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(Continued in the back of this publication)
KP or mKP

Noncommutative Mathematics of Lagrangian, Hamiltonian, and Integrable Systems
KP or mKP

Noncommutative Mathematics of Lagrangian, Hamiltonian, and Integrable Systems

Boris A. Kupershmidt
ABSTRACT. The book develops noncommutative theories of: dynamical systems in general and integrable systems in particular; Hamiltonian formalism; and variational calculus, both in continuous spaces and discrete spaces. The book is self-contained, and everything in it is developed from scratch. Motivations for the new notions are provided, each time after an extensive analysis of many different concrete models.
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PREFACE

In our youth, we lose our commutative innocence when we run into quaternions in Mathematics or operators affiliated to physical observables in Physics. After that, the next logical encounter should have been in the realm of noncommutative differential equations and field theories, but these shadowy kingdoms had so far refused to reveal themselves clearly. They are explored and mapped out in this treatise.

In retrospect, the chief underlying reason for the delayed discovery was, I believe, the absence of a hook to which one could attach a sufficiently large number of natural questions to lure the answers out. Previous attempts include the snares of topology (M. Kontsevich [Kon 1993], I. Gel’fand and M. Smirnov [GSm 1994]) and Quaternionic Quantum Mechanics (S. Adler [Adl 1995]). I’ve chosen the lure of integrable systems.

The modern theory of integrable systems originates with the discovery, in 1967, of the soliton solutions of the Korteveg-de Vries (KdV) equation; exactly 100 years after the great invention of barbed wire. This theory has blossomed by now into a well-developed body of techniques, tricks, methods, results, rigorous and heuristic principles, rules of thumb, empirical observations, abandoned and forgotten approaches, unsolved puzzles, and hopeless problems, — in short, a versatile machine ideally suited for the purpose of coming up with the right questions whatever the environment. The first 3 Chapters of the book develop answers to such questions, — to wit, what happens with the properties of familiar differential equations when the dependent variables become quaterion-, matrix-, etc. valued? — until, in §3.11, we are forced to discover the noncommutative Hamiltonian formalism as the only way to prove an overwhelmingly impenetrable identity, by restoring to operations with notions instead of with multi-index entities; similar to the classical replacement of coordinate tensor calculations by the notions of geometry on manifolds. Most of the theory of integrable systems can be extracted from the properties of the Kadomtsev-Petviashvili (KP) and modified KP (mKP) hierarchies, — whence the title of the book. These two hierarchies, together with their specialization, variations, and generalizations, are the main actors in the unfolding noncommutative play. In particular, the Chapters 1-3 in Part A on continuous space-time, lead, through systematic examination, to the noncommutative machine which, in a first approach to truth, can be thought of as a device to make manageable some terrifyingly long and cumbersome coordinate calculations, and turn others, of acceptable length, effort, and boringness, into short and elegant trivialities. With the hindsight available, I give in §4.1 a simple few-line argument of how the right definition could have been plausibly discovered (but wasn’t), — this is for the reader not interested in why but only in how. However, the noncommutative formalism, routine and easy to handle once you get used to it, is not intuitively clear at the first glance if presented as a fait accompli, without an actual account of developments leading to it.

Asked to define jazz for a nonaficionado, Louis Armstrong replied: “Man, if you’ve got to ask you’ll never know.” He might have been describing the style of current mathematical literature. This book is written in the old-fashioned 18th-
century tradition of vigorously avoiding any practice of sadomathematics but instead providing all necessary motivations and reproducing faithfully the actual process by which the results presented have been discovered. I half-way followed Voltaire’s admonition that the best way to be boring is to leave nothing out. The text is self-contained (although it hasn’t been set to music yet), with no referrals to outside sources required; but a lot of parallel material has been moved out into Exercises, of which the book has a large number, – mostly of “verify the preceding calculation/extract the motivation for the forthcoming construction”—type. All Exercises are small trivial potatoes; and all, with the single exception of Exercise 19.11.41, are one- or two-line affairs.

To give a sun-eye view of the Noncommutative Universe, here are two representative samples. First, the noncommutative KdV equation

\[ u_t = (3u^2 + u_{xx})_x \]

looks similar to its commutative counterpart, while for the modified KdV (mKdV) equation we now have two different noncommutative versions:

\[ v_t = -3(v_x v^2 + v^2 v_x) + v_{xxx}, \]

\[ w_t = -6w v_x w + 3[w, w_{xx}] + w_{xxx}; \]

and there is still more variety in the Potential form of these equations; – this is the subject of §3.9.

Second, while most (though not all) of the familiar commutative results have their noncommutative counterparts (this is an experimental fact with a so-far unknown general principle behind it), there exist purely noncommutative phenomena without any commutative predecessors. One such beauty is what I call the Kontsevich-type formula: If \( u_i \)'s are the field variables and \( H \) is a Hamiltonian then

\[ \sum_i [u_i, \frac{\delta H}{\delta u_i}] \text{ is a “divergence”;} \]

in particular, the sum is zero in Classical Mechanics, when the herd of independent variables disappears.

A word on proofs. Interestingly, it turns out to be in general easier to prove a noncommutative formula than its commutative counterpart, chiefly because noncommutative monomials of distinctly different nature coalesce into indistinguishable clusters in the commutative version. For example, most of the results in Part C on discrete space-time dynamics are new, having been left open in the commutative case as too difficult. The noncommutative point of view has turned out to be a potent tool, precluding outright a multitude of otherwise plausible directions and steering all possible attempts into narrow channels.

Readers of varying and dissimilar interests will find some pieces to satisfy their thirst for knowledge, I hope: Applied physicists and engineers interested in matrix-valued differential or difference equations, can safely assume that every dependent variable is a matrix. Those
who are curious about Poisson brackets but not about noncommutative adventures, will be pleased to meet, in Appendices A1, A2, and A4 respectively, blueprints of how to complexify a Hamiltonian system, make an intelligent asymptotic expansion out of it, and relate a pair of such systems in a proper Poisson correspondence (not necessarily of one-to-one type.) I'm aware that some readers may feel menaced by the algebraic terrorism and may even have, as children, liked Richelier more than D'Artagnan, — but instead of shedding crocodile tears for the under-privileged, I've written §A4.1 providing ample geometric motivations for algebraic definitions and constructions;

Theoretical physicists, interested in noncommutativity but not in integrable systems, can satisfy their inquisitiveness by reading Chapters 4,5 or 10,11 — depending upon whether they deal in continuous or discrete currency;

Part C, on discrete space-time, is especially recommended to those who occasionally entertain at least a shadow of suspicion that the standard differencing methods of numerical analysis and computational physics are too crude and simplistic to reflect properly the nontrivial properties of the continuous models;

Model builders, symmetry calculators, singularity analysts, etc., will find most of the scalar equations and hierarchies scattered throughout the text gathered under one roof in Appendix 8;

Excitement-challenged graduate students will similarly find all open problems and conjectures from the book collected in Appendix 9;

Connoisseurs of Classical Mechanics will find a noncommutative version of it in §4.6, while specialists in Fluid Dynamics and Plasma Physics will find various noncommutative hydrodynamical systems in Chapters 14-19, and noncommutative Clebsch representations in §6.4.

The text has turned out to be a bit bulky, being in effect 4 separate, but related, books. As a space-saving device, I routinely use symbols ⇒ for: therefore, thus, hence, whence, etc; ⇐ for: which is, which is equivalent, in other words, etc; ⊥ for: Q.E.D., end of proof. Special situations and less majestic topics have been relegated to the derrière-guard as Appendices.

It will be clear to those who have gone through the text that the noncommutative surface has been barely scratched; e.g., not one of the Hamiltonians appearing in the text is of mechanical type $H = \mathbf{p}^2/2 + V(q)$. I hope that readers will find that the treatment can stand some improvement and proceed to develop the subject further.

I believe the book doesn't have mistakes, or at any rate serious ones; but I am far less optimistic about the typos, and tender my apology in advance. I would be grateful to any reader for bringing misprints to my attention, as well as for comments and suggestions.

*   *   *

In 1943, after a two-month stand-off, the German Nazi SS called up military units with armored vehicles to finish off the Warsaw Jews on April 19th, 1943.
In 1993, after a two-month stand-off, the BATF called up FBI units with armored vehicles to finish off the Branch Davidians on April 19th, 1993.

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* The views expressed here are those of the author solely, and are not endorsed by the American Mathematical Society.
ACKNOWLEDGEMENT

The publication arm of the American Mathematical Society deserves gratitude of present and future readers everywhere for having come to the view that ethical far-sighted publishers should leave to the unscrupulous ones the business of appropriating copyright from the authors of creative works.
PART D

APPENDICES

I have loved justice and hated iniquity, therefore I die in exile.

Pope Gregory VII (1085)
APPENDIX A1. COMPLEXIFICATION OF HAMILTONIAN SYSTEMS

All our wants, beyond those which a very moderate income will supply, are purely imaginary.  
Henry St. John Bollinbroke, in a letter to Swift (March 17, 1719)

In this Appendix it is shown that a complexification of a Hamiltonian system is a biHamiltonian system.

Given a Hamiltonian system

\[ \partial_t(q) = B \left( \frac{\delta H}{\delta q} \right), \]  

(A1.1)

suppose we want to complexify it. This means that we substitute

\[ q = u + iv, \quad i^2 = -1, \]  

(A1.2)

into each side of the equality (A1.1) and then equate the real and imaginary parts in the resulting equation. What kind of system will we obtain?

To get a handle on the problem, let us consider a simple $KdV$–like system

\[ \partial_t(q) = \partial(q^{\Gamma} + q^{(2)}), \quad \Gamma \in \mathbb{N}, \]  

(A1.3)

\[ q^{\Gamma} + q^{(2)} = \frac{\delta}{\delta q}(H), \quad H = \frac{1}{\Gamma + 1} q^{\Gamma+1} - \frac{1}{2} q^{(1)2}. \]  

(A1.4)

Set

\[ (u + iv)^\Gamma = \varphi_\Gamma + i\psi_\Gamma, \quad \Gamma \in \mathbb{Z}_+, \quad \varphi_\Gamma, \psi_\Gamma \in \mathbb{Z} < u, v >. \]  

(A1.5)

In this notation, substituting

\[ q = u + iv \]  

(A1.6)

into equation (A1.3), we find

\[ \partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \partial \begin{pmatrix} \varphi_\Gamma + u^{(2)} \\ \psi_\Gamma + v^{(2)} \end{pmatrix}. \]  

(A1.7)

On the other hand, formulae (A1.4,5) imply

\[ H = Re(H) + iIm(H), \]  

(A1.8)
\[ \text{Re}(H) = \frac{1}{\Gamma + 1} \varphi_{\Gamma+1} - \frac{1}{2}(u^{(1)})^2 - v^{(1)} \], \quad (A1.9) \\
\[ \text{Im}(H) \approx \frac{1}{\Gamma + 1} \psi_{\Gamma+1} - u^{(1)}v^{(1)}, \quad (A1.10) \]

\[ \frac{\delta \varphi_{\Gamma}}{\delta u} = \Gamma \varphi_{\Gamma-1}, \quad \frac{\delta \varphi_{\Gamma}}{\delta v} = -\Gamma \psi_{\Gamma-1}, \quad (A1.11) \]
\[ \frac{\delta \psi_{\Gamma}}{\delta u} = \Gamma \psi_{\Gamma-1}, \quad \frac{\delta \psi_{\Gamma}}{\delta v} = \Gamma \varphi_{\Gamma-1}, \quad (A1.12) \]

whence

\[ \delta (\text{Re}(H)) = \begin{pmatrix} \varphi_{\Gamma} + u^{(2)} \\ -\psi_{\Gamma} - v^{(2)} \end{pmatrix}, \quad (A1.13) \]

\[ \delta (\text{Im}(H)) = \begin{pmatrix} \psi_{\Gamma} + u^{(2)} \\ \varphi_{\Gamma} + u^{(2)} \end{pmatrix}. \quad (A1.14) \]

Therefore, the complexified motion equations (A1.7) can be recast into \textit{two different Hamiltonian forms}:

\[ \partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \partial & 0 \\ 0 & -\partial \end{pmatrix} \begin{pmatrix} \delta/\delta u \\ \delta/\delta v \end{pmatrix} (\text{Re}(H)) = \]
\[ = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix} \begin{pmatrix} \delta/\delta u \\ \delta/\delta v \end{pmatrix} (\text{Im}(H)). \quad (A1.15a) \]

\[ (A1.15b) \]

The effects observed on this simple example are general: the complexification of a Hamiltonian system is a biHamiltonian system. To see how this comes about, let

\[ B|q=\textbf{u}+\textbf{v} = B_1 + iB_2, \quad (A1.16a) \]
\[ \frac{\delta H}{\delta \textbf{q}} |_{q=\textbf{u}+i\textbf{v} = \textbf{F}_1 + i\textbf{F}_2}. \quad (A1.16b) \]

The complexification of the motion equations (A1.1) is then:

\[ \partial_t (\textbf{u} + i\textbf{v}) = (B_1 + iB_2)(\textbf{F}_1 + i\textbf{F}_2) = [B_1(\textbf{F}_1) - B_2(\textbf{F}_2)] + i[B_1(\textbf{F}_2) + B_2(\textbf{F}_1)] \Rightarrow \]
\[ \partial_t \begin{pmatrix} \textbf{u} \\ \textbf{v} \end{pmatrix} = \begin{pmatrix} B_1 & B_2 \\ B_2 & -B_1 \end{pmatrix} \begin{pmatrix} \textbf{F}_1 \\ \textbf{F}_2 \end{pmatrix}. \quad (A1.17) \]

