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Quasiconformal Teichmüller Theory

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Erratum

Frederick P. Gardiner and Nikola Lakic, *Quasiconformal Teichmüller Theory*, Mathematical Surveys and Monographs, vol. 76, Amer. Math. Soc., Providence, RI, 2000.

During the editing of the manuscript for this book, we inadvertently omitted reference to the following source for Chapter 9: V. Božin, N. Lakic, V. Marković, and M. Mateljević, Unique extremality, *J. Anal. Math.* **75** (1998), 299–338. We apologize for this omission.

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ABSTRACT. This book provides background for applications of Teichmüller theory to dynamical systems and, in particular, to iteration of rational maps and conformal dynamics, to Kleinian groups and three-dimensional manifolds, to Fuchsian groups and Riemann surfaces, and to one-dimensional dynamics. The focus is on new developments in the theory in both finite and infinite dimensional cases. An exposition of the main theorems is given regardless of dimensionality by using techniques that apply generally. The hope is to provide background for more applications.

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To Joanie and Liki

CONTENTS

PREFACE

The goal of this book is to provide background for applications of Teichmüller theory to dynamical systems and in particular to iteration of rational maps and conformal dynamics, to Kleinian groups and three-dimensional manifolds, to Fuchsian groups and Riemann surfaces, and to one-dimensional dynamics. Although Teichmüller theory is a theory of two-dimensional objects, it naturally impinges on three-dimensional topology through its relationship to Kleinian groups and on one-dimensional dynamics through the quasisymmetric boundary action of a quasiconformal self-map of a disc.

Teichmüller space is a universal classification space for complex structures on a surface of given quasiconformal type. It turns out that the space itself has a natural complex structure, and in applications operators on Teichmüller space are constructed that turn out to be holomorphic and contracting. There are, for example, Thurston's skinning map for the construction of hyperbolic structures on three-manifolds, Thurston's orbit finder for the construction of a rational map in the class of a critically finite map, and various proofs of rigidity for renormalization. None of these topics is dealt with in this book. Rather we focus on new developments in the theory in both the finite and infinite dimensional cases. Our program is to give an exposition of the main known theorems regardless of dimensionality, emphasizing techniques that apply generally, and hoping to provide background for more applications.

In order to give an overview, we need to give several definitions, and we begin with the definition of a Riemann surface. A Riemann surface is an oriented topological surface together with a system of local parameters or charts whose domains of definition cover the surface and that map these domains homeomorphically onto open sets in the complex plane. The charts have the property that for any two with overlapping domains of definition, ϕ_1 and ϕ_2, the transition map, $w = \phi_2 \circ \phi_1^{-1}(z)$, mapping the plane to the plane, is holomorphic.

Holomorphic homeomorphisms of plane domains are precisely those that are differentiable in the complex sense at every point. Alternatively, they are orientation-preserving and conformal. A map f is conformal if, on an infinitesimal level, it takes any family of ellipses with given inclination and eccentricity centered at the origin in the tangent space at z to another family with the same eccentricity and possibly different inclination centered at the origin in the tangent space at $f(z)$. This infinitesimal property has implications for the local behavior of the map; shapes but not necessarily sizes of tiny objects are nearly preserved by f, and in the limit as the objects become more and more tiny, shapes are exactly preserved.

A quasiconformal map allows these shapes to be distorted, but the distortion measured as a ratio of shape measurements is uniformly bounded. Although qua-

siconformal maps need not be everywhere differentiable, the quasiconformality condition is easiest to describe when they are. Any C^1 diffeomorphism between plane domains on an infinitesimal level maps a small circle to a small ellipse. The local dilatation K_z at a point z is the ratio of the length of the major axis of this ellipse to the length of its minor axis. The global dilatation or simply the dilatation of the map is the supremum K of the values K_z for all z in the domain. An orientation-preserving map for which K is finite is by definition quasiconformal. In order to make this definition apply to maps that are not of class C^1, one needs a considerable amount of analysis and the theory of derivatives in the sense of distributions. A summary of this analysis is presented in Chapter 1. Here we mention three essential facts. First of all, a quasiconformal map f is differentiable almost everywhere, and if $K(f) = 1$, f is conformal. Secondly, if f is K-quasiconformal, then it is Hölder continuous with Hölder exponent $1/K$, that is to say, $|f(z_1) - f(z_2)| \leq C|z_1 - z_2|^{1/K}$, where the constant C depends on a normalization. Although no assumption is made in the definition about the distortion of size, this Hölder exponent yields some control. Thirdly, any quasiconformal map f has a Beltrami coefficient $\mu(z) = f_{\overline{z}}(z)/f_z(z)$, and up to postcomposition by a conformal map f is uniquely determined by μ. The absolute value of $\mu(z)$ determines $K_z(f)$ by the formula $K_z(f) = \frac{1+|\mu(z)|}{1-|\mu(z)|}$. Moreover, μ can be an arbitrary complex-valued L_∞-function with $||\mu||_\infty < 1$.

Since Riemann surfaces have conformal structure, it makes sense to speak of the conformality and quasiconformality of maps between Riemann surfaces. In particular, if $w = f(z)$, the constant $K_z(f)$ is defined independently of the selected charts ϕ_1 defined in a neighborhood of z in R_1 and ϕ_2 defined in a neighborhood of w in R_2. Therefore, we can define $K(f)$ to be the essential supremum over all z in R_1 of the quantity $K_z(f)$.

Riemann surfaces R_1 and R_2 are called conformal if there is a conformal homeomorphism from R_1 onto R_2. In general, Riemann surfaces can be quasiconformal without being conformal, and for a given Riemann surface, its quasiconformal deformation theory is the study of the conformal equivalence classes of Riemann surfaces in the same quasiconformal class. The space of such conformally distinct surfaces in the quasiconformal class of R is called moduli space $\mathcal{M}(R)$. If R is compact, $\mathcal{M}(R)$ is a finite dimensional complex variety, but not a manifold.

The study of moduli is simplified by introducing an equivalence relation on a larger set of objects. The larger set of objects is the set of orientation-preserving quasiconformal maps from a fixed base Riemann surface onto a variable Riemann surface. The equivalence relation is easiest to describe if we assume the fixed surface is compact and without boundary. Two maps f_0 and f_1 from R to R_0 and to R_1 are equivalent if there is a conformal map c from R_0 onto R_1 such that $c \circ f_0$ is homotopic to f_1 by a homotopy g_t consisting of quasiconformal maps. Since a quasiconformal map f is, up to postcomposition by a conformal map, uniquely determined by its Beltrami coefficient $\mu = f_{\overline{z}}/f_z$, this equivalence relation also induces an equivalence relation on the space of complex-valued L_∞-Beltrami differentials μ with $||\mu||_\infty < 1$.

The set $[f]$ of quasiconformal maps equivalent to a given map f is called a Teichmüller equivalence class, and the space of all equivalence classes is the quasiconformal Teichmüller space $T(R)$. Because of basic properties of quasiconformal maps, within any equivalence class $[f]$ there is always a representative f_0

such that $K_0 = K(f_0)$ is minimal. If two equivalence classes $[f]$ and $[g]$ are given, the distance in Teichmüller's metric between these two classes is $\frac{1}{2}\log K_0$, where K_0 is the minimal dilatation of a map in the class of $f \circ g^{-1}$. In the compact case it is superfluous to modify the term *Teichmüller space* with the word *quasiconformal* because any two homeomorphic compact Riemann surfaces of the same genus are automatically quasiconformal (even diffeomorphic). Teichmüller showed that when R is compact $T(R)$ is a complete metric space homeomorphic to a cell of real dimension $6g - 6$ if the genus g of R is more than 1. When the genus is 1, it has real dimension 2.

