Quasiconformal Teichmüller Theory

Frederick P. Gardiner
Nikola Lakic

American Mathematical Society
Erratum


During the editing of the manuscript for this book, we inadvertently omitted reference to the following source for Chapter 9: V. Božin, N. Lakic, V. Marković, and M. Mateljević, Unique extremality, *J. Anal. Math.* 75 (1998), 299–338. We apologize for this omission.

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<table>
<thead>
<tr>
<th>Title</th>
<th>Author(s)</th>
<th>Year</th>
</tr>
</thead>
<tbody>
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</tr>
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</tr>
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</tr>
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</tr>
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</tr>
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</tr>
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</tr>
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</tr>
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Quasiconformal
Teichmüller Theory

Frederick P. Gardiner
Nikola Lakic

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ABSTRACT. This book provides background for applications of Teichmüller theory to dynamical systems and, in particular, to iteration of rational maps and conformal dynamics, to Kleinian groups and three-dimensional manifolds, to Fuchsian groups and Riemann surfaces, and to one-dimensional dynamics. The focus is on new developments in the theory in both finite and infinite dimensional cases. An exposition of the main theorems is given regardless of dimensionality by using techniques that apply generally. The hope is to provide background for more applications.

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To Joanie and Liki
# CONTENTS

PREFACE xiii

ACKNOWLEDGEMENTS xix

1. QUASICONFORMAL MAPPING 1
   1.1 From Conformal to Quasiconformal 1
   1.2 Linear Quasiconformality 2
   1.3 Analytic Quasiconformality 4
   1.4 Geometric Quasiconformality 7
   1.5 Solving the Beltrami Equation 10
   1.6 Holomorphic Motions 12
   1.7 Lebesgue Measure and Hausdorff Dimension 13

2. RIEMANN SURFACES 17
   2.1 Conformal Structure 18
   2.2 Examples and Uniformization 18
   2.3 Extremal Length 21
   2.4 Teichmüller Space 24
   2.5 Metrics of Constant Curvature 26
   2.6 Thrice-Punctured Spheres 30
   2.7 Fuchsian Groups 31
   2.8 Types of Elements of $PSL(2, \mathbb{R})$ 37
   2.9 Fundamental Domains 38
   2.10 Dimension of Quadratic Differentials 41

3. QUADRATIC DIFFERENTIALS, PART I 43
   3.1 Integrable Quadratic Differentials 46
   3.2 Poincaré Theta Series 49
   3.3 Predual Space 52
   3.4 Closed Sets 54
   3.5 The Teichmüller Infinitesimal Norm 58
   3.6 Cross-Ratio Norm on $Z(\Lambda)$ 59
   3.7 Approximation by Rational Functions 62
   3.8 Rational Quadratic Differentials 66
   3.9 The Equivalence Theorem 67

vii
### CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.10</td>
<td>Vanishing Elements of $Z(\Lambda)$</td>
<td>70</td>
</tr>
<tr>
<td>Appendix, Proof of the Equivalence Theorem</td>
<td></td>
<td>73</td>
</tr>
</tbody>
</table>

#### 4. QUADRATIC DIFFERENTIALS, PART II | 83 |
| 4.1 | Horizontal Trajectories | 84 |
| 4.2 | Geodesic Trajectories | 86 |
| 4.3 | The Minimal Norm Property | 89 |
| 4.4 | The Reich-Strebel Inequality | 93 |
| 4.5 | Surfaces of Infinite Analytic Type | 94 |
| 4.6 | The Main Inequality and Uniqueness | 95 |
| 4.7 | The Frame Mapping Theorem | 96 |
| 4.8 | Infinitesimal Frame Mapping | 99 |
| 4.9 | The Fundamental Inequalities | 101 |
| 4.10 | Teichmüller Contraction | 102 |
| 4.11 | Strebel Points | 104 |
| 4.12 | Teichmüller’s Infinitesimal Metric | 106 |

#### 5. TEICHMÜLLER EQUIVALENCE | 109 |
| 5.1 | Conformally Natural Extension | 109 |
| 5.2 | Quasiconformal Isotopies | 116 |
| 5.3 | Isotopies over Plane Domains | 119 |
| 5.4 | Proof of Lemma 2 | 121 |

#### 6. THE BERS EMBEDDING | 125 |
| 6.1 | Cross-Ratios and Schwarzian Derivatives | 126 |
| 6.2 | Schwarzian Distortion | 131 |
| 6.3 | The Bers Embedding | 132 |
| 6.4 | The Manifold Structure | 135 |
| 6.5 | The Infinitesimal Theory | 138 |
| 6.6 | Infinitesimally Trivial Beltrami Differentials | 140 |
| 6.7 | Hamilton-Krushkal Necessary Condition | 141 |

