse scattering method and applications to the Toda lattice see [**61**] and [**62**]. In addition, there are also very nice lecture notes by Deift [**57**] which give an introduction to the Riemann-Hilbert problem and its applications.

Chapter 14. The Kac-van Moerbeke equation has been first introduced in [**142**]. A connection between the Kac-van Moerbeke and Toda systems was already known to Hénon in 1973. The Bäcklund transformation connecting the Toda and the Kac-van Moerbeke equations has first been considered in [**234**] and [**239**]. (see also [**56**], [**151**], [**152**]). The results presented here are mainly taken from [**35**], [**117**] and [**220**]. An alternative approach to the modified Toda hierarchy, using the discrete analog of the formal pseudo differential calculus, can be found in [**161**], Ch. 4.

For applications of the Kac-van Moerbeke equation I refer to(e.g.) [**25**] (Ch. 8), [**55**], [**66**], [**130**], [**159**], [**183**], or [**246**]

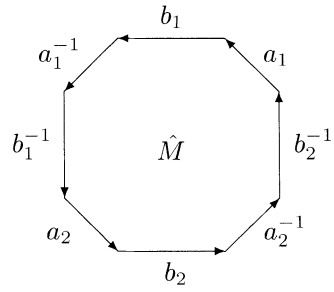
Appendix A

Compact Riemann surfaces – a review

The facts presented in this section can be found in almost any book on Riemann surfaces and their theta functions (cf., e.g. [**74**], [**81**], [**82**], [**86**], [**154**], [**178**], [**181**]). However, no book contains all results needed. Hence, in order to save you the tedious work of looking everything up from different sources and then translate results from one notation to another, I have included this appendix. In addition, I have worked out some facts on hyperelliptic surfaces in more detail than you will find them in the references. Most of the time we will closely follow [**81**] in the first sections.

A.1. Basic notation

Let M be a **compact Riemann surface** (i.e., a compact, second countable, connected Hausdorff space together with a holomorphic structure) of **genus** $g \in \mathbb{N}$. We restrict ourselves to $g \geq 1$ since most of our definitions only make sense in this case. We will think of our Riemann surface M often as it's **fundamental polygon** (e.g., for $g = 2$):



We denote the closure of the polygon by $\hat{M}$ and its boundary by $\partial\hat{M} = \prod_{\ell=1}^{g}(a_\ell b_\ell a_\ell^{-1} b_\ell^{-1})$, where $\{a_\ell\,, b_\ell\}_{1\le\ell\le g}$ are (fixed) representatives of a **canonical homology basis** for M. We get M from $\hat{M}$ by identifying corresponding points, that is, $M = \hat{M}/\simeq$.

We denote the set of **meromorphic functions** on M by $\mathcal{M}(M)$, the set of **meromorphic differentials** by $\mathcal{M}^1(M)$, and the set of **holomorphic differentials** by $\mathcal{H}^1(M)$. If we expand $f \in \mathcal{M}(M)$ respectively $\omega \in \mathcal{M}^1(M)$ in a local chart (U, z) centered at $p \in M$ we get $(z(p) = 0)$

$$\text{(A.1)} \qquad f = \sum_{\ell=m}^{\infty} c_\ell z^\ell, \quad \text{respectively} \quad \omega = \Big(\sum_{\ell=m}^{\infty} c_\ell z^\ell\Big)dz, \qquad c_m \neq 0,\ m \in \mathbb{Z}.$$

The number m is independent of the chart chosen and so we can define $\operatorname{ord}_p f = m$ respectively $\operatorname{ord}_p \omega = m$. The number m is called the **order** of f, respectively ω, at p. We will also need the **residue** $\operatorname{res}_p \omega = c_{-1}$, which is well-defined and satisfies

$$\text{(A.2)} \qquad \sum_{p\in M} \operatorname{res}_p \omega = 0.$$

The above sum is finite since ω can only have finitely many poles (M is compact!). Next, we define the **branch number** $b_f(p) = \operatorname{ord}_p(f - f(p)) - 1$ if $f(p) \neq \infty$ and $b_f(p) = -\operatorname{ord}_p(f) - 1$ if $f(p) = \infty$. There is a connection between the **total branching number** $B = \sum_{p\in M} b_f(p)$ (the sum is again finite), the **degree** n of f

$$\text{(A.3)} \qquad n = \operatorname{degree}(f) = \sum_{p\in f^{-1}(c)} (1 + b_f(p)), \qquad c \in \mathbb{C} \cup \{\infty\}$$

(we will show that n is independent of c), and the genus g, the **Riemann-Hurwitz relation**

$$\text{(A.4)} \qquad 2g - 2 = B - 2n.$$

We omit the proof which is not difficult but involves a lot of definitions which are of no further use for our purpose. (Triangulate $\mathbb{C} \cup \{\infty\}$ such that each pole and each zero of f is a vertex. Lift the triangulation to M via f and calculate the Euler characteristic ...)

A.2. Abelian differentials

Now consider two closed differentials θ, φ on M and let us try to compute

$$\iint_M \theta \wedge \varphi. \tag{A.5}$$

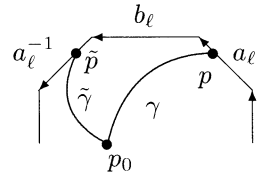
We define

$$\hat{A}(p) = \int_{p_0}^{p} \hat{\varphi} \qquad p_0, p \in \hat{M}, \tag{A.6}$$

where the hat indicates that we regard φ as a differential on to $\hat{M}$. $\hat{A}$ is well-defined since $\hat{M}$ is simply connected and $d\hat{\varphi} = 0$ (of course we require the path of integration to lie in $\hat{M}$). Using Stokes theorem we obtain

$$\begin{aligned} \iint_M \theta \wedge \varphi &= \iint_{\hat{M}} \theta \wedge d\hat{A} = -\iint_{\hat{M}} d(\theta \hat{A}) = -\int_{\partial \hat{M}} \theta \hat{A} \\ &= -\sum_{\ell=1}^{g} \left(\int_{a_\ell} \theta \hat{A} + \int_{b_\ell} \theta \hat{A} + \int_{a_\ell^{-1}} \theta \hat{A} + \int_{b_\ell^{-1}} \theta \hat{A} \right). \end{aligned} \tag{A.7}$$

Now, what can we say about $\hat{A}(p)$ and $\hat{A}(\tilde{p})$ if p, $\tilde{p}$ are corresponding points on a_ℓ, a_ℓ^{-1}, respectively? If $\gamma, \tilde{\gamma}$ are paths (inside $\hat{M}$) from p_0 to p, $\tilde{p}$, respectively, we have the following situation:



Thus we have $\hat{A}(\tilde{p}) = \hat{A}(p) + \int_{b_\ell} \varphi$ (the parts on a_ℓ and a_ℓ^{-1} cancel). A similar argument for corresponding points p, $\tilde{p}$ on b_ℓ, b_ℓ^{-1} yields $\hat{A}(\tilde{p}) = \hat{A}(p) - \int_{a_\ell} \varphi$ and we finally get

$$\iint_M \theta \wedge \varphi = \sum_{\ell=1}^{g} \left(\int_{a_\ell} \theta \int_{b_\ell} \varphi - \int_{a_\ell} \varphi \int_{b_\ell} \theta \right). \tag{A.8}$$

We will mainly be interested in meromorphic differentials $\omega \in \mathcal{M}^1(M)$, also called **abelian differentials**. We call ω of the **first kind** if ω is holomorphic, of the **second kind** if all residues vanish, and of the **third kind** else.

Unfortunately, formula (A.8) makes only sense in the special case of holomorphic differentials. Nevertheless there is one thing we can do. Let $\zeta \in \mathcal{H}^1(M)$, $\omega \in \mathcal{M}^1(M)$ and assume that ω has no poles on the boundary – which can always be achieved by slightly (continuously) deforming the (representatives of the) homology basis. The residue theorem now yields

$$2\pi \mathrm{i} \sum_{p \in M} \mathrm{res}_p \hat{A}\omega = \int_{\partial \hat{M}} \hat{A}\omega = \sum_{\ell=1}^{g} \left(\int_{a_\ell} \zeta \int_{b_\ell} \omega - \int_{a_\ell} \omega \int_{b_\ell} \zeta \right). \tag{A.9}$$

For $\zeta \in \mathcal{H}^1(M)$ we define

$$\|\zeta\|^2 = \mathrm{i} \iint_M \zeta \wedge \overline{\zeta} = \mathrm{i} \sum_{\ell=1}^{g} \left(\int_{a_\ell} \zeta \int_{b_\ell} \overline{\zeta} - \int_{a_\ell} \overline{\zeta} \int_{b_\ell} \zeta \right), \tag{A.10}$$

(the bar denotes complex conjugation) which is indeed a norm (in local coordinates we get: $\mathrm{i}\zeta \wedge \overline{\zeta} = 2|\eta|^2 dx \wedge dy$ with $\zeta = \eta\, dz$, $z = x + \mathrm{i}y$).

Therefore we conclude that $\zeta \equiv 0$ if and only if (e.g.) all the a-periods are zero and further that the map from the space of holomorphic differentials $\mathcal{H}^1(M)$ (which is g-dimensional) to the a-periods ($\cong \mathbb{C}^g$) is an isomorphism. Thus we may choose a basis $\{\zeta_j\}_{j=1}^g$ for $\mathcal{H}^1(M)$ fulfilling

$$\int_{a_k} \zeta_j = \delta_{jk}, \qquad j,k = 1,\dots,g. \tag{A.11}$$

For the b-periods we get

$$\int_{b_k} \zeta_j = \tau_{j,k}, \qquad j,k = 1,\dots,g, \tag{A.12}$$

with $\tau_{j,k} \in \mathbb{C}$. Now (A.9) implies

$$\tau_{j,k} = \tau_{k,j} \tag{A.13}$$

and (for $\zeta = \sum_{\ell=1}^{g} c_\ell \zeta_\ell \in \mathcal{H}^1(M)\backslash\{0\}$ with $c_\ell \in \mathbb{C}$)

$$\begin{aligned} 0 < \|\zeta\|^2 &= \mathrm{i} \sum_{j,k} c_j \overline{c_k} \sum_{\ell} \left(\int_{a_\ell} \zeta_j \int_{b_\ell} \overline{\zeta_k} - \int_{b_\ell} \zeta_j \int_{a_\ell} \overline{\zeta_k} \right) \\ &= 2 \sum_{j,k} c_j \overline{c_k} \mathrm{Im}(\tau_{j,k}). \end{aligned} \tag{A.14}$$

Especially for $c_\ell = \delta_{\ell j}$ we have $\mathrm{Im}(\tau_{j,j}) > 0$. Thus $\underline{\tau}$ is symmetric with positive definite imaginary part.

We will now ensure the existence of meromorphic differentials (without proof, cf. [**81**], II.5.3):

Theorem A.1. *For every set $\{(p_j, (c_{j,\ell})_{1\le\ell\le n_j})\}_{1\le j\le m}$ with distinct $p_j \in M$, $c_{j.\ell} \in \mathbb{C}$, and $m, n_j \in \mathbb{N}$ we can find a meromorphic differential ω with poles of the form*

$$\omega = \Big(\sum_{\ell=1}^{n_j} \frac{c_{j,\ell}}{z_j^\ell} + O(1)\Big) dz_j, \tag{A.15}$$

near each p_j in a given chart (U_j, z_j) centered at p_j, $1 \le j \le m$, and is holomorphic elsewhere, if and only if

$$\sum_{p\in M} \mathrm{res}_p \omega = \sum_{j=1}^{m} c_{j,1} = 0. \tag{A.16}$$

Hence, for a meromorphic differential we can prescribe the poles and the Laurent principal parts at each pole with the only restriction, that the sum of the residues has to be zero. Two differentials constructed in that way may only differ by a holomorphic differential. From now on we will assume that no poles lie on

the boundary – otherwise just deform the homology basis – so that we can make it unique by the following normalization

$$\int_{a_\ell} \omega = 0, \qquad \ell = 1, \ldots, g. \tag{A.17}$$

We are interested in some special cases. First let $\omega_{p,n}$ be holomorphic in $M\backslash\{p\}$ and let (U, z) be a chart centered at p

$$\omega_{p,n} = \left(\frac{1}{z^{2+n}} + O(1)\right)dz, \qquad \int_{a_\ell} \omega_{p,n} = 0, \quad \ell = 1, \ldots, g, \quad n \in \mathbb{N}_0. \tag{A.18}$$

This definition is obviously depending on the chart around p. The b-periods follow easily from (A.9) with $\zeta = \zeta_\ell$ and $\omega = \omega_{p.n}$

$$\int_{b_\ell} \omega_{p.n} = 2\pi\mathrm{i} \sum_{\tilde{p} \in M} \mathrm{res}_{\tilde{p}} \hat{A}\omega_{p.n} = \frac{2\pi\mathrm{i}}{(n+1)!} \left(\frac{d}{dz}\right)^n \eta_\ell(z)\Big|_{z=0}, \tag{A.19}$$

where $\zeta_\ell = \eta_\ell(z)dz$. Second suppose ω_{pq} is holomorphic in $M\backslash\{p, q\}$ (with $p \neq q$) and let

$$\begin{array}{ll} \mathrm{ord}_p\omega_{pq} = -1 & \mathrm{ord}_q\omega_{pq} = -1 \\ \mathrm{res}_p\omega_{pq} = +1 & \mathrm{res}_q\omega_{pq} = -1 \end{array}, \qquad \int_{a_\ell} \omega_{pq} = 0, \qquad \ell = 1, \ldots, g. \tag{A.20}$$

This definition depends on the specific homology basis chosen. Because if we deform, for example, a_ℓ we may cross a pole, thus changing the value of $\int_{a_\ell} \omega_{pq}$. The b-periods again follow from (A.9) with $\zeta = \zeta_\ell$ and $\omega = \omega_{pq}$

$$\int_{b_\ell} \omega_{pq} = 2\pi\mathrm{i} \sum_{\tilde{p} \in M} \mathrm{res}_{\tilde{p}} \hat{A}\omega_{pq} = 2\pi\mathrm{i}(\hat{A}(p) - \hat{A}(q)) = 2\pi\mathrm{i} \int_q^p \hat{\zeta}_\ell. \tag{A.21}$$

Recall that the path of integration in the last integral lies in $\hat{M}$.

A.3. Divisors and the Riemann-Roch theorem

This section is also valid for $g = 0$. A **divisor** $\mathcal{D}$ is a map

$$\begin{array}{rccc} \mathcal{D}: & M & \to & \mathbb{Z} \\ & p & \mapsto & \mathcal{D}(p) \end{array} \tag{A.22}$$

being nonzero only for a finite number of points in M. The set of all divisors $\mathrm{Div}(M)$ becomes a module over the ring $\mathbb{Z}$ by defining each operation in the natural way (i.e., pointwise). The point divisors $\mathcal{D}_p$, $p \in M$,

$$\mathcal{D}_p(p) = 1, \qquad \mathcal{D}_p(q) = 0, \qquad q \in M\backslash\{p\} \tag{A.23}$$

form a basis for $\mathrm{Div}(M)$. We can also establish a group homomorphism by assigning each divisor a **degree**

$$\begin{array}{rccc} \deg: & \mathrm{Div}(M) & \to & \mathbb{Z} \\ & \mathcal{D} & \mapsto & \deg(\mathcal{D}) = \sum\limits_{p \in M} \mathcal{D}(p) \end{array} \tag{A.24}$$

(the sum is finite and thus well-defined). We associate a divisor with every nonzero meromorphic function and with every nonzero meromorphic differential on M

(A.25)
$$\begin{array}{rccc} (\,): & \mathcal{M}(M)\backslash\{0\} & \to & \mathrm{Div}(M) \\ & f & \mapsto & (f) \end{array}, \qquad \begin{array}{rccc} (\,): & \mathcal{M}^1(M)\backslash\{0\} & \to & \mathrm{Div}(M) \\ & \omega & \mapsto & (\omega) \end{array},$$

where $(f)(p) = \mathrm{ord}_p f$ and $(\omega)(p) = \mathrm{ord}_p \omega$.

It is easy to verify the following formulas

$$(f_1\, f_2) = (f_1) + (f_2), \quad (\frac{1}{f_1}) = -(f_1), \quad f_1, f_2 \in \mathcal{M}(M)\backslash\{0\}, \tag{A.26}$$

implying $(f_1) = (f_2) \Leftrightarrow (f_1/f_2) = 0 \Leftrightarrow f_1 = c\, f_2$, $c \in \mathbb{C}$ since the only holomorphic functions on a compact Riemann surface are constants. Similarly,

$$(f\, \omega_1) = (f) + (\omega_1), \quad (\frac{\omega_1}{\omega_2}) = (\omega_1) - (\omega_2), \quad \omega_1, \omega_2 \in \mathcal{M}^1(M)\backslash\{0\} \tag{A.27}$$

implying $(\omega_1) = (\omega_2) \Leftrightarrow (\omega_1/\omega_2) = 0 \Leftrightarrow \omega_1 = c\, \omega_2$, $c \in \mathbb{C}$.

We claim

$$\begin{aligned} \deg((f)) &= 0, & f &\in \mathcal{M}(M)\backslash\{0\}, \\ \deg((\omega)) &= 2g-2, & \omega &\in \mathcal{M}^1(M)\backslash\{0\}. \end{aligned} \tag{A.28}$$

To obtain the first equation observe that $\deg((f)) = \sum_{p\in M} \mathrm{res}_p df/f = 0$. This implies that f attains every value exactly n times for a certain integer n called the degree of f (counting multiplicities). In fact, if 0 is attained n times the same is true for ∞ (otherwise $\deg((f))$ would not be zero). Now consider the function $f - c$ with $c \in \mathbb{C}$. It attains ∞ as often as f and thus zero as often as f.

For the second equation observe $\deg((\omega_1)) = \deg((\omega_2))$ which follows from $\omega_1/\omega_2 \in \mathcal{M}(M)\backslash\{0\}$. Now let f be a meromorphic function of degree n and let us calculate (df/f). At points p with $f(p) \notin \{0, \infty\}$, $\mathrm{ord}_p df/f$ is just the branch number $b_f(p)$ and at each other point it is -1. The desired result now follows from the Riemann-Hurwitz relation (A.4) and

$$\begin{aligned} \deg((\frac{df}{f})) &= \sum_{p\in M} b_f(p) - \sum_{p\in f^{-1}(0)} (1 + b_f(p)) - \sum_{p\in f^{-1}(\infty)} (1 + b_f(p)) \\ &= B - 2n. \end{aligned} \tag{A.29}$$

Divisors corresponding to meromorphic functions $\mathcal{D} = (f)$ are called **principal**, divisors corresponding to meromorphic differentials $\mathcal{D} = (\omega)$ **canonical**. It follows from the above considerations that the principal divisors form a subgroup. It is clearly normal (since all groups in sight are abelian) and we may consider its factor group induced by the following equivalence relation

$$\mathcal{D}_1 \simeq \mathcal{D}_2 \quad \Leftrightarrow \quad \mathcal{D}_1 - \mathcal{D}_2 = (f). \tag{A.30}$$

The factor group is called **divisor class group** and the equivalence classes $[\mathcal{D}]$ divisor classes. Observe, that all canonical divisors lie in the same divisor class (by (A.27)). The subset formed by the divisors of degree zero $\mathrm{Div}_0(M)$ is a subgroup of $\mathrm{Div}(M)$ and it's factor group is called **Picard group** $\mathrm{Pic}(M)$.

The set $\mathrm{Div}(M)$ can be partially ordered by defining

$$\mathcal{D}_1 \geq \mathcal{D}_2 \quad \Leftrightarrow \quad \mathcal{D}_1(p) \geq \mathcal{D}_2(p), \quad p \in M. \tag{A.31}$$

We write $\mathcal{D}_1 > \mathcal{D}_2$ if in addition $\mathcal{D}_1 \neq \mathcal{D}_2$. Note: $\mathcal{D}_1 \geq \mathcal{D}_2 \Rightarrow \deg(\mathcal{D}_1) \geq \deg(\mathcal{D}_2)$ and $\mathcal{D}_1 > \mathcal{D}_2 \Rightarrow \deg(\mathcal{D}_1) > \deg(\mathcal{D}_2)$.

A divisor is called **positive** if $\mathcal{D} \geq 0$ and **strictly positive** if $\mathcal{D} > 0$. Two divisors $\mathcal{D}_1$, $\mathcal{D}_2$ are called **relatively prime** if $\mathcal{D}_1(p)\mathcal{D}_2(p) = 0$, $p \in M$. A divisor can always be uniquely written as the difference of two relatively prime positive divisors

$$\mathcal{D} = \mathcal{D}^+ - \mathcal{D}^-, \qquad \mathcal{D}^+ \geq 0,\ \mathcal{D}^- \geq 0. \tag{A.32}$$

For a meromorphic function $(f)^+$ and $(f)^-$ are called the **divisor of zeros** and the **divisor of poles** of f, respectively.

For any divisor $\mathcal{D}$ we now define

$$\begin{aligned} \mathcal{L}(\mathcal{D}) &= \{f \in \mathcal{M}(M) | f = 0 \text{ or } (f) \geq \mathcal{D}\}, \\ \mathcal{L}^1(\mathcal{D}) &= \{\omega \in \mathcal{M}^1(M) | \omega = 0 \text{ or } (\omega) \geq \mathcal{D}\}. \end{aligned} \tag{A.33}$$

Hence $\mathcal{L}(\mathcal{D})$ is the set of all meromorphic functions, which are holomorphic at points $p \in M$ with $\mathcal{D}(p) \geq 0$, have zeros of order at least $\mathcal{D}(p)$ at points p with $\mathcal{D}(p) > 0$ and poles of order at most $-\mathcal{D}(p)$ at points p with $\mathcal{D}(p) < 0$. Similarly for $\mathcal{L}^1(\mathcal{D})$. Obviously we have

$$\mathcal{D}_1 \geq \mathcal{D}_2 \quad \Rightarrow \quad \begin{cases} \mathcal{L}(\mathcal{D}_1) \subset \mathcal{L}(\mathcal{D}_2) \\ \mathcal{L}^1(\mathcal{D}_1) \subset \mathcal{L}^1(\mathcal{D}_2) \end{cases} \tag{A.34}$$

and $\mathcal{L}(\mathcal{D})$, $\mathcal{L}^1(\mathcal{D})$ inherit the vector space structure from $\mathcal{M}(M)$, $\mathcal{M}^1(M)$, respectively. What can we say about their dimensions

$$r(\mathcal{D}) = \dim \mathcal{L}(\mathcal{D}), \qquad i(\mathcal{D}) = \dim \mathcal{L}^1(\mathcal{D}). \tag{A.35}$$

$i(\mathcal{D})$ is called **index of specialty** of $\mathcal{D}$ and $\mathcal{D}$ is called **nonspecial** if $i(\mathcal{D}) = 0$.

First of all we have

$$r(0) = 1, \qquad i(0) = g, \tag{A.36}$$

since a holomorphic function on a compact Riemann surface must be constant and since $\mathcal{H}^1(M)$ is g-dimensional.

If we take two divisors in the same divisor class, we can easily establish an isomorphism

$$\begin{aligned} g \in \mathcal{L}(\mathcal{D}) &\Rightarrow f g \in \mathcal{L}(\mathcal{D} + (f)), \\ g \in \mathcal{L}(\mathcal{D} + (f)) &\Rightarrow f^{-1} g \in \mathcal{L}(\mathcal{D}), \end{aligned} \tag{A.37}$$

saying

$$\mathcal{L}(\mathcal{D}) \cong \mathcal{L}(\mathcal{D} + (f)), \qquad r(\mathcal{D}) = r(\mathcal{D} + (f)). \tag{A.38}$$

Proceeding in a similar manner, we also obtain

$$\begin{aligned} \mathcal{L}^1(\mathcal{D}) &\cong \mathcal{L}^1(\mathcal{D} + (f)), & i(\mathcal{D}) &= i(\mathcal{D} + (f)), \\ \mathcal{L}^1(\mathcal{D}) &\cong \mathcal{L}(\mathcal{D} - (\omega)), & i(\mathcal{D}) &= r(\mathcal{D} - (\omega)). \end{aligned} \tag{A.39}$$

If $\deg(\mathcal{D}) > 0$ it follows from $(f) \geq \mathcal{D} \Rightarrow \deg((f)) \geq \deg(\mathcal{D}) > 0$, that $r(\mathcal{D})$ has to be zero. And using (A.39) and (A.28) we get

$$\begin{aligned} \deg(\mathcal{D}) > 0 &\Rightarrow r(\mathcal{D}) = 0, \\ \deg(\mathcal{D}) > 2g - 2 &\Rightarrow i(\mathcal{D}) = 0. \end{aligned} \tag{A.40}$$

If $\deg(\mathcal{D}) = 0$ and $f \in \mathcal{L}(\mathcal{D})$ it follows $(f) = \mathcal{D}$ (since $(f) > \mathcal{D}$ is not possible by the same argument as above) and thus

$$\deg(\mathcal{D}) = 0 \Leftrightarrow \begin{cases} \mathcal{L}(\mathcal{D}) = \operatorname{span}\{f\} & \text{if } \mathcal{D} = (f) \text{ is principal} \\ \mathcal{L}(\mathcal{D}) = \{0\} & \text{if } \mathcal{D} \text{ is not principal} \end{cases},$$

$$\text{(A.41)} \quad \deg(\mathcal{D}) = 2g-2 \Leftrightarrow \begin{cases} \mathcal{L}^1(\mathcal{D}) = \operatorname{span}\{\omega\} & \text{if } \mathcal{D} = (\omega) \text{ is canonical} \\ \mathcal{L}^1(\mathcal{D}) = \{0\} & \text{if } \mathcal{D} \text{ is not canonical} \end{cases}.$$

Next, let us assume $r(-\mathcal{D}) \leq 1 + \deg(\mathcal{D})$ for all $\mathcal{D}$ with $\deg(\mathcal{D}) \geq 0$ (we already know it for $\deg(\mathcal{D}) = 0$). If we subtract a point divisor we get

$$\text{(A.42)} \qquad r(-\mathcal{D}) \leq r(-\mathcal{D} - \mathcal{D}_p) \leq 1 + r(-\mathcal{D})$$

by (A.34) and because if f is new in our space, there is for every $g \in \mathcal{L}(-\mathcal{D} - \mathcal{D}_p)$ a $c \in \mathbb{C}$ with $g - c\,f \in \mathcal{L}(\mathcal{D})$. Hence using induction we infer

$$\begin{aligned} r(-\mathcal{D}) &\leq 1 + \deg(\mathcal{D}), & \deg(\mathcal{D}) &\geq 0, \\ \text{(A.43)} \qquad i(-\mathcal{D}) &\leq 2g - 1 + \deg(\mathcal{D}), & \deg(\mathcal{D}) &\geq 2 - 2g. \end{aligned}$$

By virtue of Theorem A.1 and (A.36) we have

$$\text{(A.44)} \qquad i(\mathcal{D}) = g - 1 - \deg(\mathcal{D}), \qquad \mathcal{D} < 0.$$

If we add a point divisor $\mathcal{D}_p$ to $\mathcal{D}$, we obtain from (A.39) and (A.42)

$$\text{(A.45)} \qquad i(\mathcal{D}) - 1 \leq i(\mathcal{D} + \mathcal{D}_p) \leq i(\mathcal{D}).$$

Thus, using induction we have

$$\text{(A.46)} \qquad i(\mathcal{D}) \geq g - 1 - \deg(\mathcal{D}),$$

which is by (A.39) and (A.28) equivalent to (**Riemann inequality**)

$$\text{(A.47)} \qquad r(-\mathcal{D}) \geq 1 - g + \deg(\mathcal{D}).$$

Now we will prove the famous **Riemann-Roch Theorem**:

Theorem A.2. (Riemann-Roch) *Let $\mathcal{D} \in \operatorname{Div}(M)$, then we have*

$$\text{(A.48)} \qquad r(-\mathcal{D}) = \deg(\mathcal{D}) + 1 - g + i(\mathcal{D}).$$

Proof. By (A.44) we know that it is true for $\mathcal{D} \leq 0$. So what happens if we add a point divisor to $\mathcal{D}$? We already know (A.42) and (A.45) but can we have $i(\mathcal{D} + \mathcal{D}_p) = i(\mathcal{D}) - 1$ and $r(-\mathcal{D} - \mathcal{D}_p) = 1 + r(-\mathcal{D})$? If this were the case, we would have a function $f \in \mathcal{L}(\mathcal{D} + \mathcal{D}_p)$ with a pole of order $(\mathcal{D}(p) + 1)$ at p and a differential $\omega \in \mathcal{L}^1(\mathcal{D})$ with a zero of order $\mathcal{D}(p)$ at p. Now consider the differential $f\,\omega$. It would have a pole of order 1 at p and would be holomorphic elsewhere, $(f\,\omega) = (f) + (\omega) \geq -\mathcal{D} - \mathcal{D}_p + \mathcal{D} = -\mathcal{D}_p$, contradicting our assumption. Thus we have by induction

$$\text{(A.49)} \qquad r(-\mathcal{D}) \leq \deg(\mathcal{D}) + 1 - g + i(\mathcal{D}).$$

Using (A.39) and (A.28):

$$\begin{aligned} i(\mathcal{D}) &= r(\mathcal{D} - (\omega)) \leq \deg((\omega) - \mathcal{D}) + 1 - g + i((\omega) - \mathcal{D}) \\ \text{(A.50)} \qquad &= -\deg(\mathcal{D}) + g - 1 + r(-\mathcal{D}) \end{aligned}$$

completes the proof of the Riemann-Roch theorem. □

Corollary A.3. *A Riemann surface of genus 0 is isomorphic to the Riemann sphere* ($\cong \mathbb{C} \cup \{\infty\}$).