Substituting \( q = \textbf{u} + i\textbf{v} \) into the defining variational relation (10.1.60), in the form

\[ d(H) \approx dq \frac{\delta H}{\delta q}, \quad (A1.18) \]

we get

\[ d(\text{Re}(H)) + id(\text{Im}(H)) \approx (d\textbf{u} + id\textbf{v})^t(\textbf{F}_1 + i\textbf{F}_2) = \]
\[ = (d\textbf{u}^t\textbf{F}_1 - d\textbf{v}^t\textbf{F}_2) + i(d\textbf{u}^t\textbf{F}_2 + d\textbf{v}^t\textbf{F}_1), \quad (A1.19) \]
whence

\[ d(Re(H)) \approx du^t F_1 - dv^t F_2, \]  
\[ d(Im(H)) \approx du^t F_2 + dv^t F_1, \] 

so that

\[
\left( \frac{\delta}{\delta u} \right) (Re(H)) = \begin{pmatrix} F_1 \\ -F_2 \end{pmatrix} = \begin{pmatrix} Re \end{pmatrix} \left( \frac{\delta H}{\delta q} \right),
\]

\[
\left( \frac{\delta}{\delta v} \right) (Im(H)) = \begin{pmatrix} F_2 \\ F_1 \end{pmatrix} = \begin{pmatrix} Im \end{pmatrix} \left( \frac{\delta H}{\delta q} \right). \]

Substituting formulae (A1.21) into the complexified motion equations (A1.17), we find the biHamiltonian representation

\[
\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} B_1 & B_2 \\ B_2 & -B_1 \end{pmatrix} \left( \frac{\delta}{\delta u} \right) (Re(H)) = \partial_t (q) = X(\{q\}), \tag{A1.22a}
\]

\[
= \begin{pmatrix} -B_2 & B_1 \\ B_1 & B_2 \end{pmatrix} \left( \frac{\delta}{\delta v} \right) (Im(H)). \tag{A1.22b}
\]

It remains to show that formulae (A1.22) describe Hamiltonian systems. This can be done by a straightforward calculation, but it would be not illuminating. A much better way, and easily generalizable as well, consists in working in coordinates

\[ q = u + iv, \quad p = \bar{q} = u - iv \]  

rather than \( u, v \). In these coordinates, a dynamical system

\[ \partial_t(q) = X(\{q\}) \]  

becomes, when complexified, a pair

\[ \partial_t(q) = X(\{q\}), \quad \partial_t(q) = X(\{\bar{q}\}). \]  

(A1.24')

A Hamiltonian system (A1.1) becomes, when complexified, a biHamiltonian system

\[ \partial_t \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} B_q & 0 \\ 0 & \pm B_p \end{pmatrix} \left( \frac{\delta}{\delta q} \right) (H_q \pm H_p), \]  

where

\[ B(\ldots) = B|_q=\ldots, \quad H(\ldots) = H|_q=\ldots. \]  

Let us verify that the biHamiltonian formulae (A1.22) and (A1.25) are equivalent. From formulae (A1.23) we get

\[ B_q = B_1 + iB_2, \quad B_p = B_1 - iB_2, \]  
\[ H_q = Re(H) + iIm(H), \quad H_p = Re(H) - iIm(H), \]  
\[ H_q + H_p = 2Re(H), \quad H_q - H_p = 2iIm(H), \]  
\[ u = (q + p)/2, \quad v = (q - p)/2i, \]  

(A1.27)  

(A1.28)  

(A1.29)  

(A1.30)
\[ J = \frac{D(u,v)}{D(q,p)} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ i1 & -i1 \end{pmatrix}, \quad J^\dagger = \frac{1}{2} \begin{pmatrix} 1 & i1 \\ 1 & i1 \end{pmatrix}. \]  

Hence,
\[ J \begin{pmatrix} B_q & 0 \\ 0 & B_p \end{pmatrix} J^\dagger = \frac{1}{2} \begin{pmatrix} B_1 & B_2 \\ B_2 & -B_1 \end{pmatrix}, \]
\[ J \begin{pmatrix} B_q & 0 \\ 0 & -B_p \end{pmatrix} J^\dagger = \frac{1}{2i} \begin{pmatrix} -B_2 & B_1 \\ B_1 & B_2 \end{pmatrix}. \]  

Formulae (A1.25,29,32) are the same as formulae (A1.22).

**Exercise A1.33.** Denote by
\[ B^1 = \begin{pmatrix} B_1 & B_2 \\ B_2 & -B_1 \end{pmatrix}, \quad B^2 = \begin{pmatrix} -B_2 & B_1 \\ B_1 & B_2 \end{pmatrix}, \]  

the Hamiltonian matrices of the complexified system (A1.22).

(i) Show that
\[ B^1 \delta(Im(H)) = B^2 \delta(-Re(H)); \]  

(ii) Show that formula (A1.35) is the R.H.S. of the complexification \( q = u + iv \) of the motion equation
\[ \partial_t(q) = -iB \begin{pmatrix} \delta H \\ \delta q \end{pmatrix}; \]  

(iii) Show that
\[ \{H,F\} \approx \{Re(H),Re(F)\}_1 + i\{Im(H),Im(F)\}_2, \quad \forall H,F, \]  

where \( \{ \ , \ \} \) is the Poisson bracket w.r.t. the matrix \( B^j \) (A1.34), \( j = 1,2 \);  

(iv) Show that if \( F \) is an integral of a dynamical system then \( Re(F) \) and \( Im(F) \) are integrals of the complexified system.

**Exercise A1.38.** Show that a complexification of a Lax equation is again a Lax equation.

There is nothing sacred about extensions by \( \sqrt{-1} \). One could equally well consider extensions by an element \( \theta \), subject to relations
\[ \theta^n = \text{const} \]  

or even
\[ \theta^n = p(\theta), \quad p(\theta) = \sum_{i=1}^{n-1} p_i \theta^i. \]  

What matters is the number of distinct roots of the polynomial \( \theta^n - p(\theta) \). The case (A1.39), with \( \text{const} \neq 0 \), is the simplest and most straightforward one, with \( n \) distinct roots \( \theta \) providing \( n \) complex conjugate analogs of the formulae (A1.24-26). The resulting extension of a Hamiltonian system has \( n \) Hamiltonian structures. Hamiltonian maps behave naturally under such extensions, etc. On the opposite end of the spectrum are the extensions of the type
\[ \theta^n = 0. \]
These are related to asymptotic expansions and we shall treat them in the next Appendix.

**APPENDIX A2. ASYMPTOTIC EXPANSIONS OF HAMILTONIAN SYSTEMS**

In this Appendix we determine the effect of asymptotic expansions on Hamiltonian systems. Briefly stated, an infinite asymptotic expansion of a Hamiltonian system is no longer Hamiltonian, but every finite cut-off of such an expansion is a Hamiltonian system whose Hamiltonian data vary nontrivially with the size of the cut.

**§A2.1 MOTIVATION FROM AN EXAMPLE: THE KdV EQUATION**

If there’s one major cause for the spread of mass illiteracy, it’s the fact that everybody can read and write.

Peter De Vries

In this Section we state the problem of how asymptotic expansions interact with the Hamiltonian formalism and, by analyzing a concrete example of the KdV equation, guess a solution to this problem. This guess will be verified in §§A2.2,3.

Suppose we have a Hamiltonian system

\[ \partial_t(q) = B \left( \frac{\delta H}{\delta q} \right) \]  

(A2.1.1)

in which the Hamiltonian structure \( B \) and the Hamiltonian \( H \) depend regularly upon a (“small”) formal parameter \( \epsilon \). If we substitute

\[ q = \sum_{s=0}^{\infty} u_s \epsilon^s \]  

(A2.1.2)

into both parts of the equation (A2.1.1) and equate the like-powers of \( \epsilon \), we get

\[ \partial_t(u_i) = \sum_{k+\ell=1} B_{(k)} \left( \frac{\delta H}{\delta q} \right)_{(\ell)}, \quad i \in \mathbb{Z}_+ \]  

(A2.1.3)

where

\[ B^* = \sum_{k \geq 0} B_{(k)} \epsilon^k, \]  

(A2.1.4)

\[ \left( \frac{\delta H}{\delta q} \right)^* = \sum_{\ell \geq 0} \left( \frac{\delta H}{\delta q} \right)_{(\ell)} \epsilon^\ell, \]  

(A2.1.5)
and * denotes the result of the substitution (A2.1.2):

\[(...)^* = (\ldots)] q = \sum u_s \epsilon^s. \tag{A2.1.6}\]

The question is: How much of the Hamiltonian character of the original Hamiltonian system (A2.1.1) survives in the asymptotic expansion (A2.1.3)? To get a handle on this problem, consider a simple example of the KdV equation

\[\partial_t(q) = \partial(3q^2 + q^{(2)}) = \partial \frac{\delta}{\delta q}(q^3 - \frac{1}{2}q^{(1)2}). \tag{A2.1.7}\]

There is no \(\epsilon\)-dependence either in the Hamiltonian matrix

\[B = \partial \tag{A2.1.8}\]

or the Hamiltonian

\[H = q^3 - \frac{1}{2}q^{(1)2}, \tag{A2.1.9}\]

and this is a particular case of a regular dependence upon \(\epsilon\). Substituting

\[q = \sum_{s \geq 0} u_s \epsilon^s \tag{A2.1.10}\]

into the KdV equation (A2.1.7), we obtain

\[\partial_t(u_i) = \partial(3 \sum_{s=0}^i u_su_{i-s} + u_i^{(2)}), \ i \in \mathbb{Z}_+. \tag{A2.1.11}\]

On the other hand,

\[H^* = (\sum u_s \epsilon^s)^3 - \frac{1}{2}(\sum u_s^{(1)} \epsilon^s)^2, \tag{A2.1.12}\]

so that

\[H(s) = \sum_{i+j+k=s} u_{i} u_{j} u_{k} - \frac{1}{2} \sum_{i=0}^s u_{i}^{(1)} u_{s-i}^{(1)}, \tag{A2.1.13}\]

whence

\[\frac{\delta H(s)}{\delta u_r} = 3 \sum_{j+k=s-r} u_j u_k + u_{s-r}^{(2)}, \ s \geq r \geq 0, \tag{A2.1.14a}\]

\[\frac{\delta H(s)}{\delta u_r} = 0, \ 0 \leq s < r. \tag{A2.1.14b}\]

Comparing formulae (A2.1.14) and (A2.1.11) we see that the latter can be rewritten as

\[\partial_t(u_i) = \partial \left( \frac{\delta H(i+r)}{\delta u_r} \right), \ r = r(i), \ i \in \mathbb{Z}_+. \tag{A2.1.15}\]

If the asymptotic expansion (A2.1.11) were to be Hamiltonian, it would need to have a specific fixed Hamiltonian, \(H(\ldots)\), say. This is plainly impossible if the index
$i$ is allowed to run freely over the whole $\mathbb{Z}_+$. We are thus forced to cut off the infinite tail of the $u_i$’s and to impose the condition

$$
\epsilon^{N+1} = 0, \quad \text{some fixed} \quad N \in \mathbb{Z}_+.
$$

(A2.1.16)

The asymptotic expansion (A2.1.11) then becomes:

$$
\partial_t(u_i) = \partial(3 \sum_{s=0}^{i} u_s u_{i-s} + u_i^{(2)}), \quad 0 \leq i \leq N,
$$

(A2.1.17)

while formula (A2.1.15) with $r(i) = N - i$ provides a Hamiltonian representation for the system (A2.1.17):

$$
\partial_t(u_i) = \partial \left( \frac{\delta H(N)}{\delta u_{N-i}} \right), \quad 0 \leq i \leq N,
$$

(A2.1.18a)

$$
H(N) = \sum_{i+j+k=N} u_i u_j u_k - \frac{1}{2} \sum_{i=0}^{N} u_i^{(1)} u_{N-i}^{(1)}.
$$

(A2.1.18b)

To sum up, starting with the scalar Hamiltonian matrix $B = \partial$ (A2.1.8) in the $q$-space we have arrived at the Hamiltonian matrix

$$
B^N = \begin{pmatrix}
\partial & \cdots & \partial \\
\partial & \ddots & \partial \\
\partial & \cdots & \partial 
\end{pmatrix}.
$$

(A2.1.19)

in the $(u_0, ..., u_N)$-space. What could be the algorithm transforming the matrix $B$ into the matrix $B^N$? To get a handle on this problem, let us consider the dispersionless version of the KdV equation:

$$
\partial_t(q) = \partial(3q^2) = (\partial + \lambda a_d q) \frac{\delta}{\delta q}(q^3), \quad \lambda = \text{const.}
$$

(A2.1.20)

Now

$$(\partial + \lambda a_d)^s = (\partial + \lambda a_d u_0) + \sum_{s=1}^{N} \lambda a_d u_s \epsilon^s,$$

(A2.1.21)

so that

$$B_{(0)} = \partial + \lambda a_d u_0; \quad B_{(s)} = \lambda a_d u_s, \quad 1 \leq s \leq N.
$$

(A2.1.22)

**Lemma A2.1.23.** The dispersionless version of the system (A2.1.17)

$$
\partial_t(u_i) = \partial(3 \sum_{s=0}^{i} u_s u_{i-s}), \quad 0 \leq i \leq N,
$$

(A2.1.24)

can be recast into the form

$$
\partial_t(u_i) = \sum_{j=0}^{i} B_{(j)} \left( \frac{\delta H(N)}{\delta u_{N-i+j}} \right), \quad 0 \leq i \leq N,
$$

(A2.1.25)
which is the component form of the system

\[
\partial_t(u) = B^N \left( \frac{\delta H(N)}{\delta u} \right), \quad (A2.1.26)
\]

\[
B^N = \begin{pmatrix}
B(0) & \cdots & B(1) \\
\vdots & \ddots & \vdots \\
B(0) & \cdots & B(N)
\end{pmatrix}. \quad (A2.1.27)
\]

**Proof.** By formula (A2.1.14),

\[
\frac{\delta H(N)}{\delta u_{N-r}} = 3 \sum_{s=0}^{\infty} u_s u_{r-s}, \quad 0 \leq r \leq N. \quad (A2.1.28)
\]

Hence,

\[
B(0) \left( \frac{\delta H(N)}{\delta u_N} \right) = (\partial + \lambda ad_{u_0})(3u_0^2) = \partial(3u_0^2),
\]

and this is the same as the \(i = 0\)-case of the dispersionless system (A2.1.24). This proves formula (A2.1.25) for \(i = 0\). For \(i > 0\), the R.H.S. of formula (A2.1.25) yields

\[
B(0) \left( \frac{\delta H(N)}{\delta u_N} \right) + \sum_{j=1}^{i} B(j) \left( \frac{\delta H(N)}{\delta u_{N-1+j}} \right) \quad \text{[by (A2.1.22, 28)]} =
\]

\[
= (\partial + \lambda ad_{u_0})(3 \sum_{s=0}^{i} u_s u_{i-s}) + i \lambda ad_{u_j} \left( 3 \sum_{s=0}^{i-j} u_s u_{i-j-s} \right) =
\]

\[
= \partial(3 \sum_{s=0}^{i} u_s u_{i-s}) +
\]

\[
+ 3\lambda \sum_{j=0}^{i} u_j, \sum_{s=0}^{i-j} u_s u_{i-j-s}. \quad (A2.1.29a)
\]

The expression (A2.1.29a) is the same as the R.H.S. of the equation (A2.1.24). The expression (A2.1.29b) vanishes, since

\[
\sum_{j=0}^{i} u_j, \sum_{s=0}^{i-j} u_s u_{i-j-s} = \sum_{j+s+k=i} [u_j, u_s u_k] = \sum_{j+s+k=i} u_j u_k u_j - \sum_{s} u_s u_k u_j = 0. \quad \Box
\]

The matrix \(B^N\) (A2.1.27) is the desired Hamiltonian structure of the finite \(N\)-cut of the asymptotic expansion of the system (A2.1.1). The next 2 Sections are devoted to a verification of this claim. Specifically, in the next Section we prove that formulae (A2.1.25) and (A2.1.3) are equivalent: this amounts to the identity

\[
\left( \frac{\delta H}{\delta q} \right)_{(i-j)} = \frac{\delta H(N)}{\delta u_{N+j-i}}, \quad (A2.1.30)
\]
or, what is the same,
\[
\frac{\delta H_{(N)}}{\delta u_r} = \left( \frac{\delta H}{\delta q} \right)_{(N-r)}.
\]
(A2.1.31)

The Hamiltonian property of the matrix \( B^N \) (A2.1.27) will be proven in §A2.3.