#### 7. KOBAYASHI’S METRIC ON TEICHMÜLLER SPACE | 145 |
| 7.1 | Kobayashi’s Metric | 145 |
| 7.2 | Teichmüller’s and Kobayashi’s Metrics | 147 |
| 7.3 | The Lifting Theorem | 149 |
| 7.4 | Uniqueness of Geodesics | 151 |

#### 8. ISOMORPHISMS AND AUTOMORPHISMS | 155 |
| 8.1 | Global to Local | 155 |
| 8.2 | Automorphisms of Teichmüller Discs | 157 |
| 8.3 | Rotational Transitivity | 159 |
| 8.4 | Adjointness Theorem | 161 |
| 8.5 | Isometries of Teichmüller Spaces | 162 |
| 8.6 | The Isometry Property | 163 |
## CONTENTS

8.7 Nonsmoothness of the Norm 164  
8.8 Isometry Theorem for Genus Zero 166  
8.9 Riemann Surfaces of Finite Genus 170  

9. TEICHMÜLLER UNIQUENESS 177  
9.1 Infinitesimal Main Inequality 178  
9.2 Constant Absolute Value 179  
9.3 Teichmüller Differentials 183  
9.4 Delta Inequalities 186  
9.5 Infinitesimal Teichmüller Uniqueness 189  
9.6 Unique Holomorphic Motions 191  

10. THE MAPPING CLASS GROUP 195  
10.1 $MCG$ for the Covering Group 196  
10.2 Moduli Sets 196  
10.3 The Length Spectrum 199  
10.4 Discreteness of Orbits 201  
10.5 Automorphism Groups 203  

11. JENKINS-STREBEL DIFFERENTIALS 207  
11.1 Admissible Systems 208  
11.2 An Extremal Problem 209  
11.3 Weyl’s Lemma 211  
11.4 Prescribing Heights 212  
11.5 Uniqueness 214  
11.6 Examples 215  
11.7 Differentials with Two Directions 219  

12. MEASURED FOLIATIONS 223  
12.1 Definition of a Measured Foliation 225  
12.2 Continuity of the Heights Mapping 228  
12.3 Convergence 229  
12.4 Intersection Numbers 230  
12.5 Projectivizations 233  
12.6 The Heights Mapping 234  
12.7 Variation of the Dirichlet Norm 236  

13. OBSTACLE PROBLEMS 241  
13.1 Extremal Problem for the Disc 242  
13.2 Extremal Problem for a Surface 244  
13.3 Smoothing the Contours 245  
13.4 Boundedness of the Norm 245  
13.5 Schiffer and Beltrami Variations 248  
13.6 Existence 250  
13.7 Uniqueness 251
13.8 Slit Mappings 253
13.9 Trajectories around the Obstacle 254

14. ASYMPTOTIC TEICHMÜLLER SPACE 257
14.1 The Infinitesimal Theory 258
14.2 Harmonic Beltrami Differentials 260
14.3 The Earle-Nag Reflection 263
14.4 Generalized Ahlfors-Weil Sections 266
14.5 Bers' \( \mathcal{L} \)-Operators 268
14.6 Inverse Operators 269
14.7 Manifold Structure 271
14.8 Inequalities for Boundary Dilatation 275
14.9 Contraction 276
14.10 Extremality in \( AT \) 281
14.11 Teichmüller’s Metric 282

15. ASYMPTOTICALLY EXTREMAL MAPS 285
15.1 Weighted Beltrami Differentials 286
15.2 Asymptotic Beltrami Differentials 288
15.3 Weighted Beltrami Coefficients 290
15.4 Asymptotic Beltrami Coefficients 296

16. UNIVERSAL TEICHMÜLLER SPACE 299
16.1 Quasisymmetric Homeomorphisms 299
16.2 Partial Topological Groups 301
16.3 Symmetric Homeomorphisms 302
16.4 Beurling-Ahlfors Extension 305
16.5 Welding 307
16.6 Zygmund Spaces 315
16.7 The Hilbert Transform 319
16.8 Global Coordinates for QS mod S 320