The group of homotopy classes of quasiconformal self-maps of the base surface R is called the mapping class group, $MCG(R)$. It acts naturally as a group of isometries on $T(R)$ and identifies points that correspond to conformally equivalent Riemann surfaces. Thus, factoring Teichmüller space by the mapping class group yields moduli space; $\mathcal{M}(R)$ is equal to $T(R)$ factored by $MCG(R)$.

A Riemann surface is of finite analytic type if it can be obtained by removing a finite number of points from a compact Riemann surface. We note that this property is quasiconformally invariant. That is, if f is a quasiconformal map from such a Riemann surface, R, to another, $f(R)$, then necessarily $f(R)$ is also of finite analytic type. A surface of infinite analytic type can be obtained by taking a compact surface and deleting any infinite closed set. For example, an interesting infinite-type Riemann surface is the Riemann sphere minus the standard middle-thirds Cantor set in the unit interval. Any surface of infinite genus has infinite analytic type. It turns out that $T(R)$ is infinite dimensional if and only if R is of infinite analytic type.

Defining the equivalence relation on quasiconformal maps f from a Riemann surface R of infinite analytic type to a variable Riemann surface $f(R)$ is a technical matter. The essential idea is that two maps f_0 and f_1 are equivalent if there is a conformal map $c : f_0(R) \to f_1(R)$ such that $c \circ f_0$ and f_1 are homotopic by a homotopy that pins down the boundary points of R. The difficulty is in how to define boundary points and how to determine whether a quasiconformal homotopy extends to those points. One way to handle the difficulty is to use hyperbolic geometry and the uniformization theorem. One obtains the so-called ideal boundary for any Riemann surface whose universal covering group is Fuchsian. The question of how to define boundary points natural for quasiconformal deformations is an interesting problem.

We also study a new type of Teichmüller space that concerns only infinite-type Riemann surfaces and that explicitly deals with the asymptotic geometrical behavior of the Riemann surface at the boundary. The definition of asymptotic Teichmüller space $AT(R)$ is the same as that of ordinary Teichmüller space except in the definition of equivalence classes the word *conformal* is replaced by *asymptotically conformal*. A class $[f]$ of maps in $T(R)$ is asymptotically conformal if one can make $K_z(f_0)$ arbitrarily close to 1 for z in $R \setminus C$ by choosing a suitable representative f_0 of $[f]$ and by choosing a sufficiently large compact subset C of R. On surfaces of finite analytic type all Teichmüller classes of quasiconformal maps are asymptotically conformal, and thus in this case $AT(R)$ consists of just one point. In all other cases $AT(R)$ is infinite dimensional. Many of the results concerning asymptotic Teichmüller space presented in the text represent our joint work with Cliff Earle.

Study of $AT(R)$ automatically leads to the notion of the boundary dilatation $H([f])$ of a class $[f]$ in $T(R)$. For any compact set $C \subset R$ one looks for a

representative f_0 of the class $[f]$ such that the essential supremum of $K_z(f_0)$ for z in $R \setminus C$ is as small as possible. The boundary dilatation $H([f])$ of $[f]$ is the infimum of all of these numbers over all compact subsets C of R. It can happen that $H([f]) = K_0([f])$. The Teichmüller metric on $AT(R)$ is given by boundary dilatation; the distance between two classes in $AT(R)$ represented by maps f and g is $\frac{1}{2}\log H([f \circ g^{-1}])$.

$T(R)$ has a natural basepoint, which is the equivalence class of the identity map on R, and the tangent space to $T(R)$ at this basepoint is isomorphic to the Banach dual space of the complex Banach space $A(R)$, the space of integrable holomorphic quadratic differentials $\varphi(z)dz^2$ on R. The norm of φ in $A(R)$ is given by $||\varphi|| = \iint_R |\varphi(z)|dxdy$.

The pairing $(\mu, \varphi) = \iint_R \mu(z)\varphi(z)\ dxdy$ induces a second notion of equivalence of Beltrami coefficients. Two Beltrami coefficients μ and ν are called infinitesimally equivalent if $(\mu - \nu, \varphi) = 0$ for all φ in $A(R)$ and μ is infinitesimally trivial if μ is infinitesimally equivalent to 0. When in addition $||\mu||_\infty$ and $||\nu||_\infty$ are both less than 1, μ and ν are globally equivalent if the quasiconformal maps f^μ and f^ν with Beltrami coefficients μ and ν represent the same point in $T(R)$. We say μ is globally trivial (or just trivial) if f^μ is equivalent to the identity map. Finally, μ is called extremal in its global class if f^μ has minimal dilatation in its Teichmüller class, and similarly, μ is extremal in its infinitesimal class if $||\mu||_\infty$ takes the smallest possible value in its infinitesimal class. We will see that the interplay of the infinitesimal and global equivalence relations on Beltrami coefficients is an essential element of Teichmüller theory.

Some of the main results we prove are:

- For any Riemann surface R, $T(R)$ and $AT(R)$ are complex manifolds modeled on Banach spaces.
- $T(R)$ and $AT(R)$ are complete with respect to Teichmüller's metric; $AT(R)$ is a quotient space of $T(R)$ with quotient metric induced by the natural projection from $T(R)$ to $AT(R)$, and this metric is given by boundary dilatation.
- These metrics have infinitesimal forms and are equal to the integrals of their infinitesimal forms.
- Teichmüller's metric on $T(R)$ is equal to Kobayashi's metric, a metric defined purely in terms of the family of holomorphic functions from the unit disc into $T(R)$.
- When R has finite genus with either finite or infinite analytic type, every holomorphic automorphism of $T(R)$ is induced by a quasiconformal self-map of R. Moreover, except for a few low-dimensional Teichmüller spaces, all occurring in genus 2 and lower, $MCG(R)$ acts as the full automorphism group of $T(R)$. For compact surfaces this result was proved by Royden.
- A quasiconformal map f has minimal dilatation in its Teichmüller class if and only if its Beltrami coefficient μ is extremal in its infinitesimal class. This result is called the Hamilton-Krushkal, Reich-Strebel necessary-and-sufficient condition for extremality.
- A quasiconformal map f is nearly extremal in its Teichmüller class if and only if its Beltrami coefficient μ is nearly extremal in its infinitesimal class.

This result is called Teichmüller contraction. There is also a version of Teichmüller contraction for $AT(R)$.

- A quasiconformal map f is the uniquely extremal representative in its Teichmüller class if and only if its Beltrami coefficient μ is uniquely extremal in its infinitesimal class.
- When R is the Riemann sphere $\overline{\mathbb{C}}$ minus a closed set, there is an alternative notion of equivalence on Beltrami coefficients. The resulting Teichmüller space is a complex manifold universal for holomorphic motions of the closed set.
- When R is a plane domain, the boundary dilatation of any class $[f]$ in $T(R)$ is realized at some point of the set-theoretic boundary of R. When R is the unit disc, this result is called Fehlmann's theorem on the existence of substantial points.

The foundation for these results is the study of the space $A(R)$ of integrable holomorphic quadratic differentials. Both the space $A(R)$ and each one of its elements play a geometric rôle. The geometric rôle of the space enters through the Teichmüller infinitesimal Banach norm on $A(R)$ and the main variational lemma that relates infinitesimally trivial Beltrami coefficients to trivial ones. The lemma says μ is infinitesimally trivial if, and only if, there is a curve μ_t of trivial Beltrami coefficients with the property that $||\mu_t - t\mu||_\infty/t$ approaches zero as $t \to 0$.

