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Classification and Orbit Equivalence Relations

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ABSTRACT. This book is a contribution to the general theory of equivalence relation, especially the orbit equivalence relations induced by Polish group actions. A theory is developed regarding when such equivalence relations allow countable structures considered up to isomorphism as complete invariants.

This book would be of interest to mathematicians in a variety of areas.

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To Noela and Bob Hjorth

Contents

Preface

What does it mean to *classify* the equivalence classes of some equivalence relation?

We have some tangible space whose points are very definite. Let X be the space. Maybe it is the reals. Or the complex numbers. Or perhaps all groups with underlying set $\mathbb{N}$. In any case, a concrete object.

In addition there is an equivalence relation E and we consider the quotient X/E. When should we judge this quotient object consisting of all equivalence classes to be comprehensible? When should we allow that the set of all equivalence classes, $\{[x]_E : x \in X\}$, is classifiable?

There is no absolute agreement on what constitutes a good classification theorem. It is necessarily a vague concept. But even granting its vagueness, there are probably some attempts one could dismiss out of hand.

For instance, assigning the equivalence class of x as a complete invariant for each point in the space is hardly satisfactory, since it provides us with no progress at all. The complete invariants should be objects we feel to be reasonably well understood – or at least, less mysterious than the equivalence classes with which we began.

Similarly we do have some standards concerning how the invariants can be produced. Simply appealing to the axiom of choice to well order the equivalence classes of E, and then using some corresponding well order of $\mathbb{R}$ to assign real numbers as complete invariants does not constitute a satisfactory system of classification, even though the complete invariants in this case are indeed well understood. The system of classification should assign invariants to the points in the space X based on their intrinsic properties; even though the properties we may use could be highly subtle or extremely complex, we would have much greater respect for a system of classification that is fantastically difficult than one that just pulls down the axiom of choice and then goes home to bed.

In general terms a complete classification of E should consist of a *reasonably intrinsic* or *definable* function

$$\theta : X \to I$$

from X to a *reasonably well understood* collection of invariants I so that for all x and y in X

$$xEy \Leftrightarrow \theta(x) = \theta(y).$$

That much is a platitude. Beyond this there is great disagreement.

EXAMPLE 0.1. **Low dimensional topology** For compact orientable surfaces a complete invariant can be obtained by simply counting the number of handles, and moreover, this invariant can be produced in a *recursive* or *computable* fashion from a finite triangularization of the manifold; for non-orientable surfaces the structure theorem is more complicated, but again the invariant can be taken to be a finite object – indeed it can be represented by a natural number – and may be computed from a triangularization using a purely finite search. ([**12**], [**46**].) In higher dimensions it is known from [**64**] that the isomorphism relation is not computable in the sense of Turing machines, and this is *sometimes* taken by topologists as indicating that there is no satisfactory system of complete invariants for higher dimension manifolds.

This is an extremely restrictive definition. Here a classification is a computable function

$$\theta : X \to \mathbb{N}$$

where X is a collection of finite triangularizations of manifolds so that $x, y \in X$ code homeomorphic manifolds if and only if $\theta(x) = \theta(y)$. This is the most stringent notion of classification: The invariants are finite, the process by which we assign them finitary.

EXAMPLE 0.2. **Linear algebra** A rather different example is suggested by linear algebra. We may reasonably regard two unitary operators over a finite dimensional (complex) Hilbert space as somehow equivalent if there is a third that conjugates them – so in this sense we obtain an orbit equivalence relation on the space of all unitary operators on $(\mathbb{C}^n, \cdot)$. A complete invariant for an operator is given by its finite set of eigenvalues, considered up to multiplicity, and it is always possible to encode finitely many complex numbers by a single real; thus we can assign in a Borel fashion to each element of U_n, the group of unitary operators over $(\mathbb{C}^n, \cdot)$, a real number as a complete invariant.

Equivalence relations which in a Borel fashion allow real numbers as complete invariants are known as *smooth* or *tame*; thus E_G on X is smooth if there is a Borel

$$\theta : X \to \mathbb{R}$$

that reduces E_G to equality on $\mathbb{R}$, in the sense

$$xE_G y \Leftrightarrow \theta(x) = \theta(y).$$

[**30**], like [**14**] that it followed, takes *classifiable* to mean smooth.