17. SUBSTANTIAL BOUNDARY POINTS 323
17.1 Local Dilatation 323
17.2 Unit Disc Case 325
17.3 Boundary Dilatation 327
17.4 Infinitesimally Substantial Points 329
17.5 Local Boundary Seminorms 330
17.6 Local Boundary Dilatation 331
17.7 Asymptotic Hamilton Sequences 333

18. EARTHQUAKE MAPPINGS 337
18.1 Finite Earthquakes 337
18.2 General Earthquakes 343
18.3 Simple Earthquakes and Bends 346
<table>
<thead>
<tr>
<th>CONTENTS</th>
<th>xi</th>
</tr>
</thead>
<tbody>
<tr>
<td>18.4 The Linear Theory</td>
<td>348</td>
</tr>
<tr>
<td>18.5 Infinitesimal Earthquakes</td>
<td>350</td>
</tr>
<tr>
<td><strong>Bibliography</strong></td>
<td>357</td>
</tr>
<tr>
<td><strong>Index</strong></td>
<td>369</td>
</tr>
</tbody>
</table>
PREFACE

The goal of this book is to provide background for applications of Teichmüller theory to dynamical systems and in particular to iteration of rational maps and conformal dynamics, to Kleinian groups and three-dimensional manifolds, to Fuchsian groups and Riemann surfaces, and to one-dimensional dynamics. Although Teichmüller theory is a theory of two-dimensional objects, it naturally impinges on three-dimensional topology through its relationship to Kleinian groups and on one-dimensional dynamics through the quasisymmetric boundary action of a quasiconformal self-map of a disc.

Teichmüller space is a universal classification space for complex structures on a surface of given quasiconformal type. It turns out that the space itself has a natural complex structure, and in applications operators on Teichmüller space are constructed that turn out to be holomorphic and contracting. There are, for example, Thurston's skinning map for the construction of hyperbolic structures on three-manifolds, Thurston's orbit finder for the construction of a rational map in the class of a critically finite map, and various proofs of rigidity for renormalization. None of these topics is dealt with in this book. Rather we focus on new developments in the theory in both the finite and infinite dimensional cases. Our program is to give an exposition of the main known theorems regardless of dimensionality, emphasizing techniques that apply generally, and hoping to provide background for more applications.

In order to give an overview, we need to give several definitions, and we begin with the definition of a Riemann surface. A Riemann surface is an oriented topological surface together with a system of local parameters or charts whose domains of definition cover the surface and that map these domains homeomorphically onto open sets in the complex plane. The charts have the property that for any two with overlapping domains of definition, \( \phi_1 \) and \( \phi_2 \), the transition map, \( w = \phi_2 \circ \phi_1^{-1}(z) \), mapping the plane to the plane, is holomorphic.

Holomorphic homeomorphisms of plane domains are precisely those that are differentiable in the complex sense at every point. Alternatively, they are orientation-preserving and conformal. A map \( f \) is conformal if, on an infinitesimal level, it takes any family of ellipses with given inclination and eccentricity centered at the origin in the tangent space at \( z \) to another family with the same eccentricity and possibly different inclination centered at the origin in the tangent space at \( f(z) \). This infinitesimal property has implications for the local behavior of the map; shapes but not necessarily sizes of tiny objects are nearly preserved by \( f \), and in the limit as the objects become more and more tiny, shapes are exactly preserved.

A quasiconformal map allows these shapes to be distorted, but the distortion measured as a ratio of shape measurements is uniformly bounded. Although qua-
siconformal maps need not be everywhere differentiable, the quasiconformality condition is easiest to describe when they are. Any $C^1$ diffeomorphism between plane domains on an infinitesimal level maps a small circle to a small ellipse. The local dilatation $K_z$ at a point $z$ is the ratio of the length of the major axis of this ellipse to the length of its minor axis. The global dilatation or simply the dilatation of the map is the supremum $K$ of the values $K_z$ for all $z$ in the domain. An orientation-preserving map for which $K$ is finite is by definition quasiconformal. In order to make this definition apply to maps that are not of class $C^1$, one needs a considerable amount of analysis and the theory of derivatives in the sense of distributions. A summary of this analysis is presented in Chapter 1. Here we mention three essential facts. First of all, a quasiconformal map $f$ is differentiable almost everywhere, and if $K(f) = 1$, $f$ is conformal. Secondly, if $f$ is $K$-quasiconformal, then it is Hölder continuous with Hölder exponent $1/K$, that is to say, $|f(z_1) - f(z_2)| \leq C|z_1 - z_2|^{1/K}$, where the constant $C$ depends on a normalization. Although no assumption is made in the definition about the distortion of size, this Hölder exponent yields some control. Thirdly, any quasiconformal map $f$ has a Beltrami coefficient $\mu(z) = f_z(z)/f_{\bar{z}}(z)$, and up to postcomposition by a conformal map $f$ is uniquely determined by $\mu$. The absolute value of $\mu(z)$ determines $K_z(f)$ by the formula $K_z(f) = \frac{|1 + \mu(z)|}{|1 - \mu(z)|}$. Moreover, $\mu$ can be an arbitrary complex-valued $L_\infty$-function with $||\mu||_\infty < 1$.