§A2.2. VECTOR FIELDS, DIFFERENTIAL FORMS, VARIATIONAL DERIVATIVES

In this Section we determine how the procedure of asymptotic expansion interacts with the basic objects enumerated in the title of the Section.

In the notation of Chapter 10, let \( C = C_q = R < q_i^{(g|v)} > \) be the basic differential-difference ring. For the remainder of this Appendix, we denote
\[
\tilde{(\cdot)} = (\cdot) [[\epsilon]],
\]
(A2.2.1)

whatever the object \((\cdot)\) is. The asymptotic expansion is the homomorphism
\[
\Phi : C_q \rightarrow \tilde{C}_q \rightarrow \tilde{C}_u
\]
given by the formula
\[
\Phi(q) = \sum_{s \geq 0} u_s \epsilon^s,
\]
(A2.2.3a)
or, in components,
\[
\Phi(q_i) = \sum_{s \geq 0} u_{i|s} \epsilon^s, \quad i \in \mathcal{I}.
\]
(A2.2.3b)

In the notation (A2.1.6), this can be rewritten as
\[
(q_i)(s) = u_{i|s}, \quad q(s) = u_s, \quad i \in \mathcal{I}, \quad s \in \mathbb{Z}_+.
\]
(A2.2.4)

It is convenient to think of \( \tilde{C}_q \) as
\[
\tilde{R} < q_i^{(g|v)} >= \tilde{(\cdot)}_q.
\]
(A2.2.5)

This allows us to use the embedding
\[
D^{ev}(C) \hookrightarrow D^{ev}(\tilde{C}) \overset{\text{iso}}{=} D^{ev}(C)^\sim,
\]
(A2.2.6)
and similarly
\[
\text{Der}(C) \hookrightarrow \text{Der}(\tilde{C}) \overset{\text{iso}}{=} \text{Der}(C)^\sim.
\]
(A2.2.7)

We shall also employ the following notation:

If \( C_q = C \) then \( C_u = C_\infty \).

(A2.2.8)

We now define how the asymptotic expansion affects various algebraic constructions. First, the vector fields. If \( X \in D^{ev}(C) = D^{ev}(C_q) \), we assign to this \( X \) the vector field \( X^A \in D^{ev}(C_u) = D^{ev}(C_\infty) \) defined by the equation
\[
\Phi X = X^A \Phi : \quad C \rightarrow \tilde{C}_\infty.
\]
(A2.2.9)
Applying this equality to \( q \), we find

\[
\Phi X(q) = \Phi(X) = \sum_{s \geq 0} X_s e^s = X^A \Phi(q) = X^A \left( \sum u_s e^s \right), \tag{A2.2.10}
\]

whence

\[
X^A(u_s) = X(s), \quad s \in \mathbb{Z}_+; \tag{A2.2.11a}
\]

or

\[
X^A(u_i s) = (X_i)(s), \quad X_i = X(q_i), \quad i \in I, \quad s \in \mathbb{Z}_+. \tag{A2.2.11b}
\]

Thus, the evolution derivation \( X^A \in D^{ev}(C_u) \) is well-defined. We denote by the same letter \( \Phi \) the map \( D^{ev}(C) \to D^{ev}(C_\infty) \), \( X \mapsto X^A \). Obviously, this map is injective.

**Proposition A2.2.12.** The asymptotic expansion map (A2.2.9) \( \Phi : D^{ev}(C) \to D^{ev}(C_\infty) \) is a Lie-algebra homomorphism.

**Proof.** We have to check that

\[
[X, Y]^A = [X^A, Y^A], \quad \forall X, Y \in D^{ev}(C). \tag{A2.2.13}
\]

In checking this identity we employ the following

**Principle A2.2.14.** Two vector fields \( U, V \in D^{ev}(C_\infty) \) are equal iff

\[
U \Phi(q) = V \Phi(q). \tag{A2.2.15}
\]

This is so because the equality (A2.2.15) reads:

\[
\sum_s U(u_s) e^s = \sum_s V(u_s) e^s, \tag{A2.2.16a}
\]

which is true iff

\[
U(u_s) = V(u_s), \quad \forall s \in \mathbb{Z}_+. \tag{A2.2.16b}
\]

With the help of this Principle, the equality (A2.2.13) is readily verified:

\[
[X, Y]^A \Phi \text{ [by (A2.2.9)]} = \Phi[X, Y] = \Phi(XY - YX) =
\]

\[
= X^A \Phi Y - Y^A \Phi X = X^A Y^A \Phi - Y^A X^A \Phi = [X^A, Y^A] \Phi. \]

Next, the differential forms. If \( \omega \in \Omega^1(C) \),

\[
\omega = \sum \psi_i^{g|\nu} dq_i^{(g|\nu)} \varphi_i^{g|\nu}, \quad \psi_i^{g|\nu}, \varphi_i^{g|\nu} \in C, \tag{A2.2.17}
\]

then \( \Phi(\omega) \in \Omega^1(C_\infty) \) is defined by the rule

\[
\Phi \left( \sum \psi_i^{g|\nu} dq_i^{(g|\nu)} \varphi_i^{g|\nu} \right) = \sum \Phi(\psi_i^{g|\nu}) d\Phi(q_i^{(g|\nu)}) \Phi(\varphi_i^{(g|\nu)}). \tag{A2.2.18}
\]

In other words, \( \Phi \) is uniquely extended from a map \( \Phi : C \to \tilde{C}_\infty \) into the map \( \Phi : \Omega^1(C) \to \Omega^1(\tilde{C}_\infty) \) by the requirement that \( \Phi \) commutes with the differential \( d \):

\[
\Phi d_q = d_u \Phi : C \to \Omega^1(\tilde{C}_\infty). \tag{A2.2.19}
\]
Obviously, the map $\Phi$ commutes with the Flip map $\Omega^1 \to \Omega^{1^+}$ (10.1.30), and $\Phi$ sends trivial elements in $C$ and $\Omega^{1^+}(C_u)$ into trivial elements in $C_u$ and $\Omega^{1}(C_u)$ respectively. Therefore, applying the map $\Phi$ to the formula (10.1.60):

$$d(H) \approx dq^\ell \frac{\delta H}{\delta q}$$  \hspace{1cm} (A2.2.20)

we find that

$$\Phi(dq^\ell)\Phi\left(\frac{\delta H}{\delta q}\right) = \left(\sum_s \epsilon^s du^t_s\right) \sum_{\ell} \left(\frac{\delta H}{\delta q}\right) (\ell) \epsilon^\ell = \sum_{\ell} \epsilon^{s+\ell} du^t_s \left(\frac{\delta H}{\delta q}\right) (\ell) \approx$$

$$\approx \Phi d(H) = d(\Phi(H)) \approx d\left(\sum_r H(r)\epsilon^r\right) \approx \sum_{rs} du^t_s \frac{\delta H(r)}{\delta u_s} \epsilon^r. \hspace{1cm} (A2.2.21)$$

Hence,

$$\frac{\delta H(r)}{\delta u_s} = \left(\frac{\delta H}{\delta q}\right)_{(rs)} \hspace{1cm} (A2.2.22)$$

This is the desired formula (A2.1.31).

The asymptotic expansion procedure treated in this Appendix can be shown to be natural. Since we won't need this fact here, its Proof is left to the reader as an Exercise.

In the next Section we shall need the following formula:

$$\frac{DH(r)}{Du_s} = \left(\frac{DH}{Dq}\right)_{(rs)}. \hspace{1cm} (A2.2.23)$$

This can be proved as follows. By the Definition (10.2.5) of the Fréchet derivative,

$$\frac{DH}{Dq}(X) = X(H), \quad \forall X \in D^{eu}(C). \hspace{1cm} (A2.2.24)$$

Applying the map $\Phi$ (A2.2.2) to both parts of this equality and extending $\Phi$ to act on operators coefficients-wise, we obtain

$$\Phi\left(\frac{DH}{Dq}(X)\right) = \left[\Phi\left(\frac{DH}{Dq}\right)\right] \Phi(X) =$$

$$= \sum_{\ell} \left(\frac{DH}{Dq}\right) (\ell) \epsilon^\ell \left(\sum_s X(\epsilon^s)\right) [\text{by (A2.2.11a)}] =$$

$$= \sum_{\ell s} \left(\frac{DH}{Dq}\right) (\ell) \epsilon^\ell (X^A(u_s)) \epsilon^s = \hspace{1cm} (A2.2.25a)$$

$$= \Phi X(H) [\text{by (A2.2.9)}] = X^A \sum r H(r) \epsilon^r =$$

$$= \sum_{rs} \frac{DH(r)}{Du_s} (X^A(u_s)) \epsilon^r. \hspace{1cm} (A2.2.25b)$$
Since $X$ is arbitrary, the equality (A2.2.25) yields the desired formula (A2.2.23). (A few more lines are needed to account for the fact that even though $X$ is arbitrary, \{X^A(u_s)\} is not arbitrary. The reader is invited to supply the missing argument. The obvious formula

$$\frac{DH_{r(e)}}{D\mathbf{u}_r} = \frac{DH}{D\mathbf{q}} \bigg|_{\mathbf{q} = \mathbf{u}_0}$$  \hspace{1cm} (A2.2.26)

might help.)

§A2.3. HAMILTONIAN STRUCTURES

In this Section we prove that the matrix $B^N$ (A2.1.27) is Hamiltonian given that the matrix $B$ is.

According to the main result of the Hamiltonian formalism, Exercise 11.1.24, a skewsymmetric matrix $B$ is Hamiltonian iff

$$B^\delta_{\mathbf{q}} [\mathbf{Y}^t B(\mathbf{X})] = D\mathbf{q}[B(\mathbf{Y})][B(\mathbf{X})] - D\mathbf{q}[B(\mathbf{X})]B(\mathbf{Y})]$$  \hspace{1cm} (A2.3.1)

for arbitrary $q$-independent vectors $\mathbf{X}, \mathbf{Y}$. Therefore, the matrix $B^N$ (A2.1.27):

$$B^N = \begin{pmatrix}
    & \cdots & B^{(0)} \\
    & \cdots & \\
    \cdots & \cdots & \\
    \vdots & \cdots & B^{(1)} \\
    B^{(0)} & B^{(1)} & \cdots & B^{(N)}
\end{pmatrix},$$  \hspace{1cm} (A2.3.2)

is Hamiltonian iff

$$B^\delta_{\mathbf{u}} [\mathbf{Y}^t B^N(\mathbf{X})] = D\mathbf{u}[B^N(\mathbf{Y})][B^N(\mathbf{X})] - D\mathbf{u}[B^N(\mathbf{X})][B^N(\mathbf{Y})],$$  \hspace{1cm} (A2.3.3)

for arbitrary $\mathbf{u}$-independent vectors $\mathbf{X}, \mathbf{Y}$.