EXAMPLE 0.3. **Ergodic theory** Consider the classification problem for measure preserving transformations of the unit interval. It is natural to say that $\pi_1, \pi_2 : [0,1] \to [0,1]$ are *equivalent* or *isomorphic* if there is some measure preserving bijection $\sigma : [0,1] \to [0,1]$ with

$$\sigma \circ \pi_1 \circ \sigma^{-1} = \pi_2 \text{ a.e.}$$

This equivalence relation arises from a group action – the action of this group on itself by conjugation. In two special cases there are classification theorems.

Ornstein's classification of Bernoulli shifts in [**71**] provides a proof that isomorphism on this class of measure preserving transformations is smooth. One can assign to each Bernoulli shift its *entropy* – a real number that completely classifies a Bernoulli shift up to isomorphism. This is one of the most celebrated theorems of ergodic theory, but it is not true that the *only* notion of classification in ergodic theory is that of being smooth or reducible to the equality relation on $\mathbb{R}$. A rather more generous notion of classification is suggested by a classic paper of Halmos and von Neumann.

In [**29**] Halmos and von Neumann show that for *discrete spectrum* measure preserving transformations we can assign a countable collection $\{c_i(\pi) : i \in \mathbb{N}\}$

of complex numbers that completely describe the equivalence class of π. This assignment is indeed natural, since it arises from taking the eigenvalues of the form $\lambda \in \mathbb{C}$ for which there is some non-zero $f \in L^2([0,1])$ with $f \circ \pi = \lambda f$ a.e.

The notion of classification here cannot be reduced to that of 0.2. For instance [**20**] shows that there is no reasonable method for representing *countably infinite* sets of real or complex numbers by single points in $\mathbb{R}$; indeed without appeal to the axiom of choice we may find it impossible to produce *any* injection from $\mathcal{P}_{\aleph_0}(\mathbb{R})$ (the set of all countable collections of reals) to $\mathbb{R}$. The conjugacy relation on discrete spectrum measure preserving transformations is non-smooth, and yet the perspective of say [**82**] would uphold this as a complete classification for discrete spectrum measure preserving transformations.

EXAMPLE 0.4. **Locally compact group actions** In [**49**] Kechris proves that the orbit equivalence relations induced by locally compact Polish groups are all reducible to countable equivalence relations. If we have locally compact G acting continuously on Polish X, with orbit equivalence relation E_G, then we can find a Borel equivalence relation F all of whose equivalence classes are countable such that we may assign to each point $x \in X$ some corresponding $\theta(x)$ so that

$$x_1 E_G x_2 \Leftrightarrow \theta(x_1) F \theta(x_2).$$

This result can be viewed as a classification theorem for orbit equivalence relations induced by locally compact group actions. We may assign to each $x \in X$ the countable set $\{y : yF\theta(x)\}$ to obtain an invariant similar in structure to the Halmos-von Neumann spectral invariants.

EXAMPLE 0.5. **Point set topology** The Cantor-Bendixson derivation as described in [**52**] can be used to provide a classification for countable compact metric spaces. At the first stage we remove the isolated points. At the next we remove the isolated points from the remaining space, and so on, through however many countable ordinals as are needed. This analysis provides a complete invariant of the space consisting of two parts: The ordinal length of this process, along with the number of points left standing before the termination at the final stage. Two countable compact metric spaces will be homeomorphic if and only if their Cantor-Bendixson derivations require the same ordinal number of steps and at the penultimate moment they share the same finite number of points remaining.

EXAMPLE 0.6. **Abelian group theory** In [**21**] ordinals also enter stage in the famed *Ulm invariants* from abelian group theory. In essence the Ulm invariants are bounded subsets of $\aleph_1$, the first uncountable ordinal, that completely describe the isomorphism type of a countable *torsion* abelian group.

EXAMPLE 0.7. **Topological dynamics** The authors of [**24**] classify so called *minimal Cantor systems* up to *strong orbit equivalence* by assigning countable ordered abelian groups. Two continuous

$$\varphi_1 : X_1 \to X_1,$$
$$\varphi_2 : X_2 \to X_2$$

which are *minimal* in the sense of having no non-trivial closed invariant sets and are *Cantor* in the sense of X_1, X_2 being compact, uncountable, and zero-dimensional metric spaces, are said to be *strong orbit equivalent* if there is a homeomorphism $F : X_1 \to X_2$ which respects the orbits structure set wise and with the resulting conjugation suffering at most a single discontinuity – so that if $m : X_1 \to \mathbb{Z}$, $n : X_2 \to \mathbb{Z}$ are defined by

$$F \circ \varphi_1(x) = \varphi_2^{n(x)}(F(x)),$$
$$F \circ \varphi_1^{m(x)}(x) = \varphi_2(F(x)),$$

then n, m are continuous on $X \setminus \{x_0\}$ for some $x_0 \in X$.