Proof. Let $p \in M$. Since $r(\mathcal{D}_p) = 2$, there exists a (non constant) meromorphic function with a single pole. It is injective (its degree is 1) and surjective (since every non constant holomorphic map between compact Riemann surfaces is). Hence it provides the desired isomorphism. □

Corollary A.4. ($g \geq 1$) *There are no meromorphic functions with a single pole of order 1 on M or equivalently, there is no point in M where all $\zeta \in \mathcal{H}^1(M)$ vanish.*

Proof. If f were such a function with pole at p we would have $r(-n\mathcal{D}_p) = n + 1$ for all $n \in \mathbb{N}_0$ (consider f^n) and from Riemann-Roch $i(n\mathcal{D}_p) = g$. Hence all holomorphic differentials would vanish of arbitrary order at p which is impossible. Thus $r(-\mathcal{D}_p) = 1$ for all $p \in M$ or equivalently $i(\mathcal{D}_p) = g - 1$, $p \in M$. □

If we have a sequence of points $\{p_\ell\}_{\ell \in \mathbb{N}} \subset M$ and define $\mathcal{D}_0 = 0$ and $\mathcal{D}_n = \mathcal{D}_{n-1} + \mathcal{D}_{p_n}$ for $n \in \mathbb{N}$, then there are exactly g ($g \geq 1$) integers $\{n_\ell\}_{\ell=1}^g$ such that $i(\mathcal{D}_{n_\ell}) = i(\mathcal{D}_{n_\ell - 1}) - 1$ (have a look at the proof of Riemann-Roch and note $i(\mathcal{D}_0) = g$) or equivalently, there exists no meromorphic function f with $(f)^- = \mathcal{D}_{n_\ell}$. Moreover,

$$1 = n_1 < \cdots < n_g < 2g. \tag{A.51}$$

(The first equality follows from our last corollary and the second from (A.41).) The numbers $\{n_\ell\}_{\ell=1}^g$ are called **Noether gaps** and in the special case where $p_1 = p_2 = \ldots$ **Weierstrass gaps**.

A surface admitting a function of lowest possible degree (i.e., 2 for $g \geq 1$) is called **hyperelliptic**. Note that if $g \leq 2$, M is always hyperelliptic ($g = 0, 1$ is obvious, for $g = 2$ choose $0 \neq \omega \in \mathcal{L}^1(0)$ and calculate $r((\omega)) = 2 + 1 - 2 + 1 = 2$).

A point $p \in M$ with $i(g\mathcal{D}_p) > 0$ is called a **Weierstrass point** (there are no Weierstrass points for $g \leq 1$). We mention (without proof) that the number of Weierstrass points W is finite and (for $g \geq 2$) satisfies $2g + 2 \leq W \leq g^3 - g$. The first inequality is attained if and only if at each Weierstrass point the gap sequence is $1, 3, \ldots, 2g - 1$ (which is the case if and only if M is hyperelliptic).

A.4. Jacobian variety and Abel's map

We start with the following (discrete) subset of $\mathbb{C}^g$

$$L(M) = \{\underline{m} + \underline{\tau}\,\underline{n} \,|\, \underline{m}, \underline{n} \in \mathbb{Z}^g\} \subset \mathbb{C}^g \tag{A.52}$$

and define the **Jacobian variety** of M

$$J(M) = \mathbb{C}^g / L(M). \tag{A.53}$$

It is a compact, commutative, g-dimensional, complex Lie group. Further we define **Abel's map** (with base point p_0):

$$\begin{array}{rccl} \underline{A}_{p_0}: & M & \to & J(M) \\ & p & \mapsto & [\int_{p_0}^p \underline{\zeta}] = [(\int_{p_0}^p \zeta_1, \ldots, \int_{p_0}^p \zeta_g)] \end{array}, \tag{A.54}$$

where $[\underline{z}] \in J(M)$ denotes the equivalence class of $\underline{z} \in \mathbb{C}^g$. It is clearly holomorphic and well-defined as long as we choose the same path of integration for all ζ_ℓ ($[\gamma] - [\gamma'] = \underline{m}[\underline{a}] + \underline{n}[\underline{b}]$) since

$$\int_\gamma \underline{\zeta} - \int_{\gamma'} \underline{\zeta} = \underline{m} + \underline{\tau}\,\underline{n} \in L(M), \qquad \underline{m}, \underline{n} \in \mathbb{Z}^g. \tag{A.55}$$

Observe that $d\underline{A}_{p_0}$ is of maximal rank (i.e., 1) by Corollary A.4 (compare also (A.86)) and that a change of base point results in a global shift of the image. Note: $\underline{A}_{p_0}(p) = -\underline{A}_p(p_0)$.

We will also need a slight modification of $\underline{A}_{p_0}$

$$\begin{array}{rccl} \underline{\hat{A}}_{p_0}: & \hat{M} & \to & \mathbb{C}^g \\ & p & \mapsto & \displaystyle\int_{p_0}^{p} \underline{\hat{\zeta}} = \Big(\int_{p_0}^{p} \hat{\zeta}_1, \ldots, \int_{p_0}^{p} \hat{\zeta}_g\Big) \end{array}, \tag{A.56}$$

with $p_0 \in \hat{M}$ and the path of integration lying in $\hat{M}$. We can easily extend Abel's map to the set of divisors $\mathrm{Div}(M)$

$$\begin{array}{rccl} \underline{\alpha}_{p_0}: & \mathrm{Div}(M) & \to & J(M) \\ & \mathcal{D} & \mapsto & \displaystyle\sum_{p \in M} \mathcal{D}(p)\underline{A}_{p_0}(p) \end{array}. \tag{A.57}$$

The sum is to be understood in $J(M)$ and multiplication with integers is also well-defined in $J(M)$. A similar extension $\underline{\hat{\alpha}}_{p_0}$ is defined for $\underline{\hat{A}}_{p_0}$. The natural domain for $\underline{\hat{\alpha}}_{p_0}$ would be $\mathrm{Div}(\hat{M})$, if we want to define it on $\mathrm{Div}(M)$ we have to make it unique on the boundary. But we will avoid this problem by requiring that divisors in $\mathrm{Div}(\hat{M})$ vanish on the boundary. We can now state **Abel's theorem**.

Theorem A.5. (Abel) $\mathcal{D} \in \mathrm{Div}(M)$ *is principal if and only if*

$$\underline{\alpha}_{p_0}(\mathcal{D}) = [0] \qquad \textit{and} \qquad \deg(\mathcal{D}) = 0. \tag{A.58}$$

Note that a change of the base point p_0 to p_1 amounts to adding a constant $\underline{A}_{p_1}(p) = \underline{A}_{p_0}(p) - \underline{A}_{p_0}(p_1)$, which yields $\underline{\alpha}_{p_1}(\mathcal{D}) = \underline{\alpha}_{p_0}(\mathcal{D}) - \deg(\mathcal{D})\,\underline{A}_{p_0}(p_1)$. Thus $\underline{\alpha}_{p_0}$ is independent of p_0 for divisors with $\deg(\mathcal{D}) = 0$.

Proof. Throughout this whole proof we choose $p_0 \notin \partial\hat{M}$ such that $\mathcal{D}(p_0) = 0$. Moreover, we assume that $\mathcal{D}(p) = 0$, $p \in \partial\hat{M}$ (otherwise deform $\{a_\ell, b_\ell\}_{1 \le \ell \le g}$ slightly).

Let $\mathcal{D} = (f)$ be the divisor of the meromorphic function f. Since df/f has only simple poles with residues $\mathrm{res}_p df/f = \mathrm{ord}_p f \in \mathbb{Z}$, we can use (A.20) to write

$$\frac{df}{f} = \sum_{p \in M} \mathcal{D}(p)\omega_{pp_0} + \sum_{\ell=1}^{g} c_\ell \zeta_\ell. \tag{A.59}$$

Note that the poles at p_0 cancel. Using that the a- and b-periods must be integer multiples of $2\pi\mathrm{i}$ by the residue theorem and (A.21) we infer

$$\begin{aligned}\int_{a_j}\frac{df}{f} &= c_j = 2\pi\mathrm{i}m_j,\\ \int_{b_j}\frac{df}{f} &= 2\pi\mathrm{i}\sum_{p\in M}\mathcal{D}(p)\int_{p_0}^{p}\zeta_j + \sum_{\ell=1}^{g}c_\ell\tau_{\ell,j} = 2\pi\mathrm{i}\hat{\alpha}_{p_0,j}(\mathcal{D}) + \sum_{\ell=1}^{g}c_\ell\tau_{\ell,j}\\ &= 2\pi\mathrm{i}\,n_j, \qquad \text{(A.60)}\end{aligned}$$

with $m_j, n_j \in \mathbb{Z}$. So we finally end up with

$$\underline{\alpha}_{p_0}(\mathcal{D}) = [\underline{\hat{\alpha}}_{p_0}(\mathcal{D})] = [\underline{n} - \underline{\tau}\,\underline{m}] = [0]. \qquad \text{(A.61)}$$

To prove the converse let $\mathcal{D}$ be given and consider (motivated by (A.59))

$$f(q) = \exp\left(\sum_{p\in M}\mathcal{D}(p)\int_{q_0}^{q}\omega_{pp_0} + \sum_{\ell=1}^{g}\tilde{c}_\ell\int_{q_0}^{q}\zeta_\ell\right), \qquad \tilde{c}_\ell\in\mathbb{C}, \quad q\in\hat{M}. \qquad \text{(A.62)}$$

where $p_0 \neq q_0 \notin \partial\hat{M}$ and $\mathcal{D}(q_0) = 0$. The constants $\tilde{c}_\ell$ are to be determined. The function f is a well-defined meromorphic function on $\hat{M}$ since a change of path (within $\hat{M}$) in the first integral amounts to a factor $2\pi\mathrm{i}\,n$ ($n\in\mathbb{Z}$) which is swallowed by the exponential function. Clearly we have $(f) = \mathcal{D}$. For f to be a meromorphic function on M its value has to be independent of the path (in M) chosen

$$\begin{aligned}\sum_{p\in M}\mathcal{D}(p)\int_{a_j}\omega_{pp_0} + \sum_{\ell=1}^{g}\tilde{c}_\ell\int_{a_j}\zeta_\ell &= \tilde{c}_j,\\ \sum_{p\in M}\mathcal{D}(p)\int_{b_j}\omega_{pp_0} + \sum_{\ell=1}^{g}\tilde{c}_\ell\int_{b_j}\zeta_\ell &= 2\pi\mathrm{i}\hat{\alpha}_{p_0,j}(\mathcal{D}) + \sum_{\ell=1}^{g}\tilde{c}_\ell\omega_{\ell,j}. \qquad \text{(A.63)}\end{aligned}$$

Thus if $\underline{\hat{\alpha}}_{p_0}(\mathcal{D}) = \underline{m} + \underline{\tau}\,\underline{n}$, we have to choose $\tilde{c}_\ell = -2\pi\mathrm{i}n_\ell \in \mathbb{Z}$. □

Corollary A.6. *The Picard group is isomorphic to the Jacobian variety*

$$\mathrm{Pic}(M) \cong J(M). \qquad \text{(A.64)}$$

Proof. The map $\underline{\alpha}_{p_0}$ provides an isomorphism since it is injective by Abel's theorem and surjective by Corollary A.15 below. □

Corollary A.7. $\underline{A}_{p_0}$ *is injective.*

Proof. Let $p, q \in M$ with $\underline{A}_{p_0}(p) = \underline{A}_{p_0}(q)$. By Abel's theorem $\mathcal{D}_p - \mathcal{D}_q$ would be principal which is impossible because of Corollary A.4. □

Thus $\underline{A}_{p_0}$ is an embedding of M into $J(M)$. For $g = 1$ it is even an isomorphism (it is surjective since every non constant map between compact Riemann surfaces is). For $g \geq 1$ we mention (without proof) **Torelli's theorem**:

Theorem A.8. (Torelli) *Two Riemann surfaces M and M' are isomorphic if and only if $J(M)$ and $J(M')$ are,*

$$M \cong M' \quad \Leftrightarrow \quad J(M) \cong J(M'). \qquad \text{(A.65)}$$

A.5. Riemann's theta function

We first define the **Riemann theta function** associated with M

$$\begin{aligned} \theta: \ \mathbb{C}^g &\to \mathbb{C} \\ \underline{z} &\mapsto \sum_{\underline{m}\in\mathbb{Z}^g} \exp 2\pi\mathrm{i}\left(\langle \underline{m}, \underline{z}\rangle + \frac{\langle \underline{m}, \underline{\tau}\,\underline{m}\rangle}{2}\right), \end{aligned} \tag{A.66}$$

where $\langle \underline{z}, \underline{z}'\rangle = \sum_{\ell=1}^g \overline{z_\ell} z'_\ell$. It is holomorphic since the sum converges nicely due to (A.14). We will now show some simple properties of $\theta(\underline{z})$. First observe that

$$\theta(-\underline{z}) = \theta(\underline{z}) \tag{A.67}$$

and second let $\underline{n}, \underline{n}' \in \mathbb{Z}^g$, then

$$\begin{aligned} \theta(\underline{z} + \underline{n} + \underline{\tau}\,\underline{n}') &= \sum_{\underline{m}\in\mathbb{Z}^g} \exp 2\pi\mathrm{i}\left(\langle \underline{m}, \underline{z} + \underline{n} + \underline{\tau}\,\underline{n}'\rangle + \frac{\langle \underline{m}, \underline{\tau}\,\underline{m}\rangle}{2}\right) \\ &= \sum_{\underline{m}\in\mathbb{Z}^g} \exp 2\pi\mathrm{i}\Bigg(\langle \underline{m} + \underline{n}', \underline{z}\rangle - \langle \underline{n}', \underline{z}\rangle + \\ &\qquad + \frac{\langle \underline{m} + \underline{n}', \underline{\tau}(\underline{m} + \underline{n}')\rangle - \langle \underline{n}', \underline{\tau}\,\underline{n}'\rangle}{2}\Bigg) \\ &= \exp 2\pi\mathrm{i}\left(-\langle \underline{n}', \underline{z}\rangle - \frac{\langle \underline{n}', \underline{\tau}\,\underline{n}'\rangle}{2}\right)\theta(\underline{z}). \end{aligned} \tag{A.68}$$

(One has to use $\langle \underline{m}, \underline{\tau}\,\underline{n}'\rangle = \langle \underline{n}', \underline{\tau}\,\underline{m}\rangle$ which follows from symmetry of $\underline{\tau}$.) As special cases we mention

$$\begin{aligned} \theta(z + \delta_j) &= \theta(z), \\ \theta(z + \tau_j) &= \mathrm{e}^{2\pi\mathrm{i}(-z_j - \tau_{j,j}/2)}\theta(z). \end{aligned} \tag{A.69}$$

Here τ_j denotes the j'th row of $\underline{\tau}$. Notice that if we choose $\underline{z} = -(\underline{n} + \underline{\tau}\,\underline{n}')/2$, we get $\theta(-\underline{z}) = \exp(\mathrm{i}\pi\langle \underline{n}, \underline{n}'\rangle)\theta(\underline{z})$, implying that $\theta(-(\underline{n}+\underline{\tau}\,\underline{n}')/2) = 0$ if $\langle \underline{n}, \underline{n}'\rangle$ is odd.

Let $\underline{z} \in \mathbb{C}^g$. We will study the (holomorphic) function $\hat{F}$

$$\begin{aligned} \hat{F}: \ \hat{M} &\to \mathbb{C} \\ p &\mapsto \theta(\underline{\hat{A}}_{p_0}(p) - \underline{z}) \end{aligned}. \tag{A.70}$$

If $\hat{F}$ is not identically zero, we can compute the number of zeros in the following manner

$$\begin{aligned} \frac{1}{2\pi\mathrm{i}}\int_{\partial\hat{M}} \frac{d\hat{F}}{\hat{F}} &= \frac{1}{2\pi\mathrm{i}}\sum_{\ell=1}^g \left(\int_{a_\ell} \frac{d\hat{F}}{\hat{F}} + \int_{b_\ell} \frac{d\hat{F}}{\hat{F}}\right. \\ &\qquad \left. + \int_{a_\ell^{-1}} \frac{d\hat{F}}{\hat{F}} + \int_{b_\ell^{-1}} \frac{d\hat{F}}{\hat{F}}\right). \end{aligned} \tag{A.71}$$

Here and in what follows we will always assume that no zero of $\hat{F} \not\equiv 0$ lies on the boundary – otherwise deform the representatives of the homology basis. From (A.69) together with a similar argument as in (A.8) we get

$$\hat{A}_{p_0,j}(\tilde{p}) = \hat{A}_{p_0,j}(p) + \tau_{\ell,j} \ \Rightarrow \ \hat{F}(\tilde{p}) = \mathrm{e}^{2\pi\mathrm{i}(z_\ell - \hat{A}_{p_0,\ell}(p) - \tau_{\ell,\ell}/2)}\hat{F}(p) \tag{A.72}$$

if p, $\tilde{p}$ are corresponding points on a_ℓ, a_ℓ^{-1} and

$$\hat{A}_{p_0,j}(\tilde{p}) = \hat{A}_{p_0,j}(p) - \delta_{\ell j} \quad \Rightarrow \quad \hat{F}(\tilde{p}) = \hat{F}(p) \tag{A.73}$$

if p, $\tilde{p}$ are corresponding points on b_ℓ, b_ℓ^{-1}. Using this we see

$$\frac{1}{2\pi\mathrm{i}} \int_{\partial \hat{M}} \frac{d\hat{F}}{\hat{F}} = \sum_{\ell=1}^{g} \int_{a_\ell} d\hat{A}_{p_0,\ell} = \sum_{\ell=1}^{g} \int_{a_\ell} \zeta_\ell = g. \tag{A.74}$$

Next, let us compute $\underline{\hat{\alpha}}_{p_0}((\hat{F}))$ (Since we have assumed that no zero lies on the boundary we may write $(\hat{F}) \in \mathrm{Div}(M)$.)

$$\begin{aligned} \underline{\hat{\alpha}}_{p_0}((\hat{F})) &= \frac{1}{2\pi\mathrm{i}} \int_{\partial \hat{M}} \underline{\hat{A}}_{p_0} \frac{d\hat{F}}{\hat{F}} \\ &= \frac{1}{2\pi\mathrm{i}} \sum_{\ell=1}^{g} \left(\int_{a_\ell} \underline{\hat{A}}_{p_0} \frac{d\hat{F}}{\hat{F}} + \int_{b_\ell} \underline{\hat{A}}_{p_0} \frac{d\hat{F}}{\hat{F}} \right. \\ &\quad \left. + \int_{a_\ell^{-1}} \underline{\hat{A}}_{p_0} \frac{d\hat{F}}{\hat{F}} + \int_{b_\ell^{-1}} \underline{\hat{A}}_{p_0} \frac{d\hat{F}}{\hat{F}} \right). \end{aligned} \tag{A.75}$$

Proceeding as in the last integral (using (A.72), (A.73)) yields

$$\hat{\alpha}_{p_0,j}((\hat{F})) = \frac{1}{2\pi\mathrm{i}} \sum_{\ell=1}^{g} \left(\int_{a_\ell} \left(2\pi\mathrm{i}(\hat{A}_{p_0,j} + \tau_{\ell,j})\zeta_\ell - \tau_{\ell,j} \frac{d\hat{F}}{\hat{F}} \right) + \delta_{\ell j} \int_{b_\ell} \frac{d\hat{F}}{\hat{F}} \right). \tag{A.76}$$

Hence we have to perform integrals of the type $\int_p^{\tilde{p}} d\ln(\hat{F})$, which is just $\ln(\hat{F}(\tilde{p})) - \ln(\hat{F}(p))$. Of course we do not know which branch we have to choose, so we have to add a proper multiple of $2\pi\mathrm{i}$. Computing $\hat{F}(p)$, $\hat{F}(\tilde{p})$ as in (A.8) we get

$$\begin{aligned} &\frac{1}{2\pi\mathrm{i}} \sum_{\ell=1}^{g} \left(\tau_{\ell,j} \int_{a_\ell} \frac{d\hat{F}}{\hat{F}} + \delta_{\ell j} \int_{b_\ell} \frac{d\hat{F}}{\hat{F}} \right) = \\ &\qquad - \hat{A}_{p_0,j}(p) + z_j - \frac{\tau_{j,j}}{2} + m_j + \sum_{\ell=1}^{g} \tau_{\ell,j} n_\ell, \end{aligned} \tag{A.77}$$

with $\underline{m}, \underline{n} \in \mathbb{Z}^g$ and p is the point in $\hat{M}$ that a_ℓ and b_ℓ have in common – we have omitted the dependence of p on ℓ since $\hat{A}_{p_0,j}(p)$ is independent of ℓ. As the last (unknown) factor is in $L(M)$, we can get rid of it by changing to $J(M)$

$$\alpha_{p_0,j}((\hat{F})) = [\hat{\alpha}_{p_0,j}((\hat{F}))] = [-\hat{A}_{p_0,j}(p) + z_j - \frac{\tau_{j,j}}{2} + \sum_{\ell=1}^{g} \int_{a_\ell} \hat{A}_{p_0,j}\zeta_\ell]. \tag{A.78}$$

There is one more thing we can do,

$$\begin{aligned} \int_{a_\ell} \hat{A}_{p_0,\ell}\zeta_\ell &= \frac{1}{2}(\hat{A}^2_{p_0,\ell}(p) - \hat{A}^2_{p_0,\ell}(\tilde{p})) \\ &= \frac{1}{2}(\hat{A}^2_{p_0,\ell}(p) - (\hat{A}_{p_0,\ell}(p) - 1)^2) = \hat{A}_{p_0,\ell}(p) - \frac{1}{2}. \end{aligned} \tag{A.79}$$

(Compare the picture on page 2.) Thus our final result is

$$\underline{\alpha}_{p_0}((\hat{F})) = [\underline{z}] - \underline{\Xi}_{p_0}, \tag{A.80}$$

with

$$\Xi_{p_0,j} = [\frac{1+\tau_{j,j}}{2} - \sum_{\ell\neq j}\int_{a_\ell} \hat{A}_{p_0,j}\zeta_\ell] \tag{A.81}$$

the **vector of Riemann constants**. Finally, we mention that evaluating the integral

$$\frac{1}{2\pi\mathrm{i}}\int_{\partial\hat{M}} f\frac{d\hat{F}}{\hat{F}}, \tag{A.82}$$

where $f \in \mathcal{M}(M)$ is an arbitrary meromorphic function yields the useful formula

$$\sum_{\ell=1}^{g}\int_{a_\ell} f\,\zeta_\ell = \sum_{\ell=1}^{g} f(p_j) + \sum_{p\in f^{-1}(\infty)} \mathrm{res}_p\Big(f\,d\ln\hat{F}\Big), \tag{A.83}$$

where $\{p_j\}_{j=1}^{g}$ are the zeros of the function $\hat{F}$ which (for simplicity) are away from the poles of f.

A.6. The zeros of the Riemann theta function

We begin with some preparations. Let M_n be the set of positive divisors of degree $n \in \mathbb{N}_0$. (The case $n = 0$ is of course trivial, but it is practical to include it.) We can make M_n a complex (n-dimensional) manifold if we identify M_n with $\sigma^n M$ the n-th **symmetric power** of M. If $\mathcal{D}_n \in M_n$ and if the points $\{\tilde{p}_j\}_{1\le j\le n}$ with $\mathcal{D}_n(p) \neq 0$ are distinct (i.e., $\mathcal{D}_n(p) \le 1$, $p \in M$), a local chart is given by

$$\begin{aligned} U &= \{[(p_1,\dots,p_n)] | p_j \in U_j\}, \quad \text{with } U_j \cap U_k = \emptyset \text{ for } j \neq k \\ [(p_1,\dots,p_n)] &\mapsto (z_1(p_1),\dots,z_n(p_n)), \quad \text{with } p_j \in U_j, \end{aligned} \tag{A.84}$$

where (U_j, z_j) are charts in M centered at $\tilde{p}_j$. If $\mathcal{D}(p) \ge 2$ for some points, one has to choose more sophisticated charts (elementary symmetric functions). Note, that $\underline{\alpha}_{p_0} : M_n \to J(M)$ is a holomorphic mapping. If $\mathcal{D}_n \in M_n$ as above, one can easily compute

$$\{d\underline{\alpha}_{p_0}(\mathcal{D}_n)\}_{jk} = \zeta_j(\tilde{p}_k), \qquad 1 \le j \le g, \quad 1 \le k \le n \tag{A.85}$$

(with some abuse of notation: $\zeta_j(\tilde{p}_k)$ means the value of $\eta_j(z(\tilde{p}_k))$ if $\zeta_j = \eta_j(z)dz$). What can we say about $\dim \mathrm{Ker}(d\underline{\alpha}_{p_0})$? If $\zeta = \sum_{j=1}^{g} c_j\zeta_j \in \mathcal{L}^1(\mathcal{D}_n)$ we have $\zeta(\tilde{p}_k) = \sum_{j=1}^{g} c_j\zeta_j(\tilde{p}_k) = 0$. But we can even reverse this argument, ending up with

$$i(\mathcal{D}_n) = \dim \mathrm{Ker}(d\underline{\alpha}_{p_0}|_{\mathcal{D}_n}) = g - \mathrm{rank}(d\underline{\alpha}_{p_0}|_{\mathcal{D}_n}). \tag{A.86}$$

(We remark that this formula remains true for arbitrary $\mathcal{D}_n \in M_n$.)

Now we will prove a small (but useful) lemma:

Lemma A.9. *Let $\mathcal{D}_1, \mathcal{D}_2 \in M_g$ with $\underline{\alpha}_{p_0}(\mathcal{D}_1) = \underline{\alpha}_{p_0}(\mathcal{D}_2)$ and $i(\mathcal{D}_2) = 0$, then $\mathcal{D}_1 = \mathcal{D}_2$.*

Proof. By Abel's theorem $\mathcal{D}_1 - \mathcal{D}_2$ is principal and thus $r(\mathcal{D}_1 - \mathcal{D}_2) = 1$. From $r(-\mathcal{D}_2) = 1 + i(\mathcal{D}_2) = 1$ (Riemann-Roch) we conclude $\mathcal{L}(\mathcal{D}_1 - \mathcal{D}_2) = \mathcal{L}(-\mathcal{D}_2) \overset{(\sim)}{=} \mathbb{C}$ and therefrom $\mathcal{D}_1 - \mathcal{D}_2 = 0$. □

Next, we will show that $i(\mathcal{D}) = 0$ is no serious restriction for $\mathcal{D} \in M_g$.

Lemma A.10. *Let $\mathcal{D}_n \in M_n$ $(0 \le n \le g)$, then in every neighborhood of $\mathcal{D}_n$ there is a $\mathcal{D}'_n \in M_n$ with $i(\mathcal{D}'_n) = g - n$. We may even require $\mathcal{D}'_n(p) \le 1$, $p \in M$.*

Proof. Our lemma is true for $n = 1$, so let us assume it is also true for some $n - 1 \ge 1$. Let $\mathcal{D}_n = \mathcal{D}_{n-1} + \mathcal{D}_p$ and let $\{\omega_\ell\}_{1 \le \ell \le g-n+1}$ be a basis for $\mathcal{L}^1(\mathcal{D}'_{n-1})$. We can always find a neighborhood U_p of p such that none of the basis elements vanishes on $U_p \backslash \{p\}$ - so just take $\mathcal{D}'_n = \mathcal{D}'_{n-1} + \mathcal{D}_{\tilde{p}}$ with a suitable $\tilde{p} \in U_p \backslash \{p\}$. □

We have shown, that the divisors $\mathcal{D}'_n$ with $i(\mathcal{D}'_n) = g - n$ and $\mathcal{D}'_n(p) \le 1$, $p \in M$, are dense in M_n. Hence $\underline{\alpha}_{p_0} : M_g \to J(M)$ has maximal rank for such divisors and $\theta(\hat{\underline{\alpha}}_{p_0}(.) - \underline{z})$ does not vanish identically on any open subset of M_g. We set $W_n^{p_0} = \underline{\alpha}_{p_0}(M_n) \subset J(M)$ and $\hat{W}_n^{p_0} = \hat{\underline{\alpha}}_{p_0}(M_n) \subset \mathbb{C}^g$. (Our last statement now reads $\theta(\hat{W}_g^{p_0} - \underline{z}) \not\equiv 0$, where $\hat{W}_g^{p_0} - \underline{z}$ stands for $w - \underline{z}$ with $w \in \hat{W}_g^{p_0}$.) We observe $W_n^{p_0} \subset W_{n+1}^{p_0}$ (just add $\mathcal{D}_{p_0}$ to each divisor in $W_n^{p_0}$) and $\hat{W}_n^{p_0} \subset \hat{W}_{n+1}^{p_0}$.

We will now characterize the set of zeros of $\theta(\underline{z})$:

Theorem A.11. *Let $\underline{z} \in \mathbb{C}^g$. Then $\theta(\underline{z}) = 0$ if and only if $[\underline{z}] \in W_{g-1}^{p_0} + \underline{\Xi}_{p_0}$.*

Note: Though θ is not well-defined on $J(M)$ the set of zeros is a well-defined subset of $J(M)$. Observe also that $[\underline{z}] \in W_{g-1}^{p_0} + \underline{\Xi}_{p_0}$ is not dependent on the base point p_0 chosen!

Proof. Choose $\mathcal{D} \in M_{g-1}$ and $\tilde{p} \in M$ such that $i(\mathcal{D} + \mathcal{D}_{\tilde{p}}) = 0$. Choose a $\underline{z} \in \mathbb{C}^g$ with $[\underline{z}] = \underline{\alpha}_{p_0}(\mathcal{D} + \mathcal{D}_{\tilde{p}}) + \underline{\Xi}_{p_0}$ and consider (still assuming that $\hat{F}$ has no zeros on the boundary)

$$\hat{F}(p) = \theta(\hat{\underline{A}}_{p_0}(p) - \underline{z}). \tag{A.87}$$

If $\hat{F}$ is not identically zero, we have $\underline{\alpha}_{p_0}((\hat{F})) = [\underline{z}] - \underline{\Xi}_{p_0} = \underline{\alpha}_{p_0}(\mathcal{D} + \mathcal{D}_{\tilde{p}})$. Using Lemma A.9 we get $(\hat{F}) = \mathcal{D} + \mathcal{D}_{\tilde{p}}$ and thus $\hat{F}(\tilde{p}) = 0$. As this is trivially true if $\hat{F}$ vanishes identically, we are ready with the first part because the divisors $\mathcal{D}$ under consideration are dense and because $\hat{\underline{\alpha}}_{p_0} : M_{g-1} \to \mathbb{C}^g$ is continuous.