The geometric rôle of each element of $A(R)$ enters through the length-area method, also called Grötzsch's argument. This method shows how any holomorphic quadratic differential defined on a subdomain of R can be used to measure shape distortion of a quasiconformal map. The argument applied to the horizontal trajectories of a globally defined integrable holomorphic quadratic differential yields the main inequality of Reich and Strebel. Eventually, one learns that elements of $A(R)$ yield well-defined functionals on $T(R)$ and measure shape distortion of a Teichmüller class.

In later chapters we explore a variety of topics including measured foliations, heights mappings, a generalization of classical slit mapping theorems, and a construction of earthquakes based in the idea of applying a limiting process to finite earthquakes.

Much of the material in Chapters 1 and 2 on quasiconformal mapping and Riemann surfaces is presented without proof. Where results in these chapters are not fully proved, it is hoped the reader will agree to work with the theorems as stated and proceed directly to the subject at hand.

At the end of each chapter we provide exercises and sometimes open problems. There is a long list of results known for finite dimensional Teichmüller spaces but not known for infinite dimensional cases, and another list known for infinite dimensional cases but not for asymptotic Teichmüller spaces. Many research problems are posed by this observation.

The purpose of the bibliography is to provide pointers to persons wishing to pursue related subjects further. Although quite long, it is by no means complete. It is meant only to steer persons to the large and expanding body of literature in the field.

FREDERICK P. GARDINER AND NIKOLA LAKIC

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BIBLIOGRAPHY

[AB61] L. V. Ahlfors and L. Bers. Riemann's mapping theorem for variable metrics. *Annals of Math.*, 72:345–404, 1961.

[Abi75] W. Abikoff. On boundaries of Teichmüller spaces and on Kleinian groups III. *Acta Math*, 134:211–237, 1975.

[Abi80] W. Abikoff. *The Real Analytic Theory of Teichmüller space*, volume 820 of *Springer Lecture Notes in Mathematics*. Springer-Verlag, 1980.

[Abi88] W. Abikoff. Kleinian groups, geometrically finite and geometrically perverse. In *Contemp. Math. Geometry of Group Representations bf 74*. AMS, 1988.

[Aga68] S. Agard. Distortion of quasiconformal mappings. *Ann. Acad. Sci. Fenn.*, 413:1–7, 1968.

[AH91] J. M. Anderson and A. Hinkkanen. Univalence criteria and quasiconformal extensions. *Trans. Am. Math. Soc.*, 324:823–842, 1991.

[Ahl53] L. V. Ahlfors. *Complex Analysis*. McGraw-Hill, New York, 1953.

[Ahl54] L. V. Ahlfors. On quasiconformal mappings. *J. Anal. Math.*, 4, 1954.

[Ahl64] L. V. Ahlfors. Quasiconformal reflections. *Acta Math.*, 109:291–301, 1964.

[Ahl65] L. V. Ahlfors. Finitely generated Kleinian groups. *American Journal of Math.*, 86, 87:413–429, 759, 1964, 1965.

[Ahl66a] L. V. Ahlfors. Fundamental polyhedrons and limit point sets of Kleinian groups. *Proc. Nat. Acad. Sci.*, 55:183–187, 1966.

[Ahl66b] L. V. Ahlfors. *Lectures on Quasiconformal Mapping*, volume 10 of *Van Nostrand Studies*. Van Nostrand-Reinhold, Princeton, N. J., 1966.

[Ahl69] L. V. Ahlfors. The structure of finitely generated Kleinian groups. *Acta Math.*, 122:1–17, 1969.

[Ahl73] L. V. Ahlfors. *Conformal Invariants: Topics in Geometric Function Theory*. McGraw-Hill, New York, 1973.

[Aka72] T Akaza. (3/2)-dimensional measures of singular sets of some Kleinian groups. *J. Math. Soc. Japan*, 24:448–464, 1972.

[And70a] E. M. Andreev. Convex polyhedra of finite volume in Lobacevskii space. *Mat. Sb.*, 83 (125):256–260, 1970. English Transl. in Math. USSR Sb. **12**(1970):255–259.

[And70b] E. M. Andreev. On convex polyhedra in Lobacevskii space. *Mat. Sb.*, 81 (123):445–478, 1970. English Transl. in Math. USSR Sb. **12**(1970):413–440.

[AS60] L. V. Ahlfors and L. Sario. *Riemann Surfaces*. Princeton University Press, Princeton, 1960.

[Ast94] K. Astala. Area distortion of quasiconformal mappings. *Acta Math.*, 173:37–60, 1994.

[AW62] L. V. Ahlfors and G. Weill. A uniqueness theorem for Beltrami equations. *Proc. Am. Math. Soc.*, 13:975–978, 1962.

[AY97] W. Abikoff and T. Ye. Computing the Douady-Earle extension. In *Contemporary Mathematics,* **211**, pages 1–8. AMS, 1997.

[Bas97] A. Basmajian. Selected problems in Kleinian groups and Hyperbolic geometry. In *Contemporary Mathematics,* **211**, pages 9–16. AMS, 1997.

[Bea83] A. Beardon. *The Geometry of Discrete Groups.* Graduate Texts in Mathematics. Springer-Verlag, New York, Berlin, 1983.

[Bea91] A. Beardon. *Iteration of Rational Functions.* Graduate Texts in Mathematics. Springer, New York, 1991.

[Bei16] L. Beiberbach. Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln. *Sitz. Ber. Preuss. Akad. Wiss.*, 138:940–955, 1916.

[Ber60] L. Bers. Quasiconformal mappings and Teichmüller's theorem, in Analytic Functions. *Princeton University Press, Princeton, N. J.*, pages 89–119, 1960.

[Ber65] L. Bers. An approximation theorem. *J. d'Analyse Math.*, 14:1–4, 1965.

[Ber66] L. Bers. A non-standard integral equation with applications to quasiconformal mapping. *Acta Math.*, 116:113–134, 1966.

[Ber68] L. Bers. Simultaneous uniformization. *Bulletin of the AMS*, 54:311–315, 1968.

[Ber73] L. Bers. Fibre spaces over Teichmüller spaces. *Acta Math.*, 130:89–126, 1973.

[Ber81] L. Bers. Finite dimensional Teichmüller spaces and their generalizations. *Bull. Amer. Math. Soc.*, 5:131–172, 1981.

[BG86] L. Bers and F. P. Gardiner. Fricke spaces. *Adv. in Math.*, 62(3):249–284, 1986.

[BH95] M. Bestvina and M. Handel. Train tracks for surface homeomorphisms. *Topology*, 34(1):249–284, 1995.

[BM81] R. Brooks and J. P. Matelski. The dynamics of 2-generator subgroups of PSL(2,C). In *Riemann Surfaces and Related Topics: Proc. of the 1978 Stony Brook Conference*, volume 97, pages 65–72, 1981.

[Boj57] B. Bojarski. Generalized solutions of a system of differential equations of first order and elliptic type with discontinuous coefficients. *Math. Sbornik*, 85:451–503, 1957.

[Bon88] F. Bonahon. The geometry of Teichmüller space via geodesic currents. *Invent. Math.*, 92:139–162, 1988.

[Bon97] F. Bonahon. Geodesic laminations and transverse Hölder distributions. *Ann. scient. Ëc. Norm. Sup.*, 30:205–240, 1997.

[Bow79] R. Bowen. Hausdorff dimension of quasi-circles. *Publ. Math. IHES*, 50:11–24, 1979.