The countable group associated to (φ, X) itself arises as a homomorphic image of $C(X, \mathbb{Z})$, the space of all continuous maps from X to $(\mathbb{Z}, +)$ with the product topology. As important as the details of the construction may be it is also remarkable that there is *any* reasonable way to assign a countable structure as a complete invariant.

EXAMPLE 0.8. **Stone spaces** Perhaps this preceding example is reminiscent of the duality theorem of [**75**] for compact separable zero-dimensional Hausdorff spaces. To each such space we can assign a countable Boolean algebra with two spaces homeomorphic if and only if there exists an algebraic isomorphism between the Boolean algebras. Similarly Pontryagin duality, as it is found in [**31**], allows us to completely classify compact abelian metric groups by their countable discrete dual groups.

While it is true that complete invariants are provided this is not to say that these dualities are only or even primarily theorems of classification.

EXAMPLE 0.9. $\mathbf{Hom}^+([0,1])$ Let $\mathrm{Hom}^+([0,1])$ be the group of all orientation preserving ($\pi(0) = 0$, $\pi(1) = 1$) homeomorphisms of the unit interval. The natural equivalence relation is that of conjugation: Two homeomorphisms, π_1, π_2 are equivalent if a homeomorphic "relabeling" of the underlying space transforms one to the other, so that there is some $\sigma \in \mathrm{Hom}^+([0,1])$ with

$$\pi_1 = \sigma^{-1} \circ \pi_2 \circ \sigma.$$

Parallel to 0.3, the equivalence relation arises by the self-action of $\mathrm{Hom}^+([0,1])$ through conjugation.

It is sometimes felt that homeomorphisms of the unit interval are completely understood since we may represent each transformation *symbolically* by indicating the maximal regions on which we have either $\pi(x) > x$, $\pi(x) = x$, or $\pi(x) < x$. This can be made more precise by providing a classification of elements of $\mathrm{Hom}^+([0,1])$ by *countable models*. We naturally assign to each $\pi \in \mathrm{Hom}^+([0,1])$ a countable model $\mathcal{M}(\pi)$ such that for all $\pi_1, \pi_2 \in \mathrm{Hom}^+([0,1])$

$$\exists \sigma \in \mathrm{Hom}^+([0,1])(\sigma \circ \pi_1 \circ \sigma^{-1} = \pi_2) \Leftrightarrow \mathcal{M}(\pi_1) \cong \mathcal{M}(\pi_2).$$

The model $\mathcal{M}(\pi)$ consists of the maximal open intervals on which π displays one of the three possible behaviors indicated above. The language of $\mathcal{M}(\pi)$ encodes the linear ordering between these intervals and indicates which of the three possibilities hold. We will have an ordering $\leq$, and predicates P_-, P_+, and $P_=$. For $I_1 = (a_1, b_1)$, $I_2 = (a_2, b_2)$ maximal open intervals on which the behavior of π is unvarying, we have:

$$I_1 \leq^{\mathcal{M}(\pi)} I_2 \Leftrightarrow a_1 \leq a_2;$$
$$P_-^{\mathcal{M}(\pi)}(I_1) \Leftrightarrow \forall x \in I_1(\pi(x) < x);$$
$$P_+^{\mathcal{M}(\pi)}(I_1) \Leftrightarrow \forall x \in I_1(\pi(x) > x);$$
$$P_=^{\mathcal{M}(\pi)}(I_1) \Leftrightarrow \forall x \in I_1(\pi(x) = x).$$

The outcome is similar for the homeomorphism group of Cantor space – $(\{0,1\})^{\mathbb{N}}$ – in the product topology. Since the homeomorphism group of the Cantor space is isomorphic to a closed subgroup of the infinite symmetric group it follows from [**4**] (see 2.39 below) that we may classify these homeomorphisms by countable models.

This is more than enough examples to be impressed by the diversity. But despite the variation, there are some common themes.

In the above we have natural numbers, real numbers, countable sets of complex numbers, countable ordinals, countable sets of countable ordinals, and various kinds of countable structures considered up to isomorphism being used as complete invariants. The connection between these examples is that in *every one* we may take a countable structure as a complete invariant; as in §2.3 below, we may code complex numbers, countable sets of complex numbers, countable ordinals, and countable sets of countable ordinals by appropriately chosen models. This suggests a notion of classification found at the opposing end of the spectrum to that of 0.1 and which is extreme in its generosity.