By formula (A2.3.2),

$$B^N_{ij} = B_{(i+j-N)}, \hspace{0.5cm} 0 \leq i, j \leq N,$$  \hspace{1cm} (A2.3.4)

where it is understood that

$$(...)_i = 0, \hspace{0.5cm} \text{for} \hspace{0.5cm} i < 0.$$  \hspace{1cm} (A2.3.5)

The entries of the matrix $B^N$ (A2.3.2) are themselves matrices; similarly, the entries of the underlined vectors $\mathbf{u}$, $\mathbf{X}$, $\mathbf{Y}$ in formula (A2.3.3) are themselves vectors. To simplify considerably the forthcoming calculations, we shall reverse the numbering of components of the vectors $\mathbf{X}$ and $\mathbf{Y}$:

$$\mathbf{X} = \begin{pmatrix}
    X_N \\
    \vdots \\
    X_0
\end{pmatrix}, \hspace{0.5cm} \mathbf{Y} = \begin{pmatrix}
    Y_N \\
    \vdots \\
    Y_0
\end{pmatrix}.$$  \hspace{1cm} (A2.3.6)
Then, by formula (A2.3.4),

\[ [B^N(Y)]_s = \sum_i B^N_{si}(Y_{N-i}) = \sum B_{(s+i-N)}(Y_{N-i}) = \sum_{\ell+j=s} B_{(\ell)}(Y_j). \]  

Therefore, the \( u \)-component of the R.H.S. of formula (A2.3.3) is:

\[ \sum_{i,\nu,\mu+j=s} \frac{D[B_{(\nu)}(Y_j)]}{D[u_{\nu+i}]} [B_{(\nu)}(X_i)] = \sum_{j,\nu,\mu+i=s} \frac{D[B_{(\mu)}(X_j)]}{D[u_{\nu+j}]} [B_{(\nu)}(Y_j)]. \]  

On the other hand,

\[ Y^t B^N(X) = \sum_{i,j} Y^t_i [B^N(X)]_{N-j} = \sum_{i,j} Y^t_i B_{(N-j-i)}(X_i), \]  

so that for the \( u \)-component of the L.H.S. of formula (A2.3.3) we obtain

\[ \sum_{r \in i,j} B_{(s+r-N)} \left\{ \frac{\delta}{\delta u_r} [Y^t B_{(N-j-i)}(X_i)] \right\}. \]  

We have to verify the equality \((A2.3.8) = (A2.3.10)\). In the notation

\[ X_i = X, \ Y_j = Y, \]  

this equality is equivalent to the equality

\[ \sum_r B_{(s+r-N)} \left\{ \frac{\delta}{\delta u_r} [Y^t B_{(N-i-j)}(X)] \right\} = \]  

\[ = \sum_{\nu} \left( \frac{D[B_{(s-j)}(Y)]}{D[u_{\nu+i}]} [B_{(\nu)}(X)] - \frac{D[B_{(s-i)}(X)]}{D[u_{\nu+j}]} [B_{(\nu)}(Y)] \right). \]  

The R.H.S. of this equality can be rewritten, by formula (A2.2.23), as

\[ \left( \frac{D[B(Y)]}{D[q]} \right)_{(s-j-i)} [B(X)]_{(\nu)} \leftrightarrow \left\{ \frac{D[B(Y)]}{D[q]} [B(X)]_{(\nu)} \right\}_{(s-j-i)} \]  

\[ = \left\{ B \frac{\delta}{\delta q} [Y^t B(X)] \right\}_{(s-j-i)} \]  

\[ \leftrightarrow \left\{ B \frac{\delta}{\delta q} [Y^t B_{(s-j-i)}(X)] \right\}_{(s-j-i)} \]  

and this is the same as the L.H.S. of the equality (A2.3.12). Recall that the notation “\( - \leftrightarrow \)” stands for: “minus the same expression with \( X \) and \( Y \) interchanged”.

**APPENDIX A3. VARIATIONAL CALCULUS OVER NONCOMMUTATIVE RINGS**

An eloquent mathematician must, from the nature of things, ever remain as rare a phenomenon as a talking fish, and it is certain that the more anyone gives himself up to the
study of oratorical effect the less will he find himself in a fit
state of mind to mathematicize. It is the constant aim of the
mathematician to reduce all his expressions to their lowest
terms, to retrench every superfluous word and phrase, and
to condense the Maximum of meaning into the Minimum of
language.
Sylvester, “Address on commemoration day at John Hopkins
University” (Feb. 22, 1877)

In this Appendix we develop an algebraic calculus of variations over associative
rings. The main result is the determination, in §A3.2, of the Image and Kernel of
the variational (Euler-Lagrange) operator $\delta$.

§A3.1. THE BASIC OBJECTS

In this Section we introduce the basic objects of the calculus of variations:
evolution derivations, differential 1-forms, variational derivatives and their trans-
formation properties, formula for the first variation. The logical development is as
parallel as possible to that of the purely abelian case in the first half of §2.1 in [Kup
1992].

Let $R$ be an associative ring with unity and a $\mathbb{Q}$-algebra (or $\mathcal{F}$-algebra, for
a field $\mathcal{F}$ of characteristic zero.) Denote by $\text{Der}(R)$ the Lie algebra of derivations
of $R$ (over $\mathbb{Q}$ or $\mathcal{F}$). Pick a derivation $Z \in \text{Der}(R)$. If $\tilde{R}$ is a $R$-bimodule then an
additive map $\tilde{Z} : \tilde{R} \to \tilde{R}$ is called a derivation of $\tilde{R}$ covering $Z$ if

$$\tilde{Z}(k_1\tilde{k}k_2) = Z(k_1)\tilde{k}k_2 + k_1\tilde{Z}(\tilde{k})k_2 + k_1\tilde{k}Z(k_2), \quad \forall k_1, k_2 \in R, \tilde{k} \in \tilde{R}. \quad (A3.1.1)$$

**Exercise A3.1.2.** Show that if $\tilde{Z}_i$ covers $Z_i$, $i = 1, 2$, then $[\tilde{Z}_1, \tilde{Z}_2]$ covers $[Z_1, Z_2]$.

An additive map $\rho : \tilde{R} \to \tilde{R}$ is called a derivation of $\tilde{R}$ into $\tilde{R}$ if

$$\rho(k_1k_2) = \rho(k_1)k_2 + k_1\rho(k_2), \quad \forall k_1, k_2 \in \tilde{R}. \quad (A3.1.3)$$

The set of all such $\rho$'s is denoted $\text{Der}(\tilde{R}, \tilde{R})$.

**Exercise A3.1.4.** Show that if $\rho \in \text{Der}(\tilde{R}, R)$ then so is

$$[Z, \rho] := \tilde{Z}\rho - \rho Z. \quad (A3.1.5)$$

Let $\partial_1, ..., \partial_m : \tilde{R} \to \tilde{R}$ be $m$ mutually commuting derivations of $\tilde{R}$ (over $\mathbb{Q}$
or $\mathcal{F}$, as the case may be; for aesthetic simplification, we shalln’t mention such
futilely simple trifles anymore.) Such a $\tilde{R}$ is called a differential ring. Let $\mathcal{I}$ be an
index set. Let $C = C(q) = R < q_{\alpha}^{(\sigma)} >, \alpha \in \mathcal{I}, \sigma \in \mathbb{Z}_+^m$, be the free associative ring
generated over $R$ by the generators $q_{\alpha}^{(\sigma)}$. We can make $C$ into a differential ring by
extending the derivations $\partial_i$'s from $R$ onto $C$ by their actions on the generators
$q_{\alpha}^{(\sigma)}$'s of $C$ via the rule:

$$\partial_i(q_{\alpha}^{(\sigma)}) = q_{\alpha}^{(\sigma + 1)_i}; \quad (A3.1.6)$$

here $1_i \in \mathbb{Z}_+^m$ is the multiindex $(0, ..., 1, ..., 0)$ (1 at the $i^{th}$ place).
Exercise A3.1.7. Show that the derivations $\partial_i$'s on $C$ also commute.

For a multiindex $\sigma = (\sigma_1, ..., \sigma_m) \in \mathbb{Z}_+^m$, denote

$$\partial^\sigma = \partial_1^{\sigma_1} ... \partial_m^{\sigma_m}, \quad (-\partial)^\sigma = (-\partial_1)^{\sigma_1} ... (-\partial_m)^{\sigma_m} = (-1)^{\sigma} \partial^\sigma,$$  \hspace{0.5cm} (A3.1.8)

where

$$(-1)^{\sigma} = (-1)^{|\sigma|}, \quad |\sigma| = \sigma_1 + ... + \sigma_m.$$  \hspace{0.5cm} (A3.1.9)

Denote

$$q_\alpha = q_\alpha^{(0)}.$$  \hspace{0.5cm} (A3.1.10)

Exercise A3.1.11. Show that

$$q_\alpha^{(\sigma)} = \partial^\sigma(q_\alpha).$$  \hspace{0.5cm} (A3.1.12)

Denote by $\text{Der}(C)$ the Lie algebra of derivations of $C$ over $\mathbb{R}$. Any element $Z \in \text{Der}(C)$ is uniquely defined by its values on the generators $q_\alpha^{(\sigma)}$'s of $C$: If $Z(q_\alpha^{(\nu)}) = Z_\alpha^{(\nu)}$ then we can suggestively write

$$Z = \sum Z_\alpha^{(\nu)} \partial_{q_\alpha^{(\nu)}},$$  \hspace{0.5cm} (A3.1.13)

where $\frac{\partial}{\partial q_\alpha^{(\nu)}}$ is a derivation of $C$ acting on the generators of $C$ by the rule

$$\frac{\partial}{\partial q_\alpha^{(\nu)}} : q_\beta^{(\sigma)} \mapsto \delta_{\beta\alpha}^\sigma.$$  \hspace{0.5cm} (A3.1.14)

Remark A3.1.15. In the object $\text{Der}(C)$ we meet the 1st instance of divergence between abelian and nonabelian notions: $\text{Der}(C)$ is no longer a left $C$-module. This means that, given $a \in C$ and $Z \in \text{Der}(C)$, there doesn’t exist an element $(a * Z) \in \text{Der}(C)$ such that

$$(a * Z)(b) = a(Z(b)), \quad \forall b \in C.$$  \hspace{0.5cm} (A3.1.16)

(The same observation applies to $\text{Der}(\mathbb{R})$.) Such instances will multiply from now on.

For $k \in \mathbb{R}$, denote by $\hat{L}_k$ and $\hat{R}_k$ the operators of left and right multiplication by $k$, respectively, on any $\mathbb{R}$-bimodule. The associative ring generated by such operations is denoted $\text{Opo}(\mathbb{R})$:

$$\text{Opo}(\mathbb{R}) = \{ \sum \hat{L}_{k(s)} \hat{R}_{\ell(s)} \mid \text{finite sums; } k(s), \ell(s) \in \mathbb{R} \}. \hspace{0.5cm} (A3.1.17)$$

We encounter an example of such zero-order operators in the standard object $\frac{\partial H}{\partial q_\alpha^{(\nu)}}$, where $H \in C$:

$$\frac{\partial H}{\partial q_\alpha^{(\nu)}}(r) := \left( r - \frac{\partial}{\partial q_\alpha^{(\nu)}} \right)(H), \quad \forall r \in C; \hspace{0.5cm} (A3.1.18)$$
here \( r \frac{\partial}{\partial q_\alpha^{(\nu)}} \) is a derivation (over \( \hat{R} \)) of \( C \) acting by the rule:

\[
\left( r \frac{\partial}{\partial q_\alpha^{(\nu)}} \right) \left( q_\beta^{(\sigma)} \right) = r \delta_\alpha^\sigma.
\]  

(A3.1.19)

Thus, \( \frac{\partial H}{\partial q_\alpha^{(\nu)}} \) is no longer an element of \( C \) but an operator.

**Exercise A3.1.20.** (i) Show that

\[
\frac{\partial H}{\partial q_\alpha^{(\nu)}}(r) = \frac{d}{de} \bigg|_{e=0} (H|_{q_\alpha^{(\nu)}=q_\alpha^{(\nu)}+er});
\]

(A3.1.21)

(ii) Show that

\[
\frac{\partial (HF)}{\partial q_\alpha^{(\nu)}} = \hat{R}_F \frac{\partial H}{\partial q_\alpha^{(\nu)}} + \hat{L}_H \frac{\partial F}{\partial q_\alpha^{(\nu)}}, \; \forall H, F \in C.
\]

(A3.1.22)

An element \( X \in \text{Der}(C) \) is called an *evolution derivation*, or *evolution field*, if it commutes with \( \partial_1, \ldots, \partial_m \):

\[
X(q_\alpha^{(\sigma)}) = X \partial^\sigma(q_\alpha) = \partial^\sigma X(q_\alpha) =: (X)_\alpha.
\]

(A3.1.23)

The Lie algebra of evolution derivations is denoted \( D^{ev} = D^{ev}(C) \).

Next come differential \((1-)\) forms. The \( C \)-bimodule of 1-forms, \( \Omega^1(C) \), is defined as

\[
\Omega^1(C) = \left\{ \sum \psi_\alpha^{(\nu)} dq_\alpha^{(\nu)} \varphi_\alpha^{(\nu)} \right\} \text{ finite sums; } \psi_\alpha^{(\nu)}, \varphi_\alpha^{(\nu)} \in C,
\]

(A3.1.24)

with the natural \( C \)-bimodule structure

\[
\hat{L}_a \hat{R}_b(\psi dq_\alpha^{(\nu)} \varphi) = a \psi dq_\alpha^{(\nu)} \varphi b.
\]

(A3.1.25)

The \( C \)-bimodule of *reduced* forms, \( \Omega^1_0(C) \), is defined as

\[
\Omega^1_0(C) = \left\{ \sum \psi_\alpha^{(\nu)} dq_\alpha \varphi_\alpha \right\} \text{ finite sums; } \psi_\alpha^{(\nu)}, \varphi_\alpha \in C.
\]

(A3.1.26)

Let us define the following natural pairing \( < (\cdot), (\cdot) > \) between \( \Omega^1(C) \) and \( \text{Der}(C) \):

\[
< \psi dq_\alpha^{(\nu)} \varphi, \sum Z_\beta^{(\sigma)} \frac{\partial}{\partial q_\alpha^{(\sigma)}} > = \psi Z_\alpha^{(\nu)} \varphi.
\]

(A3.1.27)

We shall also use the notation \( \omega(Z) \) instead of \( < \omega, Z > \).

Let \( d : C \to \Omega^1(C) \) be the natural derivation (over \( \hat{R} \)) taking \( q_\alpha^{(\nu)} \) into \( dq_\alpha^{(\nu)} \).
**Exercise A3.1.28.** Show that
\[ Z(H) = < d(H), Z >, \quad \forall H \in C, \quad \forall Z \in \text{Der}(C). \]  
(A3.1.29)

Since \( \frac{\partial H}{\partial q^{(\nu)}_\alpha} \) is an element of \( O\rho_0(C) \), it acts not only on \( C \) but on every \( C \)-bimodule; in particular, it acts on \( \Omega^1(C) \).