Since Riemann surfaces have conformal structure, it makes sense to speak of the conformality and quasiconformality of maps between Riemann surfaces. In particular, if $w = f(z)$, the constant $K_z(f)$ is defined independently of the selected charts $\phi_1$ defined in a neighborhood of $z$ in $R_1$ and $\phi_2$ defined in a neighborhood of $w$ in $R_2$. Therefore, we can define $K(f)$ to be the essential supremum over all $z$ in $R_1$ of the quantity $K_z(f)$.

Riemann surfaces $R_1$ and $R_2$ are called conformal if there is a conformal homeomorphism from $R_1$ onto $R_2$. In general, Riemann surfaces can be quasiconformal without being conformal, and for a given Riemann surface, its quasiconformal deformation theory is the study of the conformal equivalence classes of Riemann surfaces in the same quasiconformal class. The space of such conformally distinct surfaces in the quasiconformal class of $R$ is called moduli space $\mathcal{M}(R)$. If $R$ is compact, $\mathcal{M}(R)$ is a finite dimensional complex variety, but not a manifold.

The study of moduli is simplified by introducing an equivalence relation on a larger set of objects. The larger set of objects is the set of orientation-preserving quasiconformal maps from a fixed base Riemann surface onto a variable Riemann surface. The equivalence relation is easiest to describe if we assume the fixed surface is compact and without boundary. Two maps $f_0$ and $f_1$ from $R$ to $R_0$ and to $R_1$ are equivalent if there is a conformal map $c$ from $R_0$ onto $R_1$ such that $c \circ f_0$ is homotopic to $f_1$ by a homotopy $g_t$ consisting of quasiconformal maps. Since a quasiconformal map $f$ is, up to postcomposition by a conformal map, uniquely determined by its Beltrami coefficient $\mu = f_z/f_{\bar{z}}$, this equivalence relation also induces an equivalence relation on the space of complex-valued $L_\infty$-Beltrami differentials $\mu$ with $||\mu||_\infty < 1$.

The set $[f]$ of quasiconformal maps equivalent to a given map $f$ is called a Teichmüller equivalence class, and the space of all equivalence classes is the quasiconformal Teichmüller space $T(R)$. Because of basic properties of quasiconformal maps, within any equivalence class $[f]$ there is always a representative $f_0$.
such that \( K_0 = K(f_0) \) is minimal. If two equivalence classes \([f]\) and \([g]\) are given, 
the distance in Teichmüller’s metric between these two classes is \( \frac{1}{2} \log K_0 \), where 
\( K_0 \) is the minimal dilatation of a map in the class of \( f \circ g^{-1} \). In the compact case it is superfluous to modify the term Teichmüller space with the word quasiconformal because any two homeomorphic compact Riemann surfaces of the same genus are automatically quasiconformal (even diffeomorphic). Teichmüller showed that when \( R \) is compact \( T(R) \) is a complete metric space homeomorphic to a cell of real dimension \( 6g - 6 \) if the genus \( g \) of \( R \) is more than 1. When the 
genus is 1, it has real dimension 2.

The group of homotopy classes of quasiconformal self-maps of the base surface \( R \) is called the mapping class group, \( \text{MCG}(R) \). It acts naturally as a group of isometries on \( T(R) \) and identifies points that correspond to conformally equivalent Riemann surfaces. Thus, factoring Teichmüller space by the mapping class group yields moduli space; \( \mathcal{M}(R) \) is equal to \( T(R) \) factored by \( \text{MCG}(R) \).

A Riemann surface is of finite analytic type if it can be obtained by removing a finite number of points from a compact Riemann surface. We note that this property is quasiconformally invariant. That is, if \( f \) is a quasiconformal map from such a Riemann surface, \( R \), to another, \( f(R) \), then necessarily \( f(R) \) is also of finite analytic type. A surface of infinite analytic type can be obtained by taking a compact surface and deleting any infinite closed set. For example, an interesting infinite-type Riemann surface is the Riemann sphere minus the standard middle-thirds Cantor set in the unit interval. Any surface of infinite genus has infinite analytic type. It turns out that \( T(R) \) is infinite dimensional if and only if \( R \) is of infinite analytic type.