[BP78] J. Becker and C. Pommerenke. Über die quasikonforme Fortsetzung schlichter Funktionen. *Math. Zeitschrift*, 161:69–80, 1978.

[BP92] R. Benedetti and C. Petronio. *Lectures on Hyperbolic Geometry.* Springer-Verlag, New York, 1992.

[BR86] L. Bers and H. Royden. Holomorphic families of injections. *Acta Math.*, 157:259–286, 1986.

[Bro86] R. Brooks. On the deformation theory of classical Schottky groups. *Invent. Math.*, 86:461–469, 1986.

[BS85] J. Birman and C. Series. Geodesics with bounded intersection numbers on surfaces are sparsely distributed. *Topology*, 24:217–225, 1985.

[Bul99] A. Bulatovic. *Forgetful mappings between Teichmüller spaces.* PhD thesis, Graduate School & University Center of CUNY, New York, NY, June 1999.

[Bus93] P. Buser. *Geometry Spectra of Compact Riemannian Manifolds*, volume 106. Springer-Verlag, New York, 1993.

[Car67] L. Carleson. On mappings, conformal at the boundary. *J. d'Analyse Math.*, 19:1–13, 1967.

[CG93] L. Carleson and T. Gamelin. *Complex Dynamics.* Springer-Verlag, Berlin, Heidelberg, 1993.

[CO94] M. Chuaqui and B. Osgood. Ahlfors-Weill extensions of conformal mappings and critical points of the Poincaré metric. *Comment. Math. Helv.*, 69:659–668, 1994.

[CO99] M. Chuaqui and B. Osgood. General univalence criteria in the disc: extensions and extremal functions. *Ann. Acad. Sci. Fenn.*, to appear 1999.

[Cui98] G. Cui. Circle expanding maps and symmetric structures. *Ergod. Th. & Dynamical Sys.*, 18:831–842, 1998.

[dC99] A. de Carvalho. Pruning fronts and the formation of horseshoes. *Erg. Th. & Dyn. Syst.*, 19:1–44, 1999.

[Dev87] R. L. Devaney. *An Introduction to Chaotic Dynamical Systems.* Addison-Wesley, New York, 1987.

[dF92] E. de Faria. *Proof of universality for critical circle mappings.* PhD thesis, Graduate School and University Center of City University of University of New York, March 1992.

[dF96] E. de Faria. Quasisymmetric distortion and rigidity of expanding endomorphisms of S^1. *Proc. of Amer. Math. Soc.*, 124(6):1949–1957, 1996.

[dMVS93] W. de Melo and S. Van Strein. *One-Dimensional Dynamics.* Springer-Verlag, Berlin, Heidelberg, 1993.

[Ear69] C. J. Earle. On holomorphic cross-sections in Teichmüller spaces. *Duke Math. J.*, 36:409–416, 1969.

[Ear74] C. J. Earle. On the Carathéodory metric in Teichmüller spaces. In *Discontinuous Groups and Riemann Surfaces*, pages 99–103. Annal of Math Study **79**, Princ. Univ. Press, 1974.

[Ear83] C. J. Earle. The integrable automorphic forms as a dual space. In T. Metzger, editor, *Modular Functions in Analysis and Number Theory*, Lecture Notes in Mathematics and Statistics, **5**, pages 30–40, 1983.

[Ear89] C. J. Earle. The integrable automorphic forms as a dual space. *Complex Variables: Theory & Appl.*, 12:153–158, 1989.

[Ear97] C. J. Earle. Some maximal holomorphic motions. In *Lipa's Legacy: Proceedings of the Bers Colloquium.* AMS Contemp. Math. Ser. bf 211, 1997.

[Ear98] C. J. Earle, 1998. oral communication.

[EE67] C. J. Earle and J. Eells. On the differential geometry of Teichmüller spaces. *J. Anal. Math.*, 19:35–52, 1967.

[EG95] C. J. Earle and F. P. Gardiner. Teichmüller disks and Veech's F-structures. *Contemporary Mathematics*, 201:165–189, 1995.

[EGL95] C. J. Earle, F. P. Gardiner, and N. Lakic. Teichmüller spaces with asymptotic conformal equivalenc. IHES preprint, Bures-sur-Yvette, June 1995.

[EGL97] C. J. Earle, F. P. Gardiner, and N. Lakic. Vector fields for holomorphic motions of closed sets. In *Contemporary Mathematics* **211**, pages 193–225. AMS, 1997.

[EGL98a] C. J. Earle, F. P. Gardiner, and N. Lakic. Asymptotic Teichmüller spaces. to appear, November, 1998.

[EGL98b] C. J. Earle, F. P. Gardiner, and N. Lakic. Asymptotic Teichmüller space, Part I. *Trans. Am. Math. Soc.*, submitted, November, 1998.

[EK74a] C. J. Earle and I. Kra. On holomorphic mappings between Teichmüller spaces. In L. V. Ahlfors et al., editor, *Contributions to Analysis*, pages 107–124, New York, 1974. Academic Press.

[EK74b] C. J. Earle and I. Kra. On isometries between Teichmüller spaces. *Duke Math. J.*, 41:583–591, 1974.

[EKK94] C. J. Earle, I. Kra, and S. L. Krushkal. Holomorphic motions and Teichmüller spaces. *Trans. Am. Math. Soc.*, 343(2):927–948, 1994.

[EL98] C. J. Earle and N. Lakic. Teichmüller shift domains. preprint, 1998.

[EL99] C. J. Earle and Zhong Li. Extremal quasiconformal mappings in plane domains. In *Celebration of Fred Gehring's 70th birthday.* Spring-Verlag, 1999.

[ELar] C. J. Earle and Zhong Li. Isometrically embedded polydisks in infinite dimensional Teichmüller spaces. *Journal of Geometric Analysis*, to appear.

[EM] C. J. Earle and S. Mitra. Analytic dependence of conformal invariants on parameters. Cornell University, preprint, 1999.

[EM87] D. B. A. Epstein and A. Marden. Convex hulls in hyperbolic space, a theorem of Sullivan, and measured pleated surfaces. In D. B. A. Epstein, editor, *Analytical and Geometric Aspects of Hyperbolic Space*, pages 112–253. London Math Society **111**, Cambridge University Press, 1987.

[EM88] C. J. Earle and C. T. McMullen. Quasiconformal isotopies. In *Holomorphic Functions and Moduli*, volume I, pages 143–154. Springer-Verlag, 1988.

[EN86] C. J. Earle and S. Nag. Conformally natural reflections in Jordan curves with applications to Teichmüller spaces. In *Holomorphic Functions and Moduli, II*, volume II, pages 179–194. Springer-Verlag, 1986.

[Eps66] D. B. A. Epstein. Curves on 2-manifolds. *Acta Math.*, 115:83–107, 1966.

[Fay77] J. D. Fay. Fourier coefficients and the resolvent for a Fuchsian group. *J. d. reine u. angew. Math.*, 293:142–203, 1977.

[Feh81] R. Fehlmann. Über extremale quasikonforme Abbildungen. *Comment. Math. Helvetici*, 56:581–598, 1981.

[Fen89] W. Fenchel. *Elementary Geometry in Hyperbolic Space.* de Gruyter, 1989.

[FG95] R. Fehlmann and F. P. Gardiner. Extremal problems for quadratic differentials. *Mich. Math. J.*, 43(2):573–591, 1995.

[FK80] H. Farkas and I. Kra. *Riemann Surfaces.* Graduate Texts in Mathematics. Springer-Verlag, New York, 1980.