**Exercise A3.1.30.** Show that
\[ Z(H) = d(H), \quad Z, \quad V^G C, \quad V Z^G \text{Der}(C). \]  
(A3.1.29)

We next extend the actions of the \( \partial 's \) and \( \text{Der}(C) \) from \( C \) into \( \Omega^1(C) \) in such a way that: 1) these actions cover the corresponding actions on \( C \), and 2) they commute with \( d' \):
\[ \partial_i(\psi d^{(\nu)}_\alpha \varphi) = \partial_i(\psi) d^{(\nu)}_\alpha \varphi + \psi d^{(\nu+1)}_\alpha \varphi + \psi d^{(\nu)}_\alpha \partial_i(\varphi), \quad \psi \in C. \]  
(A3.1.32)

\[ Z(\psi d^{(\nu)}_\alpha \varphi) = Z(\psi) d^{(\nu)}_\alpha \varphi + \psi d(Z^{(\nu)}_\alpha) \varphi + \psi d^{(\nu)}_\alpha Z(\varphi), \quad \forall Z \in \text{Der}(C). \]  
(A3.1.33)

**Exercise A3.1.34.** Show that \( D e^\nu(C) \) and the \( \partial 's \)'s commute when acting on \( \Omega^1(C) \).

**Exercise A3.1.35.** Show that
\[ [Z(\varphi)](Y) = Z(\varphi(Y)) - \varphi([Z, Y]), \quad \forall \varphi \in \Omega^1(C), \quad \forall Y \in \text{Der}(C). \]  
(A3.1.36)

Let us make the associative ring \( O\rho_0(\tilde{R}) \) into a differential ring by the rule
\[ \partial_i(\tilde{L}_k \tilde{R}_{\ell}) = \tilde{L}_{\partial_i(k)} \tilde{R}_{\ell} + \tilde{L}_k \tilde{R}_{\partial_i(\ell)}, \quad k, \ell \in \tilde{R}. \]  
(A3.1.37)

Obviously, \( \partial_i \) is a derivation and
\[ [\partial_i, \partial_j] = 0 \quad \text{on} \quad O\rho_0(\tilde{R}). \]  
(A3.1.38)

We shall often employ the notation \( (\cdot)^{(\sigma)} \) instead of \( \partial^\sigma(\cdot) \), where \( (\cdot) \) is any element of a space on which the \( \partial 's \)'s act.

**Exercise A3.1.39.** Show that for any \( \mathcal{O} \in O\rho_0(\tilde{R}) \) and any \( \tilde{R} \)-bimodule \( \tilde{R} \) on which the \( \partial 's \)'s act,
\[ [\partial_i(\mathcal{O}) - \mathcal{O}(\partial_i)] = \mathcal{O}(\partial_i(\tilde{k})), \quad \forall k \in \tilde{R}. \]  
(A3.1.39)

Denote by
\[ \mathcal{J} = \sum_{i=1}^m \text{Im} \partial_i \]  
(A3.1.41)

the subspace of "trivial" elements ("divergencies"). We write \( (\cdot) \sim (\cdot) \) when \([\cdot] - (\cdot) \in \mathcal{J}; (\cdot) \) and \( (\cdot) \) are then called equivalent.

Let \( \delta : \Omega^1(C) \rightarrow \Omega^0_1(C) \) be the following projection:
\[ \delta(\psi d^{(\nu)}_\alpha \varphi) = \delta(\tilde{L}_\psi \tilde{R}_\varphi (d^{(\nu)}_\alpha \varphi)) = \begin{pmatrix} \begin{array}{c} \delta \varphi \\ \delta \psi \\
 \end{array} \end{pmatrix}, \quad \delta(\psi d^{(\nu)}_\alpha \varphi) = \begin{pmatrix} \begin{array}{c} \delta \varphi \\ \delta \psi \\
 \end{array} \end{pmatrix}. \]  
(A3.1.42)

**Exercise A3.1.43.** Show that
\[ \delta(\varphi) \sim \varphi, \quad \forall \varphi \in \Omega^1(C). \]
The Euler-Lagrange operator $\delta : C \to \Omega^1_0(C)$ is defined by the rule:

$$\delta = \delta d : C \to \Omega^1_0(C). \quad (A3.1.44)$$

Since any $\omega \in \Omega^1(C)$ can be written as

$$\omega = \sum \omega^\nu_\alpha(dq^\nu_\alpha), \quad \omega^\nu_\alpha \in Op_0(C), \quad (A3.1.45)$$

and likewise

$$\forall \omega \in \Omega^1_0(C) \Rightarrow \omega = \sum \omega_\alpha(dq_\alpha), \quad \omega_\alpha \in Op_0(C), \quad (A3.1.46)$$

for any $H \in C$ we have

$$\delta(H) = \sum_\alpha [\delta(H)]_\alpha(dq_\alpha) =: \sum_\alpha \frac{\delta H}{\delta q^\nu_\alpha}(dq_\alpha). \quad (A3.1.47)$$

The variational derivatives $\frac{\delta H}{\delta q_\alpha}$ are defined thereby. In contrast to the familiar abelian situation, $\frac{\delta H}{\delta q_\alpha}$ is now an operator. Let us calculate its form. We have:

$$\delta(H) = \hat{\delta}d(H) \quad \text{[by (A3.1.31)]} = \hat{\delta} \left[ \sum \frac{\partial H}{\partial q^\nu_\alpha}(dq^\nu_\alpha) \right] \quad \text{[by (A3.1.42)]} =$$

$$= \left[ \sum\left(-\partial^\nu \left( \frac{\delta H}{\delta q^\nu_\alpha} \right) \right)(dq_\alpha), \quad (A3.1.48)$$

whence

$$\frac{\delta H}{\delta q_\alpha} = \sum_\nu (-\partial^\nu \left( \frac{\partial H}{\partial q^\nu_\alpha} \right). \quad (A3.1.49)$$

This is the familiar abelian form, but the substance is quite dissimilar.

**Proposition A3.1.50.** $\hat{\delta}(J) = \{0\}$.

**Proof.** We have,

$$\hat{\delta}[\partial_i(\psi dq^\nu_\alpha \varphi)] = \hat{\delta}[\partial_i(\psi) dq^\nu_\alpha \varphi + \psi dq^{\nu+1}_\alpha \varphi + \psi dq^\nu_\alpha \partial_i(\varphi)] =$$

$$= \left[ (-\partial^\nu (L_{\partial_i(\psi)} \hat{R}_\varphi) + \hat{L}_\psi \hat{R}_{\partial_i(\psi)}) + (-\partial_i)(-\partial^\nu (L_\psi \hat{R}_\varphi))(dq_\alpha) = 0, \right.$$

since

$$(-\partial_i)(-\partial^\nu (L_\psi \hat{R}_\varphi) = -(-\partial^\nu \partial_i(\hat{L}_\psi \hat{R}_\varphi) = -(-\partial^\nu (L_{\partial_i(\psi)} \hat{R}_\varphi) + \hat{L}_\psi \hat{R}_{\partial_i(\psi))}. \quad \blacksquare$$

**Corollary A3.1.51.**

$$H \sim 0 \Rightarrow \frac{\delta H}{\delta q_\alpha} = 0, \quad \forall \alpha \in I. \quad (A3.1.52)$$

**Proof.** If $H = \partial_i(F)$ for some $F \in C$ then

$$\delta(H) = \hat{\delta}d(H) = \hat{\delta}d\partial_i(F) = \hat{\delta}\partial_i[d(F)] = 0$$
by Proposition A3.1.50. Hence,
\[
\frac{\delta H}{\delta q} = 0,
\]
where \(\frac{\delta H}{\delta q}\) is the vector with the components.

\[
\left(\frac{\delta H}{\delta q}\right)^\alpha = \frac{\delta H}{\delta q^\alpha}.
\]

**Exercise A3.1.54.** (DuBois-Reymond Lemma.) If \(\mathcal{O} \in Op_0(C)\) is such that \(\mathcal{O}(C) \sim 0\) then \(\mathcal{O} = 0\).

*Hint:* Abelian-type arguments will do the job.

**Lemma A3.1.55.** If \(\omega \in \Omega^1_0(C)\) and \(<\omega, Der(C) > \sim 0\) then \(\omega = 0\).

**Proof.** Let \(\omega = \sum_{\alpha} \omega_\alpha(dy^\alpha)\). Then \(<\sum_{\alpha} \omega_\alpha(dy^\alpha), C_{\beta}^{\partial} > = \omega_\beta(C) \sim 0\). Hence, by the DuBois-Reymond Lemma, \(\omega_\beta = 0\). Thus, \(\omega = 0\).

**Exercise A3.1.56.** Show that
\[
\{\omega \in \Omega^1_0(C) \& <\omega, D^e(C) > \sim 0\} \Rightarrow \omega = 0.
\]

The next Theorem establishes the link between the developed above differential-forms-approach to the calculus of variations and the physico-geometrical (in spirit) approach based on “variations”, i.e., on the actions of \(\Omega^1(C)\) and \(\Omega^1_0(C)\) on \(D^e(C)\).

**Theorem A3.1.58.** (a) If \(\omega \in \Omega^1_0(C)\) and \(\omega \sim 0\) then \(\omega = 0\);

(b) If \(\omega \in \Omega^1(C)\) and \(\omega \sim 0\) then \(<\omega, D^e > \sim 0\);

(c) If \(\omega \in \Omega^1\) and \(<\omega, D^e > \sim 0\) then \(\omega \sim 0\);

(d) The projection \(\delta : \Omega^1 \rightarrow \Omega^1_0\) can be uniquely defined by the relation <
\[
\sim <\omega, X >, \ \forall X \in D^e;
\]

(e) \(Ker\delta = J\) in \(\Omega^1\).

**Proof.** (b) \(<\partial_i(\omega), X > = <\partial_i(\psi dq^{(\nu)}) \varphi, X >=
\]
\[
= <\partial_i(\psi)dq^{(\nu)} \varphi + \psi dq^{(\nu)} \partial_i(\varphi), \sum \partial^\sigma(X_\beta) \frac{\partial}{\partial q^\alpha} >
\]
\[
= \partial_i(\psi)\partial^\nu(X_\alpha) \varphi + \psi \partial^\nu \partial_i(X_\alpha) \varphi + \psi \partial^\nu \partial_i(\varphi) =
\]
\[
= \partial_i[\psi \partial^\nu(X_\alpha) \varphi] = \partial_i <(\omega, X ) > \sim 0;
\]

(a) If \(\omega \sim 0\) then \(<\omega, D^e > \sim 0\) by (b). Hence, \(\omega = 0\) by (A3.1.57);

(d) Uniqueness follows from (A3.1.57), and existence follows from (A3.1.43) and (b);

(e) If \(\omega \in Ker\delta\), i.e., \(\delta(\omega) = 0\), then \(\omega \sim \delta(\omega) = 0\), so that \(\omega \sim 0\). Thus, \(Ker\delta \subset J\).

Conversely, if \(\omega \in J\), i.e., \(\omega \sim 0\), then \(\delta(\omega) \sim \omega \sim 0\), so that, by (a), \(\delta(\omega) = 0\). Thus, \(J \subset Ker\delta\);

(c) Since \(\omega \sim \delta(\omega)\), we have, by (b): \(0 \sim <\omega, D^e > \sim <\delta(\omega), D^e >\). Hence, \(\delta(\omega) = 0\) by (A3.1.57). Thus, \(\omega \sim \delta(\omega) = 0\).

**Corollary A3.1.59.** (Formula for the first variation.) For any \(H \in C\) and any \(X \in D^e(C)\),
\[
X(H) \sim \frac{\delta H}{\delta q^i}(X) := \sum_{\alpha} \frac{\delta H}{\delta q_\alpha}(X_\alpha),
\]

(A3.1.60)
and this relation uniquely defines the vector (of operators) \( \frac{\delta H}{\delta q} \).

**Proof.** Uniqueness follows from (A3.1.57). Next,

\[
X(H) \text{ [by (A3.1.29)] = } <d(H), X> \text{ [by Theorem A3.1.58(b)] } \sim \\
\sim <\delta d(H), X> = <\delta(H), X> = \sum_\alpha \frac{\delta H}{\delta q_\alpha}(dq_\alpha), X> = \sum_\alpha \frac{\delta H}{\delta q_\alpha}(X_\alpha).
\]

Let \( \bar{R} \subset \bar{R}' \) be a differential extension. This means that \( \bar{R} \) is a subring of \( \bar{R}' \), that there are \( m \) commuting derivations \( \partial'_1, \ldots, \partial'_m \) acting on \( \bar{R}' \), and these actions satisfy the compatibility condition \( \partial_i|_{\bar{R}} = \partial'_i|_{\bar{R}}, i = 1, \ldots, m \). Thus, we can drop off the accents on the \( \partial_i \)'s and keep for them the old notation \( \partial_i \). Every differential extension \( \bar{R} \subset \bar{R}' \) induces the corresponding extension \( C \subset C' = \bar{R}' < q^{(\nu)} > \). Denote

\[
D^e_v(C') = \{ X \in D^e_v(C') | X_\alpha \in \bar{R}', \ \forall \alpha \in \mathcal{I} \}. \quad (A3.1.61)
\]

We shall need in the next Section the following version of the formula for the 1st variation:

**Lemma A3.1.62.** Pick \( H \in C \). Then

\[
X(H) \sim \frac{\delta H}{\delta q^v}(X), \ \forall X \in D^e_v(C'), \ \forall \bar{R}' \supset \bar{R}, \quad (A3.1.63)
\]

and this formula uniquely defines the vector \( \frac{\delta H}{\delta q^v} \).