Conversely, suppose $\theta(\underline{z}) = 0$ and let s be the integer with $\theta(\hat{W}_{s-1}^{p_0} - \hat{W}_{s-1}^{p_0} - \underline{z}) \equiv 0$ and $\theta(\hat{W}_s^{p_0} - \hat{W}_s^{p_0} - \underline{z}) \not\equiv 0$. (Here $\hat{W}_g^{p_0} - \hat{W}_s^{p_0} - \underline{z}$ stands for $w_1 - w_2 - \underline{z}$ with $w_1, w_2 \in \hat{W}_g^{p_0}$.) We have $1 \le s \le g$. Thus we may choose two divisors $\mathcal{D}_s^1, \mathcal{D}_s^2 \in M_s$ such that $\theta(\hat{\underline{\alpha}}_{p_0}(\mathcal{D}_s^1) - \hat{\underline{\alpha}}_{p_0}(\mathcal{D}_s^2) - \underline{z}) \ne 0$ and $\mathcal{D}_s^2(p) \le 1$, $p \in M$. Denote $\mathcal{D}_{s-1}^1 = \mathcal{D}_s^1 - \mathcal{D}_{\tilde{p}}$ with $\tilde{p}$ such that $\mathcal{D}_s^1(\tilde{p}) \ne 0$ and consider the function

$$\hat{F}(p) = \theta(\hat{\underline{A}}_{p_0}(p) + \hat{\underline{\alpha}}_{p_0}(\mathcal{D}_{s-1}^1) - \hat{\underline{\alpha}}_{p_0}(\mathcal{D}_s^2) - \underline{z}). \tag{A.88}$$

Observe that $\hat{F}(\tilde{p}) \ne 0$ and that $\hat{F}(p) = 0$ if $\mathcal{D}_s^2(p) = 1$. Thus $(\hat{F})$ may be written as $\mathcal{D}_s^2 + \mathcal{D}_{g-s}$ with $\mathcal{D}_{g-s} \in M_{g-s}$. And calculating $\underline{\alpha}_{p_0}((\hat{F}))$ yields

$$\underline{\alpha}_{p_0}((\hat{F})) = -\underline{\alpha}_{p_0}(\mathcal{D}_{s-1}^1) + \underline{\alpha}_{p_0}(\mathcal{D}_s^2) + [\underline{z}] - \underline{\Xi}_{p_0} = \underline{\alpha}_{p_0}(\mathcal{D}_s^2) - \underline{\alpha}_{p_0}(\mathcal{D}_{g-s}). \tag{A.89}$$

Thus $[\underline{z}] = \underline{\alpha}_{p_0}(\mathcal{D}_{s-1}^1 + \mathcal{D}_{g-s}) + \underline{\Xi}_{p_0}$. □

Before we can proceed we need:

Lemma A.12. *Let $\mathcal{D}$ be an positive divisor on M. $r(-\mathcal{D}) \ge s \ge 1$ if and only if given any positive divisor $\mathcal{D}'$ of degree $< s$, there is an positive divisor $\mathcal{D}''$ such that $\mathcal{D}' + \mathcal{D}'' - \mathcal{D}$ is principal. For the if part it suffices to restrict $\mathcal{D}'$ to an open subset $U \subset M_{s-1}$.*

Proof. Since $-\mathcal{D}+\mathcal{D}'+\mathcal{D}''=(f)$ we have $f\in\mathcal{L}(-\mathcal{D}+\mathcal{D}')$. But if $r(\mathcal{D})<s$ we can always find a $\mathcal{D}'\in U$ with $r(-\mathcal{D}+\mathcal{D}')=0$ (Let $\{f_\ell\}_{1\le\ell\le d}$ be a basis for $\mathcal{L}(-\mathcal{D})$. Start with a suitable point p_1 where none of the functions f_ℓ vanishes, construct a new basis $\tilde{f}_\ell$ such that $\tilde{f}_\ell(p_1)=0$ for $\ell\ge 2$ – repeat $r(\mathcal{D})$ times and set $\mathcal{D}'=\sum_{\ell=1}^{d}\mathcal{D}_{p_\ell}$.).

Now assume $r(-\mathcal{D})=s$ and let $\mathcal{D}'\in M_{s-1}$. It follows that $r(\mathcal{D}'-\mathcal{D})\ge s-(s-1)=1$ (by a simple induction argument – compare (A.42)) and we can choose $\mathcal{D}''=(f)+\mathcal{D}-\mathcal{D}'$ for a nonzero $f\in\mathcal{L}(\mathcal{D}-\mathcal{D}')$. □

Note: $\mathcal{D}$ and $\mathcal{D}'+\mathcal{D}''$ have the same image under Abel's map and $r(-\mathcal{D})=i(\mathcal{D})+1$ for $\mathcal{D}\in M_g$ (by Riemann-Roch).

Now let $[\underline{z}]=\underline{\alpha}_{p_0}(\mathcal{D})+\underline{\Xi}_{p_0}$ with $\mathcal{D}\in M_{g-1}$ and $i(\mathcal{D})=s(\ge 1)$. Any point $\underline{\xi}\in(\hat{W}_{s-1}^{p_0}-\hat{W}_{s-1}^{p_0}-\underline{z})$ can be written as $\underline{\hat{\alpha}}_{p_0}(\mathcal{D}_{s-1}^1)-\underline{\hat{\alpha}}_{p_0}(\mathcal{D}_{s-1}^2)-\underline{z}$ with $\mathcal{D}_{s-1}^1,\mathcal{D}_{s-1}^2\in M_{s-1}$. Due to our lemma we can also write $[\underline{z}]$ as $\underline{\alpha}_{p_0}(\mathcal{D}_{s-1}^1+\mathcal{D}''_{g-s})+\underline{\Xi}_{p_0}$ and we get $[\underline{\xi}]=-\underline{\alpha}_{p_0}(\mathcal{D}_{s-1}^2+\mathcal{D}''_{g-s})-\underline{\Xi}_{p_0}\in-(W_{g-1}^{p_0}+\underline{\Xi}_{p_0})$ or equivalently $\theta(\underline{\xi})=0$.

On the other hand we may also vary the divisor $\mathcal{D}_s^1$ of the last theorem in a sufficiently small neighborhood getting $\mathcal{D}_{s-1}^{1\prime}$ and a corresponding $\mathcal{D}'_{g-s}$ also fulfilling $[\underline{z}]=\underline{\alpha}_{p_0}(\mathcal{D}_{s-1}^{1\prime}+\mathcal{D}'_{g-s})+\underline{\Xi}_{p_0}$. From Abel's theorem we conclude that $(\mathcal{D}_{s-1}^1+\mathcal{D}_{g-s})-(\mathcal{D}_{s-1}^{1\prime}+\mathcal{D}'_{g-s})$ is principal and from our last lemma: $i(\mathcal{D}_{s-1}^1+\mathcal{D}_{g-s})\ge s$. Thus we have proved most of **Riemann's vanishing theorem**:

Theorem A.13. (Riemann) *Let $\underline{z}\in\mathbb{C}^g$ and $s\ge 1$. $\theta(\hat{W}_{s-1}^{p_0}-\hat{W}_{s-1}^{p_0}-\underline{z})\equiv 0$ but $\theta(\hat{W}_{s}^{p_0}-\hat{W}_{s}^{p_0}-\underline{z})\not\equiv 0$ is equivalent to $[\underline{z}]=\underline{\alpha}_{p_0}(\mathcal{D})+\underline{\Xi}_{p_0}$ with $\mathcal{D}\in M_{g-1}$ and $i(\mathcal{D})=s$ which again is equivalent to the vanishing of all partial derivatives of θ of order less than s at $\underline{z}$ and to the non-vanishing of at least one derivative of order s at $\underline{z}$.*

Note: $s\ge 1$ implies $\theta(\underline{z})=0$, we always have $s\le g$ and $\hat{W}_{s-1}^{p_0}-\hat{W}_{s-1}^{p_0}$ is independent of the base point p_0.

Proof. We omit the proof of the part concerning the derivatives (cf. [**81**]), but we will prove the other part: In the one direction we already know the first part and $i(\mathcal{D})\ge s$. If $i(\mathcal{D})>s$ we would get $\theta(\hat{W}_{s}^{p_0}-\hat{W}_{s}^{p_0}-\underline{z})\equiv 0$ from the first statement above. In the other direction we would get from $\theta(\hat{W}_{s}^{p_0}-\hat{W}_{s}^{p_0}-\underline{z})\equiv 0$ that $i(\mathcal{D})>s$ using the first direction. □

With some simple changes in the proof of the last theorem, we can also get another one:

Theorem A.14. *Let $\underline{z}\in\mathbb{C}^g$. s is the least integer such that $\theta(\hat{W}_{s+1}^{p_0}-\hat{W}_{s}^{p_0}-\underline{z})\not\equiv 0$ is equivalent to $[\underline{z}]=\underline{\alpha}_{p_0}(\mathcal{D})+\underline{\Xi}_{p_0}$ with $\mathcal{D}\in M_g$ and $i(\mathcal{D})=s$.*

Note: We now have $0\le s\le g-1$. Since $\underline{z}$ was arbitrary we have solved **Jacobi's inversion problem**:

Corollary A.15. (Jacobi) $\underline{\alpha}_{p_0}: M_g\to J(M)$ *is surjective.*

Using these last theorems we can say a lot about the function

$$\hat{F}(p)=\theta(\underline{\hat{A}}_{p_0}(p)-\underline{z}),\qquad \underline{z}\in\mathbb{C}^g,\quad p\in\hat{M}. \tag{A.90}$$

Theorem A.16. *Let $\underline{z} \in \mathbb{C}^g$ and let s be the least integer such that $\theta(\hat{W}_{s+1}^{p_0} - \hat{W}_s^{p_0} - \underline{z}) \not\equiv 0$. Then $\hat{F} \not\equiv 0$ is equivalent to $s = 0$. We can write $[\underline{z}] = \underline{\alpha}_{p_0}(\mathcal{D}) + \underline{\Xi}_{p_0}$ with $\mathcal{D} \in M_g$ and $i(\mathcal{D}) = s$. Moreover, if $\hat{F} \not\equiv 0$ we have $(\hat{F}) = \mathcal{D}$ (and hence $i((\hat{F})) = 0$).*

Proof. Observe, that both sides are equivalent to $\hat{F}(p) = \theta(\underline{\hat{A}}_{p_0}(p) - \underline{z}) \not\equiv 0$ and $\mathcal{D}(p_0) = 0$. Uniqueness follows from Lemma A.9. □

Corollary A.17. *$\theta(\underline{z}) \neq 0$ is equivalent to $[\underline{z}] = \underline{\alpha}_{p_0}(\mathcal{D}) + \underline{\Xi}_{p_0}$ with a unique $\mathcal{D} \in M_g$, $i(\mathcal{D}) = 0$ and $\mathcal{D}(p_0) = 0$.*

Finally we prove Abel's theorem for differentials:

Theorem A.18. *$\mathcal{D} \in \mathrm{Div}(M)$ is canonical if and only if*

$$\underline{\alpha}_{p_0}(\mathcal{D}) = -2\,\underline{\Xi}_{p_0} \qquad \text{and} \qquad \deg(\mathcal{D}) = 2g - 2. \tag{A.91}$$

Proof. Since all canonical divisors lie in the same divisor class their image under Abel's map is a constant (by Abel's theorem). To determine this constant let $\underline{z}$ be a zero of θ. Hence $-\underline{z}$ is a zero too, and we conclude that $[\underline{z}] = \underline{\alpha}_{p_0}(\mathcal{D}_1) + \underline{\Xi}_{p_0}$ and $[-\underline{z}] = \underline{\alpha}_{p_0}(\mathcal{D}_2) + \underline{\Xi}_{p_0}$ with $\mathcal{D}_1, \mathcal{D}_2 \in M_{g-1}$. Combining these equations we get $\underline{\alpha}_{p_0}(\mathcal{D}_1 + \mathcal{D}_2) = -2\underline{\Xi}_{p_0}$. Since $\mathcal{D}_1$ is arbitrary (because $\underline{z}$ is) our last lemma yields $r(-\mathcal{D}_1 - \mathcal{D}_2) \geq g$ and from Riemann Roch $i(\mathcal{D}_1 + \mathcal{D}_2) \geq 1$. Thus $\mathcal{D}_1 + \mathcal{D}_2$ is the divisor of a holomorphic differential (since $\deg(\mathcal{D}_1 + \mathcal{D}_2) = 2g - 2$).

Conversely, suppose $\mathcal{D}$ is of degree $2g - 2$ and $\underline{\alpha}_{p_0}(\mathcal{D}) = -2\underline{\Xi}_{p_0}$. Let ω be a meromorphic differential. From Abel's theorem we get that $\mathcal{D} - (\omega)$ is the divisor of a meromorphic function f. Thus $\mathcal{D}$ is the divisor of $f\omega$. □

Corollary A.19. *$2\,\underline{\Xi}_{p_0} = [0]$ if and only if $(2g-2)\mathcal{D}_{p_0}$ is canonical.*

A.7. Hyperelliptic Riemann surfaces

Finally, we want to give a constructive approach to hyperelliptic Riemann surfaces. Let

$$\{E_n\}_{0 \leq n \leq 2g+1} \subset \mathbb{R}, \qquad E_0 < E_1 < \cdots < E_{2g+1}, \quad g \in \mathbb{N}_0, \tag{A.92}$$

be some fixed points. Using them we may define

$$\Pi = \mathbb{C} \backslash \Big(\bigcup_{j=0}^{g} [E_{2j}, E_{2j+1}] \Big), \tag{A.93}$$

which is obviously a domain (connected open subset of $\mathbb{C}$) and a holomorphic function

$$\begin{aligned} R_{2g+2}^{1/2}(.) : \ \Pi &\to \mathbb{C} \\ z &\mapsto -\prod_{n=0}^{2g+1} \sqrt{z - E_n}\ , \end{aligned} \tag{A.94}$$

where the square root is defined as follows,

$$\sqrt{z} = |\sqrt{z}| \mathrm{e}^{\mathrm{i}\arg(z)/2}, \qquad \arg(z) \in (-\pi, \pi]. \tag{A.95}$$

We first extend $R^{1/2}_{2g+2}(z)$ to the whole of $\mathbb{C}$ by

$$R^{1/2}_{2g+2}(z) = \lim_{\varepsilon \downarrow 0} R^{1/2}_{2g+2}(x + \mathrm{i}\varepsilon) \tag{A.96}$$

for $z \in \mathbb{C}\backslash\Pi$. This implies $R^{1/2}_{2g+2}(z) = \mathrm{i}^{2g-n}|R^{1/2}_{2g+2}(z)|$ for $z \in (E_{n-1}, E_n)$, $0 \leq n \leq 2g+2$, if we set $E_{-1} = -\infty$ and $E_{2g+2} = \infty$.

We may now define the following set

$$M = \{(z, \sigma R^{1/2}_{2g+2}(z)) | z \in \mathbb{C},\ \sigma \in \{-1, +1\}\} \cup \{\infty_+, \infty_-\} \tag{A.97}$$

and call $B = \{(E_n, 0)\}_{0 \leq n \leq 2g+1}$ the set of branch points.

Defining the following charts (ζ_{p_0}, U_{p_0}), we can make the set M into a Riemann surface. Abbreviate

$$p_0 = (z_0, \sigma_0 R^{1/2}_{2g+2}(z_0)),\ p = (z, \sigma R^{1/2}_{2g+2}(z)) \in U_{p_0} \subset M, \tag{A.98}$$

$U'_{p_0} = \zeta_{p_0}(U_{p_0}) \subset \mathbb{C}$.

If $p_0 \notin B$ we set:

$U_{p_0} = \{p \in M | |z - z_0| < C \text{ and } \sigma R^{1/2}_{2g+2}(z) \xrightarrow{\gamma} \sigma_0 R^{1/2}_{2g+2}(z_0)\}$, $U'_{p_0} = \{\zeta \in \mathbb{C} | |\zeta| < C\}$, $C = \min_n |z_0 - E_n| > 0$,

$$\begin{array}{lllllll} \zeta_{p_0}: & U_{p_0} & \to & U'_{p_0} & \zeta^{-1}_{p_0}: & U'_{p_0} & \to & U_{p_0} \\ & p & \mapsto & z - z_0 & & \zeta & \mapsto & (\zeta + z_0, \sigma R^{1/2}_{2g+2}(\zeta + z_0)) \end{array}, \tag{A.99}$$

where $\sigma R^{1/2}_{2g+2}(z) \xrightarrow{\gamma} \sigma_0 R^{1/2}_{2g+2}(z_0)$ means that $\sigma R^{1/2}_{2g+2}(z)$ is the branch reached by analytic continuation along γ, the straight line from z to z_0.

If $p_0 = (E_m, 0)$ we set:

$U_{p_0} = \{p \in M | |z - E_m| < C\}$, $U'_{p_0} = \{\zeta \in \mathbb{C} | |\zeta| < \sqrt{C}\}$, $C = \min_{n \neq m} |E_m - E_n| > 0$,

$$\begin{array}{l} \zeta_{p_0}: U_{p_0} \to U'_{p_0} \\ \qquad p \mapsto \sigma(z - E_m)^{1/2}, \\ \zeta^{-1}_{p_0}: U'_{p_0} \to U_{p_0} \\ \qquad \zeta \mapsto (\zeta^2 + E_m, \zeta \prod\limits_{n \neq m} (\zeta^2 + E_m - E_n)^{1/2}), \end{array} \tag{A.100}$$

where the left root is defined as $(z - E_m)^{1/2} = \sqrt{|z - E_m|} \exp(\mathrm{i} \arg(z - E_m)/2)$, with $0 \leq \arg(z - E_m) < 2\pi$ if m is even and $-\pi < \arg(z - E_m) \leq \pi$ if m is odd. The right root is holomorphic on U_{p_0} with the sign fixed by

$$\begin{aligned} \prod_{n \neq m} (\zeta^2 + E_m - E_n)^{1/2} &= \mathrm{i}^{2g-m-1} \Big| \prod_{n \neq m} \sqrt{E_m - E_n} \Big| \times \\ &\times \Big(1 - \frac{1}{2} \sum_{n \neq m} \frac{1}{E_m - E_n} \zeta^2 + O(\zeta^4)\Big). \end{aligned} \tag{A.101}$$

If $p_0 = \infty_\pm$ we set:

$$U_{p_0} = \{p \in M \,||z| > C\} \quad U'_{p_0} = \{\zeta \in \mathbb{C} \,||\zeta| < \tfrac{1}{C}\} \quad C = \max_n |E_n| < \infty$$

$$\begin{array}{cccccccc} \zeta_{p_0}: & U_{p_0} & \to & U'_{p_0} & \zeta_{p_0}^{-1}: & U'_{p_0} & \to & U_{p_0} \\ & p & \mapsto & \frac{1}{z} & & \zeta & \mapsto & (\frac{1}{\zeta}, \frac{\pm\prod_n (1-\zeta E_n)^{1/2}}{\zeta^{g+1}}) \\ & \infty_\pm & \mapsto & 0 & & 0 & \mapsto & \infty_\pm \end{array} \tag{A.102}$$

where the right root is holomorphic on U_{p_0} with the sign fixed by

$$\prod_n (1-\zeta E_n)^{1/2} = -1 + \frac{1}{2} \sum_n E_n \zeta + O(\zeta^2). \tag{A.103}$$

Let us take two subsets

$$\Pi_\pm = \{(z, \pm R^{1/2}_{2g+2}(z)) | z \in \Pi\} \subset M, \tag{A.104}$$

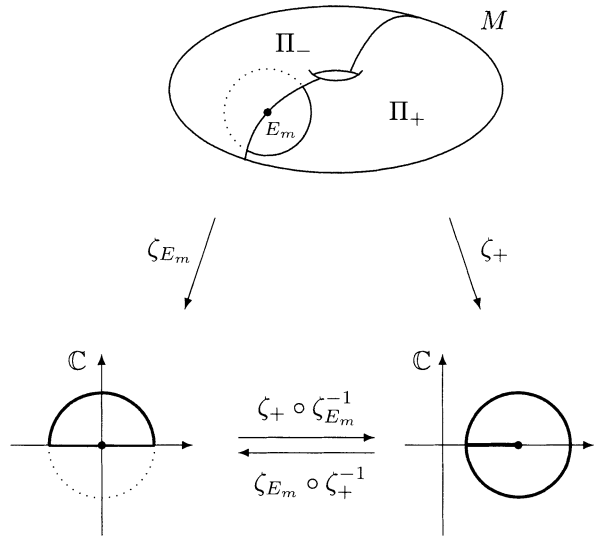
and define two more quite useful charts

$$\begin{array}{cccc} \zeta_\pm: & \Pi_\pm & \to & \Pi \\ & p & \mapsto & z \end{array}. \tag{A.105}$$

It is not hard to verify, that the transition functions of all these charts are indeed holomorphic, for example,

$$\begin{array}{cccc} \zeta_\pm \circ \zeta_{E_m}^{-1}: & U'_{E_m} \cap \zeta_{E_m}(\Pi_\pm) & \to & \Pi \cap \zeta_\pm(U_{E_m}), \\ & \zeta & \mapsto & \zeta^2 + E_m \\ \zeta_{E_m} \circ \zeta_\pm^{-1}: & \Pi \cap \zeta_\pm(U_{E_m}) & \to & U'_{E_m} \cap \zeta_{E_m}(\Pi_\pm). \\ & \zeta & \mapsto & \pm\sqrt{\zeta - E_m} \end{array} \tag{A.106}$$

Compare the following picture ($g = 1$, m odd) and note, that the branch cut of the square root does not belong to its domain!



The topology induced by these charts is Hausdorff (take the U's from above and modify C a little if necessary) and second countable (since we only need countably (in particular even finitely) many of these charts to cover M). Consider $\Pi_\pm$, they are connected (since they are homeomorphic to Π). Their closures $\overline{\Pi_\pm}$ are still connected, have points in common and fulfill $M = \overline{\Pi_+} \cup \overline{\Pi_-}$. Thus M is connected and a Riemann surface. But they tell us even more: $\overline{\Pi_\pm}$ is just a sphere with $g+1$ holes, and if we want to get M, we have to identify corresponding points on the boundaries ending up with a sphere having g handles.

We will now define three maps on M. First the sheet exchange map,

$$\begin{array}{rlcl} * : & M & \to & M \\ & (z, \sigma R^{1/2}_{2g+2}(z)) & \mapsto & (z, \sigma R^{1/2}_{2g+2}(z))^* = (z, -\sigma R^{1/2}_{2g+2}(z)) \\ & \infty_\pm & \mapsto & \infty_\mp \end{array} . \tag{A.107}$$

It is clearly holomorphic, since it is locally just the identity (up to a sign – using the charts of the previous page). And second the projection

$$\begin{array}{rlcl} \pi : & M & \to & \mathbb{C} \cup \{\infty\} \\ & (z, \sigma R^{1/2}_{2g+2}(z)) & \mapsto & z \\ & \infty_\pm & \mapsto & \infty \end{array} \tag{A.108}$$

and the evaluation map

$$\begin{array}{rlcl} R^{1/2}_{2g+2}(.) : & M & \to & \mathbb{C} \cup \{\infty\} \\ & (z, \sigma R^{1/2}_{2g+2}(z)) & \mapsto & \sigma R^{1/2}_{2g+2}(z) \\ & \infty_\pm & \mapsto & \infty \end{array} . \tag{A.109}$$

Obviously, both are meromorphic functions (use again the local charts and their inversions). π has poles of order 1 at $\infty_\pm$ and two simple zeros at $(0, \pm R^{1/2}_{2g+2}(0))$ if $R^{1/2}_{2g+2}(0) \neq 0$ respectively one double zero at $(0,0)$ if $R^{1/2}_{2g+2}(0) = 0$. $R^{1/2}_{2g+2}(.)$

has poles of order $g+1$ at $\infty_\pm$ and $2g+2$ simple zeros at $(E_n, 0)$. Notice also $\pi(p^*) = \pi(p)$ and $R_{2g+2}^{1/2}(p^*) = -R_{2g+2}^{1/2}(p)$.

Thus we may conclude that M is a two-sheeted, ramified covering of the Riemann sphere ($\cong \mathbb{C} \cup \{\infty\}$), that M is compact (since π is open and $\mathbb{C} \cup \{\infty\}$ is compact), and finally, that M is hyperelliptic (since it admits a function of degree two).

Now consider

$$\frac{d\pi}{R_{2g+2}^{1/2}}. \tag{A.110}$$

Again using local charts we see that $d\pi/R_{2g+2}^{1/2}$ is holomorphic everywhere and has zeros of order $g-1$ at $\infty_\pm$. So we may conclude that

$$\frac{\pi^{j-1}d\pi}{R_{2g+2}^{1/2}}, \qquad 1 \le j \le g, \tag{A.111}$$

form a basis for the space of holomorphic differentials. As a consequence we obtain the following result.

Lemma A.20. *Let $\mathcal{D} \ge 0$ and denote by $\tilde{\mathcal{D}}$ the divisor on $\mathbb{C} \cup \{\infty\}$ defined by $\tilde{\mathcal{D}}(z) = \max_{p \in \pi^{-1}(z)} \mathcal{D}(p)$, $z \in \mathbb{C} \cup \{\infty\}$. Then*

$$i(\mathcal{D}) = \max\{0, g - \deg(\tilde{\mathcal{D}})\}. \tag{A.112}$$

Proof. Each $\zeta \in \mathcal{L}^1(\mathcal{D})$ must be of the form

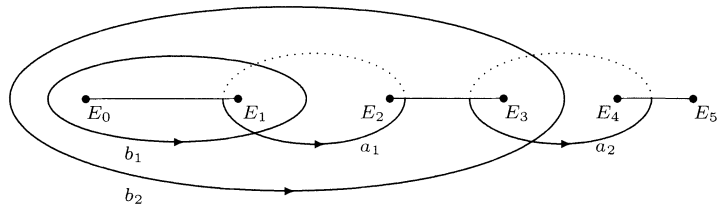
$$\zeta = \frac{(P \circ \pi)d\pi}{R_{2g+2}^{1/2}}, \tag{A.113}$$

where $P(z)$ is a polynomial of degree at most $g - \tilde{\mathcal{D}}(\infty) - 1$. Moreover, $P(z)$ must vanish of order $\tilde{\mathcal{D}}(z)$ at $z \in \mathbb{C}$. Hence we have $\max\{0, g - \deg(\tilde{\mathcal{D}})\}$ free constants for $P(z)$. □

Next, we will introduce the representatives $\{a_\ell, b_\ell\}_{\ell=1}^g$ of a canonical homology basis for M. For a_ℓ we start near $E_{2\ell-1}$ on Π_+, surround $E_{2\ell}$ thereby changing to Π_- and return to our starting point encircling $E_{2\ell-1}$ again changing sheets. For b_ℓ we choose a cycle surrounding $E_0, E_{2\ell-1}$ counter-clock-wise (once) on Π_+. The cycles are chosen so that their intersection matrix reads

$$a_j \circ b_k = \delta_{jk}, \qquad 1 \le j, k \le g. \tag{A.114}$$

Visualizing for $g = 2$:



Here the solid lines indicate the parts on Π_+ and the doted ones the parts on Π_-.

Now we can construct a canonical basis $\{\zeta_j\}_{j=1}^g$ for the space of holomorphic differentials in the following way: If we introduce the constants $\underline{c}(.)$ via

$$c_j(k) = C_{jk}^{-1}, \qquad C_{jk} = \int_{a_k} \frac{\pi^{j-1}d\pi}{R_{2g+2}^{1/2}} = 2\int_{E_{2k-1}}^{E_{2k}} \frac{z^{j-1}dz}{R_{2g+2}^{1/2}(z)} \in \mathbb{R}. \tag{A.115}$$

(C is invertible since otherwise there would be a nonzero differential with vanishing a-periods.) the differentials

$$\underline{\zeta} = \sum_{j=1}^{g} \underline{c}(j) \frac{\pi^{j-1}d\pi}{R_{2g+2}^{1/2}} \tag{A.116}$$

fulfill

$$\int_{a_k} \zeta_j = \delta_{j,k} \tag{A.117}$$

and are thus a basis of the required form. The matrix of b-periods

$$\tau_{j,k} = \int_{b_k} \zeta_j = -\sum_{j=1}^{g} c_j(m) \sum_{\ell=0}^{k-1} 2\int_{E_{2\ell}}^{E_{2\ell+1}} \frac{z^{m-1}dz}{R_{2g+2}^{1/2}(z)} \tag{A.118}$$

satisfies $\mathrm{Re}(\tau_{j,k}) = 0$ since we have $\mathrm{Im}(c_j(k)) = 0$. In the chart $(U_\pm, w) = (U_{\infty_\pm}, \zeta_{\infty_\pm})$ we have

$$\begin{aligned} \underline{\zeta} &= \pm \sum_{j=1}^{g} \underline{c}(n) \frac{w^{g-j}\, dw}{\sqrt{\prod_m (1 - wE_m)}} \\ &= \pm\Big(\underline{c}(g) + \Big(\frac{\underline{c}(g)}{2} \sum_{j=1}^{2g+2} E_j + \underline{c}(g-1)\Big)w + O(w^2)\Big)dw. \end{aligned} \tag{A.119}$$

Finally, we will show how to evaluate some integrals on $\hat{M}$. We fix a base point $p_0 = (E_0, 0)$. Then we have, for example,

$$\underline{A}_{p_0}(\infty_+) = -\sum_{j=1}^{g} \underline{c}(n) \int_{-\infty}^{E_0} \frac{z^{j-1}dz}{R_{2g+2}^{1/2}(z)} - \sum_{j=1}^{g} \underline{\delta}_j. \tag{A.120}$$

To see this, take the straight line segment from E_0 to $-\infty$ and lift it to Π_+. Unfortunately, we cannot take this lift as our path of integration for it intersects all b cycles, whereas on the other hand we are required to stay in $\hat{M}$. Hence, rather than crossing b_j we will follow the boundary of $\hat{M}$ along b_j, a_j^{-1}, b_j^{-1} until

we hit the corresponding point on b_j^{-1}. The parts on b_j, b_j^{-1} cancel (compare the picture on page 275) but the part on a_j^{-1} contributes. In other words, for each b-cycle we cross, we have to subtract the associated a-period. Similarly, $\underline{\hat{A}}_{p_0}(\infty_-) = -\underline{\hat{A}}_{p_0}(\infty_+) - \sum_{j=1}^{g} \underline{\delta}_j$.

Next, let us try to compute $\underline{\hat{\Xi}}_{p_0}$. As before one shows

$$\hat{A}_{p_0,j}((E_{2\ell-1},0)) = -\frac{\tau_{j,\ell}}{2} \tag{A.121}$$

and hence

$$\int_{a_\ell} \hat{A}_{p_0,j}\zeta_\ell = -\frac{\tau_{j,\ell}}{2} + \int_{a_\ell} \hat{A}_{(E_{2\ell-1},0),j}\zeta_\ell. \tag{A.122}$$

The last integral can be split up into two parts, one from $E_{\ell-1}$ to $E_{2\ell}$ on Π_+ and one from E_ℓ to $E_{2\ell-1}$ on Π_- . Since these two parts cancel we infer

$$\hat{\Xi}_{p_0,j} = \frac{1}{2} - \frac{1}{2}\sum_{k=1}^{g} \tau_{j,k}. \tag{A.123}$$

Herglotz functions

The results stated in this appendix are collected from [**16**], [**17**], [**68**], [**73**], [**119**], [**198**], [**199**], [**200**], [**202**], [**203**], [**206**].