QUESTION 0.10. Let E be an equivalence relation on a space X. When can we assign countable *models* or *structures* considered up to isomorphism as complete invariants?

Recall that HC is the collection of all hereditarily countable sets, and may be defined as the smallest collection of sets containing the natural numbers and closed under the operation of taking a countable subset. These therefore include all countable subsets of $\aleph_1$, all countable sets of subsets of $\mathbb{N}$, and – appropriately understood – all real numbers, all countable sets of real numbers, and so on.

In virtue of the Scott analysis of [**59**] we may equivalently ask:

QUESTION 0.11. For which equivalence relations can we assign elements of HC as complete invariants?

[**4**] allows one more reworking of the question:

QUESTION 0.12. Let E be an equivalence relation on a space X. When can we find a Polish space Y on which the infinite symmetric group S_∞ acts continuously and a reasonable function $\theta : X \to Y$ so that for all $x_1, x_2 \in X$

$$x_1 E x_2 \Leftrightarrow \exists g \in S_\infty(g \cdot \theta(x_1) = \theta(x_2))?$$

Of course we clearly need to have an assumption that the function θ or the assignments of models or HC sets be *reasonable.* On the whole I will take *reasonable* to mean Borel in an appropriate Borel structure, the technicalities of which are addressed in §2.1, §2.2, §2.3, and §3.1.[1] But I should stress that relatively little change occurs if we extend to much broader classes of functions and far more generous methods of reduction. A point made in the course of §6.2 and §9.1-2 is that if there is any remotely definable assignment of countable models or HC sets in the context of Polish group actions, then we may find a reduction that is at worst only slightly more complicated than Borel.

This monograph can be viewed as part of a broad project to understand *effective* cardinalities in the sense raised by Luzin in [**62**], in the sense which reappears

[1]In the context of Borel reduction, 0.10 and 0.12 are known to be equivalent problems from 2.7.3 of [**4**]. Provided we are willing to countenance reductions somewhat more complicated than Borel the Scott analysis of [**59**] shows both equivalent to 0.11. §6.1 and §9.1 return to these points.

briefly in the opening parts of [**11**], but which finds its most forthright statement in modern works of descriptive set theory such as [**60**] and [**50**]. Here the concept is to calculate cardinalities using only functions that lay some claim on being *reasonable* or *definable*. Formally $\mathbb{R}$ and $\mathbb{R}/\mathbb{Q}$ both have cardinality $2^{\aleph_0}$; a well ordering of $\mathbb{R}$ will enable us to find a bijection. *Effectively* $\mathbb{R}$ is smaller than $\mathbb{R}/\mathbb{Q}$, since we may find reasonable injections of $\mathbb{R}$ into $\mathbb{R}/\mathbb{Q}$ (2.59, 2.63 below), but not the converse (3.8).

Papers such as [**14**], [**63**], and [**25**] have previously addressed the question of which naturally occuring objects have effective cardinality no greater than that of $\mathbb{R}$. In a great many specific instances the answer has been determined, and one finds in [**30**] and [**14**] a kind of theory, recounted in §3.1 and §7.1, regarding when a reduction exists and why in certain cases it cannot. In turn this monograph tries to understand which objects have effective cardinality below HC and develop a parallel theory of why some do not.

We will only be concerned with the case that E arises from a Polish group action. Admittedly this may seem very restrictive, and it would certainly be desirable to have an analysis for all Borel or even $\underset{\sim}{\Sigma}{}^1_1$ equivalence relations. On the other hand most naturally occurring examples can be subsumed under an appropriately chosen Polish group action, and the impression left by chapter 8 is that the exceptions are somehow pathological. This is the point of question 10.9 near the end of the book, and even if that conjecture should fail it seems plausible that a similar outlook is justified.