**Proof.** Existence of formula (A3.1.63) was proven in Corollary A3.1.59, and uniqueness follows from the following Proposition:

**Proposition A3.1.64.** If \( \omega \in \Omega^1_0(C) \) is such that

\[
<\omega, D^e_v(C') > \sim 0, \ \forall \bar{R}' \supset \bar{R} \Rightarrow \omega = 0. \quad (A3.1.65)
\]

**Proof.** Take \( \bar{R}' = \bar{R} < Q^{(\sigma)} > \). Then, for each \( \alpha \in \mathcal{I} \), we have

\[
0 \sim <\omega, \sum_\alpha Q^{(\sigma)} \frac{\partial}{\partial q^{(\sigma)}} > = \omega_\alpha(Q), \text{ so that, treating } C' = \bar{R}' < q^{(\nu)} > \text{ as } \bar{R} < q^{(\nu)}, Q^{(\nu)}, \text{ we get } \frac{\delta}{\delta Q} [\omega_\alpha(Q)] = 0, \text{ and hence}
\]

\[
\omega_\alpha(X_\alpha) = [\frac{\delta}{\delta Q} [\omega_\alpha(Q)]](X_\alpha) = 0, \ \forall X_\alpha \in C. \quad \blacksquare
\]

We next discuss, as briefly as possible, the notions of an operator and its adjoint. The main reason for brevity is that these notions, in contrast to the abelian case, are cumbersome and open-ended, and we try to get by with the minimal amount of theory necessary to determine the \( \text{Ker} \delta \) and \( \text{Im} \delta \) in the next Section.

So, operators first. Those acting on \( C \) with values in \( C \) are, by definition, finite linear combinations of the expressions

\[
\hat{L}_a \hat{R}_b \partial^\sigma = \hat{L}_a \hat{R}_b \hat{L}_{\partial^\sigma}, \quad a, b \in C, \quad (A3.1.66)
\]
each one acting on $C$ (or any differential $C$-bimodule) via the rule

$$(\hat{L}_a \hat{R}_b \partial^\sigma)(r) = ar^{(\sigma)}b.$$  \hfill (A3.1.67)

Operators mapping the ring $Op_0(C)$ into itself are of the same form, with the action rule being

$$(\hat{L}_a \hat{R}_b \partial^\sigma)(\hat{L}_\psi \hat{R}_\varphi)(r) = \hat{L}_a \hat{R}_b (\hat{L}_\psi \hat{R}_\varphi)^{\sigma}.$$  \hfill (A3.1.68)

Next, the evolution fields act on $C$, $Op_0(C)$, operators, operator-valued operators, etc., component-wise by the inductive rule

$$[X(Ob)](\cdot) = X(Ob(\cdot)) - Ob(X(\cdot)),$$  \hfill (A3.1.69)

where $Ob$ is an object which $X \in D^{eu}(C)$ differentiates, and $(\cdot)$ is any element on which the object $Ob$ acts.

**Exercise A3.1.70.** Show that

$$X(\hat{L}_a \hat{R}_b \partial^\sigma) = (\hat{L}_X(a) \hat{R}_b + \hat{L}_a \hat{R}_{X(b)}) \partial^\sigma.$$  \hfill (A3.1.71)

For any such object $Ob$, the Fréchet derivative

$$\frac{D(Ob)}{Dq} = \left( \frac{D(Ob)}{Dq_\alpha} \right)$$

is defined via the rule

$$\sum_\alpha \frac{D(Ob)}{Dq_\alpha} (X_\alpha) =: \frac{D(Ob)}{Dq} (X) = X(Ob), \quad \forall X \in D^{eu}(C).$$  \hfill (A3.1.72)

Finally, the adjoint operators. These most conveniently act to the left. We shall need only the following set-up. Suppose $A : C \to C$ is an operator. Its adjoint with respect to the pairing $Op_0(C) \times C \to C$,

$$O \times r \mapsto O(r), \quad O \in Op_0(C), \quad r \in C,$$  \hfill (A3.1.73)

is the operator

$$A^\dagger : Op_0(C) \to Op_0(C),$$

defined by the relation

$$[A^\dagger(O)](r) \sim O(A(r)), \quad \forall O \in Op_0(C), \quad \forall r \in C.$$  \hfill (A3.1.74)

From the DuBois-Reymond Lemma it follows that the adjoint operator is unique. Its existence follows from the following calculation: If $O = \hat{L}_\psi \hat{R}_\varphi$, and $A = \hat{L}_a \hat{R}_b \partial^\sigma$ then

$$O(A(r)) = \hat{L}_\psi \hat{R}_\varphi (\hat{L}_a \hat{R}_b \partial^\sigma(r)) = \psi a r^{(\sigma)}b \varphi = \hat{L}_\psi a \hat{R}_b \partial^\sigma (r) \sim$$

$$\sim [(-\partial)^{\sigma}(\hat{L}_\psi a \hat{R}_b \varphi)](r) = [\hat{L}_\psi \hat{R}_\varphi \hat{L}_a \hat{R}_b (- \partial)^{\sigma}] (r),$$  \hfill (A3.1.75)

where the arrow “$\sim$” over $\partial$ indicates that $\partial$ acts to the left. Thus,

$$(\hat{L}_a \hat{R}_b \partial^\sigma)^\dagger = \hat{L}_a \hat{R}_b (- \partial)^{\sigma}.$$  \hfill (A3.1.76)
So far all the operators have been scalar. In the matrix case, all the same
definitions apply trivially modified, resulting in the familiar formula for the matrix
elements of the adjoint operator:

$$(A^\dagger)_{\alpha\beta} = (A_{\beta\alpha})^\dagger. \quad (A3.1.77)$$

We are now ready to derive the transformation formula for variational derivatives.

Let $C_1 = \tilde{R} < u^{(\nu)}_\eta >$ be another differential ring. A (differential) homomorphism $\Phi : C \rightarrow C_1$ is a homomorphism over $\tilde{R}$ commuting with the actions of the $\partial_i$’s. Thus,

$$\Phi(u^{(\nu)}_\eta) = \Phi(\partial^{\nu}_\eta(u_\alpha)) = \partial^{\nu}_\eta(\Phi(u_\alpha)),$$

so that $\Phi$ is uniquely defined by the vector

$$\Phi = (\Phi_\alpha), \quad \Phi_\alpha = \Phi(u_\alpha) \in C_1. \quad (A3.1.78)$$

We extend $\Phi$ to the map $\Omega^1(C) \rightarrow \Omega^1(C_1)$ by the rule

$$\Phi(\psi d\xi^{(\nu)}_\alpha \varphi) = \Phi(\psi) d[\Phi(\xi^{(\nu)}_\alpha)] \Phi(\varphi). \quad (A3.1.79)$$

Thus,

$$\Phi(d\xi_\alpha) = \partial(\Phi_\alpha) = \sum \frac{\partial \Phi_\alpha}{\partial u^{(\nu)}_\eta}(du^{(\nu)}_\eta) \sim \sum \frac{\partial \Phi_\alpha}{\partial u_\eta}(du_\eta) = \delta_1(\Phi_\alpha), \quad (A3.1.80)$$

where the subscript “1” is used to distinguish operations in $C_1$ from those in $C$.

**Exercise A3.1.81.** Show that $\Phi$ commutes with $d$:

$$\Phi d\psi_\alpha = d\psi_\alpha \Phi : \quad C \rightarrow \Omega^1(C_1). \quad (A3.1.82)$$

**Lemma A3.1.83.** $\delta_1 \Phi \delta = \delta_1 \Phi$.

**Proof.** We have to show that

$$\delta_1 \Phi(\delta - 1) = 0. \quad (A3.1.84)$$

Since $\Phi$ commutes with the actions of the $\partial_i$’s,

$$\Phi(\mathcal{J}) \subset \mathcal{J}_1. \quad (A3.1.85)$$

By (A3.1.43), $Im(\delta - 1) \subset \mathcal{J}$. Hence $Im(\delta - 1) \subset \mathcal{J}_1$. Therefore, $\delta_1 \Phi(\delta - 1) = 0$, which is (A3.1.84).

**Theorem A3.1.86.** For any $H \in C$,

$$\frac{\delta \Phi(H)}{\delta u_\eta} = \sum \Phi(\frac{\delta H}{\partial q_\alpha})(\frac{D\Phi_\alpha}{D u_\eta})^\dagger, \quad (A3.1.87)$$

where: $\Phi(\frac{\delta H}{\partial q_\alpha})$ is determined by the natural extension of $\Phi$ into a homomorphism
\[ O\rho_0(C) \to O\rho_0(C_1); \left( \frac{D\Phi_\alpha}{Du_\eta} \right)^\dagger \] acts to the left; and
\[ \frac{D\Phi_\alpha}{Du_\eta} = \sum_\nu \frac{\partial\Phi_\alpha}{\partial u_\eta^{(\nu)}} \partial^\nu. \tag{A3.1.88} \]

**Proof.** Notice that formula (A3.1.88) results from the definition (3.1.72): \( \forall X \in D^{ev}(C_1) \),
\[ X(\Phi_\alpha) = \left( \sum X_\eta^{(\sigma)} \frac{\partial}{\partial u_\eta^{(\sigma)}} \right)(\Phi_\alpha) = \sum \frac{\partial\Phi_\alpha}{\partial u_\eta^{(\sigma)}} (X_\eta^{(\sigma)}) = \sum_{\eta} \left( \sum_\sigma \frac{\partial\Phi_\alpha}{\partial u_\eta^{(\sigma)}} \partial^\sigma \right)(X_\eta). \]

Next,
\[ \sum_\eta \frac{\delta\Phi(H)}{\delta u_\eta} (du_\eta) = \delta_1\Phi(H) = \delta_1 d\Phi(H) = \delta_1\Phi d(H) \quad \text{[by Lemma A3.1.83]} = \]
\[ \delta_1 \Phi \delta d(H) = \delta_1 \Phi \delta(H) = \delta_1 \Phi \left( \sum_\alpha \frac{\delta H}{\delta q_\alpha} (dq_\alpha) \right) = \delta_1 \left( \sum_\alpha \Phi \left( \frac{\delta H}{\delta q_\alpha} \right) \partial^\alpha \right) = \]
\[ \sum_\alpha \Phi \left( \frac{\delta H}{\delta q_\alpha} \right) \left( \sum_\nu \frac{\partial\Phi_\alpha}{\partial u_\eta^{(\nu)}} \right) (du_\eta) \Rightarrow \frac{\delta\Phi(H)}{\delta u_\eta} = \sum_\alpha \left( -\partial \right)^\alpha \Phi \left( \frac{\delta H}{\delta q_\alpha} \right) \left( \sum_\nu \frac{\partial\Phi_\alpha}{\partial u_\eta^{(\nu)}} \partial^\nu \right) = \]
\[ \sum_\alpha \Phi \left( \frac{\delta H}{\delta q_\alpha} \right) \left( \sum_\nu \frac{\partial\Phi_\alpha}{\partial u_\eta^{(\nu)}} \partial^\nu \right)^\dagger = \sum_\alpha \Phi \left( \frac{\delta H}{\delta q_\alpha} \right) \left( \frac{D\Phi_\alpha}{Du_\eta} \right)^\dagger. \tag{A3.1.89} \]

**Exercise A3.1.90.** Suppose \( m = 1, C = \bar{R} < q^{(n)} > \). Show that
\[ \frac{\delta([q, q^{(n)}])}{\delta q} = [-1 + (-1)^n] a d q^{(n)}. \tag{A3.1.91} \]

§A3.2. THE IMAGE AND KERNEL OF THE VARIATIONAL OPERATOR \( \delta \)

In this Section we determine \( \text{Im}\delta \) and \( \text{Ker}\delta \), in that order. This is done with the help of the variational complex.

Let \( \partial_{m+1} : \bar{R} \to 0 \) be a new derivation acting trivially on \( \bar{R} \). Denote \( \bar{C} = \bar{R} < q_\alpha^{(\nu)} >, \alpha \in I, \bar{\nu} \in \mathbb{Z}_+^m \). Writing \( \bar{\nu} = (\nu|m) \), \( \nu \in \mathbb{Z}_+^m \), \( n \in \mathbb{Z}_+ \), we have \( \bar{C} = \bar{R} < q_\alpha^{(\nu|n)} >; C \) is thereby identified with the subring \( \bar{R} < q_\alpha^{(\nu|0)} > \) of \( \bar{C} \). \( \bar{C} \) is a \( \bar{C} \)-bimodule, and also a \( \bar{C} \)-bimodule.

Denote by \( \tau : \Omega^1(C) \to \bar{C} \) the following homomorphism of \( C \)-bimodules:
\[ \tau(\psi d q_\alpha^{(\nu)} \varphi) = \psi q_\alpha^{(\nu|1)} \varphi. \tag{A3.2.1} \]
Exercise A3.2.2. Show that $\tau$ commutes with the actions of the $\partial_i$’s. Hence,
\[ \tau(\mathcal{J}(\Omega^1(C))) \subset \mathcal{J}(\bar{C}). \tag{A3.2.3} \]

Lemma A3.2.4. For any $H \in C$,
\[ \tau d(H) = \partial_{m+1}(H). \tag{A3.2.5} \]

Proof. We have,
\[
\tau d(H) = \tau\left( \sum \frac{\partial H}{\partial q^{(\nu)}_\alpha} (dq^{(\nu)}_\alpha) \right) \text{ [by (A3.2.1)]} = \sum \frac{\partial H}{\partial q^{(\nu)}_\alpha} (q^{(\nu+1)}_\alpha) = \\
= \sum \frac{\partial H}{\partial q^{(\nu)}_\alpha} (\partial_{m+1}(q^{(\nu)}_\alpha)) = \partial_{m+1}(H). \]

To distinguish operations in $C$ and $\bar{C}$, we use bar over $\delta$: $\bar{\delta}$, whenever we are in $\bar{C}$.

Theorem A3.2.6. The sequence
\[ C \xrightarrow{\delta} \Omega^1_0(C) \xrightarrow{\bar{\delta}} \Omega^1_0(\bar{C}) \quad \tag{A3.2.7} \]
is a complex, i.e.
\[ \bar{\delta} \tau \delta = 0. \tag{A3.2.8} \]

Proof. For any $H \in C$, $\delta(H) = \hat{\delta} d(H)$ [by (A3.1.43)] $\sim d(H)$. Hence, by (A3.2.3), $\tau \delta(H) \sim \tau d(H)$ [by (A3.2.5)] $= \partial_{m+1}(H) \sim 0$. Thus, $\delta \tau \delta(H) = 0$.