In this chapter an important role will be played by **Borel measures** ρ on $\mathbb{R}$. Hence we fix some notation first. For any Borel set $B \subseteq \mathbb{R}$ we will denote by $\rho(B) = \int_B d\rho$ its ρ-measure. Moreover, to each Borel measure ρ on $\mathbb{R}$ corresponds a monotone increasing function (also denoted by ρ for simplicity) which we normalize as

$$\rho(\lambda) = \begin{cases} -\rho\big((\lambda, 0]\big), & \lambda < 0 \\ 0, & \lambda = 0 \\ \rho\big((0, \lambda]\big), & \lambda > 0 \end{cases}. \tag{B.1}$$

By definition, $\rho(\lambda)$ is right continuous.

Associated to ρ is the (separable) Hilbert space $L^2(\mathbb{R}, d\rho)$ with scalar product

$$\langle f, g\rangle = \int_{\mathbb{R}} \overline{f(\lambda)} g(\lambda) d\rho(\lambda). \tag{B.2}$$

It has the following properties.

Lemma B.1. *Suppose $\rho(\mathbb{R}) < \infty$. The set of all continuous functions is dense in $L^2(\mathbb{R}, d\rho)$. Moreover, if ρ is compactly supported ($\rho(K) = \rho(\mathbb{R})$ for a compact set $K \subseteq \mathbb{R}$), then the set of polynomials is also dense.*

Proof. First of all, bounded functions are dense. In fact, consider

$$f_n(\lambda) = \begin{cases} f(\lambda), & |f(\lambda)| \le n \\ 0 \quad , & |f(\lambda)| > n \end{cases}. \tag{B.3}$$

Then $|f(\lambda) - f_n(\lambda)| \to 0$ a.e. since $|f(\lambda)| < \infty$ a.e.. Since $|f(\lambda) - f_n(\lambda)| \le |f(\lambda)|$ we obtain $\int |f - f_n|^2 d\rho \to 0$ by the dominated convergence theorem.

So it remains to show that bounded elements can be approximated by continuous ones. By Lusin's theorem, there are continuous functions f_n such that

$\|f_n\|_\infty \le \|f\|_\infty$ and

$$\rho(B_n) \le \frac{1}{n}, \quad B_n = \{\lambda | f(\lambda) \neq f_n(\lambda)\}. \tag{B.4}$$

Here $\|f\|_\infty$ denotes the essential supremum of f, that is, the smallest constant M such that $|f(\lambda)| \le M$ a.e.. Hence

$$\|f - f_n\|^2 = \int_{B_n} |f_n(\lambda) - f(\lambda)|^2 d\rho(\lambda) \le \frac{4\|f\|_\infty^2}{n} \to 0 \tag{B.5}$$

and the first claim follows.

To show the second, it suffices to prove that a continuous element f can be approximated by polynomials. By the Stone-Weierstrass Theorem there is a polynomial P_n approximating f in the sup norm on K such that

$$\|f - P_n\|^2 \le \rho(\mathbb{R})\|f - P_n\|_\infty^2 \le \frac{\rho(\mathbb{R})}{n} \to 0 \tag{B.6}$$

and we are done. □

If $\rho(\mathbb{R}) < \infty$ we can define the **Borel transform** (also **Cauchy transform**) of ρ by

$$F(z) = \int_{\mathbb{R}} \frac{d\rho(\lambda)}{z - \lambda}, \qquad z \in \mathbb{C}_\pm, \tag{B.7}$$

where $\mathbb{C}_\pm = \{z \in \mathbb{C} | \ \pm \mathrm{Im}(z) > 0\}$. In addition, we can also define the **Fourier transform**

$$T(\lambda) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\lambda t} d\rho(t), \qquad \lambda \in \mathbb{R}, \tag{B.8}$$

which is a bounded function and is related to the Borel transform via

$$F(z) = \begin{cases} \mathrm{i}\int_{-\infty}^0 \mathrm{e}^{-\mathrm{i}z\lambda} T(\lambda) d\lambda, & \mathrm{Im}(z) > 0 \\ -\mathrm{i}\int_0^\infty \mathrm{e}^{-\mathrm{i}z\lambda} T(\lambda) d\lambda, & \mathrm{Im}(z) < 0 \end{cases} \tag{B.9}$$

as can be verified using Fubini's theorem.

The Borel transform has the interesting property that it maps the upper half plane $\mathbb{C}_+$ into itself as can be seen from

$$\mathrm{Im}\big(F(z)\big) = \mathrm{Im}(z) \int_{\mathbb{R}} \frac{d\rho(\lambda)}{|z - \lambda|^2}. \tag{B.10}$$

In general, a holomorphic function $F : \mathbb{C}_+ \to \mathbb{C}_+$ is called a **Herglotz function** (sometimes also Pick or Nevanlinna-Pick function). It is no restriction to assume that F is defined on $\mathbb{C}_- \cup \mathbb{C}_+$ satisfying $F(\bar{z}) = \overline{F(z)}$.

The following theorem shows that all Herglotz functions arise as Borel transform (in an extended sense) of some unique measure.

Theorem B.2. *F is a Herglotz function if and only if*

$$F(z) = a + b\,z + \int_{\mathbb{R}} \Big(\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2}\Big) d\rho(\lambda), \quad z \in \mathbb{C}_\pm, \tag{B.11}$$

where $a, b \in \mathbb{R}$, $b \ge 0$, and ρ is a nonzero measure on $\mathbb{R}$ which satisfies $\int_{\mathbb{R}}(1 + \lambda^2)^{-1} d\rho(\lambda) < \infty$.

Moreover, the triple a, b, and ρ is unambiguously determined by F using

$$a = \mathrm{Re}\big(F(\mathrm{i})\big), \qquad \lim_{\substack{z\to\infty \\ \mathrm{Im}(z)\geq\varepsilon>0}} \frac{F(z)}{z} = b \geq 0, \tag{B.12}$$

and **Stieltjes inversion formula**

$$\rho\big((\lambda_0,\lambda_1]\big) = \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_{\lambda_0+\delta}^{\lambda_1+\delta}\mathrm{Im}\big(F(\lambda+\mathrm{i}\varepsilon)\big)d\lambda. \tag{B.13}$$

The content of Stieltjes inversion formula can be strengthened.

Lemma B.3. *Let F be a Herglotz function with associated measure ρ. Then the measure $\pi^{-1}\mathrm{Im}(F(\lambda+\mathrm{i}\varepsilon))d\lambda$ converges weakly to $d\rho(\lambda)$ in the sense that*

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_{\mathbb{R}} f(\lambda)\mathrm{Im}\big(F(\lambda+\mathrm{i}\varepsilon)\big)d\lambda = \int_{\mathbb{R}} f(\lambda)d\rho(\lambda) \tag{B.14}$$

for all continuous functions f with $|f| \leq const(1+\lambda^2)^{-1}$. In addition,

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_{\lambda_0}^{\lambda_1} f(\lambda)\mathrm{Im}\big(F(\lambda+\mathrm{i}\varepsilon)\big)d\lambda &= \int_{(\lambda_0,\lambda_1)} f(\lambda)d\rho(\lambda) \\ &+ \frac{f(\lambda_1)\rho(\{\lambda_1\}) - f(\lambda_0)\rho(\{\lambda_0\})}{2}.\end{aligned} \tag{B.15}$$

The following result shows that a different analytic continuation of F from $\mathbb{C}_+$ to $\mathbb{C}_-$ can exist in some cases.

Lemma B.4. *Let $(\lambda_1,\lambda_2) \subseteq \mathbb{R}$ and suppose $\lim_{\varepsilon\downarrow 0}\mathrm{Re}(F(\lambda+i\varepsilon)) = 0$ for a.e. $\lambda \in (\lambda_1,\lambda_2)$. Then F can be analytically continued from $\mathbb{C}_+$ into $\mathbb{C}_-$ through the interval (λ_1,λ_2). The resulting function $\tilde{F}(z)$ coincides with $F(z)$ on $\mathbb{C}_+$ and satisfies*

$$\overline{\tilde{F}(z)} = -\tilde{F}(\overline{z}). \tag{B.16}$$

In addition, $\mathrm{Im}(F(\lambda+i0)) > 0$, $\mathrm{Re}(F(\lambda+i0)) = 0$ for all $\lambda \in (\lambda_1,\lambda_2)$.

The measure ρ is called spectral measure of F. The set of all **growth points**, that is,

$$\sigma(\rho) = \{\lambda \in \mathbb{R} | \rho((\lambda-\varepsilon,\lambda+\varepsilon)) > 0 \text{ for all } \varepsilon > 0\}, \tag{B.17}$$

is called the spectrum of ρ. Invoking Morea's together with Fubini's theorem shows that $F(z)$ is holomorphic for $z \in \mathbb{C}\backslash\sigma(\rho)$. The converse following from Stieltjes inversion formula. Moreover, we have

$$\frac{dF(z)}{dz} = -\int_{\mathbb{R}}\frac{d\rho(\lambda)}{(\lambda-z)^2}. \tag{B.18}$$

In particular, if $\rho\big((\lambda_0,\lambda_1)\big) = 0$, then $F(\lambda)$, $\lambda \in (\lambda_0,\lambda_1)$, is decreasing and hence tends to a limit (in $\mathbb{R}\cup\{\infty\}$) as $\lambda \downarrow \lambda_0$ or $\lambda \uparrow \lambda_1$.

The following result explains why $\sigma(\rho)$ is called the spectrum of ρ.

Lemma B.5. *The set $\sigma(\rho)$ is precisely the spectrum $\sigma(\tilde{H})$ of the multiplication operator $\tilde{H}f(\lambda) = \lambda f(\lambda)$, $\mathfrak{D}(\tilde{H}) = \{f \in L^2(\mathbb{R},d\rho) | \lambda f(\lambda) \in L^2(\mathbb{R},d\rho)\}$.*

Proof. If $\lambda \in \sigma(\rho)$, then the sequence $f_n = \rho((\lambda - \frac{1}{n}, \lambda + \frac{1}{n}))^{-1/2}\chi_{(\lambda-\frac{1}{n},\lambda+\frac{1}{n})}$ satisfies $\|f_n\| = 1$, $\|(\tilde{H} - \lambda)f_n\| \to 0$ and hence $\lambda \in \sigma(\tilde{H})$. Conversely, if $z \notin \sigma(\rho)$, then $(\tilde{H} - z)^{-1}f(\lambda) = (\lambda - z)^{-1}f(\lambda)$ exists and is bounded, implying $z \notin \sigma(\tilde{H})$. □

Our main objective is to characterize $\sigma(\rho)$, and various decompositions of $\sigma(\rho)$, in terms of $F(z)$. More precisely, this will be done by investigating the boundary behavior of $F(\lambda + \mathrm{i}\varepsilon)$, $\lambda \in \mathbb{R}$, as $\varepsilon \downarrow 0$.

First we recall the unique decomposition of ρ with respect to Lebesgue measure,

$$d\rho = d\rho_{ac} + d\rho_s, \tag{B.19}$$

where ρ_{ac} is **absolutely continuous** with respect to Lebesgue measure (i.e., we have $\rho_{ac}(B) = 0$ for all B with Lebesgue measure zero) and ρ_s is **singular** with respect to Lebesgue measure (i.e., ρ_s is supported, $\rho_s(\mathbb{R}\backslash B) = 0$, on a set B with Lebesgue measure zero). The singular part ρ_s can be further decomposed into a **(singular) continuous** and a **pure point** part,

$$d\rho_s = d\rho_{sc} + d\rho_{pp}, \tag{B.20}$$

where ρ_{sc} is continuous on $\mathbb{R}$ and ρ_{pp} is a step function.

By the **Radon-Nikodym Theorem** we have

$$d\rho_{ac}(\lambda) = f(\lambda)d\lambda \tag{B.21}$$

for a locally integrable positive function f which is unique (a.e.). Moreover, we note that the singular part can be characterized by the following lemma.

Lemma B.6. *For any Borel set B we have*

$$\rho_s(B) = \lim_{n\to\infty} \sup_{I\in\mathcal{I},|I|<1/n} \rho(B \cap I), \tag{B.22}$$

where $\mathcal{I}$ is the (countable) family of finite unions of open intervals, each of which has rational endpoints, and $|I|$ denotes the Lebesgue measure of I.

Proof. Since the supremum decreases as n increases, the limit exists. Denote it by $\tilde{\rho}(B)$. Using $d\rho_{ac}(\lambda) = f(\lambda)d\lambda$ we see

$$\rho(B \cap I) \leq \rho_s(B) + R|I| + \rho_{ac}(f^{-1}((R, \infty))). \tag{B.23}$$

implying $\tilde{\rho}(B) \leq \rho_s(B) + \lim_{R\to\infty} \rho_{ac}(f^{-1}((R, \infty))) = \rho_s(B)$.

Conversely, pick a support B_s for ρ_s of Lebesgue measure $|B_s|$ zero. By $0 = |B_s| = \sup\{|O|\,|B_s \subset O, O \text{ open}\}$ there exists a sequence O_n such that $B_s \subset O_n$ and $|O_n| < 1/n$. Moreover, we can find a sequence $I_m \in \mathcal{I}$ such that $I_m \subseteq I_{m+1}$ and $\bigcup I_m = O_n$ (hence $\rho(I_m) \to \rho(O_n)$). Using $\rho(B \cap I_m) \geq \rho(B \cap O_n) - \rho(O_n \backslash I_m)$ concludes the proof

$$\sup_{I\in\mathcal{I},|I|<1/n} \rho(B \cap I) \geq \rho(B \cap O_n) \geq \rho(B \cap B_s) \geq \rho_s(B). \tag{B.24}$$

□

Now we turn to the concept of a **minimal support** (sometimes also essential support) of a measure. A set M is called a minimal support for ρ if M is a support

(i.e., $\rho(\mathbb{R}\backslash M) = 0$) and any subset $M_0 \subseteq M$ which does not support M (i.e, $\rho(M_0) = 0$) has Lebesgue measure zero. Let

$$\operatorname{Im}(F(\lambda)) = \limsup_{\varepsilon \downarrow 0} \operatorname{Im}(F(\lambda + \mathrm{i}\varepsilon)) \tag{B.25}$$

and let $L(\rho)$ be the set of all $\lambda \in \mathbb{R}$ for which $\lim_{\varepsilon\downarrow 0} \operatorname{Im}(F(\lambda + \mathrm{i}\varepsilon))$ exists (finite or infinite). Then we have

Lemma B.7. *Minimal supports M, M_{ac}, M_s for ρ, ρ_{ac}, ρ_s, respectively, are given by*

$$\begin{aligned} M &= \{\lambda \in L(\rho) | 0 < \operatorname{Im}(F(\lambda)) \leq \infty\}, \\ M_{ac} &= \{\lambda \in L(\rho) | 0 < \operatorname{Im}(F(\lambda)) < \infty\}, \\ M_s &= \{\lambda \in L(\rho) | \operatorname{Im}(F(\lambda)) = \infty\}. \end{aligned} \tag{B.26}$$

In particular, the ρ-measure and the Lebesgue measure of $\mathbb{R}\backslash L(\rho)$ are both zero. This also says that if $\operatorname{Im}(F(\lambda)) < \infty$ for all $\lambda \in (\lambda_1, \lambda_2)$, then ρ is purely absolutely continuous on (λ_1, λ_2). Furthermore, M_s has Lebesgue measure zero and might be nonempty even though $\rho_s = 0$. However, note that if M_s is a countable set, then $\rho_{sc} = 0$.

It is interesting to know when the minimal supports determine the corresponding spectra.

The spectrum of ρ is given by

$$\sigma(\rho) = \overline{M}. \tag{B.27}$$

To see this observe that F is real holomorphic near $\lambda \notin \sigma(\rho)$ and hence $\operatorname{Im}(F(\lambda)) = 0$ in this case. Thus $M \subseteq \sigma(\rho)$ and since $\sigma(\rho)$ is closed we even have $\overline{M} \subseteq \sigma(\rho)$. Conversely, if $\lambda \in \sigma(\rho)$, then $0 < \rho((\lambda - \varepsilon, \lambda + \varepsilon)) = \rho((\lambda - \varepsilon, \lambda + \varepsilon) \cap M)$ for all $\varepsilon > 0$ and we can find a sequence $\lambda_n \in (\lambda - 1/n, \lambda + 1/n) \cap M$ converging to λ from inside M. This is the remaining part $\sigma(\rho) \subseteq \overline{M}$.

Moreover, $\sigma(\rho_{ac})$ can be recovered from the essential closure of M_{ac}, that is,

$$\sigma(\rho_{ac}) = \overline{M}_{ac}^{ess}, \tag{B.28}$$

where

$$\overline{M}_{ac}^{ess} = \{\lambda \in \mathbb{R} | |(\lambda - \varepsilon, \lambda + \varepsilon) \cap M_{ac}| > 0 \text{ for all } \varepsilon > 0\}. \tag{B.29}$$

Note that $\overline{M}_{ac}^{ess}$ is closed, whereas we might have $M_{ac} \not\subset \overline{M}_{ac}^{ess}$. To prove (B.28) we use that $0 < \rho_{ac}((\lambda - \varepsilon, \lambda + \varepsilon)) = \rho_{ac}((\lambda - \varepsilon, \lambda + \varepsilon) \cap M_{ac})$ is equivalent to $|(\lambda - \varepsilon, \lambda + \varepsilon) \cap M_{ac}| > 0$. One direction follows from the definition of absolute continuity and the other from minimality of M_{ac}.

Next we define the derivative

$$D\rho(\lambda) = \limsup_{\varepsilon \downarrow 0} \frac{\rho\big((\lambda - \varepsilon, \lambda + \varepsilon)\big)}{2\varepsilon}. \tag{B.30}$$

Now we can say more about how to extend a Herglotz function F to the real axis.

Theorem B.8. *Let F be a Herglotz function and ρ its spectral measure.*
(i). For all $\lambda \in \mathbb{R}$, $D\rho(\lambda)$ and $\operatorname{Im}(F(\lambda))$ are either both zero, both in $(0, \infty)$, or both infinite. Moreover, $D\rho(\lambda)$ exists as ordinary limit if $\lambda \in L(\rho)$ and

$$D\rho(\lambda) = \frac{1}{\pi} \operatorname{Im}\big(F(\lambda)\big), \qquad \lambda \in L(\rho). \tag{B.31}$$

Moreover, $d\rho_{ac}(\lambda) = D\rho(\lambda)d\lambda$.
(ii). For almost all $\lambda_0 \in \mathbb{R}$ *with respect to* ρ *and Lebesgue measure, the derivative* $D\rho(\lambda_0)$ *and*

$$R_F(\lambda_0) = a + b\lambda_0 + \lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}\setminus(\lambda_0-\varepsilon,\lambda_0+\varepsilon)} \Big(\frac{1}{\lambda-\lambda_0} - \frac{\lambda}{1+\lambda^2}\Big)\, d\rho(\lambda) \tag{B.32}$$

both exist (as ordinary limit) and are finite. For those λ_0 *we have*

$$F(\lambda_0 + \mathrm{i}0) = \lim_{\varepsilon\downarrow 0} F(\lambda_0 + \mathrm{i}\varepsilon) = R_F(\lambda_0) + \mathrm{i}\pi D\rho(\lambda_0). \tag{B.33}$$

(iii). If $F(z)$ *and* $G(z)$ *are Herglotz functions and the boundary values coincide* $F(\lambda+\mathrm{i}0) = G(\lambda+\mathrm{i}0)$ *for* λ *in a set of positive Lebesgue measure, then* $F(z) = G(z)$.

Next, we want to generalize the decomposition of ρ by taking the α-dimensional Hausdorff measure h^α rather than the Lebesgue measure. The splitting $d\rho = d\rho_{ac} + d\rho_s$ will correspond to the case $\alpha = 1$.

For any given Borel set $B \subseteq \mathbb{R}$ and $\alpha \in [0,1]$ we define

$$h^\alpha_\varepsilon(B) = \inf\Big\{ \sum_{j\in\mathbb{N}} |I_j|^\alpha \,\Big|\, |I_j| < \varepsilon,\ B \subseteq \bigcup_{j\in\mathbb{N}} I_j \Big\}, \quad \varepsilon > 0, \tag{B.34}$$

the infimum over all countable covers by intervals I_j of length at most ε. Since $h^\alpha_\varepsilon(B)$ is increasing with respect to ε (the number of covers decreases) we can define the **α-dimensional Hausdorff measure** of B as

$$h^\alpha(B) = \lim_{\varepsilon\downarrow 0} h^\alpha_\varepsilon(B). \tag{B.35}$$

Note that $h^{\alpha_1}(B) \geq h^{\alpha_2}(B)$ if $\alpha_1 \leq \alpha_2$ and h^0, h^1 correspond to counting, Lebesgue measure, respectively. The **Hausdorff dimension** $\alpha(B)$ of B is the unique number for which $h^\alpha(B) = \infty$, $\alpha < \alpha(B)$ and $h^\alpha(B) = 0$, $\alpha > \alpha(B)$.

Now any measure can be uniquely decomposed with respect to the Hausdorff measure h^α, that is,

$$d\rho = d\rho_{\alpha c} + d\rho_{\alpha s}, \tag{B.36}$$

where $\rho_{\alpha c}$ is absolutely continuous with respect to h^α (i.e., $\rho_{\alpha c}(B) = 0$ for all B with $h^\alpha(B) = 0$) and $\rho_{\alpha s}$ is singular with respect to h^α (i.e., $\rho_{\alpha s}$ is supported on a set B with $h^\alpha(B) = 0$).

Next, define the **α-derivative** of ρ by

$$D^\alpha\rho(\lambda) = \limsup_{\varepsilon\downarrow 0} \frac{\rho\big((\lambda-\varepsilon,\lambda+\varepsilon)\big)}{(2\varepsilon)^\alpha}. \tag{B.37}$$

Theorem B.9. *Let* F *be a Herglotz function with associated measure* $d\rho$.
(i). Let $C^\alpha(\rho) = \{\lambda \in \mathbb{R} |\, D^\alpha\rho(\lambda) < \infty\}$, $\alpha \in [0,1]$, *then we have* $d\rho_{\alpha c} = \chi_{C^\alpha(\rho)}d\rho$ *and* $d\rho_{\alpha s} = (1 - \chi_{C^\alpha(\rho)})d\rho$.
(ii). Set

$$\begin{aligned} Q^\alpha\rho(\lambda) &= \limsup_{\varepsilon\downarrow 0} \varepsilon^{1-\alpha}\mathrm{Im}F(\lambda+\mathrm{i}\varepsilon), \\ R^\alpha\rho(\lambda) &= \limsup_{\varepsilon\downarrow 0} \varepsilon^{1-\alpha}|F(\lambda+\mathrm{i}\varepsilon)|. \end{aligned} \tag{B.38}$$

Then $D^\alpha\rho(\lambda)$ *and* $Q^\alpha\rho(\lambda)$, $\alpha \in [0,1]$, *are either both zero, both in* $(0,\infty)$, *or both infinite.*

If $\alpha \in [0,1)$, $Q^\alpha\rho(\lambda)$ can be replaced by $R^\alpha\rho(\lambda)$ and for any $\alpha \in [0,1]$ we have $2^{\alpha-1}D^\alpha\rho(\lambda) \le Q^\alpha\rho(\lambda) \le R^\alpha\rho(\lambda)$.

In particular,

$$C^\alpha(\rho) = \{\lambda \in \mathbb{R} | \, Q^\alpha\rho(\lambda) < \infty\} \supseteq \{\lambda \in \mathbb{R} | \, R^\alpha\rho(\lambda) < \infty\}, \tag{B.39}$$

where all three sets are equal if $\alpha \in [0,1)$. For $\alpha = 1$ this is not true in general, as the example $F(\lambda) = \ln(\lambda)$ (see below) shows. For $\alpha = 0$ we have $C^0(\rho) = \mathbb{R}$.

Again there is a value

$$\alpha(\lambda) = \liminf_{\varepsilon \downarrow 0} \frac{\ln \rho\big((\lambda-\varepsilon, \lambda+\varepsilon)\big)}{\ln \varepsilon} \tag{B.40}$$

such that $D^\alpha\rho(\lambda) = 0$ for $\alpha < \alpha(\lambda)$ and $D^\alpha\rho(\lambda) = \infty$ for $\alpha > \alpha(\lambda)$.

Lemma B.10. *We have*

$$\lim_{\varepsilon \downarrow 0} \frac{\varepsilon}{\mathrm{i}} F(\lambda + \mathrm{i}\varepsilon) = \rho(\{\lambda\}) \tag{B.41}$$

and hence $Q^0\rho(\lambda) = R^0\rho(\lambda) = \rho(\{\lambda\})$. Moreover, if $\rho((\lambda_0, \lambda_1)) = 0$, we also have

$$\lim_{\varepsilon \downarrow 0} (-\varepsilon) F(\lambda_0 + \varepsilon) = \rho(\{\lambda_0\}) \text{ and } \lim_{\varepsilon \downarrow 0} \varepsilon F(\lambda_1 - \varepsilon) = \rho(\{\lambda_1\}). \tag{B.42}$$

Proof. We only prove the first identity of (B.42), the remaining claims being similar. After splitting the integral in (B.11) into a part over $(\lambda_0 - \delta, \lambda_1)$ and one over $\mathbb{R}\backslash(\lambda_0 - \delta, \lambda_1)$, we see that it suffices to consider only the first one (since the second one is holomorphic near λ_0). By our assumption $\rho((\lambda_0, \lambda_1)) = 0$, the integral is only taken over $(\lambda_0 - \delta, \lambda_0]$ and because of $|\frac{-\varepsilon}{\lambda_0 + \varepsilon - \lambda}| \le 1$, $\lambda \in (\lambda_0 - \delta, \lambda_0]$, the desired result follows from the dominated convergence theorem. □

Now, we want to consider an alternate integral representation of Herglotz functions connected to the logarithm.

Let $\ln(z)$ be defined such that

$$\ln(z) = \ln|z| + \mathrm{i}\arg(z), \quad -\pi < \arg(z) \le \pi. \tag{B.43}$$

Then $\ln(z)$ is holomorphic and $\operatorname{Im}\big(\ln(z)\big) > 0$ for $z \in \mathbb{C}_+$, hence $\ln(z)$ is a Herglotz function. The representation of $\ln(z)$ according to (B.11) reads

$$\ln(z) = \int_{\mathbb{R}} \Big(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\Big) \chi_{(-\infty,0)}(\lambda)\, d\lambda, \quad z \in \mathbb{C}_\pm, \tag{B.44}$$

which can be easily verified.

The sum of two Herglotz functions is again a Herglotz function, similarly the composition of two Herglotz functions is Herglotz. In particular, if $F(z)$ is a Herglotz function, the same holds for $\ln\big(F(z)\big)$ and $-\frac{1}{F(z)}$. Thus, using the representation (B.11) for $\ln\big(F(z)\big)$, we get another representation for $F(z)$. The main feature of this new representation is that, by Lemma B.7, the corresponding measure is purely absolutely continuous since the imaginary part of $\ln(z)$ is uniformly bounded.

Theorem B.11. *A given function F is Herglotz if and only if it has the representation*

$$F(z) = \exp\Big\{c + \int_{\mathbb{R}} \Big(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\Big) \xi(\lambda)\, d\lambda\Big\}, \quad z \in \mathbb{C}_\pm, \tag{B.45}$$

where $c = \ln|F(\mathrm{i})| \in \mathbb{R}$, $\xi \in L^1(\mathbb{R}, (1+\lambda^2)^{-1}d\lambda)$ *real-valued and* ξ *is not identically zero. Moreover,*

$$\xi(\lambda) = \frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\mathrm{Im}\Big(\ln\big(F(\lambda+\mathrm{i}\varepsilon)\big)\Big) = \frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\arg\big(F(\lambda+\mathrm{i}\varepsilon)\big) \tag{B.46}$$

for a.e. $\lambda \in \mathbb{R}$, *and* $0 \le \xi(\lambda) \le 1$ *for a.e.* $\lambda \in \mathbb{R}$. *Here* $-\pi < \arg(F(\lambda+\mathrm{i}\varepsilon)) \le \pi$ *according to the definition of* $\ln(z)$.

Proof. $F(z)$ is Herglotz, therefore $\ln(F(z))$ is Herglotz with

$$|\mathrm{Im}(\ln F(z))| = |\arg(F(z))| \le \pi. \tag{B.47}$$

Hence by Theorem B.2

$$\ln(F(z)) = c + \int_{\mathbb{R}}\Big(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\Big)\,d\rho(\lambda), \qquad z \in \mathbb{C}_{\pm}, \tag{B.48}$$

where $\rho(\lambda)$ is (by Lemma B.7) absolutely continuous with respect to Lebesgue measure, that is, $d\rho(\lambda) = \xi(\lambda)d\lambda$, $\xi \in L^1_{loc}(\mathbb{R})$. According to (B.31), ξ is given by (B.46) and taking the Herglotz property of F into account immediately implies $0 \le \xi(\lambda) \le 1$ for a.e. $\lambda \in \mathbb{R}$. The converse is easy. □

Some additional properties are collected in the following lemma.

Lemma B.12. *Let* F *be a Herglotz function with spectral measure* ρ *and exponential Herglotz measure* $\xi(\lambda)d\lambda$.
(i). The set $\{\lambda \in \mathbb{R} | 0 < \xi(\lambda) < 1\}$ *is a minimal support for* ρ_{ac}. *Moreover, if there are constants* $0 < c_1 < c_2 < 1$ *such that* $c_1 \le \xi(\lambda) \le c_2$, $\lambda \in (\lambda_1, \lambda_2)$, *then* ρ *is purely absolutely continuous in* (λ_1, λ_2).
(ii). Fix $n \in \mathbb{N}$ *and set* $\xi_+(\lambda) = \xi(\lambda)$, $\xi_-(\lambda) = 1 - \xi(\lambda)$. *Then*

$$\int_{\mathbb{R}} |\lambda|^n \xi_{\pm}(\lambda)d\lambda < \infty \tag{B.49}$$

if and only if

$$\int_{\mathbb{R}} |\lambda|^n\, d\rho(\lambda) < \infty \quad \textit{and} \quad \lim_{z\to \mathrm{i}\infty} \pm F(z) = \pm a \mp \int_{\mathbb{R}} \frac{\lambda d\rho(\lambda)}{1+\lambda^2} > 0. \tag{B.50}$$

(iii). We have

$$F(z) = \pm 1 + \int_{\mathbb{R}} \frac{d\rho(\lambda)}{\lambda - z} \quad \textit{with} \quad \int_{\mathbb{R}} d\rho(\lambda) < \infty \tag{B.51}$$

if and only if

$$F(z) = \pm \exp\Big(\pm \int_{\mathbb{R}} \xi_{\pm}(\lambda)\frac{d\lambda}{\lambda - z}\Big) \quad \textit{with} \quad \xi_{\pm} \in L^1(\mathbb{R}) \tag{B.52}$$

($\xi_{\pm}$ *from above). In this case*

$$\int_{\mathbb{R}} d\rho(\lambda) = \int_{\mathbb{R}} \xi_{\pm}(\lambda)d\lambda. \tag{B.53}$$

Observe that the set

$$\tilde{M}_{ac} = \{\lambda \in \mathbb{R} | 0 < \xi(\lambda) < 1\} \tag{B.54}$$

is a minimal support for ρ_{ac}.