[FLP79] A. Fathi, F. Laudenbach, and V. Poénaru. *Travaux de Thurston sur les Surfaces*, volume 66-67 of *Asterisque.* Soc. Math. de France, Seminaire Orsay, 1979.

[For51] L. R. Ford. *Automorphic Functions.* Chelsea, 2nd edition, 1951.

[Fos81] O. Foster. *Lectures on Riemann Surfaces.* Springer-Verlag, New York, Heidelberg, Berlin, 1981.

[FRW85] C. H. Fitzgerald, B. Rodin, and S. E. Warschawski. Estimates for the harmonic measure of a continuum in the unit disc. *Trans. Amer. Math. Soc.*, 287(2):681–685, 1985.

[Gai72] D. Gaier. Estimates of conformal mapping near the boundary. *Indiana Univ. Math. J.*, 21:581–595, 1972.

[Gar68] F. P. Gardiner. An analysis of the group operation in universal Teichmülller space. *Tans. Am. Math. Soc.*, 132(2):471–486, 1968.

[Gar80] F. P. Gardiner. On the variation of Teichmüller's metric. *Proc. of the Royal Soc. of Edinburgh*, pages 143–152, 1980.

[Gar82] F. P. Gardiner. On partially Teichmüller Beltrami differentials. *Mich. Math. J.*, 29:237–242, 1982.

[Gar84a] F. P. Gardiner. Approximation of infinite dimensional Teichmüller spaces. *Trans. Amer. Math. Soc.*, 282(1):367–383, 1984.

[Gar84b] F. P. Gardiner. Measured foliations and the minimal norm property for quadratic differentials. *Acta Math.*, 152:57–76, 1984.

[Gar87] F. P. Gardiner. *Teichmüller Theory and Quadratic Differentials.* John Wiley & Sons, New York, 1987.

[Gar93] F. P. Gardiner. On Teichmüller contraction. *Proc. Am. Math. Soc.*, 118(3):865–875, 1993.

[Gar95] F. P. Gardiner. Infinitesimal bending and twisting in one-dimensional dynamics. *Trans. Am. Math. Soc.*, 347(3):915–937, 1995.

[Gil95a] J. Gilman. A discreteness condition for subgroups of $PSL(2, C)$. In *Contemporary Mathematics* **211**, pages 261–267. AMS, 1995.

[Gil95b] J. Gilman. Two-generator discrete subgroups of $PSL(2, R)$. *Memoirs of the AMS*, 117, 1995.

[GK98a] F. P. Gardiner and L. Keen. Applications of holomorphic motions to quasiconformal mappings and quasiFuchsian manifolds. In *Proceedings of the Ibero-American Conference on Geometry: Cruz del Sur*, Olmué, Chile, January, 1998. to appear.

[GK98b] F. P. Gardiner and L. Keen. Coverings of cantor sets. In *Proc. of the Ahlfors-Bers Conference*, SUNY at Stony Brook, November, 1998. to appear.

[GL99] F. P. Gardiner and N. Lakic. Comparing Poincaré densities. submitted, 1999.

[GM91a] F. P. Gardiner and H. Masur. Extremal length geometry of Teichmüller space. *Complex Variables*, 16:209–237, 1991.

[GM91b] J. Gilman and B. Maskit. An algorithm for two-generator discrete subgroups. *Mich. Math. J.*, 38(1):13–32, 1991.

[Gro15] T. H. Gronwall. Some remarks on conformal representation. *Ann. Math.*, 16:72–76, 1914–1915.

[Gru39] H. Grunsky. Koeffizientenbedingungen für schlicht abbildende meromorphe Funktionen. *Math. Z.*, 45:29–61, 1939.

[GS92] F. P. Gardiner and D. P. Sullivan. Symmetric structures on a closed curve. *Amer. J. of Math.*, 114:683–736, 1992.

[GS94] F. P. Gardiner and D. P. Sullivan. Lacunary series as quadratic differentials in conformal dynamics. In *Contemporary Mathematics* **169**, pages 307–330. AMS, 1994.

[Gue81] I. Guerrero. Automorphisms of compact Riemann surfaces and Weierstrass points. In *Annals of Math Studies* **97**, pages 215–224, 1981.

[Hal94] T. Hall. The creation of horseshoes. *Nonlinearity*, 7:861–924, 1994.

[Ham69] R. S. Hamilton. Extremal quasiconformal mappings with prescribed boundary values. *Trans. Amer. Math. Soc.*, 138:399–406, 1969.

[Har74] W. J. Harvey. Chabauty spaces of discrete groups. In *Annals of Math Studies* **79**, pages 239–246, 1974.

[Har77] W. J. Harvey. *Discrete Groups and Automorphic Functions.* Academic Press, 1977.

[Har79] L. A. Harris. Schwarz-Pick systems of pseudometrics for domains in normed linear spaces. In *Advances in Holmorphy*, volume 34, pages 345–406, Amsterdam, 1979. North-Holland Mathematical Studies.

[Har81] W. J. Harvey. Boundary structure of the modular group. In *Annals of Math Studies* **97**, pages 245–252, 1981.

[Har88] W. J. Harvey. Modular groups and representation spaces. *Contemporary Math.*, 74:205–214, 1988.

[Har92] W. J. Harvey. Modular groups in geometry and physics. In *Discrete Groups and Geometry, Lecture Notes* **173**, pages 94–103. London Mathematical Society, Cambridge University Press, 1992.

[Har93] W. J. Harvey. On certain families of compact Riemann surfaces. In *Mapping Class Groups and Moduli Spaces*, volume 150 of *Contemporary Math.*, pages 137–148. Amer. Math. Soc., 1993.

[Hei62] M. Heins. *Selected Topics in the Classical Theory of Functions of a Complex Variable.* Holt, Rinehart and Winston, New York, 1962.

[Hej78] D. Hejhal. Mondromy groups and Poincaré series. *Bull. Amer. Math. Soc.*, pages 339–376, 1978.

[Hin93] A. Hinkkanen. Julia sets of rational functions are uniformly perfect. *Math. Proc. Camb. Phil. Soc.*, 113:543–559, 1993.

[Hin94] A. Hinkkanen. Julia sets of polynomials are uniformly perfect. *Bull. Lond. Math. Soc.*, 26:153–159, 1994.

[HM79] J. Hubbard and H. Masur. Quadratic differentials and foliations. *Acta Math.*, 142:221–274, 1979.

[HM92] R. Harmelin and D. Minda. Quasi-invariant domain constants. *Israel J. of Math.*, 77:115–127, 1992.

[HM96] A. Hinkkanen and G. J. Martin. Julia sets of rational semigroups. *Math. Z.*, 222:161–169, 1996.

[Hop71] E. Hopf. Ergodic theory and geodesic flow on surfaces of constant negative curvature. *Bull. Amer. Math. Soc.*, 77:863–877, 1971.

[HS97] J. Hu and D. Sullivan. Topological conjugacy of circle diffeomorphisms. *Ergod. Th. & Dyn. Sys.*, 17:173–186, 1997.

[Hub76] J. Hubbard. Memoir. *Memoirs of the AMS*, pages 1–116, 1976.

[Jen57] J. A. Jenkins. On the existence of certain general extremal metrics. *Ann. Math.*, 66:440–453, 1957.

[Jen58] J. A. Jenkins. Univalent functions and conformal mapping. *Ergeb. Math. Grenzgeb.*, 18:1–167, 1958.

[Jen60] J. A. Jenkins. On the global structure of a positive quadratic differential, III. *J. Math.*, 48:405–412, 1960.