Defining the equivalence relation on quasiconformal maps \( f \) from a Riemann surface \( R \) of infinite analytic type to a variable Riemann surface \( f(R) \) is a technical matter. The essential idea is that two maps \( f_0 \) and \( f_1 \) are equivalent if there is a conformal map \( c : f_0(R) \to f_1(R) \) such that \( c \circ f_0 \) and \( f_1 \) are homotopic by a homotopy that pins down the boundary points of \( R \). The difficulty is in how to define boundary points and how to determine whether a quasiconformal homotopy extends to those points. One way to handle the difficulty is to use hyperbolic geometry and the uniformization theorem. One obtains the so-called ideal boundary for any Riemann surface whose universal covering group is Fuchsian. The question of how to define boundary points natural for quasiconformal deformations is an interesting problem.

We also study a new type of Teichmüller space that concerns only infinite-type Riemann surfaces and that explicitly deals with the asymptotic geometrical behavior of the Riemann surface at the boundary. The definition of asymptotic Teichmüller space \( AT(R) \) is the same as that of ordinary Teichmüller space except in the definition of equivalence classes the word conformal is replaced by asymptotically conformal. A class \([f]\) of maps in \( T(R) \) is asymptotically conformal if one can make \( K_\pm(f_0) \) arbitrarily close to 1 for \( z \) in \( R \setminus C \) by choosing a suitable representative \( f_0 \) of \([f]\) and by choosing a sufficiently large compact subset \( C \) of \( R \). On surfaces of finite analytic type all Teichmüller classes of quasiconformal maps are asymptotically conformal, and thus in this case \( AT(R) \) consists of just one point. In all other cases \( AT(R) \) is infinite dimensional. Many of the results concerning asymptotic Teichmüller space presented in the text represent our joint work with Cliff Earle.

Study of \( AT(R) \) automatically leads to the notion of the boundary dilatation \( H([f]) \) of a class \([f]\) in \( T(R) \). For any compact set \( C \subset R \) one looks for a
representative \( f_0 \) of the class \([f]\) such that the essential supremum of \( K_z(f_0) \) for \( z \) in \( R \setminus C \) is as small as possible. The boundary dilatation \( H([f]) \) of \([f]\) is the infimum of all of these numbers over all compact subsets \( C \) of \( R \). It can happen that \( H([f]) = K_0([f]) \). The Teichmüller metric on \( AT(R) \) is given by boundary dilatation; the distance between two classes in \( AT(R) \) represented by maps \( f \) and \( g \) is \( \frac{1}{2} \log H((f \circ g^{-1})) \).

\( T(R) \) has a natural basepoint, which is the equivalence class of the identity map on \( R \), and the tangent space to \( T(R) \) at this basepoint is isomorphic to the Banach dual space of the complex Banach space \( A(R) \), the space of integrable holomorphic quadratic differentials \( \varphi(z)dz^2 \) on \( R \). The norm of \( \varphi \) in \( A(R) \) is given by \( \|\varphi\| = \int_R |\varphi(z)|dxdy \).

The pairing \( (\mu, \varphi) = \int_R \mu(z)\varphi(z)\ dxdy \) induces a second notion of equivalence of Beltrami coefficients. Two Beltrami coefficients \( \mu \) and \( \nu \) are called infinitesimally equivalent if \( (\mu - \nu, \varphi) = 0 \) for all \( \varphi \) in \( A(R) \) and \( \mu \) is infinitesimally trivial if \( \mu \) is infinitesimally equivalent to 0. When in addition \( ||\mu||_\infty \) and \( ||\nu||_\infty \) are both less than 1, \( \mu \) and \( \nu \) are globally equivalent if the quasiconformal maps \( f^\mu \) and \( f^\nu \) with Beltrami coefficients \( \mu \) and \( \nu \) represent the same point in \( T(R) \). We say \( \mu \) is globally trivial (or just trivial) if \( f^\mu \) is equivalent to the identity map. Finally, \( \mu \) is called extremal in its global class if \( f^\mu \) has minimal dilatation in its Teichmüller class, and similarly, \( \mu \) is extremal in its infinitesimal class if \( ||\mu||_\infty \) takes the smallest possible value in its infinitesimal class. We will see that the interplay of the infinitesimal and global equivalence relations on Beltrami coefficients is an essential element of Teichmüller theory.