In addition to these results, we will also need some facts on matrix valued measures. Let

$$d\rho = \begin{pmatrix} d\rho_{0,0} & d\rho_{0,1} \\ d\rho_{1,0} & d\rho_{1,1} \end{pmatrix}, \quad d\rho_{0,1} = d\rho_{1,0}, \tag{B.55}$$

where $d\rho_{i,j}$ are (in general) signed measures. Associated with $d\rho$ is the trace measure $d\rho^{tr} = d\rho_{0,0} + d\rho_{1,1}$.

We require $d\rho$ to be positive, that is, the matrix $(\rho_{i,j}(B))_{0\le i,j\le 1}$ is nonnegative for any Borel set B. Equivalently, $\rho^{tr}(B) \ge 0$ and $\rho_{0,0}(B)\rho_{1,1}(B) - \rho_{0,1}(B)^2 \ge 0$ for any Borel set. This implies that $d\rho_{i,i}$ are positive measures and that

$$|\rho_{0,1}(B)| \le \frac{1}{2}\rho^{tr}(B). \tag{B.56}$$

Hence $d\rho_{i,j}$ is absolutely continuous with respect to $d\rho^{tr}$. Assuming $\rho^{tr}(\mathbb{R}) < \infty$ we can also define the Borel transform

$$F(z) = \int_{\mathbb{R}} \frac{d\rho(\lambda)}{z-\lambda}. \tag{B.57}$$

The matrix $F(\lambda)$ satisfies

$$\pm \mathrm{Im}(F(z)) \ge 0, \qquad z \in \mathbb{C}_{\pm}. \tag{B.58}$$

Next, consider the sesquilinear form

$$\langle \underline{f}, \underline{g} \rangle = \int_{\mathbb{R}} \sum_{i,j=0}^{1} \overline{f_i}(\lambda) g_j(\lambda) d\rho_{i,j}(\lambda). \tag{B.59}$$

Suppose $\underline{f}$ is a simple function, that is, $\underline{f}(\lambda) = \sum_{k=1}^{n} \chi_{B_k}(\lambda)(f_{k,1}, f_{k,2})$, where $(f_{k,1}, f_{k,2}) \in \mathbb{C}^2$ and B_k are disjoint Borel sets. Then our assumptions ensures that $\|\underline{f}\|^2 = \sum_{k=1}^{n} \sum_{i,j=0}^{1} \overline{f_{k,i}} f_{k,j} \rho_{i,j}(B_k) \ge 0$. If $\underline{f}$ is such that $f_i \in L^2(\mathbb{R}, d\rho^{tr})$ we can approximate $\underline{f}$ by simple functions implying $\|f\| \ge 0$. As a consequence we get a separable Hilbert space $L^2(\mathbb{R}, \mathbb{C}^2, d\rho)$ with the above scalar product. As before we have

Lemma B.13. *The set $\sigma(\rho^{tr})$ is precisely the spectrum $\sigma(\tilde{H})$ of the multiplication operator $\tilde{H}\underline{f}(\lambda) = \lambda \underline{f}(\lambda)$, $\mathfrak{D}(\tilde{H}) = \{\underline{f} \in L^2(\mathbb{R}, \mathbb{C}^2, d\rho) | \lambda \underline{f}(\lambda) \in L^2(\mathbb{R}, \mathbb{C}^2, d\rho)\}$.*

Proof. The proof is as like the one of Lemma B.5 except that ρ and f_n have to be replaced by ρ^{tr} and $\underline{f}_n = \rho^{tr}((\lambda - \frac{1}{n}, \lambda + \frac{1}{n}))^{-1/2} \chi_{(\lambda - \frac{1}{n}, \lambda + \frac{1}{n})}(1,1)$, respectively. □

The trace measure can even be used to diagonalize $d\rho$ as follows. Since $d\rho_{i,j}$ is absolutely continuous with respect to $d\rho^{tr}$, there is a symmetric (integrable) matrix $R(\lambda)$ such that

$$d\rho(\lambda) = R(\lambda) d\rho^{tr}(\lambda) \tag{B.60}$$

by the Radon-Nikodym theorem. Moreover, the matrix $R(\lambda)$ is nonnegative, $R(\lambda) \ge 0$, and given by

$$R_{i,j}(\lambda) = \lim_{\varepsilon \downarrow 0} \frac{\mathrm{Im}(F_{i,j}(\lambda + \mathrm{i}\varepsilon))}{\mathrm{Im}(F_{0,0}(\lambda + \mathrm{i}\varepsilon) + F_{1,1}(\lambda + \mathrm{i}\varepsilon))}. \tag{B.61}$$

Note also that we have $\mathrm{tr}(R(\lambda)) = 1$.

Next, there is a measurable unitary matrix $U(\lambda)$ which diagonalizes $R(\lambda)$, that is,

$$R(\lambda) = U(\lambda)^* \begin{pmatrix} r_1(\lambda) & 0 \\ 0 & r_2(\lambda) \end{pmatrix} U(\lambda), \tag{B.62}$$

where $0 \leq r_{1,2}(\lambda) \leq 1$ are the (integrable) eigenvalues of $R(\lambda)$. Note $r_1(\lambda)+r_2(\lambda) = 1$ by $\mathrm{tr}(R(\lambda)) = 1$. The matrix $U(\lambda)$ provides a unitary operator

(B.63)

$$\begin{array}{lcl} L^2(\mathbb{R}, \mathbb{C}^2, d\rho) & \to & L^2(\mathbb{R}, \mathbb{C}^2, \left(\begin{smallmatrix} r_1 & 0 \\ 0 & r_2 \end{smallmatrix}\right) d\rho^{tr}) = L^2(\mathbb{R}, r_1 d\rho^{tr}) \oplus L^2(\mathbb{R}, r_2 d\rho^{tr}) \\ \underline{f}(\lambda) & \mapsto & U(\lambda)\underline{f}(\lambda) \end{array}$$

which leaves $\tilde{H}$ invariant. This allows us to investigate the spectral multiplicity of $\tilde{H}$.

Lemma B.14. *Define*

$$\begin{aligned} B_1 &= \{\lambda \in \sigma(\rho^{tr}) | \det R(\lambda) = r_1(\lambda) r_2(\lambda) = 0\}, \\ B_2 &= \{\lambda \in \sigma(\rho^{tr}) | \det R(\lambda) = r_1(\lambda) r_2(\lambda) > 0\}. \end{aligned} \tag{B.64}$$

Then $\tilde{H} = \chi_{B_1}\tilde{H} \oplus \chi_{B_2}\tilde{H}$ and the spectral multiplicity of $\chi_{B_1}\tilde{H}$ is one and the spectral multiplicity of $\chi_{B_2}\tilde{H}$ is two.

Proof. It is easy to see that $\chi_{B_1}\tilde{H}$ is unitary equivalent to multiplication by λ in $L^2(\mathbb{R}, \chi_{B_1} d\rho^{tr})$. Moreover, since $r_i \chi_{B_2} d\rho^{tr}$ and $\chi_{B_2} d\rho^{tr}$ are mutually absolutely continuous, $\chi_{B_2}\tilde{H}$ is unitary equivalent to multiplication by λ in the Hilbert space $L^2(\mathbb{R}, \mathbb{C}^2, \mathbb{1}_2 \chi_{B_1} d\rho^{tr})$. □

Jacobi Difference Equations with Mathematica

The purpose of this chapter is to show how *Mathematica* can be used to make some calculations with difference equations, in particular, Jacobi difference equations. I assume that you are familiar with *Mathematica*, version 3.0. The calculations require the packages *DiscreteMath'DiffEqs'* and *DiscreteMath'JacOp'* which are available via

- `ftp://ftp.mat.univie.ac.at/pub/teschl/book-jac/DiffEqs.m`
- `ftp://ftp.mat.univie.ac.at/pub/teschl/book-jac/JacDEqs.m`

and need to be stored in the `AddOns/Applications/DiscreteMath` subfolder of your *Mathematica* folder. On multi-user systems, you can install an add-on either in the central *Mathematica* directory (provided you have access to it) or else in your individual user's Mathematica directory (usually `~/.Mathematica/3.0/`).

The *Mathematica* notebook used to make the calculations below is also available

- `ftp://ftp.mat.univie.ac.at/pub/teschl/book-jac/JacDEqs.nb`

C.1. The package *DiffEqs* and first order difference equations

We first load the package DiscreteMath'DiffEqs'.

```
In[1]:= Needs["DiscreteMath`DiffEqs`"]
```

The package *DiscreteMath'DiffEqs'* provides some basic commands for dealing with differences and difference equations.

`S[f[n],{n,j}]`	shifts the sequence `f[n]` by `j` places
`FDiff[f[n],n]`	`(=f[n+1]-f[n])` forward difference operator
`BDiff[f[n],n]`	`(=f[n-1]-f[n])` backward difference operator

`SymbSum[f[j],{j,m,n}]`	sum for symbolic manipulations
`SymbProduct[f[j],{j,m,n}]`	product for symbolic manipulations
`Casoratian[`$f_1,..,f_k$`,n]`	Casoratian of a list of sequences
`SplitSum[expr]`	splits sums into smaller parts
`SplitProduct[expr]`	splits products into smaller parts
`SplitAll[expr]`	splits sums and products into smaller parts
`TestDiff[f[n]==g[n],{n,m}]`	simple test for equality based on the fact that `f = g` if equality holds at one point `m` and `FDiff[f[n] -g[n],n]=0`

The main purpose of the built-in `Sum` command is to search for a closed form. It is, however, not suitable for manipulating sums which cannot be brought to a closed form. For example,

```
In[2]:= Simplify[Sum[f[j],{j,1,n+1}]-Sum[f[j],{j,1,n}]]
```

$$\text{Out[2]=}\quad \sum_{j=1}^{n+1} f[j] - \sum_{j=1}^{n} f[j]$$

Moreover, the built-in `Sum` is zero if the upper limit is smaller than the lower limit. This can lead to, at first sight, surprising results:

```
In[3]:= S1[n_] := SymbSum[j,{j,1,n}];
        S2[n_] := Sum[j,{j,1,n}];
        S3[n_] = Sum[j,{j,1,n}];
        {{S1[n],S1[-3]},{S2[n],S2[-3]},{S3[n],S3[-3]}}//MatrixForm
```

Out[3]//MatrixForm=

$$\begin{pmatrix} \sum_{j=1}^{n}{}^{*}\, j & 3 \\ \frac{1}{2}n(n+1) & 0 \\ \frac{1}{2}n(n+1) & 3 \end{pmatrix}$$

(note that *Mathematica* assumes n to be positive). Observe that `SymbSum` does not look for a closed form; which can be obtained by switching to the built-in `Sum`:

```
In[4]:= S1[n] /. SymbSum-> Sum
```

$$\text{Out[4]=}\quad \frac{1}{2}n(1+n)$$

Observe also the definition for situations where the upper limit is smaller than the lower limit:

```
In[5]:= {Sum[f[j],{j,0,1}]^*, Sum[f[j],{j,0,1}], Sum[f[j],{j,0,-3}]^*, Sum[f[j],{j,0,-3}]}
```

$$\text{Out[5]=}\quad \{f[0]+f[1], f[0]+f[1], -f[-2]-f[-1], 0\}$$

Similarly for symbolic products

$$\text{In[6]:=}\ \{\prod_{j=0}^{1}{}^{*}\ f[j], \prod_{j=0}^{1} f[j], \prod_{j=0}^{-3}{}^{*}\ f[j], \prod_{j=0}^{-3} f[j]\}$$

$$\text{Out[6]=}\ \{f[0]f[1], f[0]f[1], \frac{1}{f[-2]\ f[-1]}, 1\}$$

The main purpose of the commands `SplitSum` and `SplitProduct` is to simplify expressions involving sums and products (as pointed out earlier, *Mathematica*'s built-in capabilities in this respect are limited). The command `SplitSum` will try to break sums into parts and pull out constant factors such that the rest can be done with built-in commands. In particular, it will split off extra terms

$$\text{In[7]:=}\ \text{SplitSum}[\sum_{j=0}^{n+1}{}^{*}\ f[j]]$$

$$\text{Out[7]=}\ f[1+n] + \sum_{j=0}^{n}{}^{*}\ f[j]$$

expand sums and pull out constants

$$\text{In[8]:=}\ \text{SplitSum}[\sum_{j=0}^{n}{}^{*}\ (h\,f[j] + g[j])]$$

$$\text{Out[8]=}\ h\sum_{j=0}^{n}{}^{*}\ f[j] + \sum_{j=0}^{n}{}^{*}\ g[j]$$

The command `SplitProduct` will perform similar operations with products.

$$\text{In[9]:=}\ \text{SplitProduct}[\prod_{j=0}^{n-1}{}^{*}\ \frac{g}{f[j]}]$$

$$\text{Out[9]=}\ \frac{g^{n} f[n]}{\prod_{j=0}^{n}{}^{*}\ f[j]}$$

Finally, the command `SplitAll` also handles nested expressions

$$\text{In[10]:=}\text{SplitAll}[\sum_{m=0}^{n}{}^{*}\ \sum_{j=0}^{m}{}^{*}\ \left(f[j] + \prod_{i=0}^{j}{}^{*}\ x\,g[i]\right)$$

$$\text{Out[10]=}\ \sum_{m=0}^{n}{}^{*}\ \sum_{j=0}^{m}{}^{*}\ f[j] + \sum_{m=0}^{n}{}^{*}\ \sum_{j=0}^{m}{}^{*}\ x^{1+j} \prod_{i=0}^{j}{}^{*}\ g[i]$$

We can shift a sequence $f(n)$ by j places using `S[f[n],{n,j}]`. For example,

```
In[11]:=S[f[g[n^2]],{n,1}]
Out[11]= f[g[(1+n)^2]]
```

or

```
In[12]:=S[f[n,m],{n,3}]
Out[12]= f[3+n,m]
```

Similarly, we can compute forward, backward differences using `FDiff[f[n],n]`, `BDiff[f[n],n]`, respectively. For example,

```
In[13]:={FDiff[f[n],n],BDiff[f[n],n]}
Out[13]= {-f[n]+f[1+n],f[-1+n]-f[n]}
```

A simple test for equality of sequences is based on the fact that $f(n) = 0$ if $f(n)$ is constant, that is, $f(n+1) = f(n)$ and $f(n_0) = 0$ at one fixed point.

```
In[14]:=TestDiff[f[n]==0,{n,n0}]
Out[14]= -f[n]+f[1+n]==0&&f[n0]==0
```

As a first application we can verify Abel's formula (summation by parts)

```
In[15]:= Sum[g[j]FDiff[f[j],j],{j,m,n}]* ==
          (g[n]f[n+1] - g[m-1] f[m] + Sum[BDiff[g[j],j]f[j],{j,m,n}]*)//
          TestDiff[#,{n,m}]&
Out[15]= True
```

Note also the following product rules

```
In[16]:=FDiff[f[n]g[n],n]==f[n]FDiff[g[n],n]+g[n+1]FDiff[f[n],n]//
        Simplify
Out[16]= True

In[17]:=BDiff[f[n]g[n],n]==f[n]BDiff[g[n],n]+g[n-1]BDiff[f[n],n]//
        Simplify
Out[17]= True
```

Next, the product

```
In[18]:=P[n_]:= Product[f[j],{j,1,n-1}]*
```

satisfies the difference equation $P(n+1) = f(n)P(n)$ plus the initial condition $P(0) = 1$

```
In[19]:={P[0], SplitProduct[P[n + 1] − f[n]P[n]]}
Out[19]= {1, 0}
```

Moreover, the sum

$$\texttt{In[20]:=F[n\_] := } \sum_{\texttt{j=1}}^{\texttt{n-1}}{}^{*}\ \texttt{f[j]}$$

satisfies the difference equation $\partial F(n) = F(n+1) - F(n) = f(n)$ plus the initial condition $F(0) = 0$

```
In[21]:={F[0], SplitSum[FDiff[F[n], n] − f[n]]}
Out[21]= {0, 0}
```

Similarly, the general solution of the difference equation $F(n+1) = f(n)F(n)+g(n)$ with initial condition $F(0) = F_0$ is given by

$$\texttt{In[22]:=F[n\_] := P[n]}\left(\texttt{F}_0 + \sum_{\texttt{i=0}}^{\texttt{n-1}}{}^{*}\ \frac{\texttt{g[i]}}{\texttt{P[i + 1]}}\right)$$

In fact, we can easily verify this claim:

```
In[23]:={F[0], SplitAll[F[n + 1] − (f[n]F[n] + g[n])]}
Out[23]= {F0, 0}
```

C.2. The package *JacDEqs* and Jacobi difference equations

Now we come to Jacobi difference equations. The necessary commands are provided in the package *DiscreteMath'JacDEqs'* which we load first.

```
In[1]:= Needs["DiscreteMath`JacDEqs`"]
```

This will also load the package *DiscreteMath'DiffEqs'* if necessary. The following commands are provided in the package *JacDEqs* :

`JacobiDE[f[n],n]`	computes the Jacobi difference expression, associated with `a[n]`, `b[n]` of `f[n]`
`JacobiDE[k,f[n],n]`	applies the Jacobi difference expression k times.
`SolutionJacobi[u]`	tells *Mathematica* that `u[z,n]` solves `JacobiDE[u[z,n],n] = z u[z,n]`
`SolutionJacobi[u,x,y,m]`	tells *Mathematica* that `u[z,n]` solves `JacobiDE[u[z,n],n] = z u[z,n]` and satisfies the initial conditions `u[z,m]=x`, `u[z,m+1]=y`
`SolutionJacobi[u,x,y]`	tells *Mathematica* that `u[z,n,m]` solves `JacobiDE[u[z,n,m],n] = z u[z,n,m]` and

	satisfies the initial conditions u[z,m,m]=x, u[z,m+1,m]=y
SolutionJacobi[f,g]	tells *Mathematica* that f[n] satisfies JacobiDE[f[n],n] = g[n]

The Jacobi difference expression τ applied to a sequence gives the following result

```
In[2]:= JacobiDE[f[n], n]
Out[2]= a[-1 + n]f[-1 + n] + b[n]f[n] + a[n]f[1 + n]
```

This can also be written as

```
In[3]:= JacobiDE[f[n], n] == -BDiff[a[n]FDiff[f[n], n], n] + (a[n - 1] + a[n] +
        b[n])f[n]//Simplify
Out[3]= True
```

or

```
In[4]:= JacobiDE[f[n], n] == -FDiff[a[n - 1]BDiff[f[n], n], n] + (a[n - 1] + a[n] +
        b[n])f[n]//Simplify
Out[4]= True
```

The command

```
In[5]:= SolutionJacobi[u]
```

will tell *Mathematica* that $u(z,n)$ satisfies the Jacobi equation $\tau u(z) = z\,u(z)$. Let's see how this works:

```
In[6]:= JacobiDE[u[z, n], n] == z u[z, n]
Out[6]= True
```

We can also define solutions $u(z,n,m)$ satisfying the initial conditions $u(z,m,m) = u_0$ and $u(z,m+1,m) = u_1$ by

```
In[7]:= Clear[u];
        SolutionJacobi[u, u0, u1]
```

Indeed, we obtain

```
In[8]:= {JacobiDE[u[z, n, m], n] == z u[z, n, m], u[z, m, m], u[z, m + 1, m]}
Out[8]= {True, u0, u1}
```

The solutions $c(z,n,m)$ and $s(z,n,m)$ are predefined by the package

```
In[9]:= {JacobiDE[c[z, n, m], n] == z c[z, n, m], c[z, m, m], c[z, m + 1, m]}
Out[9]= {True, 1, 0}

In[10]:={JacobiDE[s[z, n, m], n] == z s[z, n, m], s[z, m, m], s[z, m + 1, m]}
Out[10]= {True, 0, 1}
```

Note that you can tell *Mathematica* that $u(n)$ satisfies $(\tau - z)u = 0$ using

```
In[11]:=Clear[u];
        SolutionJacobi[u, (z u[#])&]
```

Indeed,

```
In[12]:=JacobiDE[u[n], n] == z u[n]
Out[12]= True
```

C.3. Simple properties of Jacobi difference equations

Let us first load our package

```
In[1]:= Needs["DiscreteMath`JacDEqs`"]
```

The Wronskian of two solutions $W_n(u(z), v(z))$ is defined as $a(n)$ times the Casoratian of two solutions $C_n(u(z), v(z))$

```
In[2]:= W[u_, v_, n_] := a[n]Casoratian[u, v, n]
```

Please note that the argument n tells *Mathematica* what to consider as index. To compute the value of the Wronskian use

```
In[3]:= W[u[n], v[n], n] /. n-> 0
Out[3]= a[0](-u[1]v[0] + u[0]v[1])
```

and not

```
In[4]:= W[u[n], v[n], 0]
Out[4]= 0
```

Defining two solutions $u(z, n)$ and $v(z, n)$

```
In[5]:= Clear[u, v]; SolutionJacobi[u]; SolutionJacobi[v];
```

we can easily verify that the Wronskian is independent of n

```
In[6]:= FDiff[W[u[z, n], v[z, n], n], n]//Simplify
Out[6]= 0
```

Similarly, we can verify Green's formula

$$\texttt{In[7]:=}\ \sum_{\texttt{j=m}}^{\texttt{n}}{}^{*}\ (\texttt{f[j]JacobiDE[g[j], j]} - \texttt{g[j]JacobiDE[f[j], j]}) == \texttt{W[f[n], g[n], n]} - (\texttt{W[f[n], g[n], n]} /. \texttt{n-> m - 1})\texttt{//TestDiff[\#, n, m]\&}$$

```
Out[7]= True
```

We can also define a second solution using the formula

$$\texttt{In[8]:=}\ \texttt{Clear[v]; v[z\_, n\_]} := \sum_{\texttt{j=0}}^{\texttt{n-1}}{}^{*}\ \frac{\texttt{u[z, n]}}{\texttt{a[j]u[z, j]u[z, j + 1]}};$$

Indeed, the sequence $v(z,n)$ defined as above satisfies the Jacobi equation

```
In[9]:= SplitSum[JacobiDE[v[z,n],n] - zv[z,n]]//Simplify
Out[9]= 0
```

The Wronskian of $u(z,n)$ and $v(z,n)$ is one:

```
In[10]:=SplitSum[W[u[z,n],v[z,n],n]]//Simplify
Out[10]= 1
```

Next, recall that the equality $u(n) = v(n)$ holds if equality holds at two consecutive points $n = m$, $m+1$ and the Jacobi difference equations applied to both sides gives the same result, that is, $\tau u(z) = \tau v(z)$. We can define a simple test using this fact:

```
In[11]:=TestJac[eqn_Equal,{n_,m_}] :=
        Simplify[(JacobiDE[#,n] - z#) & /@ eqn]&&(eqn /. n-> m)&&
        (eqn /. n-> m + 1);
```

As an application we verify that any solution can be written as a linear combination of the two fundamental solutions $c(z,n,m)$ and $s(z,n,m)$ as follows

```
In[12]:=SolutionJacobi[u,u0,u1];
        u[z,n,m] == u0c[z,n,m] + u1s[z,n,m]//TestJac[#,{n,m}]&
Out[12]= True
```

The solutions $c(z,n,m)$ and $s(z,n,m)$ have some interesting properties.

```
In[13]:=s[z,n,m + 1] == -a[m + 1]/a[m] c[z,n,m] // TestJac[#,{n,m}]&
Out[13]= True
```

```
In[14]:=s[z,n,m - 1] == c[z,n,m] + (z - b[m])/a[m] s[z,n,m] // TestJac[#,{n,m}]&
Out[14]= True
```

```
In[15]:=c[z,n,m + 1] == (z - b[m + 1])/a[m] c[z,n,m] + s[z,n,m] // TestJac[#,{n,m}]&
Out[15]= True
```

```
In[16]:=c[z,n,m - 1] == -a[m - 1]/a[m] s[z,n,m] // TestJac[#,{n,m}]&
Out[16]= True
```

Unfortunately, this does not work for the following formula

```
In[17]:=c[z,m,n] == a[n]/a[m] s[z,n + 1,m] // TestJac[#,{n,m}]&
```

```
Out[17]= a[-1 + n]c[z, m, -1 + n] + (-z + b[n])c[z, m, n] + a[n] c[z, m, 1 + n] ==
         1/a[m] a[-1+n]^2 s[z, n, m] - a[n](a[n]s[z, n, m] + (-b[n] + b[1+n])s[z, 1+n, m])
```

and we can only check it for specific values of n:

```
In[18]:=Simplify[c[z, m, n] == a[n]/a[m] s[z, n + 1, m] /. n-> m + 3]

Out[18]= True
```

Similarly for the following asymptotics $w = \frac{1}{z}$

```
In[19]:=((Product_{j=m+1}^{n-1} a[j]) w^(n-m-2)/(-a[m]) c[1/w, n, m] -
         (1 - w Sum_{j=m+2}^{n-1} b[j] + O[w]^2)) /. n-> m + 3

Out[19]= O[w]^2
```

```
In[20]:=((Product_{j=m+1}^{n-1} a[j]) w^(n-m-1) s[1/w, n, m] -
         (1 - w Sum_{j=m+1}^{n-1} b[j] + O[w]^2)) /. n-> m + 3

Out[20]= O[w]^2
```

```
In[21]:=((Product_{j=n}^{m-1} a[j]) w^(m-n) c[1/w, n, m] -
         (1 - w Sum_{j=n+1}^{m} b[j] + O[w]^2)) /. n-> m + 3

Out[21]= O[w]^2
```

```
In[22]:=((Product_{j=n}^{m-1} a[j]) w^(m-n-1)/(-a[m]) s[1/w, n, m] -
         (1 - w Sum_{j=n+1}^{m-1} b[j] + O[w]^2)) /. n-> m + 3

Out[22]= O[w]^2
```

Defining the Jacobi matrix

```
In[23]:=Delta[i_, j_] := If[Simplify[i == j], 1, 0];
        J[i_, j_] := a[j]Delta[i - 1, j] + a[i]Delta[i + 1, j]
        + b[j]Delta[i, j];
        JacobiMatrix[m_, n_] := Table[J[m + i, m + j], {i, n - m - 1},
        {j, n - m - 1}]
```

```
In[24]:=MatrixForm[JacobiMatrix[m, m + 6]]
```

```
Out[24]//MatrixForm=
```

$$\begin{pmatrix} b[1+m] & a[1+m] & 0 & 0 & 0 \\ a[1+m] & b[2+m] & a[2+m] & 0 & 0 \\ 0 & a[2+m] & b[3+m] & a[3+m] & 0 \\ 0 & 0 & a[3+m] & b[4+m] & a[4+m] \\ 0 & 0 & 0 & a[4+m] & b[5+m] \end{pmatrix}$$

we can check

```
In[25]:=(s[z, #, m] ==
         Det[z IdentityMatrix[# - m - 1] - JacobiMatrix[m, #]] / Product[a[j], {j, m + 1, # - 1}]) &[m + 4]
         // Simplify
Out[25]= True
```

or compute traces

```
In[26]:=Tr[M_] := Sum[M[[i, i]], {i, Length[M]}]; (* Not need for Mathematica 4.x *)
```

```
In[27]:=Expand[(Tr[JacobiMatrix[m, #].JacobiMatrix[m, #]]
         - (2 Sum[a[j]^2, {j, m + 1, # - 2}] + Sum[b[j]^2, {j, m + 1, # - 1}])) &[m + 3]]
Out[27]= 0
```

Let us next investigate the inhomogeneous Jacobi operator. The command

```
In[28]:=SolutionJacobi[f, g];
```

tells *Mathematica* that $f(n)$ satisfies the inhomogeneous Jacobi equation $\tau f = g$. In fact,

```
In[29]:=JacobiDE[f[n], n] == g[n]
Out[29]= True
```

Next, let us express $f(n)$ in terms of $s(z, n, m)$. We define the kernel

```
In[30]:=K[z_, n_, j_] := s[z, n, j] / a[j]
```

which satisfies

```
In[31]:=Factor[{K[z, n, n - 2], K[z, n, n - 1], K[z, n, n], K[z, n, n + 1], K[z, n, n + 2]}]
```

$$\text{Out[31]= } \left\{\frac{z - b[-1+n]}{a[-2+n]a[-1+n]}, \frac{1}{a[-1+n]}, 0, -\frac{1}{a[n]}, -\frac{z - b[1+n]}{a[n]a[1+n]}\right\}$$

and the sequence $f(n)$

```
In[32]:=Clear[f];
```

$$\texttt{f[n\_]} := \sum_{\texttt{j=1}}^{\texttt{n}} {}^{*}\ \texttt{K[z,n,j]g[j]}$$

```
In[33]:=SplitSum[JacobiDE[f[n],n] - zf[n]]//Factor
Out[33]= g[n]
```

Finally, we consider the case with $a(n) = 1/2$, $b(n) = 0$.

```
In[34]:=Unprotect[a,b];
```

$$\texttt{a[n\_]} = \frac{1}{2}; \texttt{b[n\_]} = 0;$$

Then we have

```
In[35]:=JacobiDE[(z - (z + Sqrt[z^2 - 1])^n,n] == z(z + Sqrt[z^2 - 1])^n // Simplify
Out[35]= True

In[36]:=JacobiDE[(z - (z - Sqrt[z^2 - 1])^n,n] == z(z - Sqrt[z^2 - 1])^n // Simplify
Out[36]= True
```

and

```
In[37]:=s[z,n,0] == ((z + Sqrt[z^2 - 1])^n - (z - Sqrt[z^2 - 1])^n)/(2Sqrt[z^2 - 1]) // TestJac[#,{n,0}]&
Out[37]= True

In[38]:=c[z,n,0] == -s[z,n - 1,0] // TestJac[#,{n,0}]&
Out[38]= True
```