[Jen64] J. A. Jenkins. Some area theorems and a special coefficient theorem. *Ill. J. of Math.*, 8:80–99, 1964.

[Jia96] Y. Jiang. *Renormalization and Geometry in One-Dimensional Complex Dynamics.* World Scientific, Singapore, 1996.

[Jør76] T. Jørgensen. On discrete groups of Möbius transformations. *Amer. J. of Math.*, 98:739–749, 1976.

[Jør77a] T. Jørgensen. Compact 3-manifolds of constant negative curvature fibering over the circle. *Ann. of Math.*, 106:61–72, 1977.

[Jør77b] T. Jørgensen. A note on subgroups of $SL(2, C)$. *Quarterly J. Math.*, 28:209–212, 1977.

[Jør81] T. Jørgensen. Commutators in $SL(2, C)$. In *Annals of Math Studies* **97**, pages 301–301, 1981.

[Kee66] L. Keen. Canonical polygons for finitely generated Fuchsian groups. *Acta Math.*, 115:1–16, 1966.

[Kee71] L. Keen. Intrinsic moduli on Riemann surfaces. *Ann. Math.*, 84:404–420, 1971.

[Kee74] L. Keen. Collars on Riemann surfaces. *Ann Math. Studies*, 66:205–224, 1974.

[Ker83] S. Kerckhoff. The Nielsen realization problem. *Ann. Math.*, 117:235–265, 1983.

[Ker85] S. Kerckhoff. Earthquakes are analytic. *Comm. Mat. Helv.*, 60:17–30, 1985.

[KK84] S. Krushkal and R. Kuhnau. *Quasiconformal Mappings-New Methods and Applications.* Nauka, Novosibirsk, 1984.

[KMS86] S. Kerckhoff, H. Masur, and J. Smillie. Ergodicity of billiard flows. *Ann. of Math.*, 124, 1986.

[Kob70] S. Kobayashi. *Hyperbolic Manifolds and Holomorpic Mappings.* Marcel Dekker, New York, 1970.

[Kol81] C. J. Kolaski. Isometries of Bergman spaces over bounded Runge domains. *Can. J. of Math.*, 33:1157–1164, 1981.

[Kou85] C. Kourouniotis. Deformations of hyperbolic structures. *Math. Proc. Camb. Phil. Soc.*, 98:247–261, 1985.

[Kou91] C. Kourouniotis. Bending in the space of quasi-Fuchsian structures. *Glasgow Math. J.*, 33:41–49, 1991.

[Kra59] S. Kravetz. On the geometry of Teichmüller spaces. *Ann. Acad. Sci. Fenn.*, AI278:1–35, 1959.

[Kra69] I. Kra. Deformations of Fuchsian groups, I. *Duke Math. J.*, 36:537–546, 1969.

[Kra71] I. Kra. Deformations of Fuchsian groups, II. *Duke Math. J.*, 38:449–508, 1971.

[Kra72] I. Kra. *Automorphic Forms and Kleinian Groups.* Benjamin, Reading, Mass., 1972.

[Kra81a] I. Kra. Canonical mappings between Teichmüller spaces. *Bull. Amer. Math. Soc.*, 4:143–179, 1981.

[Kra81b] I. Kra. The Carathéodory metric on abelian Teichmüller disks. *J. Analyse Math.*, 40:129–143, 1981.

[Kra81c] I. Kra. On the Nielsen-Thurston-Bers type of some self-maps of Riemann surfaces. *Acta Math.*, 146:231–270, 1981.

[Kru69] S. Krushkal. Extremal quasiconformal mappings. *Sib. Math. J.*, 10:411–418, 1969.

[Kru79] S. Krushkal. *Quasiconformal Mappings and Riemann Surfaces.* translation edited by I. Kra, Winston, Washington, D. C., 1979.

[KS97] L. Keen and C. Series. How to bend punctured tori. In *Lipa's Legacy: Proceedings of the Bers Colloquium*, pages 359–387. AMS Contemp. Math. Ser. **211**, 1997.

[Lak96] N. Lakic. Infinitesimal Teichmüller geometry. *Complex Variables*, 30:1–17, 1996.

[Lak97a] N. Lakic. An isometry theorem for quadratic differentials on Riemann surfaces of finite genus. *Trans. Amer. Math. Soc.*, 349:2951–2967, 1997.

[Lak97b] N. Lakic. The minimal norm property for quadratic differentials in the disk. *Mich. Math. J.*, 44:299–316, 1997.

[Lak97c] N. Lakic. Strebel points. *Contemporary Math.*, 211:417–431, 1997.

[Lak98] N. Lakic. Geodesics in Asymptotic Teichmüller space. Preprint, November, 1998.

[Lak99] N. Lakic. Substantial boundary points and Gardiner's conjecture. to appear in Ann. Acad. Sci. Fenn., 1999.

[Leh64] J. Lehner. *Discontinuous Groups and Automorphic Functions.* Math. Surv., no. 8. Amer. Math. Soc., Providence, R. I., 1964.

[Leh87] O. Lehto. *Univalent Functions and Teichmüller Spaces.* Springer-Verlag, New York, 1987.

[Leu67] A. Leutbecher. Über Spitzen diskontinuierlichen Gruppen von lineargebrochenen Transformationen. *Math. Z.*, 100:183–200, 1967.

[Li91] Z. Li. Non-uniqueness of geodesics in infinite dimensional Teichmüller spaces. *Complex Variables*, 16:261–272, 1991.

[Li93] Z. Li. Non-uniqueness of geodesics in infinite dimensional Teichmüller spaces II. *Ann. Acad. Sci. Fenn.*, 18:355–367, 1993.

[Li96] Z. Li. Closed geodesics and non-differentiability of the Teichmüller metric in infinite dimensional Teichmüller spaces. *Proc. Amer. Math. Soc.*, 124:1459–1465, 1996.

[Lie90] G. S. Lieb. *Holomorphic motions and Teichmüller space.* PhD thesis, Cornell University, 1990.

[Lor66] G. G. Lorenz. *Approximation of Functions.* Holt, Rinehart and Winston, New York, 1966.

[Luo98] F. Luo. Geodesic length functions and Teichmüller spaces. *J. Diff. Geom.*, 48:275–317, 1998.

[Luo99] F. Luo. Characters of $SL(2,\mathbb{C})$ representations of groups. Preprint, 1999.

[LV65] O. Lehto and K. I. Virtanen. *Quasiconformal Mapping.* Springer-Verlag, New York, Berlin, 1965.

[Lyu93] M. Lyubich. Geometry of quadratic polynomials: moduli, rigidity, and local connectivity. IMS preprint, number 9, SUNY at Stony Brook, 1993.

[Mac85] A. M. Macbeath. Generic Dirichlet polygons and the modular group. *Glasgow Math. J.*, 27:129–141, 1985.

[Mar97] G. J. Martin. The distortion theorem for quasiconformal mappings, Schottky's theorem & holomorphic motions. *Proc. Amer. Math. Soc.*, 1997.

[Mas81] H. Masur. Dense geodesics in moduli space. In *Riemann Surfaces and Related Topics*, volume 97, pages 417–438. Ann. of Math. Studies, Princeton Univ. Press, 1981.

[Mas86a] B. Maskit. Parameters for Fuchsian groups. In *Holomorphic Functions and Moduli II*, volume 11, pages 417–438, New York, 1986. Springer-Verlag.

[Mas86b] H. Masur. Closed trajectories for quadratic differentials with an application to billiards. *Duke Math. J.*, 53(2):307–314, 1986.