Some of the main results we prove are:

- For any Riemann surface \( R \), \( T(R) \) and \( AT(R) \) are complex manifolds modeled on Banach spaces.
- \( T(R) \) and \( AT(R) \) are complete with respect to Teichmüller’s metric; \( AT(R) \) is a quotient space of \( T(R) \) with quotient metric induced by the natural projection from \( T(R) \) to \( AT(R) \), and this metric is given by boundary dilatation.
- These metrics have infinitesimal forms and are equal to the integrals of their infinitesimal forms.
- Teichmüller’s metric on \( T(R) \) is equal to Kobayashi’s metric, a metric defined purely in terms of the family of holomorphic functions from the unit disc into \( T(R) \).
- When \( R \) has finite genus with either finite or infinite analytic type, every holomorphic automorphism of \( T(R) \) is induced by a quasiconformal self-map of \( R \). Moreover, except for a few low-dimensional Teichmüller spaces, all occurring in genus 2 and lower, \( MCG(R) \) acts as the full automorphism group of \( T(R) \). For compact surfaces this result was proved by Royden.
- A quasiconformal map \( f \) has minimal dilatation in its Teichmüller class if and only if its Beltrami coefficient \( \mu \) is extremal in its infinitesimal class. This result is called the Hamilton-Krushkal, Reich-Strebel necessary-and-sufficient condition for extremality.
- A quasiconformal map \( f \) is nearly extremal in its Teichmüller class if and only if its Beltrami coefficient \( \mu \) is nearly extremal in its infinitesimal class.
This result is called Teichmüller contraction. There is also a version of Teichmüller contraction for $AT(R)$.

- A quasiconformal map $f$ is the uniquely extremal representative in its Teichmüller class if and only if its Beltrami coefficient $\mu$ is uniquely extremal in its infinitesimal class.

- When $R$ is the Riemann sphere $\mathbb{C}$ minus a closed set, there is an alternative notion of equivalence on Beltrami coefficients. The resulting Teichmüller space is a complex manifold universal for holomorphic motions of the closed set.

- When $R$ is a plane domain, the boundary dilatation of any class $[f]$ in $T(R)$ is realized at some point of the set-theoretic boundary of $R$. When $R$ is the unit disc, this result is called Fehlmann’s theorem on the existence of substantial points.

The foundation for these results is the study of the space $A(R)$ of integrable holomorphic quadratic differentials. Both the space $A(R)$ and each one of its elements play a geometric rôle. The geometric rôle of the space enters through the Teichmüller infinitesimal Banach norm on $A(R)$ and the main variational lemma that relates infinitesimally trivial Beltrami coefficients to trivial ones. The lemma says $\mu$ is infinitesimally trivial if, and only if, there is a curve $\mu_t$ of trivial Beltrami coefficients with the property that $||\mu_t - t\mu||_\infty / t$ approaches zero as $t \to 0$.

The geometric rôle of each element of $A(R)$ enters through the length-area method, also called Grötzsch’s argument. This method shows how any holomorphic quadratic differential defined on a subdomain of $R$ can be used to measure shape distortion of a quasiconformal map. The argument applied to the horizontal trajectories of a globally defined integrable holomorphic quadratic differential yields the main inequality of Reich and Strebel. Eventually, one learns that elements of $A(R)$ yield well-defined functionals on $T(R)$ and measure shape distortion of a Teichmüller class.

In later chapters we explore a variety of topics including measured foliations, heights mappings, a generalization of classical slit mapping theorems, and a construction of earthquakes based in the idea of applying a limiting process to finite earthquakes.

Much of the material in Chapters 1 and 2 on quasiconformal mapping and Riemann surfaces is presented without proof. Where results in these chapters are not fully proved, it is hoped the reader will agree to work with the theorems as stated and proceed directly to the subject at hand.

At the end of each chapter we provide exercises and sometimes open problems. There is a long list of results known for finite dimensional Teichmüller spaces but not known for infinite dimensional cases, and another list known for infinite dimensional cases but not for asymptotic Teichmüller spaces. Many research problems are posed by this observation.

The purpose of the bibliography is to provide pointers to persons wishing to pursue related subjects further. Although quite long, it is by no means complete. It is meant only to steer persons to the large and expanding body of literature in the field.