§3 isolates a dynamical property for analyzing which Polish group actions allow reduction to countable models:

DEFINITION 0.13. Let G be a topological group acting on a space X. The action is said to be *turbulent* if:

(i) every orbit is dense;

(ii) every orbit is meager;

(iii) for all $x, y \in X$, $U \subset X$, $V \subset G$ open with $x \in U$, $1 \in V$, there exists $y_0 \in [y]_G =_{df} G \cdot y$ and $(g_i)_{i\in\mathbb{N}} \subset V$, $(x_i)_{i\in\mathbb{N}} \subset U$ with

$$x_0 = x,$$

$$x_{i+1} = g_i \cdot x_i,$$

and for some subsequence $(x_{n(i)})_{i\in\mathbb{N}} \subset (x_i)_{i\in\mathbb{N}}$

$$x_{n(i)} \to y_0.$$

Turbulence is a sufficient condition for the orbit equivalence relation of a Polish group to refuse classification by countable structures; further: for a turbulent orbit equivalence relation any function assigning countable models up to isomorphism as invariants must be constant on a comeager set. (3.18, 3.19)

A transfinite analysis shows turbulence to be necessary for non-classification:

THEOREM 0.14. *Let G be a Polish group acting continuously on a Polish space X. Exactly one of the following holds:*

1. *the orbit equivalence relation E^X_G is reasonably reducible to isomorphism on countable models;*
2. *there is a turbulent Polish G-space Y and a continuous G-embedding from Y to X.*

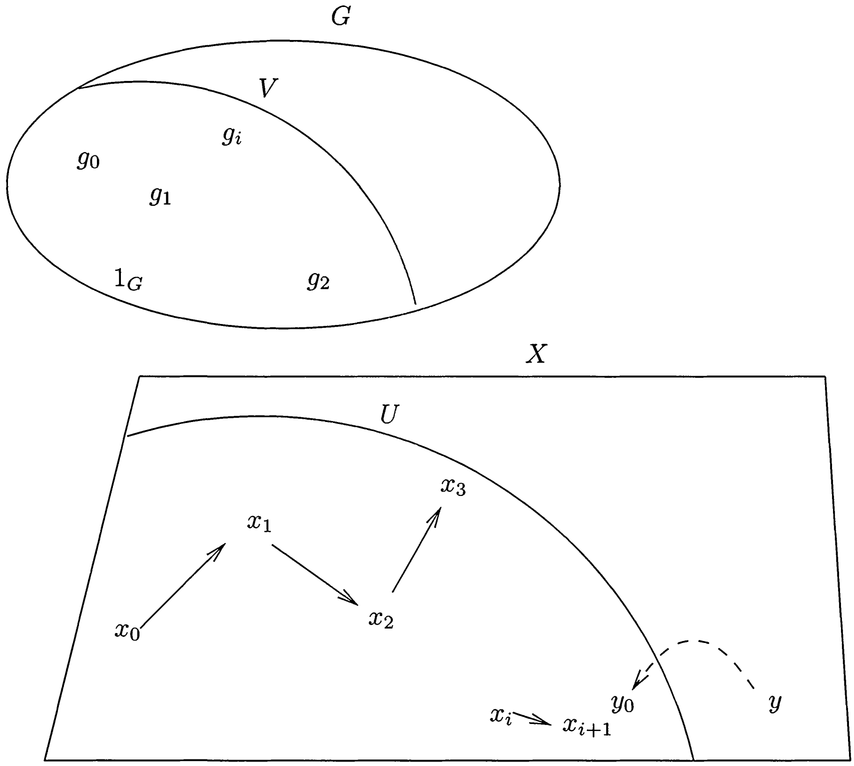

FIGURE 0.1. Turbulence

Here the notion of reduction is somewhat more complicated than Borel. In special cases, such as for G abelian or invariantly metrizable, one can obtain a reduction in 1. that is not only Borel but admits a Borel inverse up to orbit equivalence (6.40 and 6.30). The proof that 2. implies the negation of 1. is given in §3.2; the converse requires a long argument, finally concluding in chapter 9, and uses an elaboration on the Scott analysis presented in §6.2 that may have independent interest.

The form of 0.14 is intended to make it into a tool that can be easily applied in concrete cases.

APPENDIX A

Ordinals

Ordinals arise from the need to keep counting through infinitely many stages. For instance, if we define the $\underset{\sim}{\Sigma}_1^0$ subsets of $\mathbb{R}$ to be the open sets and $\underset{\sim}{\Pi}_1^0$ to be the closed, and more generally $\underset{\sim}{\Sigma}_{n+1}^0$ to be the countable unions of $\underset{\sim}{\Pi}_n^0$ and $\underset{\sim}{\Pi}_{n+1}^0$ to be the complements of $\underset{\sim}{\Sigma}_{n+1}^0$ then we obtain an initial segment of the Borel sets consisting of those with *finite* rank. The issue here is that not all Borel sets are in these classes. A Borel set of the form

$$B = \bigcup_{n \in \mathbb{N}} B_n$$

with each $B_n \in \underset{\sim}{\Sigma}_n^0$ may provide a counterexample.