C.4. Orthogonal Polynomials

We first introduce the matrices $C(k)$, $D(k)$, and $P(z,k)$. We use `Cm[k]` for the matrix $C(k)$ and `Cd[k]` for its determinant. Similar for the other three matrices.

```
In[1]:= m[0] := 1;
        Cd[0] = 1; Cd[k_] := Det Cm[k];
        Cm[k_?IntegerQ] := Table[m[i + j],{i,0,k - 1},{j,0,k - 1}];
        Dd[0] = 0; Dd[k_] := Det Dm[k];
        Dm[k_?IntegerQ] := Table[If[j == k - 1,m[i + j + 1],m[i + j]],
            {i,0,k - 1},{j,0,k - 1}];
        Pd[z_,0] = 0; Pd[z_,k_] := Det Pm[z,k];
        Pm[z_,k_?IntegerQ] := Table[If[i == k - 1,z^j,m[i + j]],
            {i,0,k - 1},{j,0,k - 1}];
```

Here is how the matrices look like.

```
In[2]:= Print[MatrixForm[Cm[3]],"    ",MatrixForm[Dm[3]],"    ",
          MatrixForm[Pm[z,3]]]
```

$$\begin{pmatrix} 1 & m[1] & m[2] \\ m[1] & m[2] & m[3] \\ m[2] & m[3] & m[4] \end{pmatrix} \quad \begin{pmatrix} 1 & m[1] & m[2] \\ m[1] & m[2] & m[3] \\ m[2] & m[3] & m[4] \end{pmatrix} \quad \begin{pmatrix} 1 & m[1] & m[2] \\ m[1] & m[2] & m[3] \\ 1 & z & z^2 \end{pmatrix}$$

The determinants of $C(k)$ and $D(k)$ appear as the first two coefficients in the expansion of the determinant of $P(z,k)$ for large z.

```
In[3]:= Simplify[(Series[Pd[1/t,k],{t,0,-1}] ==
          Cd[k-1](1/t)^(k-1) - Dd[k-1](1/t)^(k-2))/.k-> 3]
Out[3]= O[t]^0 == 0
```

The polynomials $s(z,k)$

```
In[4]:= s[z_,k_] := Pd[z,k]/Sqrt[Cd[k]Cd[k-1]];
```

are orthogonal with respect to the measure ρ whose moments are $m(k) = \int \lambda^k d\rho(\lambda)$. Moreover, defining

```
In[5]:= a[k_] := Sqrt[Cd[k+1]]Sqrt[Cd[k-1]]/Cd[k]; a[0] := 0;
        b[k_] := Dd[k]/Cd[k] - Dd[k-1]/Cd[k-1];
```

we have that $s(z,k)$ solves the corresponding Jacobi equation.

```
In[6]:= Simplify[a[k]s[z,k+1] + a[k-1]s[z,k-1] + (b[k] - z)s[z,k] /. k-> 3]
Out[6]= 0
```

Moreover, the polynomials $s(z,k)$ are orthogonal with respect to the scalar product

```
In[7]:= ScalarProduct[P_,Q_,z_] := Sum[Sum[m[i+j]
          Coefficient[P,z,i]Coefficient[Q,z,j],{j,0,Exponent[Q,z]}],
          {i,0,Exponent[P,z]}]/;
        PolynomialQ[P,z]&&PolynomialQ[Q,z]
```

which can be checked as follows

```
In[8]:= Simplify[ScalarProduct[s[z,j],s[z,k],z] ==
          If[j == k,1,0] /. {k-> 3,j-> 3}]
Out[8]= True
```

C.5. Recursions

Now we want to look at the matrix elements $g_j(n) = \langle \delta_n, H^j \delta_n \rangle$ and $h_j(n) = 2a(n)\langle \delta_{n+1}, H^j \delta_n \rangle$, which are important when dealing with high energy expansions for the resolvent of our Jacobi operator H.

```
In[1]:= Needs["DiscreteMath`JacDEqs`"]
```

For computational purpose it is most convenient to define them using the following recursions.

```
In[2]:= g[0,n_] = 1; h[0,n_] = 0;
        g[j_?IntegerQ,n_] := g[j,n] =
            1/2 (h[j-1,n] + h[j-1,n-1]) + b[n]g[j-1,n];
        h[j_?IntegerQ,n_] := h[j,n] =
            2a[n]^2 Sum[g[j-l-1,n]g[l,n+1], {l,0,j-1}] - 1/2 Sum[h[j-l-1,n]h[l,n], {l,0,j-1}];
```

These definitions can be used to compute arbitrary coefficients as needed.

```
In[3]:= Simplify[g[2,n]]
Out[3]= a[-1+n]^2 + a[n]^2 + b[n]^2
```

```
In[4]:= Simplify[h[2,n]]
Out[4]= 2a[n]^2 (a[-1+n]^2 + a[n]^2 + a[1+n]^2 + b[n]^2 + b[n]b[1+n] + b[1+n]^2)
```

In addition, we have the third sequence $\gamma_j^\beta(n) = \langle (\delta_{n+1} + \beta\delta_n), H^j(\delta_{n+1} + \beta\delta_n) \rangle$.

```
In[5]:= γ[j_,n_] := g[j,n+1] + β/a[n] h[j,n] + β^2 g[j,n];
```

But now we want to see that the recursions indeed produce the matrix elements from above. For this purpose we need a delta sequence.

```
In[6]:= Delta[n_] := If[n == 0, 1, 0];
```

Using this we have

```
In[7]:= g[j,n] == JacobiDE[j,Delta[n-m],m] /. j-> 3 /. m-> n//Simplify
Out[7]= True
```

and

```
In[8]:= h[j,n] == 2a[n]JacobiDE[j,Delta[n-m],m] /. j-> 3 /. m-> n+1//Simplify
Out[8]= True
```

Hence the recursions produce indeed the desired matrix elements.

There is another relation for g_j and h_j which can be verified for given j.

```
In[9]:= Simplify[h[j+1,n] - h[j+1,n-1] ==
            2(a[n]^2 g[j,n+1] - a[n-1]^2 g[j,n-1]) + b[n](h[j,n] - h[j,n-1]))/.j-> 4]
```

```
Out[9]= True
```

Moreover, there is a recursion involving only g_j.

```
In[10]:=Clear[g];
        g[0,n_] = 1;
        g[1,n_] = b[n];
        g[2,n_] = a[-1+n]^2 + a[n]^2 + b[n]^2;
        g[j_?IntegerQ,n_] := -a[n]^2 g[j-2,n+1] + a[n-1]^2 g[j-2,n-1]
            - b[n]^2 g[j-2,n] + 2b[n] g[j-1,n] - 1/2 Sum[k[j-1-2,n] k[1,n], {1,0,j-2}]
            + 2a[n]^2 Sum[g[j-1-2,n+1] g[1,n], {1,0,j-2}]
            - 2b[n] Sum[g[j-1-3,n] g[1,n+1], {1,0,j-3}]
            + b[n]^2 Sum[g[j-1-4,n] g[1,n+1], {1,0,j-4}];
        k[0,n_] = -b[n];
        k[j_,n_] := a[n]^2 g[j-1,n+1] - a[n-1]^2 g[j-1,n-1] + b[n]^2 g[j-1,n] -
        2b[n] g[j,n] + g[j+1,n];
```

Again, it can be checked, that it is equivalent to our other one for given j.

```
In[11]:=g[j,n] == JacobiDE[j,Delta[n-m],m] /. j-> 4 /. m-> n//Simplify
Out[11]= True
```

In particular, these relations will be used in the sections on the Toda and Kac van Moerbeke lattice.

C.6. Commutation methods

C.6.1. Single commutation method. In this section we want to consider the single commutation method. As first steps we load our package

```
In[1]:= Needs["DiscreteMath`JacDEqs`"]
```

and introduce a few solutions of our original operator

```
In[2]:= SolutionJacobi[u]; SolutionJacobi[v]; SolutionJacobi[f,g];
        W[f__,n_] := a[n] Casoratian[f,n];
```

Next, we declare the commuted operator H_σ

```
In[3]:= a_σ[n_] := Sqrt[-a[n]] Sqrt[-a[n+1]] Sqrt[u[λ,n]] Sqrt[u[λ,n+2]] / u[λ,n+1];
        b_σ[n_] := λ - a[n] (u[λ,n]/u[λ,n+1] + u[λ,n+1]/u[λ,n]);
        JacobiDE_σ[f_,n_] := a_σ[n] S[f,{n,1}] + a_σ[n-1] S[f,{n,-1}] + b_σ[n] f;
        W_σ[f__,n_] := a_σ[n] Casoratian[f,n];
```

and the operator A

```
In[4]:= v_σ[λ_, n_] := 1/(-a[n]u[λ, n]u[λ, n + 1]);
        A[f_, n_] := v_σ[λ, n]W[u[λ, n], f, n];
```

Note that v_σ is a solution of $H_\sigma u = \lambda u$

```
In[5]:= Simplify[JacobiDE_σ[v_σ[λ, n], n] == λv_σ[λ, n]]
Out[5]= True
```

Now we can verify that $Au(z)$ satisfies $H_\sigma(Au(z)) = zAu(z)$

```
In[6]:= Simplify[JacobiDE_σ[A[u[z, n], n], n] == zA[u[z, n], n]]
Out[6]= True
```

and that Af satisfies $H_\sigma(Af) = Ag$.

```
In[7]:= Simplify[JacobiDE_σ[A[f[n], n], n] == A[g[n], n]]
Out[7]= True
```

Moreover, the following relations for the Wronskians are valid:

```
In[8]:= Simplify[W_σ[A[u[z, n], n], A[v[z, n], n], n] == (z - λ)W[u[z, n], v[z, n], n]]
Out[8]= True
```

```
In[9]:= Simplify[W_σ[A[u[z, n], n], A[v[zz, n], n], n] ==
          -a[n]a[n + 1]Casoratian[u[λ, n], u[z, n], v[zz, n], n]/u[λ, n + 1]]
Out[9]= True
```

```
In[10]:=Simplify[W_σ[A[u[z, n], n], A[v[zz, n], n], n] ==
          (z - λ)W[u[z, n], v[zz, n], n] - (z - zz)v[zz, 1 + n]W[u[z, n], u[λ, n], n]/u[λ, 1 + n]]

Out[10]= True
```

For additional material see the section on the Kac-van Moerbeke lattice below.

C.6.2. Double commutation method. In this section we want to consider the double commutation method. Again we load our package

```
In[1]:= Needs["DiscreteMath`JacDEqs`"]
```

and introduce two solutions of our original operator.

```
In[2]:= SolutionJacobi[u]; SolutionJacobi[v];
        W[f__, n_] := a[n]Casoratian[f, n];
```

Next, we declare the doubly commuted operator H_γ

In[3]:= $c_\gamma[n\_] = \frac{1}{\gamma} + \sum_{j=-\infty}^{n}{}^{*}\ u[\lambda, j]^2;$

$a_\gamma[n\_] = a[n]\frac{\sqrt{c_\gamma[n-1]}\sqrt{c_\gamma[n+1]}}{c_\gamma[n]};$

$b_\gamma[n\_] = b[n] - \mathrm{BDiff}[\frac{a[n]u[\lambda,n]u[\lambda,n+1]}{c_\gamma[n]}, n];$

$\mathrm{JacobiDE}_\gamma[f\_, n\_] := a_\gamma[n]\ S[f, \{n, 1\}] + a_\gamma[n-1]S[f, n, -1] + b_\gamma[n]f;$

$W_\gamma[f\_\_, n\_] := a_\gamma[n]\mathrm{Casoratian}[f, n];$

As a first fact we note that u_γ

In[4]:= $u_\gamma[n\_] = \frac{u[\lambda, n]}{\sqrt{c_\gamma[n-1]}\sqrt{c_\gamma[n]}};$

satisfies $H_\gamma u_\gamma = \lambda u_\gamma$.

In[5]:= $\mathrm{Simplify}[\mathrm{SplitSum}[\mathrm{JacobiDE}_\gamma[u_\gamma[n], n] == \lambda u_\gamma[n]]]$

Out[5]= True

To obtain more solutions we introduce the operator A

In[6]:= $A[f\_, \{z\_, n\_\}] := \frac{\sqrt{c_\gamma[n]}}{\sqrt{c_\gamma[n-1]}}f - \frac{1}{z-\lambda}u_\gamma[n]W[u[\lambda, n], f, n];$

and set

In[7]:= $u_\gamma[z\_, n\_] = A[u[z, n], \{z, n\}];$

The sequence $u_\gamma(z)$ satisfies $H_\gamma u_\gamma(z) = z\, u_\gamma(z)$

In[8]:= $\mathrm{Simplify}[\mathrm{SplitSum}[\mathrm{JacobiDE}_\gamma[u_\gamma[z, n], n] == zu_\gamma[z, n]]]$

Out[8]= True

and for $u_\gamma(z, n)^2$ we have

In[9]:= $\mathrm{Simplify}[\mathrm{SplitSum}[u_\gamma[z, n]^2 ==$
$u[z, n]^2 + \frac{1}{(z-\lambda)^2}\mathrm{BDiff}[\frac{W[u[\lambda, n], u[z, n], n]^2}{c_\gamma[n]}, n]]]$

Out[9]= True

In addition, we note two equivalent forms for $u_\gamma(z)$

In[10]:= $\mathrm{Simplify}[\mathrm{SplitSum}[u_\gamma[z, n] ==$
$\sqrt{\frac{c_\gamma[n-1]}{c_\gamma[n]}}u[z, n] - \frac{1}{z-\lambda}u_\gamma[n]S[W[u[\lambda, n], u[z, n], n], \{n, -1\}]]]$

Out[10]= True

In[11]:= $AA[f\_, n\_] := \frac{\sqrt{c_\gamma[n]}}{\sqrt{c_\gamma[n-1]}}f - u_\gamma[n]\sum_{j=-\infty}^{n}{}^{*}\ u[\lambda, j](f\ /.\ n- > j);$

In[12]:=
$$\texttt{Simplify[AA[u}[\lambda,\texttt{n}],\texttt{n}] == \frac{\texttt{u}_\gamma[\texttt{n}]}{\gamma}]$$

Out[12]= True

and the following relations for the Wronskian:

In[13]:=
$$\texttt{Simplify[SplitSum[W}_\gamma[\texttt{u}_\gamma[\texttt{n}],\texttt{A[u[z,n]},\{\texttt{z},\texttt{n}\}],\texttt{n}]]$$

Out[13]=
$$\frac{\gamma\texttt{a[n]}(\texttt{u[z,}1+\texttt{n]u}[\lambda,\texttt{n}]-\texttt{u[z,n]u}[\lambda,1+\texttt{n}])}{1+\gamma\sum_{j=-\infty}^{n}{}^{*}\ \texttt{u}[\lambda,\texttt{j}]^2}$$

In[14]:=
$$\texttt{Simplify[SplitSum[W}_\gamma[\texttt{A[u[z,n]},\{\texttt{z},\texttt{n}\}],\texttt{A[v[zz,n]},\{\texttt{zz},\texttt{n}\}],\texttt{n}] == \texttt{W[u[z,n]},\texttt{v[zz,n]},\texttt{n}] + \frac{(\texttt{z}-\texttt{zz})\texttt{W[u}[\lambda,\texttt{n}],\texttt{u[z,n]},\texttt{n]W[u}[\lambda,\texttt{n}],\texttt{v[zz,n]},\texttt{n}]}{(\texttt{z}-\lambda)(\texttt{zz}-\lambda)\texttt{c}_\gamma[\texttt{n}]}]]$$

Out[14]= True

Finally, we turn to the solutions c_β and s_β.

In[15]:=
$$\texttt{u}[\lambda,0] = -\texttt{Sin}[\alpha];\texttt{u}[\lambda,1] = \texttt{Cos}[\alpha];$$
$$\texttt{SolutionJacobi[sb},-\texttt{Sin}[\alpha],\texttt{Cos}[\alpha],0];$$
$$\texttt{SolutionJacobi[cb},\texttt{Cos}[\alpha],\texttt{Sin}[\alpha],0];$$

They are transformed into $c_{\hat\beta,\gamma}$ and $s_{\hat\beta,\gamma}$

In[16]:=
$$\texttt{cn} = \texttt{c}_\gamma[0];$$
$$\texttt{cp} = \texttt{SplitSum}[\texttt{c}_\gamma[\texttt{n}+1]]\ /.\ \texttt{n}-> 0;$$
$$\texttt{cm} = \texttt{SplitSum}[\texttt{c}_\gamma[\texttt{n}-1]]\ /.\ \texttt{n}-> 0;$$
$$\beta = \texttt{Cot}[\alpha];\hat\beta = \frac{\sqrt{\texttt{cm}}}{\sqrt{\texttt{cp}}}\beta;$$
$$\delta = \frac{\texttt{a[0]}}{\texttt{cn}(\texttt{z}-\lambda)} - \frac{\beta}{\texttt{cp}(1+\hat\beta^2)};$$
$$\texttt{sb}_\gamma[\texttt{z\_},\texttt{n\_}] = \frac{\sqrt{\texttt{cm}}\sqrt{\texttt{cp}}}{\texttt{cn}}\texttt{A[sb[z,n]},\{\texttt{z},\texttt{n}\}];$$
$$\texttt{cb}_\gamma[\texttt{z\_},\texttt{n\_}] = \frac{\texttt{cp}}{\texttt{cn}}\frac{1+\hat\beta^2}{1+\beta^2}(\texttt{A[cb[z,n]},\{\texttt{z},\texttt{n}\}] - \delta\texttt{A[sb[z,n]},\{\texttt{z},\texttt{n}\}]);$$

as can be seen from

In[17]:=
$$\texttt{Simplify[SplitSum[}\frac{\texttt{sb}_\gamma[\texttt{z},\texttt{n}+1]}{\texttt{sb}_\gamma[\texttt{z},\texttt{n}]}]\ /.\ \texttt{n}-> 0\Big]\Big/\hat\beta$$

Out[17]= -1

and

In[18]:=
$$\texttt{Simplify[SplitSum[}\frac{\texttt{cb}_\gamma[\texttt{z},\texttt{n}+1]}{\texttt{cb}_\gamma[\texttt{z},\texttt{n}]}]\hat\beta\ /.\ \texttt{n}-> 0]$$

Out[18]= 1

C.6.3. Dirichlet commutation method. In this section we want to consider the Dirichlet commutation method. As before we load our package

```
In[1]:= Needs["DiscreteMath`JacDEqs`"]
```

and introduce two solutions of our original operator.

```
In[2]:= SolutionJacobi[u];
        SolutionJacobi[v];
        u[μ0, 0] = 0;
        W[f__, n_] := a[n]Casoratian[f, n];
```

Next, we introduce the operator $H_{(\mu,\sigma)}$

```
In[3]:= W{μ,σ}[n_] = W[u[μ0, n], u[μ, n], n] / (μ - μ0);
        a{μ,σ}[n_] := a[n]√W{μ,σ}[n - 1]√W{μ,σ}[n + 1] / W{μ,σ}[n];
        b{μ,σ}[n_] := b[n] - BDiff[a[n]u[μ0, n]u[μ, n + 1] / W{μ,σ}[n], n];
        JacobiDE{μ,σ}[f_, n_] := a{μ,σ}[n] S[f, {n, 1}] + a{μ,σ}[n - 1]S[f, {n, -1}]
            + b{μ,σ}[n]f;
        W{μ,σ}[f__, n_] := a{μ,σ}[n]Casoratian[f, n];
```

We note the alternate expression for $b_{(\mu,\sigma)}$

```
In[4]:= Simplify[b{μ,σ}[n] == μ0 - a[n]u[μ, n + 1]W{μ,σ}[n - 1] / u[μ, n]W{μ,σ}[n]
        - a[n - 1]u[μ, n - 1]W{μ,σ}[n] / u[μ, n]W{μ,σ}[n - 1]]
Out[4]= True
```

and $W_{(\mu,\sigma)}$

```
In[5]:= Simplify[SplitSum[FDiff[ Sum*_{j=0}^{n} u[μ0, j]u[μ, j] == W{μ,σ}[n], n] ]]
Out[5]= True
```

Defining the quantities

```
In[6]:= uμ[n_] := u[μ0, n] / √W{μ,σ}[n - 1]√W{μ,σ}[n];
        uμ0[n_] := u[μ, n] / √W{μ,σ}[n - 1]√W{μ,σ}[n];
        A[f_, z_, n_] := √W{μ,σ}[n] / √W{μ,σ}[n - 1] f - uμ0[n]W[u[μ0, n], f, n] / (z - μ0);
Out[6]= True
```

we get several solutions of our new operator.

In[7]:= $\texttt{Simplify[JacobiDE}_{\{\mu,\sigma\}}\texttt{[u}_{\mu}\texttt{[n],n]} == \mu\texttt{u}_{\mu}\texttt{[n]]}$

Out[7]= True

In[8]:= $\texttt{Simplify[JacobiDE}_{\{\mu,\sigma\}}\texttt{[u}_{\mu_0}\texttt{[n],n]} == \mu_0\texttt{u}_{\mu_0}\texttt{[n]]}$

Out[8]= True

In[9]:= $\texttt{Simplify[JacobiDE}_{\{\mu,\sigma\}}\texttt{[A[u[z,n],z,n],n]} == \texttt{zA[u[z,n],z,n]]}$

Out[9]= True

In addition, we note

In[10]:=
$$\texttt{Simplify[A[u[z,n],z,n]}^2 == \texttt{u[z,n]}^2 + \frac{\texttt{u}[\mu,\texttt{n}]}{(\texttt{z}-\mu_0)^2}\texttt{u}[\mu_0,\texttt{n}]\texttt{BDiff}[\frac{\texttt{W[u}[\mu_0,\texttt{n}],\texttt{u[z,n]},\texttt{n}]^2}{\texttt{W}_{\{\mu,\sigma\}}\texttt{[n]}},\texttt{n}]]$$

Out[10]= True

and the following formulas for Wronskians

In[11]:=
$$\texttt{Simplify[W}_{\{\mu,\sigma\}}\texttt{[u}_{\mu_0}\texttt{[n],A[u[z,n],z,n],n]} == \frac{\texttt{W[u}[\mu,\texttt{n}],\texttt{u[z,n]},\texttt{n}]}{\texttt{W}_{\{\mu,\sigma\}}\texttt{[n]}}]$$

Out[11]= True

In[12]:=
$$\texttt{Simplify[W}_{\{\mu,\sigma\}}\texttt{[u}_{\mu}\texttt{[n],A[u[z,n],z,n],n]} == \frac{\texttt{z}-\mu}{\texttt{z}-\mu_0}\frac{\texttt{W[u}[\mu_0,\texttt{n}],\texttt{u[z,n]},\texttt{n}]}{\texttt{W}_{\{\mu,\sigma\}}\texttt{[n]}}]$$

Out[12]= True

In[13]:=
$$\texttt{Simplify[W}_{\{\mu,\sigma\}}\texttt{[A[u[z,n],z,n],A[v[zz,n],zz,n],n]} == \frac{\texttt{z}-\mu}{\texttt{z}-\mu_0}\texttt{W[u[z,n],v[zz,n],n]} + \frac{\texttt{z}-\texttt{zz}}{(\texttt{z}-\mu_0)(\texttt{zz}-\mu_0)}\frac{\texttt{W[u}[\mu,\texttt{n}],\texttt{u[z,n]},\texttt{n}]\texttt{W[u}[\mu_0,\texttt{n}],\texttt{v[zz,n]},\texttt{n}]}{\texttt{W}_{\{\mu,\sigma\}}\texttt{[n]}}]$$

Out[13]= True

Finally, we turn to the solutions $c(z,n)$ and $s(z,n)$ which transform into

In[14]:=
$$\texttt{s}_{\{\mu,\sigma\}}\texttt{[z\_,n\_]} = \frac{\sqrt{\texttt{W}_{\{\mu,\sigma\}}\texttt{[m+1]}}}{\sqrt{\texttt{W}_{\{\mu,\sigma\}}\texttt{[m]}}}\texttt{A[s[z,n,0],z,n]} /.\ \texttt{m\_} :> 0;$$
$$\texttt{c}_{\{\mu,\sigma\}}\texttt{[z\_,n\_]} = \frac{\texttt{z}-\mu_0}{\texttt{z}-\mu}\Big(\texttt{A[c[z,n,0],z,n]} + \frac{(\mu-\mu_0)\texttt{u}[\mu,\texttt{m}+1]\texttt{A[s[z,n,0],z,n]}}{(\texttt{z}-\mu_0)}\texttt{u}[\mu,\texttt{m}]\Big) /.\ \texttt{m\_} :> 0;$$

Out[14]= True

as can be seen from

```
In[15]:=s{μ,σ}[z, 0]
Out[15]= 0

In[16]:=Simplify[s{μ,σ}[z, 1]]
Out[16]= 1

In[17]:=Simplify[c{μ,σ}[z, 0]]
Out[17]= 1

In[18]:=Simplify[c{μ,σ}[z, 1]]
Out[18]= 0
```

Note also

```
In[19]:=Simplify[W{μ,σ}[n + 1] / W{μ,σ}[n] /. n_ -> 0]
Out[19]= (a[0]u[μ, 0] + (-μ + μ0)u[μ, 1]) / (a[0]u[μ, 0])
```

C.7. Toda lattice

Now we come to the Toda lattice. Our package is needed again.

```
In[1]:= Needs["DiscreteMath`JacDEqs`"]
```

To define the Toda hierarchy, we recall the recursions for g_j and h_j found before. We only consider the homogeneous case for simplicity.

```
In[2]:= g[0, n_] = 1;
        h[0, n_] = 0;
        g[j_?IntegerQ, n_] := g[j, n] =
          1/2 (h[j - 1, n] + h[j - 1, n - 1]) + b[n]g[j - 1, n];
        h[j_?IntegerQ, n_] := h[j, n] =
          2a[n]^2 Sum[g[j - l - 1, n]g[l, n + 1], {l, 0, j - 1}] -
            1/2 Sum[h[j - l - 1, n]h[l, n], {l, 0, j - 1}];
```

The following command will compute the r'th (homogeneous) Toda equation for us

```
In[3]:= TL[r_] := Simplify[
          {∂t a[n, t] == a[n](g[r + 1, n + 1] - g[r + 1, n]),
           ∂t b[n, t] == h[r + 1, n] - h[r + 1, n - 1]}] /.
          {a[n_] -> a[n, t], b[n_] -> b[n, t]}
```

For example

```
In[4]:= TL[1]//TableForm
```

```
Out[4]//TableForm=
a^(0,1)[n,t] == -a[n,t](a[-1+n,t]^2 - a[1+n,t]^2 + b[n,t]^2 - b[1+n,t]^2)
b^(0,1)[n,t] == -2a[-1+n,t]^2(b[-1+n,t] + b[n,t])
                + 2a[n,t]^2(b[n,t] + b[1+n,t])
```

Next, we introduce the second operator in the Lax pair which will be denoted by TodaP.

```
In[5]:= TodaP[r_,f_,n_] := -JacobiDE[r+1,f,n] +
          Sum[2(a[n]g[j,n]S[JacobiDE[r-j,f,n],{n,1}]
          - h[j,n]JacobiDE[r-j,f,n]), {j,0,r}] + g[r+1,n]f;
```

Here is the first one applied to a sequence $f(n)$

```
In[6]:= TodaP[0,f[n],n]
```

```
Out[6]= -a[-1+n]f[-1+n] + a[n]f[1+n]
```

or, alternatively, in matrix form (cut off after 2 terms in both directions).

```
In[7]:= m = 2;
        Table[Coefficient[TodaP[0,f[n],n], f[n+i-j]],
          {i,-m,m},{j,-m,m}]//MatrixForm
        m = .;
```

```
Out[7]//MatrixForm=
```

$$\begin{pmatrix} 0 & -a[-1+n] & 0 & 0 & 0 \\ a[n] & 0 & -a[-1+n] & 0 & 0 \\ 0 & a[n] & 0 & -a[-1+n] & 0 \\ 0 & 0 & a[n] & 0 & -a[-1+n] \\ 0 & 0 & 0 & a[n] & 0 \end{pmatrix}$$

Similarly, the second one applied to a sequence $f(n)$

```
In[9]:= TodaP[1,f[n],n]//Simplify
```

```
Out[9]= -a[-2+n]a[-1+n]f[-2+n] - a[-1+n](b[-1+n] + b[n])f[-1+n] +
        a[n](b[n]f[1+n] + b[1+n]f[1+n] + a[1+n]f[2+n])
```

The following command will display the coefficients in a handy format.

```
In[10]:=CoeffTodaP[k_] := Block[{dummy,i,n},
          dummy = TodaP[k,f[n],n];
          TableForm[
           Table[{f[n+i]," : ",Simplify[Coefficient[dummy,f[n+i]]]},
           {i,-k-2,k+2}]]
          ];
```

For example we obtain for $r = 0$

```
In[11]:=CoeffTodaP[0]
```

```
Out[11]//TableForm=
```

$$
\begin{array}{ll}
f[-2+n] & : \quad 0 \\
f[-1+n] & : \quad -a[-1+n] \\
f[n] & : \quad 0 \\
f[1+n] & : \quad a[n] \\
f[2+n] & : \quad 0
\end{array}
$$

and $r = 1$

```
In[12]:=CoeffTodaP[1]
```

```
Out[12]//TableForm=
```

$$
\begin{array}{ll}
f[-3+n] & : \quad 0 \\
f[-2+n] & : \quad -a[-2+n]a[-1+n] \\
f[-1+n] & : \quad -a[-1+n](b[-1+n]+b[n]) \\
f[n] & : \quad 0 \\
f[1+n] & : \quad a[n](b[n]+b[1+n]) \\
f[2+n] & : \quad a[n]a[1+n] \\
f[3+n] & : \quad 0
\end{array}
$$

We can also compute the commutator of H and P_{2r+2}

```
In[13]:=Commutator[r_, f_, n_] :=
          JacobiDE[TodaP[r, f, n], n] - TodaP[r, JacobiDE[f, n], n];


In[14]:=CoeffCommutator[r_] := Block[{dummy, i, n},
          dummy = Commutator[r, f[n], n];
          TableForm[Table[{f[n + i], " : ",
           Simplify[
           Coefficient[dummy, f[n + i]]]}, {i, -2, 2}]]    ];
```

and display its matrix coefficients

```
In[15]:=CoeffCommutator[0]
```

```
Out[15]//TableForm=
```

$$
\begin{array}{ll}
f[-2+n] & : \quad 0 \\
f[-1+n] & : \quad a[-1+n](b[-1+n]-b[n]) \\
f[n] & : \quad 2(a[-1+n]^2 - a[n]^2) \\
f[1+n] & : \quad a[n](b[n]-b[1+n]) \\
f[2+n] & : \quad 0
\end{array}
$$

C.8. Kac-van Moerbeke lattice

Finally we come to the Kac van Moerbeke lattice.

```
In[1]:= Needs["DiscreteMath`JacDEqs`"]
```