Frederick P. Gardiner and Nikola Lakic
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BIBLIOGRAPHY


BIBLIOGRAPHY


BIBLIOGRAPHY


BIBLIOGRAPHY


BIBLIOGRAPHY


BIBLIOGRAPHY


INDEX

Absolute trace, 197
Absolutely continuous, 6
Adjacent points, 337
Adjointness theorem, 161
Admissible system, 207, 208
Agard’s formula, 31
Ahlfors-Weill lemma, 134
generalized, 266
Ahlfors-Weill sections, 134, 266
Allowable metrics, 21
Analytic, 1
Analytic type
   finite, 177
   infinite, 94, 177
Anharmonic ratios, 59, 126
Area element, 86
Asymptotic Teichmüller space, 257
Asymptotically extremal map, 285
Atomic decomposition, 69
Automorphism groups, 203

Banach-Alaouglu theorem, 53
Beltrami coefficient, 3, 5
Beltrami differential, 24
Beltrami equation, 5, 10, 15
Bergman kernel, 50
Bers’ L-operator, 268
Bers’ approximation theorem, 63, 166
Bers’ embedding, 125, 132
Best approximating Möbius transformations, 130
Border triangle, 338
   midpoint of, 338
Boundary dilatation, 180, 275, 327
   inequalities for, 277
   infinitesimal, 54, 82, 99, 260, 324
Boundary seminorms, 330

Cantor set, 20, 108
Cantor staircase, 16
Catalan number, 338

Characteristic topological subgroup, 302
Chimney domain, 177
Circle equivalence, 117
Classification theorem, 162
Cocycles on mappings, 126
Comb domain, 124
Comparison map, 339
Comparison principle, 23
Compatible system, 209
Compatible system of annuli, 208, 209
Complex dilatation, 3
Complex partial derivatives, 5, 14
Composition laws, 23
Conformal, 1
Conformal equivalence, 18
Conformal structure, 18
Conformally natural extension, 109, 124
Conformally natural reflection, 263
Constant absolute value, 179
Constant curvature, 26
Contraction, 276
Convergence principles, 11
Convexity
   weak uniform, 184
Core curve, 207
Critical point, 85
Cross-ratio, 126
Cross-ratio distortion, 127
Cross-ratio norm, 59
Curvature, 26

Decays geometrically, 217
Degenerating Hamilton sequence, 97, 100
Degenerating sequences, 47
Dehn twist, 204
Delta inequality, 186
Differentiable, 6
Dilatation, 3
Dirichlet principle, 92, 241
Disc template property, 309
Discontinuous, 32
Discontinuous action, 195
Discrete group, 34
Discrete orbits, 33
Distortion, 3, 127
  of cross ratio, 127
  vanishing cross-ratio, 71
Distortion of shape, 7
Distributional derivative, 5
Douady-Earle extension, 110
Earle-Nag reflection, 263
Earthquake measure
  finite, 340
Earthquakes
  finite, 337
  general, 343
  infinitesimal, 350
Eccentricity, 2
Elliptic, 37
Elliptic point, 36
Equivalence theorem, 67
  proof of, 73
Essential supremum, 53
Exceptional type, 159
Extension
  Beurling-Ahlfors, 305
  Douady-Earle, 110
  Slodkowski, 13
  Sullivan-Thurston, 67, 329
Extremal length, 21
Extremal length problems
  sufficient, 26
  well-posed, 26
Extremal length property, 308

Finite analytic type, 177
Finite earthquake theorem, 340
Finite lamination, 337
Finite left earthquake, 339, 340
Finite right earthquake, 340
Frame mapping theorem, 96
Frontier equivalence, 119
Fuchsian groups, 19, 31
Fundamental domain, 38
  Dirichlet, 38
  Ford, 39

Fundamental inequalities, 102
Gap theorem, 171
Gaps
  complete, 343
  of a lamination, 343
Generalized derivative, 5
Geodesic lamination, 343
Geodesics
  uniqueness of, 151
Geometric isomorphism, 155
Grötzsch’s argument, 8, 22
Group of isometries, 33

Hamilton sequence, 97
  degenerating, 97
Hamilton sequences
  asymptotic, 333
Hamilton-Krushkal necessary condition, 141
Hamilton-Krushkal theorem, 98
Harmonic Beltrami differential, 135, 140
Harmonic Beltrami differentials, 260
Harmonic measure, 121
Hausdorff dimension, 13
Hilbert transform, 319
Holomorphic, 1
Holomorphic motion, 12, 191
Horizontal trajectory, 84
Hyperbolic, 37
Hyperbolic metric, 4, 27
Hyperbolic surfaces, 19