So we are led to a notation for the infinite rank Borel sets. The first infinite ordinal is ω and we can let $\underset{\sim}{\Sigma}_\omega^0$ be the countable unions of finite rank Borel sets, $\underset{\sim}{\Pi}_\omega^0$ *their* complements, $\underset{\sim}{\Sigma}_{\omega+1}^0$ to be the countable unions of $\underset{\sim}{\Pi}_\omega^0$, and so on.

The above provides the motivation. The formal definition is:

Definition A.1. A set α is an *ordinal* if

(i) α is transitive (so for all $\beta \in \alpha$ and $\gamma \in \beta$ we have $\gamma \in \alpha$);

(ii) α is linearly ordered by the $\in$-relation (so for all $\beta, \gamma \in \alpha$, either $\beta \in \gamma$, or $\gamma \in \beta$, or $\gamma = \beta$).

α is a *successor* ordinal if it has a largest element, and otherwise it is a *limit*.

Example A.2. The first ordinal is the empty set, $\emptyset$, which we may identify with 0. Then

$$1 = \{0\}$$

the set whose only member is $0 (=_{df} \emptyset)$. Then

$$2 = \{0, 1\}$$

and more generally

$$n + 1 = \{0, 1, 2, ..., n\}.$$

The first infinite ordinal is

$$\omega = \{0, 1, 2, 3, ...\}$$

and we keep counting with

$$\omega + 1 = \{0, 1, 2, 3, ...\omega\},$$

$$\omega + 2 = \{0, 1, 2, 3, ...\omega, \omega + 1\},$$

reaching the next limit at

$$\omega + \omega = \{0, 1, 2, 3, ...\omega, \omega + 1, \omega + 2, ...\}.$$

Strictly speaking 0 is a limit ordinal, but it is frequently put in a class by itself. In any case, with the exception of this minor point, we can divide the ordinals into the classes of successor and limit.

At once from the definition at A.1 we have that any member of an ordinal is again an ordinal.

DEFINITION A.3. A set A is said to be *well ordered* by a relation $<$ if

(i) $<$ is well founded, in the sense that every non-empty subset of A has a $<$-least element:

$$\forall B \subset A(B \neq \emptyset \Rightarrow \exists b \in B(\forall a \in A(a < b \Rightarrow a \notin B)));$$

(ii) A is linearly ordered by the $<$-relation.

Two immediate consequences of the usual formalizations of mathematical reasoning are:

THEOREM A.4. *Every ordinal is well ordered by* $\in$.

THEOREM A.5. *If* $<$ *well orders the set* A *then there is some ordinal* α *so that* $(A, <)$ *and* $(\alpha, \in)$ *are isomorphic as linear orderings.*

For our purposes we may as well take A.4 and A.5 as axioms.

A.4 provides the foundation for arguments by *transfinite induction*. Suppose $C \subset \alpha$ includes 0, and for all $\beta \in \alpha$ it includes $\beta + 1$, and whenever $\lambda \in \alpha$ is a limit ordinal with $\lambda \subset C$ we have $\lambda \in C$, then we may conclude that C equals α: Otherwise there will be some $\in$-least $\gamma \in (\alpha \setminus C)$ by the assumption $\in$ provides a well ordering on α; but then this least γ can be neither successor nor limit by the assumptions on C.

DEFINITION A.6. A set is countable if there is an onto map

$$\pi : \mathbb{N} \twoheadrightarrow A.$$

ω_1 is the first uncountable ordinal.

Thus we allow that finite sets are countable. A set is said to have cardinality $\aleph_0$ if it is infinite and countable, and we express this with the notation

$$|A| = \aleph_0.$$

Since there is a bijection between $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$, countable unions of countable sets are countable.

LEMMA A.7.

$$\text{Borel} = \bigcup_{\alpha \in \omega_1} \utilde{\Pi}^0_\alpha$$

PROOF. $\supset$ is clear from the definition of Borel as the smallest superset of open forming a σ-algebra.

For the other direction we have that open sets are $\utilde{\Pi}^0_2$ and so should only be concerned with showing $\bigcup_{\alpha \in \omega_1} \utilde{\Pi}^0_\alpha$ forms a σ-algebra. Suppose that $B_n \in \utilde{\Pi}^0_{\alpha(n)}$ for each $n \in \mathbb{N}$. Note that since ω_1 is not the countable union of countable sets there will be some $\delta < \omega_1$ with each $\alpha(n) < \delta$. But then

$$\bigcup_{n \in \mathbb{N}} B_n \in \utilde{\Sigma}^0_\delta \subset \utilde{\Pi}^0_{\delta+1}.$$

$\square$

For a thorough treatment see [**45**] or [**68**].