Again we start with the recursions for G_j and H_j and only consider the homogeneous case for simplicity.

```
In[2]:= G[0,n_] = 1;H[0,n_] = 0;
        G[j_?IntegerQ,n_] := G[j,n] =
          1/2(H[j - 1,n] + H[j - 1,n - 2]) + (ρ[n]^2 + ρ[n - 1]^2)G[j - 1,n];
        H[j_?IntegerQ,n_] := H[j,n] =
          2(ρ[n]ρ[n+1])^2 Sum[G[j-l-1,n]G[l,n+1],{l,0,j-1}] - 1/2 Sum[H[j-l-1,n]H[l,n],{l,0,j-1}];
```

We can easily compute the first few

```
In[3]:= Table[Simplify[{G[j,n],H[j,n]}],{j,0,1}]//TableForm
```

```
Out[3]//TableForm=
1                          0
ρ[-1 + n]^2 + ρ[n]^2       2ρ[n]^2ρ[1 + n]^2
```

This command will display the r'th Kac-van Moerbeke equation

```
In[4]:= KM[r_] := Simplify[
          ∂t ρ[n,t] == ρ[n](G[r + 1,n + 1] - G[r + 1,n])] /. ρ[n_]- > ρ[n,t];
```

For example

```
In[5]:= KM[1]
```

$$\begin{aligned}\text{Out[5]=}\quad & \rho^{(0,1)}[n,t] == -\rho[n,t](\rho[-2+n,t]^2\rho[-1+n,t]^2 + \rho[-1+n,t]^4 \\ & + \rho[-1+n,t]^2\rho[n,t]^2 - \rho[1+n,t]^2(\rho[n,t]^2 + \rho[1+n,t]^2 + \rho[2+n,t]^2)\end{aligned}$$

Next, we need two pairs of sequences a_1, b_1 and a_2, b_2

```
In[6]:= a1[n_] := ρe[n]ρo[n]; b1[n_] := ρe[n]^2 + ρo[n - 1]^2;
        a2[n_] := ρe[n + 1]ρo[n]; b2[n_] := ρe[n]^2 + ρo[n]^2;
```

plus the corresponding quantities $g_{k,j}$, $h_{k,j}$, $k = 1, 2$.

```
In[7]:= gk_[0,n_] = 1; hk_[0,n_] = 0;
        gk_[j_?IntegerQ,n_] := gk[j,n] =
          1/2(hk[j - 1,n] + hk[j - 1,n - 1]) + bk[n]gk[j - 1,n];
        hk_[j_?IntegerQ,n_] := hk[j,n] =
          2ak[n]^2 Sum[j - 1gk[j - l - 1,n]gk[l,n + 1],{l,0}] -
          1/2 Sum[j - 1hk[j - l - 1,n]hk[l,n],{l,0}];
```

They satisfy several relations which can be checked for given j:

```
In[8]:= Simplify[(g2[j + 1,n] ==
          ρo[n]^2 g1[j,n + 1] + ρe[n]^2 g1[j,n] + h1[j,n]) /. j- > 5]
Out[8]= True
```

```
In[9]:= Simplify[(h2[j + 1,n] - h1[j + 1,n] == ρo[n]^2(2(ρe[n + 1]^2 g1[j,n + 1] -
        ρe[n]^2 g1[j,n]) + (h1[j,n + 1] - h1[j,n]))) /. j- > 5]
Out[9]= True
```

```
In[10]:=Simplify[(g1[j+1,n] == ρe[n]^2 g2[j,n] + ρo[n-1]^2 g2[j,n-1] + h2[j,n-
        1]) /. j-> 5]
Out[10]= True
```

```
In[11]:=Simplify[(h2[j + 1,n - 1] - h1[j + 1,n] == ρe[n]^2(-2(ρo[n]^2 g2[j,n] -
        ρo[n - 1]^2 g2[j,n - 1]) - (h2[j,n] - h2[j,n - 1]))) /. j-> 5]
Out[11]= True
```

```
In[12]:=Simplify[(h2[j,n - 1] - h1[j,n] == 2ρe[n]^2(g1[j,n] - g2[j,n])) /. j-> 5]
Out[12]= True
```

```
In[13]:=Simplify[(h1[j + 1,n] - h2[j + 1,n - 1] ==
            2(a1[n]^2 g1[j,n] - a2[n-1]^2 g2[j,n-1]) - ρo[n]^2(h2[j,n-1] - h1[j,n]) +
            ρe[n]^2(h2[j,n] - h2[j,n - 1])) /. j-> 5]
Out[13]= True
```

Now we introduce the corresponding operators A, A^*, H_k, $P_{k,2r+2}$, D, Q_{2r+2}, $k = 1, 2$.

```
In[14]:=A[f_,n_] := ρo[n]S[f,{n,1}] + ρe[n]f;
        A*[f_,n_] := ρo[n - 1]S[f,{n,-1}] + ρe[n]f;
        JacobiDEk_[f_,n_] := ak[n]S[f,{n,1}] + S[ak[n]f,{n,-1}] + bk[n]f;
        JacobiDEk_[r_?IntegerQ,f_,n_] := Nest[JacobiDEk[#,n]&,f,r];
        DiracDE[{f_,g_},n_] := A[g,n],A*[f,n];
        TodaPk_[r_,f_,n_] := -JacobiDEk[r + 1,f,n]
            + Σ_{j=0} r(2ak[n]gk[j,n]S[JacobiDEk[r - j,f,n],n,1]
            - hk[j,n]JacobiDEk[r - j,f,n]) + gk[r + 1,n]f;
        KMQ[r_,{f_,g_},n_] := {TodaP1[r,f,n],TodaP2[r,g,n]};
```

Again we can use *Mathematica* to check some simple formulas. For example $A^*A = H_1$

```
In[15]:=Simplify[A*[A[f[n],n],n] == JacobiDE1[f[n],n]]
Out[15]= True
```

or $AA^* = H_2$

```
In[16]:=Simplify[A[A*[f[n],n],n] == JacobiDE2[f[n],n]]
Out[16]= True
```

Moreover, we have $H_2^r A = AH_1^r$

```
In[17]:=Simplify[(JacobiDE2[r,A[f[n],n],n] == A[JacobiDE1[r,f[n],n],n])
            /. r-> 5]
```

```
Out[17]= True
```

respectively $A^*H_2^r = H_1^r A^*$

```
In[18]:=Simplify[(A*[JacobiDE2[r, f[n], n], n] == JacobiDE1[r, A*[f[n], n], n])
        /. r− > 5]
Out[18]= True
```

and $P_{1,2r+2}A - AP_{2,2r+2} = \rho_o(g^+_{1,r+1} - g_{2,r+1})S^+ + \rho_e(g_{2,r+1} - g_{1,r+1})$

```
In[19]:=Simplify[(TodaP2[r, A[f[n], n], n] − A[TodaP1[r, f[n], n], n] ==
        ρo[n](g1[r + 1, n + 1] − g2[r + 1, n])f[n + 1] + ρe[n](g2[r + 1, n] − g1[r +
        1, n])f[n]) /. r− > 5]
Out[19]= True
```

respectively $P_{1,2r+2}A^* - A^*P_{2,2r+2} = -\rho_o^-(g^-_{2,r+1} - g_{1,r+1})S^- + \rho_e(g_{2,r+1} - g_{1,r+1})$

```
In[20]:=Simplify[(TodaP1[r, A*[f[n], n], n] − A*[TodaP2[r, f[n], n], n] ==
        − ρo[n − 1](g2[r + 1, n − 1] − g1[r + 1, n])f[n − 1] + ρe[n](g2[r + 1, n] −
        g1[r + 1, n])f[n]) /. r− > 5]
Out[20]= True
```

Bibliography

[1] R. Abraham, J.E. Marsden, and T. Ratiu, *Manifolds, Tensor Analysis, and Applications*, 2nd edition, Springer, New York, 1983.

[2] M. Adler, *On the Bäcklund transformation for the Gel'fand-Dickey equations*, Commun. Math. Phys. **80**, 517–527 (1981).

[3] M. Adler, L. Haine, and P. van Moerbeke, *Limit matrices for the Toda flow and periodic flags for loop groups*, Math. Ann. **296**, 1–33 (1993).

[4] R. P. Agarwal, *Difference Equations and Inequalities*, Marcel Dekker, New York, 1992.

[5] C. D. Ahlbrandt, *Continued fraction representation of maximal and minimal solutions of a discrete matrix Riccati equation*, Siam J. Math. Anal. **24**, 1597–1621 (1993).

[6] C. D. Ahlbrandt, *Dominant and recessive solutions of symmetric three term recurrences*, J. Diff. Eqs. **107**, 238–258 (1994).

[7] C. D. Ahlbrandt and W. T. Patula, *Recessive solutions of block tridiagonal nonhomogenous systems*, J. Difference Eqs. Appl. **1**, 1–15 (1995).

[8] N. I. Akhiezer, *A continuous analogue of orthogonal polynomials on a system of intervals*, Sov. Math. Dokl. **2**, 1409–1412 (1961).

[9] N. Akhiezer, *The Classical Moment Problem*, Oliver and Boyd, London, 1965.

[10] N. Akhiezer, *Elements of the Theory of Elliptic Functions*, Transl. Math. Monographs, vol. 79, Amer. Math. Soc., Providence, R. I., 1990.

[11] S. J. Al'ber, *Associated integrable systems*, J. Math. Phys. **32**, 916–922 (1991).

[12] A. J. Antony and M. Krishna, *Nature of some random Jacobi matrices*, preprint (1993).

[13] A. J. Antony and M. Krishna, *Inverse spectral theory for Jacobi matrices and their almost periodicity*, Proc. Indian Acad. Sci. (Math. Sci.) **104**, 777–818 (1994).

[14] A. J. Antony and M. Krishna, *Almost periodicity of some Jacobi matrices*, Proc. Indian Acad. Sci. (Math. Sci.) **102**, 175–188 (1992).

[15] A. I. Aptekarev, A. Branquinho, and F. Marcellan, *Toda-type differential equations for the recurrence coefficients of orthogonal polynomials and Freud transformation*, J. Comput. Appl. Math. **78**, 139–160 (1997).

[16] N. Aronszajn, *On a problem of Weyl in the theory of singular Sturm–Liouville equations*, Am. J. of Math. **79**, 597–610 (1957).

[17] N. Aronszajn and W. Donoghue, *On the exponential representation of analytic functions in the upper half-plane with positive imaginary part*, J. d'Analyse Math. **5**, 321–388 (1956-57).

[18] T. Asahi, *Spectral theory of the difference equations*, Prog. Theor. Phys. Suppl., **36**, 55–96 (1966).

[19] F. Atkinson, *Discrete and Continuous Boundary Problems*, Academic Press, New York, 1964.

[20] M. Audin, *Vecteurs propres de matrices de Jacobi*, Ann. Inst. Fourier **44**, 1505–1517 (1994).

[21] J. Avron and B. Simon *Singular continuous spectrum of a class of almost periodic Jacobi operators*, Bull. Amer. Math. Soc. **6**, 81–85 (1982).

[22] J. Avron and B. Simon *Almost periodic Schrödinger operators II. The integrated density of states*, Duke Math. J. **50**, 369–391 (1983).

[23] H. F. Baker, *Note on the foregoing paper, "Commutative ordinary differential operators", by J. L. Burchnall and T. W. Chaundy*, Proc. Roy. Soc. London **A118**, 584–593 (1928).

[24] B. Beckermann and V. Kaliaguine, *The diagonal of the Pade table and the approximation of the Weyl function of second-order difference operators*, Constructive Approximation **13**, 481–510 (1997).

[25] E. D. Belokolos, A. I. Bobenko, V. Z. Enol'skii, A. R. Its, and V. B. Matveev, *Algebro-geometric Approach to Nonlinear Integrable Equations*, Springer, Berlin, 1994.

[26] M. Benammar and W.D. Evans, *On the Friedrichs extension of semi-bounded difference operators*, Math. Proc. Camb. Phil. Soc. **116**, 167–177 (1994).

[27] Yu. M. Berezanskiĭ, *Expansions in Eigenfunctions of Self-adjoint Operators*, Transl. Math. Monographs, vol. 17, Amer. Math. Soc., Providence, R. I., 1968.

[28] Yu. M. Berezanskiĭ, *Integration of nonlinear difference equations by the inverse spectral problem method*, Soviet Math. Dokl., **31** No. 2, 264–267 (1985).

[29] Yu. M. Berezanski, *The integration of semi-infinite Toda chain by means of inverse spectral problem*, Rep. Math. Phys., **24** No. 1, 21–47 (1985).

[30] Yu. M. Berezansky, *Integration of nonlinear nonisospectral difference-differential equations by means of the inverse spectral problem*, in "*Nonlinear Physics. Theory and experiment*", (eds E. Alfinito, M. Boiti, L. Martina, F. Pempinelli), World Scientific, 11–20 (1996).

[31] Yu. M. Berezansky and M. I. Gekhtman, *Inverse problem of the spectral analysis and non-Abelian chains of nonlinear equations*, Ukrain. Math. J., **42**, 645–658 (1990).

[32] Yu. Berezansky and M. Shmoish, *Nonisospectral flows on semi-infinite Jacobi matrices*, Nonl. Math. Phys., **1** No. 2, 116–146 (1994).

[33] D. Bättig, B. Grébert, J. C. Guillot, and T. Kappeler, *Fibration of the phase space of the periodic Toda lattice*, J. Math. Pures Appl. **72**, 553–565 (1993).

[34] O. I. Bogoyavlenskii, *Algebraic constructions of integrable dynamical systems–extensions of the Volterra system*, Russian Math. Surv. **46:3**, 1–64 (1991).

[35] W. Bulla, F. Gesztesy, H. Holden, and G. Teschl *Algebro-Geometric Quasi-Periodic Finite-Gap Solutions of the Toda and Kac-van Moerbeke Hierarchies*, Memoirs of the Amer. Math. Soc. **135/641**, 1998.

[36] J. L. Burchnall and T. W. Chaundy, *Commutative ordinary differential operators*, Proc. London Math. Soc. Ser. 2, **21**, 420–440 (1923).

[37] J. L. Burchnall and T. W. Chaundy, *Commutative ordinary differential operators*, Proc. Roy. Soc. London **A118**, 557–583 (1928).

[38] P. Byrd and M. Friedman *Handbook of Elliptic Integrals for Engineers and Physists*, Springer, Berlin, 1954.

[39] L. Breiman, *Probability*, Addison-Wesley, London, 1968.

[40] R. Carmona and J. Lacroix, *Spectral Theory of Random Schrödinger Operators*, Birkhäuser, Boston, 1990.

[41] R. Carmona and S. Kotani, Inverse spectral theory for random Jacobi matrices, J. Stat. Phys. **46**, 1091–1114 (1987).

[42] K. M. Case, *Orthogonal polynomials from the viewpoint of scattering theory*, J. Math. Phys. **14**, 2166–2175 (1973).

[43] K. M. Case, *The discrete inverse scattering problem in one dimension*, J. Math. Phys. **15**, 143–146 (1974).

[44] K. M. Case, *Orthogonal polynomials II*, J. Math. Phys. **16**, 1435–1440 (1975).

[45] K. M. Case, *On discrete inverse scattering problems. II*, J. Math. Phys. **14**, 916–920 (1973).

[46] K. M. Case and S. C. Chiu *The discrete version of the Marchenko equations in the inverse scattering problem*, J. Math. Phys. **14**, 1643–1647 (1973).

[47] K. M. Case and M. Kac, *A discrete version of the inverse scattering problem*, J. Math. Phys. **14**, 594–603 (1973).

[48] S. Clark and D. Hinton, *Strong nonsubordinacy and absolutely continuous spectra for Sturm-Liouville equations*, Diff. Int. Eqs. **6**, 573–586 (1993).

[49] W. Craig and B. Simon, *Subharmonicity of the Lyaponov exponent*, Duke Math. J. **50**, 551–560 (1983).

[50] H. L. Cyclon, R. G. Froese, W. Kirsch and B. Simon, *Schrödinger Operators*, Springer, Berlin, 1987.

[51] D. Damanik and D. Lenz, *Uniform spectral properties of one-dimensional quasicrystals, I. Absence of eigenvalues*, Comm. Math. Phys. (to appear).

[52] D. Damanik and D. Lenz, *Uniform spectral properties of one-dimensional quasicrystals, II. The Lyapunov exponent*, preprint (1999).

[53] E. Date and S. Tanaka *Analogue of inverse scattering theory for the discrete Hill's equation and exact solutions for the periodic Toda lattice*, Prog. Theoret. Phys. **56**, 457–465 (1976).

[54] P. Deift, *Applications of a commutation formula*, Duke Math. J. **45**, 267–310 (1978).

[55] P. A. Deift and L. C. Li, *Generalized affine Lie algebras and the solution of a class of flows associated with the QR eigenvalue algorithm*, Commun. Pure Appl. Math. **42**, 963–991 (1989).

[56] P. Deift and L. C. Li, *Poisson geometry of the analog of the Miura maps and Bäcklund-Darboux transformations for equations of Toda type and periodic Toda flows*, Commun. Math. Phys. **143**, 201–214 (1991).

[57] P. Deift, *Orthogonal Polynomials and Random Matrices: A Riemann-Hilbert Approach*, Courant Lecture Notes in Mathematics 3, Courant Institute, New York, 1999.

[58] P. Deift and R. Killip, *On the absolutely continuous spectrum of one dimensional Schrödinger operators with square summable potentials*, preprint (1999).

[59] P. Deift and K. T-R. McLaughlin, *A Continuum Limit of the Toda Lattice*, Memoirs of the Amer. Math. Soc. **131/624**, 1998.

[60] P. Deift and E. Trubowitz, *Inverse scattering on the line*, Comm. Pure Appl. Math. **32**, 121–251 (1979).

[61] P. Deift and X. Zhou, *Long-time asymptotics for integrable systems. Higher order theory.*, Commun. Math. Phys. **165**, 175–191 (1994).

[62] P. Deift and X. Zhou, *A steepest descent method for oscillatory Riemann-Hilbert problems. Asymptotics for the MKdV equation*, Ann. of Math. **137**, 295–368 (1993).

[63] P. Deift, T. Kriecherbauer, and S. Venakides, *Forced lattice vibrations. I*, Commun. Pure Appl. Math. **48**, 1187–1250 (1995).

[64] P. Deift, T. Kriecherbauer, and S. Venakides, *Forced lattice vibrations. II*, Commun. Pure Appl. Math. **48**, 1251–1298 (1995).

[65] P. Deift, L.C. Li, and C. Tomei, *Toda flows with infinitely many variables*, J. Func. Anal. **64**, 358–402 (1985).

[66] P. Deift, L. C. Li, and C. Tomei, *Matrix factorizations and integrable systems*, Commun. Pure Appl. Math. **42**, 443–521 (1989).

[67] P. Deift, S. Kamvissis, T. Kriecherbauer, and X. Zhou, *The Toda rarefaction problem*, Commun. Pure Appl. Math. **49**, 35–83 (1996).

[68] R. del Rio, S. Jitomirskaya, Y. Last, and B. Simon, *Operators with Singular Continuous Spectrum, IV. Hausdorff dimensions, rank one perturbations, and localization*, J. Anal. Math. **69**, 153–200 (1996).

[69] F. Delyon and B. Souillard, *The rotation number for finite difference operators and its properies*, Comm. Math. Phys. **89**, 415–426 (1983).

[70] F. Delyon and B. Souillard, *Remark on the continuity of the density of states of ergodic finite difference operators*, Comm. Math. Phys. **94**, 289–291 (1984).

[71] J. Dombrowski, *Absolutely continuous measures for systems of orthogonal polynomials with unbounded recurrence coefficients*, Constructive Approximation **8**, 161–167 (1992).

[72] J. Dombrowski and S. Pedersen, *Orthogonal polynomials, spectral measures, and absolute continuity*, J. Comput. Appl. Math. **65**, 115–124 (1995).

[73] W. Donoghue, *Monotone mMatrix Functions and Analytic Continuation*, Springer, Berlin, 1974.

[74] B. Dubrovin, *Theta functions and nonlinear equations*, Russian Math. Surveys. **362**, 11–92 (1981).

[75] B. Dubrovin, I. Krichever, and S. Novikov *Integrable systems I*, in "*Dynamical Systems IV*", (eds. V. Arnold, S. Novikov), 173–280 (1990).

[76] B. A. Dubrovin, V. B. Matveev, and S. P. Novikov, *Non-linear equations of Korteweg-de Vries type, finite-zone linear operators, and Abelian varieties*, Russian Math. Surv. **31:1**, 59–146 (1976).

[77] J. Edwards, *Spectra of Jacobi matrices, differential equations on the circle, and the* $su(1,1)$ *Lie algebra*, SIAM J. Math. Anal. **24**, No. 3, 824–831 (1993).

[78] S. N. Eilenberger, *An Introduction to Difference Equations*, Springer, New York, 1996.

[79] G. Elaydi, *Solitons*, Springer, Berlin, 1981.

[80] L. Faddeev and L. Takhtajan, *Hamiltonian Methods in the Theory of Solitons*, Springer, Berlin, 1987.

[81] H. Farkas and I. Kra, *Riemann Surfaces*, 2nd edition, GTM 71, Springer, New York, 1992.

[82] J. Fay, *Theta Functions on Riemann Surfaces*, LN in Math. 352, Springer, Berlin, 1973.

[83] A. Finkel, E. Isaacson, and E. Trubowitz, *An explicit solution of the inverse periodic problem for Hill's equation*, SIAM J. Math. Anal. **18**, 46–53 (1987).

[84] H. Flaschka, *On the Toda lattice. I*, Phys. Rev. B**9**, 1924–1925 (1974).

[85] H. Flaschka, *On the Toda lattice. II*, Progr. Theoret. Phys. **51**, 703–716 (1974).

[86] O. Forster, *Lectures on Riemann Surfaces*, GTM 81, Springer, New York, 1991.

[87] T. Fort, *Finite Differences and Difference Equations in the Real Domain*, Oxford University Press, London, 1948.

[88] L. Fu, *An Inverse theorem for Jacobi matrices*, Linear Algebra Appl. **12**, 55–61 (1975).

[89] L. Fu and H. Hochstadt, *Inverse theorems for Jacobi matrices* , J. Math. Anal. Appl. **47**, 162–168 (1974).

[90] H. Furstenberg and H. Kesten, *Products of random matrices*, Ann. Math. Stat. **31**, 457–469 (1960).

[91] F. R. Gantmacher, *The Theory of Matrices*, Vol. 1, Chelsa, New York, 1990.

[92] M. G. Gasymov and G. Sh. Gusejnov, *On the inverse problems of spectral analysis for infinite Jacobi matrices in the limit-circle*, Sov. Math. Dokl. **40**, 627–630 (1990).

[93] I. M. Gel'fand and B. M. Levitan, *On a simple identity for eigenvalues of a second order differential operator* Dokl. Akad. Nauk SSSR **88**, 593–596 (1953); English translation in Izrail M. Gelfand, *Collected papers*, Vol. I (S. G. Gindikin et al, eds), pp 457–461, Springer, Berlin, 1987.

[94] J. S. Geronimo, *A relation between the coefficients in the recurrence formula and the spectral function for orthogonal polynomials*, Trans. Amer. Math. Soc. **260**, 65–82 (1980).

[95] J. S. Geronimo, *On the spectra of infinite–dimensional Jacobi matrices*, J. App. Th. **53**, 251–265 (1988).

[96] J. S. Geronimo, *An upper bound on the number of eigenvalues of an infinite–dimensional Jacobi matrix*, J. Math. Phys. **23**, 917–921 (1982).

[97] J. S. Geronimo, *Scattering theory, orthogonal polynomials, and q-series*, Siam J. Anal. **25**, 392–419 (1994).

[98] J. S. Geronimo and W. Van Assche, *Orthogonal polynomials with asymptotically periodic recurrence coefficients*, J. App. Th. **46**, 251–283 (1986).

[99] K. M. Case and J. S. Geronimo, *Scattering theory and polynomials orthogonal on the unit circle*, J. Math. Phys. **20**, 299–310 (1979).

[100] J. S. Geronimo and R. A. Johnson, *Rotation number associated with difference equations satisfied by polynomials orthogonal on hte unit circle*, J. Diff. Eqs., **132**, 140–178 (1996).

[101] F. Gesztesy, *A complete spectral characterization of the double commutation method*, J. Funct. Anal. **117**, 401–446 (1993).

[102] F. Gesztesy and H. Holden, *Trace formulas and conservation laws for nonlinear evolution equations*, Rev. Math. Phys. **6**, 51–95 (1994).

[103] F. Gesztesy and H. Holden, *Hierarchies of Soliton Equations and their Algebro-Geometric Solutions*, monograph in preparation.

[104] F. Gesztesy and B. Simon, *m-functions and inverse spectral analysis for finite and semi-infinite Jacobi matrices*, J. Anal. Math. **73**, 267–297 (1997).

[105] F. Gesztesy and B. Simon, *The xi function*, Acta Math. **176**, 49–71 (1996).

[106] F. Gesztesy and B. Simon, *Uniqueness theorems in inverse spectral theory for one-dimensional Schrödinger operators*, Trans. Amer. Math. Soc. **348**, 349–373 (1996).

[107] F. Gesztesy and B. Simon, *Rank one perturbations at infinite coupling*, J. Funct. Anal. **128**, 245–252 (1995).

[108] F. Gesztesy and R. Svirsky, *(m)KdV-Solitons on the Background of Quasi-Periodic Finite-Gap Solutions*, Memoirs of the Amer. Math. Soc. **118**, 1995.

[109] F. Gesztesy and G. Teschl, *On the double commutation method*, Proc. Amer. Math. Soc. **124**, 1831–1840 (1996).

[110] F. Gesztesy and G. Teschl, *Commutation methods for Jacobi operators*, J. Diff. Eqs. **128**, 252–299 (1996).

[111] F. Gesztesy and Z. Zhao, *Critical and subcritical Jacobi operators defined as Friedrichs extensions*, J. Diff. Eq. **103**, 68–93 (1993).

[112] F. Gesztesy, H. Holden, and B. Simon, *Absolute summability of the trace relation for certain Schrödinger operators*, Com. Math. Phys. **168**, 137–161 (1995).

[113] F. Gesztesy, M. Krishna, and G. Teschl, *On isospectral sets of Jacobi operators*, Com. Math. Phys. **181**, 631–645 (1996).

[114] F. Gesztesy, R. Ratnaseelan, and G. Teschl, *The KdV hierarchy and associated trace formulas*, in "*Proceedings of the International Conference on Applications of Operator Theory*", (eds. I. Gohberg, P. Lancaster, and P. N. Shivakumar), Oper. Theory Adv. Appl., 87, Birkhäuser, Basel, 125–163 (1996).

[115] F. Gesztesy ,R. Nowell, and W. Pötz, *One-dimensional scattering for quantum systems with nontrivial spatial asymptotics*, Diff. Integral Eqs. **10**, 521–546 (1997).

[116] F. Gesztesy, B. Simon, and G. Teschl, *Zeroes of the Wronskian and renormalized oscillation Theory*, Am. J. Math. **118** 571–594 (1996).

[117] F. Gesztesy, H. Holden, B. Simon, and Z. Zhao, *On the Toda and Kac-van Moerbeke systems*, Trans. Amer. Math. Soc. **339**, 849–868 (1993).

[118] F. Gesztesy, H. Holden, B. Simon, and Z. Zhao, *Higher order trace relations for Schrödinger operators*, Rev. Math. Phys. **7**, 893–922 (1995).

[119] D. J. Gilbert, *On subordinacy and analysis of the spectrum of Schrödinger operators with two singular endpoints*, Proc. Roy. Soc. Edinburgh. **112A**, 213–229 (1989).

[120] D. J. Gilbert and D. B. Pearson, *On subordinacy and analysis of the spectrum of Schrödinger operators*, J. Math. Anal. Appl. **128**, 30–56 (1987).

[121] I. M. Glazman, *Direct Methods of Qualitative Spectral Analysis of Singular Differential Operators*, I.P.S.T., Jerusalem 1965.

[122] A. Ya. Gordon, *On the point spectrum of one-dimensional Schrödinger operators (in Russian)* Usp. Mat. Nauk **31**, 257-258 (1976).

[123] E. van Groesen and E.M. de Jager, *Mathematical Structures in Continuous Dynamical Systems*, North–Holland, Amsterdam 1994.

[124] G. S. Guseinov, *The inverse problem of scattering theory for a second-order difference equation on the whole axis*, Soviet Math. Dokl., **17**, 1684–1688 (1976).

[125] G. S. Guseinov, *The determination of an infinite Jacobi matrix from the scattering data*, Soviet Math. Dokl., **17**, 596–600 (1976).

[126] G. S. Guseinov, *Scattering problem for the infinite Jacobi matrix*, Izv. Akad. Nauk Arm. SSR, Mat. **12**, 365–379 (1977).

[127] H. Hancock, *Theory of Elliptic Functions*, Dover Publications, New York, 1958.

[128] P. Hartman, *Difference equations: Disconjugacy, principal solutions, Green's functions, complete monotonicity*, Trans. Amer. Math. Soc., **246**, 1–30 (1978).

[129] R. Hartshorne, *Algebraic Geometry*, Springer, Berlin, 1977.

[130] A. J. Heeger, S. Kivelson, J. R. Schrieffer, and W. P. Su, *Solitons in conducting polymers*, Revs. Mod. Phys. **60**, 781–850 (1988).

[131] E. Hellinger, *Zur Stieltjeschen Kettenbruch Theorie*, Math. Ann. **86**, 18–29 (1922).

[132] D. B. Hinton and R. T. Lewis, *Spectral analysis of second order difference equations*, J. Math. Anal. Appl. **63**, 421–438 (1978).

[133] J. W. Hooker and W. T. Patula, *Riccati type transformations for second-order linear difference equations*, J. Math. Anal. Appl. **82**, 451–462 (1981).

[134] J. W. Hooker, M. K. Kwong and W. T. Patula, *Oscillatory second-order linear difference equations and Riccati equations*, SIAM J. Math. Anal., **18**, 54–63 (1987).

[135] E. K. Ifantis, *A criterion for the nonuniqueness of the measure of orthogonality*, J. Approximation Theory **89**, 207–218, (1997).

[136] A. R. Its and V. B. Matveev, *Schrödinger operators with finite-gap spectrum and N-soliton solutions of the Korteweg-de Vries equation*, Theoret. Math. Phys. **23**, 343–355 (1975).

[137] C. G. T. Jacobi, *Über eine neue Methode zur Integration der hyperelliptischen Differentialgleichung und über die rationale Form ihrer vollständigen algebraischen Integralgleichungen*, J. Reine Angew. Math. **32**, 220–226 (1846).