[Mas87] B. Maskit. *Kleinian Groups.* Springer-Verlag, New York, 1987.

[Mas94] B. Maskit. Explicit matrices for Fuchsian groups. *Contemporary Math.*, pages 451–466, 1994.

[Mat82] J. P. Matelski. The classification of 2-generator subgroups of $PSL(2,R)$. *Israel J. of Math.*, 42(4):309–317, 1982.

[McM89] C. T. McMullen. Amenability, Poincaré series and quasiconformal maps. *Inv. Math.*, 97:95–127, 1989.

[McM91] C. T. McMullen. Cusps are dense. *Ann. Math.*, 133:217–247, 1991.

[McM94] C. T. McMullen. *Complex Dynamics and Renormalization*, volume 135 of *Ann. of Math. Studies*. Princeton Univ. Press, 1996, 1994.

[McM96] C. T. McMullen. *Renormalization and 3-Manifolds which Fiber over the Circle*, volume 142. Princeton Univ. Press, Princeton, N. J., 1996.

[McM98] C. T. McMullen. The moduli space of Riemann surfaces is Kähler hyperbolic. preprint, December 1998.

[Min94] Y. Minsky. On rigidity, limit sets, and end invariants of hyperbolic 3-manifolds. *J. Amer. Math. Soc.*, 7:539–588, 1994.

[Mit98] S. Mitra. Teichmüller spaces and holomorphic motions. Preprint, 1998.

[MM95] W. Ma and D. Minda. Behaviour of domain constants under conformal mappings. *Israel J. of Math.*, 91:157–171, 1995.

[MnSS83] R. Màñé, P. Sad, and D. Sullivan. On the dynamics of rational maps. *Ann. Èc. Norm. Sup.*, 96:193–217, 1983.

[MS84] A. Marden and K. Strebel. The heights theorem for quadratic differentials on Riemann surfaces. *Acta Math.*, 153:153–211, 1984.

[MS86] A. Marden and K. Strebel. Pseudo-Anosov Teichmüller mappings. *J. Analyse Math.*, 46:195–220, 1986.

[MS91] H. Masur and J. Smillie. Hausdorff dimesnion of sets of nonergodic measured foliations. *Ann. of Math.*, 134:455–543, 1991.

[MS93] A. Marden and K. Strebel. A characterization of Teichmüller differentials. *J. Diff. Geom.*, 37:1–29, 1993.

[Nag87] S. Nag. Conformally natural Ahlfors-Weill sections and Bers' reproducing formulas. *Bull. Austral. Math. Soc.*, 36:187–196, 1987.

[Nev34] R. Nevanlinna. Das harmonische Mass von Punctmengen und seine Anwendung in der Funktionentheorie. *8th Scan. Math. Congr.*, 1934.

[Nev36] R. Nevanlinna. *Eindeutige analytische Funktionen.* Springer, Berlin, 1936.

[Nic86] P. J. Nicholls. The limit set of a discrete group of hyperbolic motions. In *Holomorphic Functions and Moduli,* **II**, pages 141–164. Springer-Verlag, 1986.

[NV90] S. Nag and A. Verjovsky. *Diff*(S^1) and the Teichmüller spaces. *Comm. Math. Phys.*, 130:123–138, 1990.

[O'B71] B. O'Byrne. On Finsler geometry and applications to Teichmüller space. *Ann. Math. Stud.*, 66:317–328, 1971.

[Osg82] B. Osgood. Some properties of f''/f' and the Poincaré metric. *Indiana Univ. Math. J.*, 31:108–154, 1982.

[Pat72] D. B. Patterson. The Teichmüller spaces are distinct. *Proc. Amer. Math. Soc.*, 35:179–182, 1972.

[Pen93] R. C. Penner. Universal constructions in Teichmüller theory. *Advances in Math.*, 98:143–215, 1993.

[Pen97] R. C. Penner. The universal Ptolemy group and its completions. In L. Schneps and P. Lochak, editors, *Geometric Galois Actions 2*, volume 243. London Math Society Lecture Notes, 1997.

[Pet97] I. Petrovic. *Teichmüller theory for* $E \circ G \circ G = G \circ E$. PhD thesis, Graduate School and University Center of CUNY, New York, 1997.

[PH92] R. C. Penner and J. L. Harer. Combinatorics of train tracks. *Ann. of Math Studies*, 125, 1992.

[Pic15] G. Pick. Über die Beschränkungen analytischer Funktionen, welche durch vorgeschriebene Werte ewirkt werden. *Math. Ann.*, 77:7–23, 1915.

[Pom92] C. Pommerenke. *Boundary Behaviour of Conformal Maps.* Springer-Verlag, New York, Berlin, 1992.

[Por79] M. Porter. Computation of a boundary point of Teichmüller space. *Bol. Soc. Mat. Mexicana*, 24:15–26, 1979.

[PS95] J. R. Parker and C. Series. Bending formulae for convex hull boundaries. *J.d'Analyse Math.*, 67:165–198, 1995.

[PS99] A. A. Pinto and D. Sullivan. Dynamical systems applied to asymptotic geometry. to appear, 1999.

[Rat94] J. G. Ratcliffe. *Foundations of Hyperbolic Manifolds.* Springer-Verlag, New York, 1994.

[Ree81a] M. Rees. An alternative approach to the ergodic theory of measured foliations. *Erg. Th. & Dynam. Syst.*, 1:461–488, 1981.

[Ree81b] M. Rees. Divergence type of some subgroups of finitely generated Fuchsian groups. *Erg. Th. & Dynam. Syst.*, 1:209–221, 1981.

[Rei76] M. Reimann. Ordinary differential equations and quasiconformal mapping. *Inv. Math.*, 33:247–270, 1976.

[Rei81] E. Reich. A criterion for unique extremality of Teichmüller mappings. *Indian Univ. Math. J.*, 30(3):441–447, 1981.

[Rei83] E. Reich. Nonuniqueness of Teichmüller extremal mappings. *Proc. Amer. Math. Soc.*, 88(3):513–516, 1983.

[Rei85] M. Reimann. Invariant extension of quasiconformal deformations. *Ann. Acad. Sci. Fenn. Ser. AI*, 10:477–492, 1985.

[Rei95] E. Reich. A theorem of Fehlmann-type for extensions with bounded $\overline{\partial}$-derivative. *Complex Variables*, 26:343–351, 1995.

[Ren76] H. Renelt. Konstruktion gewisser quadratischer Differential mit von Dirichletintergralen. *Math. Nachr.*, 73:125–142, 1976.

[Ric69] S. Rickman. Removability theorem for quasiconformal mappings. *Ann. Acad. Scient. Fenn.*, 449:1–8, 1969.

[Rie97] G. Riera. Basis of quadratic differentials for Riemann surfaces with automorphisms. *Glasgow Math. J.*, 1997.

[Roy71] H. Royden. Automorphisms and isometries of Teichmüller space. In *Advances in the Theory of Riemann Surfaces*, pages 369–384, 1971.

[RR88] G. Riera and R. Rodriguez. Uniformization of surfaces of genus two with automorphisms. *Math. Ann.*, 282(1):51–67, 1988.

[RS71] E. Reich and K. Strebel. Teichmüller mappings which keep the boundary pointwise fixed. *Ann. Math. Studies*, 66:365–367, 1971.

[RS87] B. Rodin and D. Sullivan. The convergence of circle packings to the Riemann mapping. *J. Diff. Geom.*, 26:349–360, 1987.

[Rue78] D. Ruelle. *Thermodynamic Formalism: The Mathematical Structures of Classical Equilibrium Statistical Mechanics.* Addison-Wesley, Reading, Mass., 1978.