APPENDIX B

Notation

NOTATION B.1. **Sets** $\emptyset$ is the empty set, uniquely defined by the description that it has no members.

For A, B both sets, $A \subset B$ indicates that A is included in B; $A \subset B$ is *not* taken to mean that A is necessarily unequal to B, so for instance it is true that

$$\emptyset \subset \emptyset.$$

$A \setminus B$ is the set of all points in A and not in B. $A \cup B$ is the set of points in either A or B and $A \cap B$ is the set of points in both. $A \Delta B$ indicates the *symmetric difference* of these sets:

$$A \Delta B = (A \setminus B) \cup (B \setminus A).$$

For A some set and $Q(\cdot)$ some property, we use

$$\{x \in A : Q(x)\}$$

and

$$\{x \in A| \ Q(x)\}$$

interchangeably to mean the set of all points in A with the property Q. This allows some deviations, for instance if f is a function, then

$$\{f(x) : Q(x)\}$$

would indicate the set of all images under f of x's which have Q. In the case of functions it would look strange to write

$$\{f : X \to Y : Q(f)\},$$

and so we use

$$\{f : X \to Y| \ Q(f)\}$$

to indicate the set of functions from X to Y which have property Q.

$\mathcal{P}(A)$ is the set of all subsets of A.

NOTATION B.2. **Sequences** A sequence – that is to say a function from $\mathbb{N}$ or some finite initial segment of $\mathbb{N}$ – is commonly denoted by vector notation, as in $\vec{x}$; here the reader should expect that the nth term of this sequence is x_n. Alternative notations for sequences include $(\cdot, \cdot, ...)$ and $\langle \cdot, \cdot, ... \rangle$. For instance the sequence $(a_0, a_1, ..., a_{k-1})$ has k elements and its second term is a_1.

For s, t finite sequences, we denote the concatenation by $s^\frown t$. So for l the length of s and $i < l$ we have

$$(s^\frown t)(i) = s(i),$$

whilst for $j \geq l$ we let

$$(s^\frown t)(j) = t(j - l).$$

There is a variant for the case that one of the sequences has length one, where we write $s^\frown a$ or $a^\frown s$ instead of $s^\frown\langle a\rangle$ or $\langle a\rangle^\frown s$. Thus for $l =$ length of s,

$$\begin{aligned}(s^\frown a)(i) &= s(i), \qquad i < l,\\ (s^\frown a)(l) &= a,\\ (a^\frown s)(0) &= a,\\ (a^\frown s)(i+1) &= s(i), \qquad i < l.\end{aligned}$$

We also write $(x_n)_{n\in\mathbb{N}}$ to indicate the sequence $(x_0, x_1, ...)$.

For a set A, we let $A^{\mathbb{N}}$ denote the collection of all infinite sequences from A and $A^{<\mathbb{N}}$ denote the collection of all finite sequences.

Given the identification of each natural number with its predecessors, we thus have that 2^n equals the set of functions from $\{0, 1, ..., n-1\}$ to $\{0, 1\}$. So in this context, 2^1 does *not* equal 2, but rather the set $\{v_0, v_1\}$, where v_0 is the sequence of length 1 whose only term is 0 and v_1 is the sequence whose only term is 1.

Of course this may seem an insane convention, but it makes for compacting expressions which would otherwise be verbose. In context it will never create confusion, but the reader should note that 2^{-n} *always* equals the inverse of 2 to the nth power. In particular 2^{-1} equals

$$\frac{1}{2},$$

and not some exotic attempt to invert the set $\{v_0, v_1\}$ from above.

NOTATION B.3. **Functions** Given a set A and a function f with domain A and a subset $B \subset A$, we use $f|_B$ to denote the restriction of the function f to B. In the special case that

$$f : \mathbb{N} \to C$$

is a function from $\mathbb{N}$ to some set C we use $f|_n$ to denote the restriction of f to the set $\{0, 1, ..., n-1\}$. Since we understand $\mathbb{N} = \{0, 1, 2, ...\}$ and each $n = \{0, 1, ..., n-1\}$, this is entirely consistent. A^B is used to indicate the set of all functions from B to A.