Identity property, 35
Inequivalent Teichmüller spaces, 176
Infinitesimal boundary dilatation, 54, 99
Infinitesimal cross-ratio norm, 60
Infinitesimal frame mapping, 99
Infinitesimal local boundary dilatation, 324
Infinitesimal main inequality, 178
Infinitesimal metric, 106
Infinitesimally trivial Beltrami differential, 140
Isometry property, 159
Isomorphism theorem, 162
Isotopy equivalence, 117
Isotropy group, 33
INDEX

Jenkins-Strebel differential, 207
direction of, 208
two directions, 219

Kobayashi’s metric, 145

Lamination
finite, 337
goedesic, 343
Lebesgue measure, 13
Left earthquake, 340
Length distortion, 127
Length spectrum, 199
Length-area argument, 22
Lifting theorem, 149, 150
Local boundary dilatation, 331
Local dilatation, 323
Locally integrable, 5

Main inequality, 95
infinitiesimal, 178
Manifold structure, 135
Mapping class group, 195, 196
of a Fuchsian group, 196
of a surface, 196
Mapping theorem, 10, 15
Maximal shrinking, 3
Maximal stretching, 3
Measurable Riemann mapping theorem, 10
Minimal norm property, 89, 90
Moduli set, 196, 197
Moduli space, 18

Natural parameter, 85
Neighborhoods
ample, 75
uniform, 75
Noether gap theorem, 171
Non-Euclidean metric, 27
Nonlinearity, 128
Nonsmoothness of the norm, 164

Obstacle problems, 241

Parabolic, 37
Partial topological group, 301
Picard’s big theorem, 42
Picard’s little theorem, 41
Poincaré density
estimation of, 81

Poincaré metric, 27, 28
Poincaré theta series, 49
Predual space, 52
Properly discontinuous, 32, 195

Quadratic differential
initial, 96
terminal, 96
Quadratic differentials, 46
dimension of, 41
rational, 66
Quasiconformal, 1, 2
analytic definition, 5
generic definition, 9
Quasiconformal isotopies, 116
Quasisymmetric homeomorphism, 299
Quasisymmetry, 300
Quasisymmetry constant, 299

Ratio distortion, 127
Rational functions, 62
Reflection
conformally natural, 263
Reich-Strebel functional, 99
Reich-Strebel inequality, 93
Reproducing formula, 138, 261
Reverse triangle inequality, 307
Riemann surface, 17
Right earthquake, 340
Ring domain, 207
Rotational transitivity, 159, 220

Schiffer variation, 248
Schwarz-Pick lemma, 29
Schwarzian derivative, 126, 129
Schwarzian distortion, 131
Simple earthquakes, 346
Slit mapping theorem, 244
Slit mappings, 253
Slodkowski extension, 13
Sphere equivalence, 120
Stratum
of a finite earthquake, 340
of a general earthquake, 343
Strebel functional, 104
Strebel point, 98, 104, 179
Stretch map, 3
Strong reverse triangle property, 308
Substantial boundary points, 323
infinitesimal, 329
Sufficient family, 26
Symmetric homeomorphism, 143, 303

Teichmüller’s infinitesimal norm, 58
Teichmüller Beltrami coefficient, 177
Teichmüller contraction, 102
Teichmüller disc, 156
Teichmüller equivalence, 25, 109, 116
Teichmüller form, 95
Teichmüller space, 25
definition of, 116
Teichmüller’s existence theorem, 98
Teichmüller’s infinitesimal metric, 106
Teichmüller’s metric, 25, 282
Teichmüller’s uniqueness theorem, 96
Thick-thin decomposition, 77
Thrice-punctured spheres, 30
Topological group, 301
Trace identity, 198
Trajectories, 84
Trajectory
horizontal, 85
vertical, 85
Transportation cost, 241, 242
Tree of triangles, 339
Triangles of a finite lamination, 337
Trivial Beltrami coefficient, 119

Uniformization, 19
Uniquely extremal, 179, 182
Uniqueness of geodesics, 151
Universal Teichmüller space, 299

Vanishes at infinity, 259
Vanishing ratio distortion, 302
Vector field, 44, 45, 56
extension of, 59
Zygmund bounded, 45

Weak uniform convexity, 184
Weierstrass ω-function, 7, 205
Weighted Beltrami coefficients, 290
Weighted Beltrami differentials, 286
Welding, 307
Welding homeomorphism, 307
Well-posed family, 26
Weyl’s lemma, 211
Wings, 95

Zygmund spaces, 45, 315