[Sak86] K. Sakan. Necessary and sufficient conditions for extremality in certain classes of quasiconformal mappings. *J. Math. Kyoto Univ.*, 26(1):31–37, 1986.

[Ser81a] C. Series. The infinite word problem and limit sets in fuchsian groups. *Ergod. Th. & Dyn. Systs.*, 146:337–360, 1981.

[Ser81b] C. Series. Symbolic dynamics for goedesic flows. *Acta Math.*, 146:103–128, 1981.

[Shi59] H. Shimizu. On discontinuous groups operating on the product of upper half-planes. *Ann. Math.*, 77:33–77, 1959.

[Shi84] H. Shiga. On analytic and geometric properties of Teichmüller spaces. *J. Math. Kyoto Univ.*, 24:441–452, 1984.

[Sip82] P. Sipe. Roots of the canonical bundle of universal Teichmüller space and certain subgroups of the mapping class group. *Math. Ann.*, 260:67–92, 1982.

[Slo91] Z. Slodkowski. Holomorphic motions and polynomial hulls. *Proc. Amer. Math. Soc.*, 111:347–355, 1991.

[Spr57] G. Springer. *Introduction to Riemann Surfaces.* Addison-Wesley, Reading, Mass., 1957.

[SS54] M. Schiffer and D. C. Spencer. *Functionals of Finite Riemann Surfaces.* Princ. Univ. Press, Princeton, N. J., 1954.

[SS86] M. Seppälä and T. Sorvali. Parametrization of Teichmüller spaces by geodesic length functions. In *Holomorphic Functions and Moduli*, pages 267–284, 1986.

[ST86] D. P. Sullivan and W. P. Thurston. Extending holomorphic motions. *Acta Math.*, 157:243–257, 1986.

[Str74] K. Strebel. On the trajectory structure of quadratic differentials. *Ann. of Math. Studies*, 79:419–438, 1974.

[Str75] K. Strebel. On quadratic differentials and extremal quasiconformal mappings. In *Proceedings Int. Congr. Math.*, pages 223–227, Vancouver, 1975. Canadian Math Congress **2**.

[Str76] K. Strebel. On the existence of extremal Teichmüller mappings. *J. Anal. Math.*, 30:464–480, 1976.

[Str77] K. Strebel. On lifts of quasiconformal mappings. *J. Anal. Math.*, 31:191–203, 1977.

[Str78] K. Strebel. On quasiconformal mappings of open Riemann surfaces. *Comment. Math. Helv.*, 53:301–321, 1978.

[Str84] K. Strebel. *Quadratic Differentials.* Springer-Verlag, Berlin & New York, 1984.

[Str92] K. Strebel. On the geometry of quadratic differentials in the disk. *Results Math.*, 22:799–816, 1992.

[Str93] K. Strebel. The mapping by heights for quadratic differentials in the disk. *Ann. Acad. Sci. Fenn.*, 18:155–190, 1993.

[Sul78] D. Sullivan. On the ergodic theory at infinity of an arbitrary discrete group of hyperbolic motions. In *Ann. Math. Studies*, 1978.

[Sul85] D. Sullivan. Quasiconformal homeomorphisms and dynamics I: Solution of the Fatou-Julia problem on wandering domains. *Ann. of Math.*, 122:401–418, 1985.

[Sul93] D. Sullivan. Linking the universalities of Milnor-Thurston, Feigenbaum and Ahlfors-Bers. In *Topological Methods in Modern Mathematics*, pages 543–563. Publish or Perish, 1993.

[Tan92a] H. Tanigawa. Holomorphic families of geodesic discs in infinite dimensional Teichmüller spaces. *Nagoy Math. J.*, 127:117–128, 1992.

[Tan92b] M. Taniguchi. *An Introduction to Teichmüller Spaces.* Springer-Verlag, Tokyo, 1992.

[Tei38] O. Teichmüller. Untersuchungen über konforme und quasikonforme Abbildungen. *Deutsche Math.*, 3:621–678, 1938.

[Tei39a] O. Teichmüller. Eine Verschärfung des Dreikreisesatzes. *Deutsche Math.*, 4:16–22, 1939.

[Tei39b] O. Teichmüller. Extremale quasikonforme Abbildungen und quadratische Differentiale. *Abh. Preuss. Akad.*, 22:3–197, 1939.

[Tei41] O. Teichmüller. Über extremalprobleme der konformen Geometrie. *Deutsche Math.*, 6:50–77, 1941.

[Thu79] W. Thurston. *The Geometry and Topology of Three Manifolds.* Princeton Lecture notes, 1979.

[Thu85a] W. P. Thurston. Minimal stretch maps between hyperbolic surfaces. unpublished preprint, 1985.

[Thu85b] W. P. Thurston. Zippers and schlicht functions. unpublished preprint, 1985.

[Thu88] W. P. Thurston. On the geometry and dynamics of diffeomorphisms of surfaces. *Bull. Amer. Math. Soc.*, 19:417–431, 1988.

[Tsu59] M. Tsuji. *Potential Theory in Modern Function Theory Manifolds.* Maruzen, 1959.

[Tuk85] P. Tukia. Quasiconformal extension of quasisymmetric mappings compatible with a Möbius group. *Acta Math.*, 154:153–193, 1985.

[Tuk79] P. Tukia. On torsionless subgroups of infinitely generated Fuchsian groups. *Ann. Acad. Sci. Fenn.*, 4:203–205, 1978-79.

[TV81] P. Tukia and J. Väisälä. Lipschitz and quasiconformal approximation and extension. *Ann. Acad. Sci. Fenn.*, 6, 1981.

[Vee89] W. A. Veech. Teichmüller curves in moduli space, Eisenstein series and application to triangular billiards. *Invent. Math.*, 97:553–583, 1989.

[Vee90] W. A. Veech. Moduli spaces of quadratic differentials. *J. Anal. Math.*, 55:117–171, 1990.

[Vee92] W. A. Veech. The billiard in a regular polygon. *Geom. Funct. Anal.*, 2:341–379, 1992.

[Wey55] H. Weyl. *Die Idee der Riemannschen Fläche.* Teubner, Stuttgart, 3rd edition, 1955.

[Wol75] S. Wolpert. Non-completeness of the Weil-Petersson metric for Teichmüller space. *Pac. J. Math.*, 61:573–577, 1975.

[Wol77] S. Wolpert. The length as moduli. In Kra and Maskit, editors, *Riemann Surfaces and Related Topics*, volume 97, pages 515–517. Annals of Math. Studies, 1977.

[Wol79] S. Wolpert. The length spectrum as moduli for compact Riemann surfaces. *Ann. of Math.*, 109:323–351, 1979.

[Wol85] S. Wolpert. On the Weil-Petersson geometry of the moduli space of curves. *Amer. J. of Math.*, 107:969–997, 1985.

[Wol86] S. Wolpert. Thurston's Riemannian metric for Teichmüller space. *J. Diff. Geometry*, 23:143–174, 1986.

[YL96] Shen Yu-Liang. On the weak convexity of $Q(R)$. *Proc. Amer. Math. Soc.*, 124:1879–1882, 1996.

[Zhu90] K. Zhu. *Operator Theory in Function Spaces.* Marcel Dekker, New York, 1990.

[Zyg45] A. Zygmund. Smooth functions. *Duke Math. J.*, pages 47–76, 1945.

[Zyg59] A. Zygmund. *Trigonometric Series*, volume 1 and 2. Cambridge University Press, Cambridge, 2nd edition, 1959.

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