[138] S. Jitomirskaya, *Almost everything about the almost Mathieu operator, II*, in "*Proceedings of XI International Congress of Mathematical Physics, Paris 1994*", Int. Press, 373–382 (1995).

[139] S. Jitomirskaya and Y. Last, *Dimensional Hausdorff properties of singular continuous spectra*, Phys. Rev. Lett. **76**, 1765–1769 (1996).

[140] S. Jitomirskaya and B. Simon, *Operators with singular continuous spectrum, III. Almost periodic Schrödinger operators*, Commun. Math. Phys. **165**, 201–205 1994.

[141] C. Jordan, *Calculus of Finite Difference Equations*, Chelsea, New York, 1979.

[142] M. Kac and P. van Moerbeke, *On an explicitly soluble system of nonlinear differential equations, related to certain Toda lattices*, Adv. Math. **16**, 160–169 (1975).

[143] M. Kac and P. van Moerbeke, *On some periodic Toda lattices*, Proc. Nat. Acad. Sci. USA **72**, 2879–2880 (1975).

[144] M. Kac and P. van Moerbeke, *A complete solution of the periodic Toda problem*, Proc. Nat. Acad. Sci. USA **72**, 1627–1629 (1975).

[145] S. Kahn and D.B. Pearson, *Subordinacy and spectral theory for infinite matrices*, Hel. Phys. Acta **65**, 505–527 (1992).

[146] Y. Kato, *On the spectral density of periodic Jacobi matrices*, in "*Non-linear Integrable Systems–Classical Theory and Quantum Theory*", (eds., M. Jimbo and T. Miwa), World Scientific, 153–181, (1983).

[147] W. Kelley and A. Peterson, *Difference Equations: An Introduction with Applications*, Academic Press, San Diego, 1991.

[148] J. Kingman, *Subadditive ergodic theory*, Ann. Prob. **1**, 883–909 (1973).

[149] A. Kiselev, *Absolutely continuous spectrum of one-dimensional Schrödinger operators and Jacobi matrices with slowly decreasing potentials*, Comm. Math. Phys. **179**, 377-400 (1996).

[150] M. Klaus, *On bound states of the infinite harmonic crystal*, Hel. Phys. Acta **51**, 793–803 (1978).

[151] O.Knill, *Factorization of random Jacobivoperators and Bäcklund transformations*, Commun. Math. Phys. **151** , 589–605 (1993).

[152] O. Knill, *Renormalization of random Jacobi operators*, Commun. Math. Phys. **164**, 195–215 (1993).

[153] S. Kotani and M. Krishna, Almost periodicity of some random potentials, J. Funct. Anal. **78**, 390–405 (1988).

[154] A. Krazer, *Lehrbuch der Thetafunktionen*, Chelsea Publ. Comp., New York, 1970.

[155] M. G. Krein, *Perturbation determinants and a formula for the traces of unitary and self-adjoint operators*, Soviet Math. Dokl. **3**, 707–710 (1962).

[156] I. M. Krichever, *Algebraic curves and nonlinear difference equations*, Russian Math. Surveys. **334**, 255–256 (1978).

[157] I. M. Krichever, *Nonlinear equations and elliptic curves*, Rev. of Science and Technology **23**, 51–90 (1983).

[158] I. M. Krichever, *Algebro-geometric spectral theory of the Schrödinger difference operator and the Peierls model*, Soviet Math. Dokl. **26**, 194–198 (1982).

[159] I. M. Krichever, *The Peierls model*, Funct. Anal. Appl. **16**, 248–263 (1982).

[160] I. Krichever, *Algebraic-geometrical methods in the theory of integrable equations and their perturbations*, Acta Appl. Math. **39**, 93–125 (1995).

[161] B. A. Kupershmidt, *Discrete Lax equations and differential-difference calculus*, Astérisque **123**, 1985.

[162] Y. Last, *Almost everything about the almost Mathieu operator, I*, in "Proceedings of XI International Congress of Mathematical Physics, Paris 1994", Int. Press, 366–372 (1995).

[163] Y. Last and B. Simon *Eigenfunctions, transfer matrices, and absolutely continuous spectrum of one-dimensional Schrödinger operators*, Invent. Math., **135**, 329–367 (1999).

[164] P. D. Lax *Integrals of nonlinear equations of evolution and solitary waves*, Comm. Pure and Appl. Math. **21**, 467–490 (1968).

[165] F. G. Maksudov, Eh. M. Bajramov, and R. U. Orudzheva, *The inverse scattering problem for an infinite Jacobi matrix with operator elements*, Russ. Acad. Sci., Dokl., Math. **45**, No.2, 366–370 (1992).

[166] F. Mantlik and A. Schneider, *Note on the absolutely continuous spectrum of Sturm-Liouville operators*, Math. Z. **205**, 491–498 (1990).

[167] V. A. Marchenko, *Sturm–Liouville Operators and Applications*, Birkhäuser, Basel, 1986.

[168] H. P. McKean, *Integrable systems and algebraic curves*, in "*Global Analysis*", (eds., M. Grmela and J. E. Marsden), Lecture Notes in Mathematics 755, Springer, Berlin, 83–200 (1979).

[169] H. McKean, *Variation on a theme of Jacobi*, Comm. Pure Appl. Math. **38**, 669–678 (1985).

[170] H. P. McKean and P. van Moerbeke, *Hill and Toda curves*, Commun. Pure Appl. Math. **33**, 23–42 (1980).

[171] R. E. Mickens, *Difference Equations. Theory and Applications*, 2nd ed., Van Nostrand Reinhold, New York, 1990.

[172] R. M. Miura, *Korteweg-de Vries equation and generalizations. I. A remarkable explicit nonlinear transformation*, J. Math. Phys. **9**, 1202–1204 (1968).

[173] P. van Moerbeke, *The spectrum of Jacobi Matrices*, Inv. Math. **37**, 45–81 (1976).

[174] P. van Moerbeke, *About isospectral deformations of discrete Laplacians*, in "*Global Analysis*", (eds., M. Grmela and J. E. Marsden), Lecture Notes in Mathematics 755, Springer, Berlin, 313–370 (1979).

[175] P. van Moerbeke and D. Mumford *The spectrum of difference operators and algebraic curves*, Acta Math. **143**, 97–154 (1979).

[176] F. Sh. Mukhtarov, *Reconstruction of an infinite nonselfadjoint Jacobi matrix by scattering data*, Dokl., Akad. Nauk Az. SSR **40**, No.10, 10–12 (1984).

[177] D. Mumford, *An algebro-geometric construction of commuting operators and of solutions to the Toda lattice equation, Korteweg de Vries equation and related non-linear equations*, Intl. Symp. Algebraic Geometry, 115–153, Kyoto, 1977.

[178] D. Mumford, *Tata Lectures on Theta II*, Birkhäuser, Boston, 1984.

[179] S. N. Naboko and S. I. Yakovlev *The discrete Schrödinger operator. The point spectrum lying on the continuous spectrum*, St. Petersbg. Math. J. **4**, No.3, 559–568 (1993).

[180] S. N. Naboko and S. I. Yakovlev *On the point spectrum of discrete Schrödinger operators*, Funct. Anal. Appl. **26**, No.2, 145-147 (1992).

[181] R. Narasimhan, *Compact Riemann Surfaces*, Birkhäuser, Basel, 1992.

[182] R. Nevanlinna *Asymptotische Entwicklung beschränkter Funktionen und das Stieltjesche Momentenproblem*, Ann. Acad. Sci. Fenn. A **18, 5**, 52pp. (1922).

[183] A. J. Niemi and G. W. Semenoff, *Fermion number fractionization in quantum field theory*, Phys. Rep. **135**, 99–193 (1986).

[184] S. Novikov, S. V. Manakov, L. P. Pitaevskii, and V. E. Zakharov, *Theory of Solitons*, Consultants Bureau, New York, 1984.

[185] F. W. J. Olver, *Asymptotics and Special Functions*, Academic Press, New York, (1974).

[186] B. P. Osilenker, *M. G. Krein's ideas in the theory of orthogonal polynomials* (English translation) Ukr. Math. J. **46**, 75–86 (1994).

[187] L. Pastur, *Spectral properties of disordered systems in one-body approximation*, Comm. Math. Phys. **75**, 179–196 (1980).

[188] W. T. Patula, *Growth and oscillation properties of second order linear difference equations*, SIAM J. Math. Anal. **6**, 55–61 (1979).

[189] W. T. Patula, *Growth, oscillation and comparison theorems for second order linear difference equations*, SIAM J. Math. Anal. **6**, 1272–1279 (1979).

[190] A. Perelomov, *Integrable Systems of Classical Mechanics and Lie Algebras*, Birkhäuser, Basel, (1990).

[191] CH. G. Philos and Y. G. Sficas, *Positive solutions of difference equations*, Proc. of the Amer. Math. Soc. **108**, 107–115 (1990).

[192] M. Reed and B. Simon, *Methods of Modern Mathematical Physics I. Functional Analysis*, rev. and enl. edition, Academic Press, San Diego, 1980.

[193] M. Reed and B. Simon, *Methods of Modern Mathematical Physics II. Fourier Analysis, Self-Adjointness*, Academic Press, San Diego, 1975.

[194] M. Reed and B. Simon, *Methods of Modern Mathematical Physics III. Scattering Theory*, Academic Press, San Diego, 1979.

[195] M. Reed and B. Simon, *Methods of Modern Mathematical Physics IV. Analysis of Operators*, Academic Press, San Diego, 1978.

[196] W. Renger, *Toda soliton limits on general background*, in Differential equations, asymptotic analysis, and mathematical physics, (ed. M. Demuth), 287-291, Math. Res. 100, Akademie Verlag, (1997).

[197] F. S. Rofe-Beketov, *A test for the finiteness of the number of discrete levels introduced into gaps of a continuous spectrum by perturbations of a periodic potential*, Soviet Math. Dokl. **5**, 689–692 (1964).

[198] C. A. Rodgers, *Hausdorff Measures*, Cambridge Univ. Press, London, 1970.

[199] C. A. Rodgers and S. J. Taylor, *The analysis of additive set functions in Euclidean space*, Acta Math., Stock., **101**, 273–302 (1959).

[200] C. A. Rodgers and S. J. Taylor, *The analysis of additive set functions in Euclidean space*, Acta Math., Stock., **109**, 207–240 (1963).

[201] D. Ruelle, *Ergodic theory of differentiable dynamical systems.*, Publ. Math., Inst. Hautes Etud. Sci. **50**, 27–58 (1979).

[202] W. Rudin *Real and Complex Analysis*, 2nd ed., McGraw Hill, New York, 1974.

[203] B. Simon, *Trace Ideals and their Applications*, Cambridge Univ. Press, Cambridge, 1979.

[204] B. Simon, *Schrödinger semigroups*, Bulletin Amer. Math. Soc. **7**, 447-526 (1982).

[205] B. Simon, *Kotani theory for one dimensional Jacobi matrices*, Com. Math. Phys. **89**, 227–234 (1983).

[206] B. Simon, *Spectral Analysis of Rank One Perturbations and Applications*, proceedings, "*Mathematical Quantum Theory II: Schrödinger Operators*", (eds. J. Feldman, R. Froese, and L. M. Rosen), CRM Proc. Lecture Notes 8, 109–149 (1995).

[207] B. Simon, *Bounded eigenfunctions and absolutely continuous spectra for one-dimensional Schrödinger operators*, Proc. Amer. Math. Soc. **124**, 3361–3369 (1996).

[208] B. Simon, *Operators with singular continuous spectrum. I: General operators*, Ann. Math.,**141**, 131–145 (1995).

[209] B. Simon, *The classical moment problem as a self-adjoint finite difference operator*, Advances in Math. **137**, 82–203 (1998).

[210] Smirnov, *Finite-gap solutions of Abelian Toda chain of genus 4 and 5 in elliptic functions*, Teoret. Mat, Fiz., **78**, No.1, 11–21 (1989).

[211] A. Sinap and W. Van Assche,*Orthogonal matrix polynomials and applications*, J. Comput. Appl. Math. **66**, 27–52 (1996).

[212] M.L. Sodin and P.M. Yuditskiĭ, *Infinite-zone Jacobi matrices with pseudo-extendible Weyl functions and homogeneous spectrum*, Russian Acad. Sci. Dokl. Math. **49**, 364–368 (1994).

[213] M. Sodin and P. Yuditskii, *Almost periodic Jacobi matrices with homogeneous spectrum, infinite dimensional Jacobi inversion, and Hardy spaces of character-automorphic functions*, preprint, (1994).

[214] J. M. Steele, *Kingman's subadditive ergodic theorem* Ann. Inst. Henri Poincare, Probab. Stat. **25**, No.1, 93–98 (1989)

[215] G. Stolz, *Spectral Theory for slowly oscillating potentials I. Jacobi Matrices*, Man. Math. **84**, 245–260 (1994).

[216] J.C.F. Sturm *Mémoire sur les équations différentielles linéaires du second ordre*, J. Math. Pures Appl., **1**, 106–186 (1836).

[217] G. Szegö, *Orthogonal Polynomials*, Amer. Math. Soc., Providence, R.I., 1978.

[218] H. Tanabe, *Equations of Evolution*, Pitman, London, 1979.

[219] G. Teschl, *Inverse scattering transform for the Toda hierarchy*, Math. Nach. (to appear).

[220] G. Teschl, *On the Toda and Kac-van Moerbeke hierarchies*, Math. Z. (to appear).

[221] G. Teschl, *Spectral Deformations of Jacobi Operators*, J. Reine. Angew. Math. **491**, 1–15 (1997).

[222] G. Teschl, *Trace Formulas and Inverse Spectral Theory for Jacobi Operators*, Comm. Math. Phys. **196**, 175–202 (1998).

[223] G. Teschl, *Oscillation theory and renormalized oscillation theory for Jacobi operators*, J. Diff. Eqs. **129**, 532–558 (1996).

[224] G. Teschl, *On the initial value problem for the Toda and Kac-van Moerbeke hierarchies*, in "Proceedings of the International Conference on Differential Equations and Mathematical Physics, Birmingham 1999" (to appear).

[225] B. Thaller, *The Dirac Equation*, Springer, Berlin, 1991.

[226] B. Thaller, *Normal forms of an abstract Dirac operator and applications to scattering theory*, J. Math. Phys. **29**, 249–257 (1988).

[227] E. Titchmarsh, *Eigenfunction Expansions Associated with Second-Order Differential Equations, Part I*, Oxford University Press, Oxford, 1946.

[228] E. Titchmarsh, *Eigenfunction Expansions Associated with Second-Order Differential Equations, Part II*, Oxford University Press, Oxford, 1958.

[229] E. C. Titchmarsh, *The Theory of Functions*, 2nd edition, Oxford Univ. Press, Oxford, 1985.

[230] M. Toda, *Theory of Nonlinear Lattices*, 2nd enl. ed., Springer, Berlin, 1989.

[231] M. Toda, *Theory of Nonlinear Waves and Solitons*, Kluwer, Dordrecht, 1989.

[232] M. Toda, *Selected pPapers of Morikazu Toda*, ed. by M. Wadati, World Scientific, Singapore, 1993.

[233] M. Toda, *Vibration of a chain with nonlinaer interactions*, J. Phys. Soc. Jpn **22**, 431–436 (1967).

[234] M. Toda and M. Wadati, *A canonical transformation for the exponential lattice*, J. Phys. Soc. Jpn **39**, 1204–1211 (1975).

[235] S. Venakides and P. Deift, R. Oba *The Toda shock problem*, Comm. Pure Appl. Math. **44**, 1171–1242 (1991).

[236] K. Ueno and K. Takasaki, *Toda lattice hierarchy*, in "*Advanced Studies in Pure Mathematics 4*", (ed. K. Okamoto), North-Holland, Amsterdam, 1–95 (1984).

[237] K. Ueno and K. Takasaki, *Toda lattice hierarchy. I'*, Proc. Japan Acad., Ser. A **59**, 167-170 (1983).

[238] K. Ueno and K. Takasaki, *Toda lattice hierarchy. II'*, Proc. Japan Acad., Ser. A **59**, 215-218 (1983).

[239] M. Wadati and M. Toda, *Bäcklund transformation for the exponential lattice*, J. Phys. Soc. Jpn **39**, 1196–1203 (1975).

[240] H. Wall, *Analytic Theory of Continued Fractions*, Chelsa, New York, 1973.

[241] J. Weidmann, *Linear Operators in Hilbert Spaces*, Springer, New York, 1980.

[242] S. T. Welstead, *Selfadjoint extensions of Jacobi matrices of limit-circle type*, J. Math. Anal. Appl. **89**, 315–326 (1982).

[243] S. T. Welstead, *Boundary conditions at infinity for difference equations of limit-circle type*, J. Math. Anal. Appl. **89**, 442–461 (1982).

[244] H. Weyl, *Über gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörige Entwicklung willkürlicher Funktionen*, Math. Ann. **68**, 220–269 (1910).

[245] Y. Xu, *Block Jacobi matrices and zeros of multivariate orthogonal polynomials*, Trans. Am. Math. Soc. **342**, 855–866 (1994).

[246] V. E. Zakharov, S. L. Musher, and A. M. Rubenchik, *Nonlinear stage of parametric wave excitation in a plasma*, JETP Lett. **19**, 151–152 (1974).

Glossary of notations

$a_R(n)$	$= a(2n_0 - n - 1)$ reflected coefficient, 15
$\underline{A}_{p_0}$	... Abel map, 281
$\underline{\alpha}_{p_0}$	... Abel map for divisors, 282
arg(.)	... argument of a complex number, $\arg(z) \in (-\pi, \pi]$
$b_R(n)$	$= b(2n_0 - n)$ reflected coefficient, 15
$b^{(\ell)}(n)$	... expansion coefficient of $\ln g(z, n)$, 112
$b^{\beta,(\ell)}(n)$	... expansion coefficient of $\ln \gamma^{\beta}(z, n)$, 113
$B_{\pm}(\tau)$	... 67
$B_{\pm,f}(\tau)$	... 67
$c(z, n)$	$= c(z, n, 0)$
$c(z, n, n_0)$	... fundamental solution of $\tau u = zu$ vanishing at $n_0 + 1$, 6
$c_\beta(z, n)$	$= c_\beta(z, n, 0)$
$c_\beta(z, n, n_0)$	... fundamental solution of $\tau u = zu$ corresponding to general boundary conditions at n_0, 37
$\mathbb{C}$	... the set of complex numbers
$\mathbb{C}_{\pm}$	$= \{z \in \mathbb{C} \mid \pm \mathrm{Im}(z) > 0\}$
$C^{\alpha}(\rho)$	... support of the α-continuous part of ρ, 302
$C(M_1, M_2)$	... continuous functions from M_1 to M_2
$C^{\infty}(M_1, M_2)$	... smooth functions from M_1 to M_2
$\chi_{\Lambda}(.)$	... characteristic function of the set Λ
deg	... degree of a divisor (or of a polynomial), 277
δ_n	... $\delta_n(m) = \delta_{n,m}$
$\delta_{n,m}$	... Kronecker delta $\delta_{n,m} = 0$, if $n \neq m$ and $\delta_{n,n} = 1$
dim	... dimension of a linear space
dist(A, B)	... distance between two sets A, B
D^{α}	... α-derivative, 302
$\mathcal{D}$	... divisor on a Riemann surface, 277
Div(M)	... ring of divisors on M, 277
$\mathfrak{D}(.)$	... domain of an operator

e ... exponential function, $\mathrm{e}^z = \exp(z)$
$\mathbb{E}(.)$ ... expectation of a random variable, 87
$f_\pm(k,n)$ ... Jost solutions, 167
$g(z,n)$ $= G(z,n,n)$
$G_r(z,n)$ ... polynomial, 144
$G(z,n,m)$ ... Green function of H, 16
$\gamma^\pm(z)$ ... Lyapunov exponent, 7
$\gamma^\beta(z,n)$ $= g(z,n+1) + \frac{\beta}{a(n)} h(z,n) + \beta^2 g(z,n)$, 18
h^α ... Hausdorff measure, 302
$h(z,n)$ $= 2a(n)G(z,n,n+1) - 1$
H ... Jacobi operator, 13
$H^\beta_{\pm,n_0}$ ... Jacobi operator on $\ell^2(n_0, \pm\infty)$, 16
$H_{\pm,n_0}$ $= H^\infty_{\pm,n_0}$
$H_\pm$ $= H_{\pm,0}$
$H^{\beta_1,\beta_2}_{n_1,n_2}$ ... Jacobi operator on $\ell^2(n_1,n_2)$, 16
H_{n_1,n_2} $= H^{\infty,\infty}_{n_1,n_2}$
H_R ... reflected Jacobi operator, 15
$\hat{H}$ ... Helmholtz operator, 20
$\mathfrak{H}$ ... (separable) Hilbert space
$H_{r+1}(z,n)$ ... polynomial, 145
$\mathrm{hull}(c)$ ... hull of a sequence, 100
$\mathrm{Hyp}_\pm(\Phi)$ ... set of hyperbolic numbers, 7
i ... complex unity, $\mathrm{i}^2 = -1$
$i(\mathcal{D})$ $= \dim \mathcal{L}^1(\mathcal{D})$ index of specialty, 279
$\mathrm{Im}(.)$ ... imaginary part of a complex number
$\mathrm{Iso}_R(\Sigma)$ ... set of reflectionless Jacobi operators with spectrum Σ, 141
J_{n_1,n_2} ... Jacobi matrix, 12
$J(M)$ ... Jacobian variety of the Riemann surface M, 281
$\mathrm{KM}_r(\rho)$ ... Kac-van Moerbeke hierarchy, 258
$\widetilde{\mathrm{KM}}_r(\rho)$ ... homogeneous Kac-van Moerbeke hierarchy, 259
Ker ... kernel of an operator
$K^\beta_{r+1}(z,n)$ ... polynomial, 150
$\ell(I,M)$ ... set of M-valued sequences with index set $I \subseteq \mathbb{Z}$, 3
$\ell(I)$ $= \ell(I,\mathbb{C})$
$\ell^p(I,M)$ ... sequences in $\ell(I,M)$ being p summable, 3
$\ell^p(I)$ $= \ell^p(I,\mathbb{C})$
$\ell^\infty(I,M)$ ... sequences in $\ell(I,M)$ being bounded, 3
$\ell^\infty(I)$ $= \ell^\infty(I,\mathbb{C})$
$\ell_0(I,M)$ ... sequences in $\ell(I,M)$ with only finitely many nonzero values, 3
$\ell_0(I)$ $= \ell_0(I,\mathbb{C})$
$\ell^p_\pm(\mathbb{Z},M)$ ... sequences in $\ell(I,M)$ being ℓ^p near $\pm\infty$, 3
$\ell_\pm(\mathbb{Z})$ $= \ell_\pm(I,\mathbb{C})$
$\ell^2(\mathbb{Z};w)$ $=$ weighted ℓ^2 space, 20

$L^1_{loc}(x_0, x_1)$ ... space of (equivalence classes of) locally integrable functions over the interval (x_0, x_1)

$L^2(x_0, x_1)$ ... Hilbert space of (equivalence classes of) square integrable functions over the interval (x_0, x_1)

$\mathcal{L}(\mathcal{D})$ $= \{f \in \mathcal{M}(M) | f = 0 \text{ or } (f) \geq \mathcal{D}\}$, 279

$\mathcal{L}^1(\mathcal{D})$ $= \{\omega \in \mathcal{M}^1(M) | \omega = 0 \text{ or } (\omega) \geq \mathcal{D}\}$, 279

λ ... a real number

$\lambda_j^\beta(n)$ ... eigenvalues associated with general boundary conditions, 139

$m^\pm(z)$ ... Floquet multipliers, 116

$m_\pm(z, n)$ ... Weyl m-functions, 27

$m_\pm(z)$ $= m_\pm(z, 0)$

$\tilde{m}_\pm(z, n)$ ... Weyl $\tilde{m}$-functions, 29

$\tilde{m}_\pm(z)$ $= \tilde{m}_\pm(z, 0)$

$m_\pm^\beta(z)$ ... Weyl m-functions associated with general boundary conditions, 29

$\tilde{m}_\pm^\beta(z)$ ... Weyl $\tilde{m}$-functions associated with general boundary conditions, 39

M ... manifold or Riemann surface

$M(z)$ ... Weyl matrix, 46

$M^\beta(z)$ ... Weyl matrix corresponding to general boundary conditions, 46

$\mathcal{M}(M)$ ... meromorphic functions on M

$\mathcal{M}^1(M)$ ... meromorphic forms on M

$\mu_j(n)$ ... Dirichlet eigenvalue, 135

$\mathbb{N}$ ... the set of positive integers

$\mathbb{N}_0$ $= \mathbb{N} \cup \{0\}$

$N_\pm(\tau)$ ... set of λ for which no subordinate solution exists, 66

p ... point on M

$P_\Lambda(.)$ ... family of spectral projections of an operator

$\Phi(z, n, n_0)$ ... fundamental matrix, 7

$\Phi(z, n)$ $= \Phi(z, n, 0)$

$\mathbb{Q}$ ... the set of rational numbers

$q(z)$ ... Floquet momentum, 116

Q^α ... 302

$r(\mathcal{D})$ $= \dim \mathcal{L}(\mathcal{D})$, 279

Re(.) ... real part of a complex number

$\mathbb{R}$ ... the set of real numbers

$R^{1/2}_{2r+2}(z)$ ... 143

$R_j(n)$ ... residue associated with the Dirichlet eigenvalue $\mu_j(n)$, 136

$R_\pm(k)$ ... reflection coefficient(constant background), 173

$R_\pm(\lambda)$ ... reflection coefficient, 129

Ran ... range of an operator

R^α ... 302

ρ ... a Borel measure

$s(z,n)$ $= s(z,n,0)$

$s(z,n,n_0)$ ...fundamental solution of $\tau u = zu$ vanishing at n_0, 6

$s_\beta(z,n)$ $= s_\beta(z,n,0)$

$s_\beta(z,n,n_0)$ ...fundamental solution of $\tau u = zu$ corresponding to general boundary conditions at n_0, 37

$S(k)$ ...scattering matrix (constant background), 173

$S(\lambda)$ ...scattering matrix, 129

$S^\pm$ ...shift expressions, 4

$S_\pm(H)$ ...scattering data of H, 174

$S(\tau)$ ...59

$S_f(\tau)$ ...59

$S_\pm(\tau)$ ...58

$S_{f,\pm}(\tau)$ ...58

$\sigma(H)$ ...spectrum of an operator H, 15

$\sigma_{ac}(H)$ ...absolutely continuous spectrum of H, 41, 45

$\sigma_{sc}(H)$ ...singular continuous spectrum of H, 41, 45

$\sigma_{pp}(H)$ ...pure point spectrum of H, 41, 45

$\sigma_s(H)$ $= \sigma_{sc}(H) \cup \sigma_{pp}(H)$

$\sigma_p(H)$ ...point spectrum (set of eigenvalues) of H, 41

$\sigma_{ess}(H)$ ...essential spectrum of H, 60

$\sigma_{\alpha c}(H)$ ...α-continuous spectrum of H, 41

$\sigma_{\alpha s}(H)$ ...α-singular spectrum of H, 41

$\sigma_j(n)$ ...sign associated with the Dirichlet eigenvalue $\mu_j(n)$, 136

$\sigma_j^\beta(n)$ ...sign associated with the eigenvalue $\lambda_j^\beta(n)$, 151

$\operatorname{sgn}(.)$ ...sign of a real number, $\operatorname{sgn}(\lambda) = -1$ if $\lambda < 0$, $\operatorname{sgn}(0) = 0$, and $\operatorname{sgn}(\lambda) = 1$ if $\lambda > 0$

$T(k)$ ...transmission coefficient(constant background), 173

$T(\lambda)$ ...transmission coefficient, 129

τ ...second order, symmetric difference expression, 5

$\underline{\tau}$ $= (\tau_{i,j})_{i,j=1}^g$, matrix of b periods of ζ_j, 276

$\theta(\underline{z})$ ...Riemann theta function, 284

$\mathrm{TL}_r(a,b)$ ...Toda hierarchy, 227

$\widetilde{\mathrm{TL}}_r(a,b)$ ...homogeneous Toda hierarchy, 228

tr ...trace of an operator

$u(z,n)$ ...solution of $\tau u = zu$

$u_\pm(z,.)$ ...solution of $\tau u = zu$ being square summable near $\pm\infty$, 30

$U(t,s)$ ...unitary propagator, 228

$W_n(.,..)$ ...Wronskian of two sequences, 6

$\xi(\lambda,n)$ ...xi function for Dirichlet boundary conditions, 112

$\xi^\beta(\lambda,n)$ ...xi function for general boundary conditions, 113

$\underline{\Xi}_{p_0}$ ...vector of Riemann constants, 285

$\mathbb{Z}$ ...the set of integers

z ...a complex number

$\mathbb{1}$	... identity operator
$\sqrt{z}$	... standard branch of the square root, 19
$\overline{z}$	... complex conjugation
$\|.\|_p$	... norm in the Banach space ℓ^p, 3
$\|.\|$	... norm in the Hilbert space ℓ^2, 13
$\langle .,..\rangle_{\mathfrak{H}}$	... scalar product in $\mathfrak{H}$, 4
$\langle .,..\rangle$	... scalar product in $\ell^2(I)$, 13
$\oplus$	... direct sum of linear spaces or operators
∂	... forward difference, 5
∂^*	... backward difference, 5
$\sum^*$	... indefinite sum, 9
$\prod^*$	... indefinite product, 10
$\top$	... transpose of a matrix
$\#(u)$	... total number of nodes of u, 75
$\#_{(m,n)}(u)$	... number of nodes of u between m and n, 75
$\#W(u_1,u_2)$	... total number of nodes of $W_.(u_1,u_2)$, 81
$\#_{(m,n)}W(u_1,u_2)$	... number of nodes of $W_.(u_1,u_2)$ between m and n, 81
$[.]_n$	... Weyl bracket, 31
$[.]_\pm$	... upper/lower triangular part of a matrix, 5
$[.]_0$	... diagonal part of a matrix, 5
$[\![x]\!]$	$= \sup\{n \in \mathbb{Z} \mid n < x\}$, 74
$\overline{M}^{ess}$	... essential closure of a set, 301
(λ_1,λ_2)	$= \{\lambda \in \mathbb{R} \mid \lambda_1 < \lambda < \lambda_2\}$, open interval
$[\lambda_1,\lambda_2]$	$= \{\lambda \in \mathbb{R} \mid \lambda_1 \leq \lambda \leq \lambda_2\}$, closed interval

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