Remark A3.2.9. In the abelian case, the equality $\bar{\delta} \tau \delta = 0$, written in components, amounts to the self-adjointness of the Fréchet derivative $D\left(\frac{\delta H}{\delta q}\right)$:
\[ D\left(\frac{\delta H}{\delta q}\right)^\dagger = D\left(\frac{\delta H}{\delta q}\right), \quad \tag{A3.2.10} \]
which is, matrix-element wise:
\[ \left[ \sum_{\nu} \frac{\partial}{\partial q^{(\nu)}_\beta} \left( \frac{\delta H}{\delta q_\alpha} \right) \theta^{\nu} \right]^\dagger = \sum_{\nu} \frac{\partial}{\partial q^{(\nu)}_\alpha} \left( \frac{\delta H}{\delta q_\beta} \right) \theta^{\nu}. \tag{A3.2.11} \]

In the nonabelian case, it takes some effort to make sense out of the equality (A3.2.11) of tri-operators. One has to order these operators, like, e.g., in the equality
\[ D(\hat{L}_a \hat{R}_b) = \hat{R}_b^{[2]} \hat{L}_a^{[2]} D(a)_{[1]} + \hat{L}_a^{[2]} \hat{R}_b^{[2]} D(b)_{[1]}, \tag{A3.2.12} \]
indicating who acts first, second, etc. We shall avoid this route altogether and prove directly that the sequence (A3.2.7) is exact.

Theorem A3.2.13. The sequence (A3.2.7) is exact, i.e., if $\omega \in \Omega^1_0(C)$ and $\delta \tau(\omega) = 0$ then there exists $H \in C$ such that $\delta(H) = \omega$. 

**Proof.** Denote by $A_t : C \to C[t]$ (and also $\tilde{C} \to \tilde{C}[t]$) the homomorphism over $\tilde{R}$ which acts on the generators of $C$ by the rule

$$A_t(a^{(\nu)}) = t a^{(\nu)}.$$  

Extend the homomorphism $A_t$ from $C$ onto $O p_0 (C)$ naturally:

$$A_t(\tilde{L}_a \tilde{R}_b) = \tilde{L}_{A_t(a)} \tilde{R}_{A_t(b)}.$$  

Now, if $\omega = \sum_{\alpha \nu} \omega^{\nu}_\alpha (dq^{(\nu)}_\alpha)$, with some $\omega^{\nu}_\alpha \in O p_0 (C)$, we define

$$A_t(\omega) = \sum_{\alpha} [A_t(\omega^{\nu}_\alpha)](dq^{(\nu)}_\alpha).$$  

**Lemma A3.2.17.** If $\omega \in \Omega^1_0 (C)$ and $\delta \tau (\omega) = 0$ then also $\delta \tau (A_t(\omega)) = 0$.

**Proof.** This follows from the identity

$$\delta \tau A_t = A_t \delta \tau \quad \text{on } \Omega^1 (C).$$  

Indeed, since $\delta$ decreases the $q$-degree by 1,

$$\delta A_t = t A_t \delta \quad \text{on } \tilde{C}.$$  

Hence, for a typical element $\psi dq^{(\nu)}_\alpha \varphi \in \Omega^1 (C)$ we get

$$\delta \tau A_t (\psi dq^{(\nu)}_\alpha \varphi) = \delta \tau [A_t (\psi) dq^{(\nu)}_\alpha A_t (\varphi)] = \delta [A_t (\psi) pq^{(\nu)}_\alpha (1) A_t (\varphi)] =$$

$$= \delta t^{-1} A_t (\psi pq^{(\nu)}_\alpha (1) \varphi) \quad \text{by } (A3.2.19) = A_t \delta (\psi pq^{(\nu)}_\alpha (1) \varphi) = A_t \delta \tau (\psi dq^{(\nu)}_\alpha \varphi).$$  

Now, given $\omega = \sum \omega^{\nu}_\alpha (dq^{(\nu)}_\alpha) \in \Omega^1_0 (C)$ from the Kernel of $\delta \tau$: $\delta \tau (\omega) = 0$, we define

$$H = \sum_{\alpha} \int_0^1 dt A_t (\omega^{\nu}_\alpha (q_\alpha)).$$  

The claim is that this $H$ does the job:

$$\delta (H) = \omega.$$  

To prove this equality, we appeal to Proposition A3.1.64 and prove instead that

$$X(H) \sim < \omega, X > = \omega (X) = \sum \omega^{\nu}_\alpha (X^{\nu}_\alpha), \quad \forall X \in D^{1v} (C').$$  

Denoting temporarily

$$\bar{\omega} = A_t (\omega), \quad \bar{\omega}_{\alpha} = A_t (\omega^{\nu}_\alpha),$$

we have:

$$X(H) = X \left( \sum \int_0^1 dt \bar{\omega}_{\alpha} (q_\alpha) \right) = \sum \int_0^1 dt \left[ \bar{\omega}_{\alpha} (X^{\nu}_\alpha) + (X (\bar{\omega}_{\alpha})) (q_\alpha) \right].$$
Lemma A3.2.23. For any $\omega \in \Omega_0^1(C) \cap \text{Ker} \delta \tau$,

$$X(\omega_\alpha) = \frac{\delta<\omega, X>}{\delta q_\alpha}, \ \forall X \in D_{n_\nu}'(C').$$  \hfill (A3.2.24)

Granted the Lemma, we have

$$\sum \alpha [X(\bar{\omega}_\alpha)](q_\alpha) = \sum \alpha \frac{\delta[\omega(X)]}{\delta q_\alpha}(q_\alpha) \sim X^{rad}[\bar{\omega}(X)],$$  \hfill (A3.2.25)

where $X^{rad}$ is the radial evolution field:

$$X^{rad} = q : \ X^{rad} = \sum q^{(\nu)}_\alpha \frac{\partial}{\partial q^{(\nu)}_\alpha}.$$  \hfill (A3.2.26)

Now, since $X \in D_{n_\nu}'(C')$,

$$X^{rad}[\bar{\omega}(X)] = \sum \alpha [X^{rad}(\bar{\omega}_\alpha)](X_\alpha).$$  \hfill (A3.2.27)

Hence, (A3.2.25) becomes

$$\sum \alpha [X(\bar{\omega}_\alpha)](q_\alpha) \sim \sum \alpha [X^{rad}(\bar{\omega}_\alpha)](X_\alpha).$$  \hfill (A3.2.28)

Substituting this into (A3.2.22) we get

$$X(H) \sim \sum [\int_0^1 dt(1 + X^{rad})A_t(\omega_\alpha)](X_\alpha).$$  \hfill (A3.2.29)

Since the field $X^{rad}$ is an operation of $C$-degree zero,

$$X^{rad}A_t = A_tX^{rad}.$$  \hfill (A3.2.30)

Thus, (A3.2.29) becomes

$$X(H) \sim \sum <\int_0^1 dtA_t(1 + X^{rad})(\omega_\alpha), X_\alpha >.$$  \hfill (A3.2.31)

Lemma A3.2.32.

$$\int_0^1 dtA_t(1 + X^{rad}) = 1.$$  \hfill (A3.2.33)

Proof. It’s enough to prove (A3.2.33) on $C$. Take any monomial $r = kq^{(\nu_1)}_\alpha \ldots q^{(\nu_n)}_\alpha$. Then $(1 + X^{rad})(r) = (1 + n)r$; therefore

$$A_t(1 + X^{rad})(r) = (1 + n)t^n r, \ \text{and thus} \ \int_0^1 dt(1 + n)t^n = 1.$$  \hfill \blacksquare

Substituting (A3.2.33) into (A3.2.31) we arrive at (A3.2.21), which proves Theorem (A3.2.13). \hfill \blacksquare
**Proof of Lemma A3.2.23.** Let us write down in the long-hand the equality $\delta \tau(\omega) = 0$. Let

$$\omega = \sum \psi_{\alpha} dq_{\alpha} \varphi_{\alpha} \Rightarrow$$

$$0 = \delta \tau(\omega) = \delta(\sum \psi_{\alpha} q_{\alpha}^{(0)[1]} \varphi_{\alpha}) = \sum\left[ \frac{\delta}{\delta q_{\beta}} (\psi_{\alpha} q_{\alpha}^{(0)[1]} \varphi_{\alpha}) \right] (dq_{\beta}) \Rightarrow$$

$$0 = \frac{\delta}{\delta q_{\beta}} \left( \sum \psi_{\alpha} q_{\alpha}^{(0)[1]} \varphi_{\alpha} \right) = \sum (\nabla) \frac{\partial}{\partial q_{\beta}^{(\nu)}} \left( \nabla \right) (\sum \psi_{\alpha} q_{\alpha}^{(0)[1]} \varphi_{\alpha}) =$$

$$= \sum (\nabla) \frac{\partial}{\partial q_{\beta}^{(\nu)}} \left( \sum \psi_{\alpha} q_{\alpha}^{(0)[1]} \varphi_{\alpha} \right) - \partial_{m+1} \frac{\partial}{\partial q_{\beta}^{(0)[1]}} \left( \sum \psi_{\alpha} q_{\alpha}^{(0)[1]} \varphi_{\alpha} \right) \Rightarrow$$

$$\sum (\nabla) \frac{\partial}{\partial q_{\beta}^{(\nu)}} \left( <\omega, q_{\alpha}^{(0)[1]} > \right) = \sum \partial_{m+1} (\hat{L} \psi_{\beta} \hat{R} \psi_{\beta}) =$$

$$= q_{\alpha}^{(0)[1]} \left( \sum \hat{L} \psi_{\beta} \hat{R} \varphi_{\beta} \right) = q_{\alpha}^{(0)[1]} (\omega_{\beta}), \quad (A3.2.35)$$

where $q_{\alpha}^{(0)[1]}$ is the derivation $\partial_{m+1}$:

$$q_{\alpha}^{(0)[1]} = \sum q_{\gamma}^{(\sigma)[1]} \frac{\partial}{\partial q_{\gamma}^{(\sigma)}} = \partial_{m+1}. \quad (A3.2.36)$$

Since the $q_{\alpha}^{(0)[1]}$’s are differentially independent over the ring $Op_{0}(C)[\partial_{\sigma}]$, we can substitute into the equality (A3.2.35) an arbitrary $X \in D^{cv}(C)$ instead of $q_{\alpha}^{(0)[1]}$, resulting in the desired equality

$$\frac{\delta}{\delta q_{\beta}} (\omega, X) = X(\omega_{\beta}).$$

**Remark A3.2.37.** To prove the equality $\delta(H) = \omega$ (A3.2.20b), we instead proved the equivalent relation $X(H) \sim <\omega, X>$ for a very special class of $X$’s, namely these $X \in D^{cv}(C)$. Of course, this relation holds a posteriori for all $X \in D^{cv}(C)$. This amounts to the relation

$$\sum [X(\omega_{\alpha})](q_{\alpha}) \sim \sum [X^{rad}(\omega_{\alpha})](X_{\alpha}), \quad \forall \omega \in Ker \delta \tau \cap \Omega_{0}^{1}(C), \quad \forall X \in D^{cv}(C), \quad (A3.2.38)$$

which is equivalent to the equality

$$\frac{\delta(\sum_{\alpha} \omega_{\alpha}(q_{\alpha}))}{\delta q_{\beta}} = X^{rad}(\omega_{\beta}), \quad \forall \omega \in Ker \delta \tau \cap \Omega_{0}^{1}(C). \quad (A3.2.39)$$

We now can determine $Ker \delta$.

**Exercise A3.2.40.** Show that

$$X^{rad} = t(A_{t})^{-1} \frac{\partial}{\partial t} A_{t}. \quad (A3.2.41)$$
**Corollary A3.2.42.**
\[ \frac{\partial}{\partial t} A_t = t^{-1} A_t X^{\text{rad}}. \quad (A3.2.43) \]

**Theorem A3.2.44.** \( \text{Ker}\delta = J + \tilde{R}. \)

**Proof** For any \( f \in C \) we have:
\[
 f - f \bigg|_{q=0} = A_1(f) - A_0(f) = \int_0^1 dt \frac{\partial}{\partial t} A_t(f) \quad [\text{by (A3.2.43)}] =
\]
\[
 = \int_0^1 dt t^{-1} A_t X^{\text{rad}}(f) \sim \int_0^1 dt t^{-1} A_t \frac{\delta f}{\delta q}(X^{\text{rad}}). \quad (A3.2.45)
\]

If now \( f \in \text{Ker}\delta \), the last term on the right of (A3.2.45) vanishes, and we get
\[
\frac{\delta f}{\delta q} = 0 \implies f \sim f|_{q=0} \in \tilde{R}. \]

**Remark A.3.2.46.** For simplicity, the discrete degrees of freedom have been left out of this Appendix. Had they been let in, everything in §§A3.1, 2 would have been exactly the same apart from minor differences in notation. E.g., formula (A3.1.49) should be replaced by the formula
\[
\frac{\delta H}{\delta q_\alpha} = \sum_{\nu_\eta} (-\partial)^{\nu_\eta} \hat{g}^{-1} \left( \frac{\partial H}{\partial q^{(\nu_\eta)}_\alpha} \right), \quad (A3.2.47)
\]

etc.

**Remark A3.2.48** The attentive reader would have noticed that the variational calculus developed in this Appendix is different from the one developed in Parts A, B. In the latter, we calculate modulo
\[
\left\{ \sum \text{Im}(\partial_s), \sum \text{Im}(\hat{g} - \hat{e}) \right\} \text{ and commutators.} \quad (A3.2.49)
\]

We do not neglect commutators in this Appendix, following therefore the familiar abelian philosophy. (This distinction becomes apparent at once when one looks at quantum and \( q \)-integrable systems.)

**Exercise A3.2.50.** Suppose that only the discrete degrees of freedom are present, parametrized by elements \( g \) of a discrete group \( G \).