In general $f[A] = \{f(x) : x \in A\}$ whenever the set A is a subset of the domain of f, and $f^{-1}[B] = \{x : f(x) \in B\}$ when B is a subset of the range.

$$f : X \to Y$$

indicates that f is a function with domain X and range Y. An expression such as

$$f : x \mapsto \frac{x}{5} + 7$$

indicates that f is the function which takes x, divides by 5, adds seven, and spits out

$$\frac{x}{5} + 7.$$

If X is a set and $A \subset X$ a subset, then we define the characteristic function of A on X by

$$\begin{aligned}\chi_A &: X \to \{0, 1\},\\ \chi_A(x) &= 1 \text{ if } x \in A,\\ \chi_A(x) &= 0 \text{ if } x \notin A.\end{aligned}$$

If $X \times Y$ is a product of two spaces and $f : X \times Y \to B$ is a function whose domain is the product, we prefer the informal $f(x, y)$ to denote the value of f applied to the pair $(x, y) \in X \times Y$, rather than the fussier but more correct $f((x, y))$. Given $x \in X$ we let

$$f(x, \cdot) : Y \to B$$
$$y \mapsto f(x, y).$$

If A is a set, then $(x_a)_{a \in A}$ indicates the function with domain A which assumes the value x_a for any $a \in A$. So in particular in the case that $A = \mathbb{N}$ this leads to one of the usual notations, $(x_n)_{n \in \mathbb{N}}$, for a sequence.

NOTATION B.4. **Topological notions** We customarily identify a topological space with its underlying set.

If X is a topological space and $A \subset X$, then $\overline{A}$ denotes its closure, A^o denotes its interior (that is the largest open set included in A), and

$$\partial A = \overline{A} \cap \overline{X \setminus A}$$

indicates the boundary.

In the case that X is a metric space and d is the metric, we let

$$d(B) = \sup \{d(b_1, b_2) : b_1, b_2 \in B\}.$$

For $b \in X$

$$d(b, B) = \inf \{d(b, b_1) : b_1 \in B\}.$$

NOTATION B.5. **Logical quantifiers** $\exists x(P(x))$ is used to indicate that there exists some x with $P(x)$, while $\forall x(P(x))$ indicates that every x has $P(x)$. These "quantifiers" are embellished in a number of ways. For instance

$$\exists^\infty x(P(x))$$

indicates that there are infinitely many different x's with P, while

$$\forall^\infty x(P(x))$$

is used when all but finitely many x's have P – that is to say, $\forall^\infty x(P(x))$ is equivalent to

$$\neg \exists^\infty \neg x(P(x)).$$

Alternatively

$$\exists x \in X(P(x))$$

is used to assert that there is some element x of the set X with $P(x)$ whilst

$$\forall x \in X(P(x))$$

asserts that no element of X fails to have P.

The categoricity quantifiers $\exists^*$ and $\forall^*$ are defined at 2.45.

NOTATION B.6. **Unions of sets** If A is a set of sets we let

$$\bigcup A$$

be the union of all sets contained in A. If

$$x \mapsto A_x$$

is an assignment of sets to elements of a space X, then for any $Y \subset X$

$$\bigcup_{x \in Y} A_x$$

indicates the union of the set of sets $\{A_x : x \in Y.\}$.

Of course one could iterate the union operation, and we have

$$\bigcup \bigcup A$$

as the union of all sets contained in $\bigcup A$, and so on. The end result of iterating this process infinitely often and taking all sets arising this way is denoted $\mathrm{TC}(A)$. Thus $\mathrm{TC}(A)$, the *transitive closure* of A, is the collection of all sets appearing as an element somewhere in the hierarchy

$$\begin{gathered} A, \\ \bigcup A, \\ \bigcup\bigcup A, \\ \bigcup\bigcup\bigcup A, \\ \vdots \end{gathered}$$

Alternatively one may define the transitive closure of A to be the smallest transitive set including A.

Thus if we have $A = \{\{1, \{2, 4\}\}, 6, \{1, 3, 4\}\}$, the set whose members are exactly $\{1, \{2, 4\}\}, 6, \{1, 3, 4\}$, then $\mathrm{TC}(A)$ would be

$$\{0, 1, 2, 3, 4, 5, 6, \{2, 4\}, \{1, 3, 4\}, \{1, \{2, 4\}\}\}.$$

The hereditarily countable sets, HC, are all sets A for which $\mathrm{TC}(A)$ is countable.

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