(i) Show that
\[
\frac{\partial}{\partial q_\alpha} \left( \sum_{g \in G} \hat{g}(H) \right) = \frac{\delta H}{\delta q_\alpha}, \quad \forall H \in C, \quad (A3.2.51a)
\]
\[
\frac{\partial}{\partial q_\alpha^{(h)}} \left( \sum_{g \in G} \hat{g}(H) \right) = \hat{h} \left( \frac{\delta H}{\delta q_\alpha} \right), \quad \forall H \in C, \quad \forall h \in G; \quad (A3.2.51b)
\]

(ii) Similarly, show that in the variational calculus of Parts, A, B,
\[
\hat{h}^{-1} \frac{\partial}{\partial q_\alpha^{(h)}} \left( \sum_{g \in G} \hat{g}(H) \right) = \frac{\delta H}{\delta q_\alpha}, \quad \forall H \in C, \quad \forall h \in G; \quad (A3.2.52)
\]
(iii) Suppose we have a family \( \{ H(g) \in C \mid g \in G \} \) such that:

\[
V_\alpha := \hat{h}^{-1} \frac{\partial}{\partial q_\alpha^{(h)}} \sum_{g \in G} H(g) \quad \text{is } h - \text{independent, } \forall h \in G,
\]

(A3.2.53a)

\[
\delta T \left( \sum_\alpha V_\alpha(dq_\alpha) \right) = 0.
\]

(A3.2.53b)

Can one find \( H \in C \) such that

\[
\sum_{g \in G} (H(g) - \hat{g}(H)) \quad \text{is trivial?}
\]

(A3.2.53c)

\section{A3.3. THE IMAGE OF THE OPERATOR \( \partial + \epsilon ad_u \)}

It’s all right in practice but it won’t work in theory.

Bishop Stubb on abstract thinking

In §3.1 we have seen that the motion equations of the type

\[
\partial_t(v) = (\partial + \epsilon ad_u)(x), \quad \epsilon = \pm 1;
\]

(A3.3.1)

allow the lifts

\[
Pot_+ (v) = -V^{(1)} V^{-1}, \quad \partial_t(V) = -Pot_+(x)V,
\]

(A3.3.2)

\[
Pot_- (v) = -V^{-1} V^{(1)}, \quad \partial_t(V) = -VPot_-(x),
\]

(A3.3.3)

which are homomorphisms of the corresponding Lie algebras of evolution derivations. In this Section we examine a model problem of how to describe the Image of the operator \( \partial + \epsilon ad_u \).

So, let us take \( C = \tilde{R} < u^{(n)} >, \ n = Z_+ \), with \( m = 1 \). The Lie algebra \( \text{Der}(C) \) acts on \( C \), and hence it acts by derivations, in the Lie algebra sense, on the Lie algebra \( \text{Lie}(C) \), with the commutator

\[
[a, b] = ab - ba, \quad a, b \in C.
\]

(A3.3.4)

Hence, we can form the semidirect sum Lie algebra \( \text{Der}(C) \ltimes \text{Lie}(C) \), with the commutator

\[
[Z_1 + ad_u, Z_2 + ad_u] = [Z_1, Z_2] + ad_{Z_1(b) - Z_2(u) + [a, b]},
\]

(A3.3.5)

for any \( Z_1, Z_2 \in \text{Der}(C) \); the same formula applies when \( Z_1 \) and \( Z_2 \) are derivations of \( C \) not necessarily over \( \tilde{R} \). For example \( Z_2 \) can be \( \partial \).

Let us determine the centralizer \( Z(\partial + \epsilon ad_u) \) of the element \( \partial + \epsilon ad_u \) in \( \text{Der}(C) \ltimes \text{Lie}(C) \). Using formula (A3.3.5) with \( Z_2 = \partial \) and \( b = \epsilon u \) we find that \( Z_1 \in D^{ev}(C) \) and

\[
\epsilon Z_1(u) = a^{(1)} + \epsilon [u, a].
\]

(A3.3.6)
Thus,

\[ Z := Z(\partial + \epsilon ad_u) = \{ X_a = \epsilon ad_u + \sum_{n \geq 0} (a^{(1)} + \epsilon [u, a])^{(n)} \frac{\partial}{\partial u^{(n)}} \mid a \in C \}. \]  

(A3.3.7)

So far \( \epsilon \) was thought to be \( \pm 1 \). However, if we treat \( \epsilon \) as a formal deformation parameter, we see from formula (A3.3.7) that

\[
[Z(\partial + \epsilon ad_u)]_{\epsilon=0} = \{ \sum a^{(n+1)} \frac{\partial}{\partial u^{(n)}} \} \neq Z[(\partial + \epsilon ad_u)]_{\epsilon=0} = \]

\[
Z(\partial) = \{ \sum a^{(n)} \frac{\partial}{\partial u^{(n)}} \} = D^{\epsilon u}(C). \]  

(A3.3.8)

Thus, we are dealing with a singular (in \( \epsilon \)) perturbation of the calculus of variations. The latter can be thought of as a machine to construct resolutions of first-order differential operators in such a way that cohomologies are either zero or tightly controlled. Let us apply the variational procedure to the operator \( \partial + \epsilon ad_u \).

Let us introduce an equivalence relation \( \approx \) on \( C[e] \) by the rule:

\[ a \approx b \text{ iff } (a - b) \in Im(\partial + \epsilon ad_u). \]  

(A3.3.9)

Next, we define the action of \( ad_u \) on \( Op_0(C)[e] \):

\[ ad_u(\mathcal{O}) = (L_u - R_u)\mathcal{O}. \]  

(A3.3.10)

**Lemma A3.3.11.**

\[ \mathcal{O}(\partial(r)) \approx -[\mathcal{O}(\partial + \epsilon ad_u)(\mathcal{O})](r), \quad \forall \mathcal{O} \in Op_0(C)[e], \quad \forall r \in C[e]. \]  

(A3.3.12)

**Proof.** We have, for \( \mathcal{O} = \hat{L}_a \hat{R}_b \):

\[ \mathcal{O}(\partial(r)) \approx \mathcal{O}(\partial(r)) - (\partial + \epsilon ad_u)(\mathcal{O}(r)) = \]

\[ ar^{(1)}b - (ar^{(1)}b + a^{(1)}rb + arb^{(1)}) + \epsilon warb - earbu) = \]

\[ = -\mathcal{O}^{(1)}(r) - [\epsilon ad_u(\mathcal{O})](r) = -[(\partial + \epsilon ad_u)(\mathcal{O})](r). \]

**Corollary A3.3.13.**

\[ \mathcal{O}(\partial^n(r)) \approx \{ -[\mathcal{O}(\partial + \epsilon ad_u)]^n(\mathcal{O}) \}(r), \quad \forall \mathcal{O} \in Op_0(C)[e], \quad \forall r \in C[e]. \]  

(A3.3.14)

Now, for any \( H \in C \) (or \( C[e] \)) and any \( X_a \in Z \) (or \( Z[e] \)), we have

\[ X_a(H) = \epsilon ad_u(H) + \sum_{n \geq 0} \frac{\partial H}{\partial u^{(n)}} [(a^{(1)} + \epsilon [u, a])^{(n)}] \text{ [by (A3.3.14)]} \approx \]

\[ \approx -\epsilon ad_H(a) + \left( \sum (-\partial - \epsilon ad_u)^n \left( \frac{\partial H}{\partial u^{(n)}} \right) \right) (a^{(1)} + \epsilon [u, a]) \approx \]  

(A3.3.15)
\[ \approx \{-\epsilon a d_H + \sum (-\partial - \epsilon a d_u)^{n+1} \left( \frac{\partial H}{\partial u^{(n)}} \right) + \epsilon \sum (-\partial - \epsilon a d_u)^n \left( \frac{\partial H}{\partial u^{(n)}} \right) \} a d_u \{a\}. \]

This calculation defines the element \( \frac{\tilde{\delta} H}{\partial u} \in O p_0(C)[\epsilon] \):

\[ \frac{\tilde{\delta} H}{\partial u} = -\epsilon a d_H + \sum (-\partial - \epsilon a d_u)^{n+1} \left( \frac{\partial H}{\partial u^{(n)}} \right) + \epsilon \sum (-\partial - \epsilon a d_u)^n \left( \frac{\partial H}{\partial u^{(n)}} \right) \]  
with the defining relation

\[ X_a(H) \approx \frac{\tilde{\delta} H}{\partial u} (a), \ \forall a \in C \text{ (or } C[\epsilon]). \quad (A3.3.16) \]

The element \( \frac{\tilde{\delta} H}{\partial u} \) satisfying (A3.3.17) is unique thanks to the following analog of the DuBois-Reymond Lemma:

**Lemma A3.3.18.** If \( \mathcal{O} \in O p_0(C)[\epsilon] \) and

\[ \mathcal{O}(a) \approx 0, \ \forall a \in C, \quad (A3.3.19) \]

then \( \mathcal{O} = 0. \)

**Proof.** Suppose \( \mathcal{O} = \sum_{i \geq n} \mathcal{O}_i \epsilon^i, \ \mathcal{O}_i \in O p_0(C). \) We show that \( \mathcal{O}_n = 0. \) This follows by letting \( \epsilon = 0 \) in the equality

\[ \epsilon^{-n} \left( \sum \epsilon^i \mathcal{O}_i (a) \right) \approx 0, \ \forall a \in C, \]

which becomes

\[ \mathcal{O}_n(a) \approx 0, \ \forall a \in C, \]

and now the DuBois-Reymond Lemma A3.1.54 guarantees that \( \mathcal{O}_n = 0. \)

Further, if \( H \approx 0, \ H = (\partial + \epsilon a d_u)(F), \) then

\[ X_a(H) = X_a(\partial + \epsilon a d_u)(F) = (\partial + \epsilon a d_u)(X_a(F)) \approx 0 \approx \frac{\tilde{\delta} H}{\partial u} (a), \ \forall a \in C, \]

so that \( \frac{\tilde{\delta} H}{\partial u} = 0. \) In other words,

\[ H \approx 0 \Rightarrow \frac{\tilde{\delta} H}{\partial u} = 0. \quad (A3.3.20) \]

This can be rewritten as

\[ \mathcal{J} \subset K e r \tilde{\delta}, \quad (A3.3.21) \]

where

\[ \mathcal{J} = I m(\partial + \epsilon a d_u), \ \tilde{\delta} : C[\epsilon] \hookrightarrow O p_0(C)[\epsilon], \ \tilde{\delta}(H) : = \frac{\tilde{\delta} H}{\partial u}. \quad (A3.3.22) \]
Exercise A3.3.23. Show that

\[ \text{Im}(\partial + \epsilon ad_u) \bigg|_{\tilde{R}} = \text{Im}(\partial) \bigg|_{\tilde{R}} = \mathcal{J} \cap \tilde{R}, \quad (A3.3.24) \]

\[ \text{Ker}(\tilde{\delta}) \bigg|_{\tilde{R}} = Z(\tilde{R}). \quad (A3.3.25) \]

Formulae (A3.3.24,25) suggest that in analyzing the factorspace \( \text{Ker}\tilde{\delta}/\tilde{\mathcal{J}} \) one should concentrate on the case \( \tilde{R} = \mathbb{Z} \) first.

The reader is invited to determine \( \text{Ker}\tilde{\delta}/\tilde{\mathcal{J}} \).

He probably said to himself, "Must stop or I shall be getting silly." That is why there are only ten commandments.

Mrs. Patrick Campbell on Moses

APPENDIX A4. HAMILTONIAN CORRESPONDENCIES

The progression from maps of manifolds \( \varphi : M \rightarrow N \), to graphs \( \Gamma_{\varphi} \subset M \times N \), to multivalued maps, to correspondencies (as submanifolds in \( M \times N \)), - how could it be described in the Hamiltonian framework? This can be most easily achieved by passing from a geometric to an algebraic language. First, the finite-dimensional case is considered in the next Section. The general functional case is examined in §A4.2. The formalism developed in §A4.2 is then put to work in §§A4.3,4 on various examples.

§A4.1. FROM GEOMETRY TO ALGEBRA

Tradition may be explained as an extension of the franchise. Tradition means giving the vote to the most obscure of all classes, our ancestors. It is the democracy of the dead. Tradition refuses to submit to the small and arrogant oligarchy of those who merely happen to be walking about.

Chesterton

In this Section we first reformulate the condition that two tensor fields, \( T \) on \( M \) and \( \mathcal{T} \) on \( N \), are compatible with respect to a smooth map \( \varphi : M \rightarrow N \). The main idea is to replace such a map by the ideal \( \mathcal{E} \) of functions on \( M \times N \) vanishing on the graph \( \Gamma_{\varphi} \subset M \times N \) of the map \( \varphi \). When the tensors \( T \) and \( \mathcal{T} \) are of the type \((0,2)\), describing the Poisson brackets case, we establish a purely algebraic criterion for a map \( \varphi \) to be Poisson, in terms of the ring \( \mathcal{C}^\infty(M \times N) \).

As a preliminary issue, let \( S \) be a submanifold of a manifold \( \mathcal{M} \), and let \( t \) be a tensor field of \( \mathcal{M} \). We want to make sense out of and then reformulate algebraically
the condition
\[ t \big|_S = 0. \]  
(A4.1.1)

Denote by \( \mathcal{E} = \mathcal{E}(S) \) the ideal of functions of \( \mathcal{M} \) which vanish on \( S \).

If \( t \) is a function of \( \mathcal{M} \), i.e., a tensor of \((0,0)\)-type, then the condition \( t|_S = 0 \) may be rewritten as
\[ t \in \mathcal{E} \]  
(A4.1.2a)

or as
\[ [t] = 0, \]  
(A4.1.2b)

where \([t]\) is the class of the element \( t \in C^\infty(\mathcal{M}) \) in the factor-ring \( C^\infty(\mathcal{M})/\mathcal{E} \).

If \( t \) is a 1-form on \( \mathcal{M} \), \( t \in \Lambda^1(\mathcal{M}) \), then the condition \( t|_S = 0 \) can be rewritten as
\[ t(X) \in \mathcal{E}, \ \forall X \in \text{Norm}(\mathcal{E}) \]  
(A4.1.3a)

or as
\[ [t(X)] = 0, \ \forall X \in \text{Norm}(\mathcal{E}), \]  
(A4.1.3b)

where \( X \in \text{Norm}(\mathcal{E}) \) denotes a vector field on \( \mathcal{M} \) tangent to \( S \); in algebraic terms this means that
\[ X(\mathcal{E}) \subset \mathcal{E}, \]  
(A4.1.4)

i.e., \( X \) belongs to the normalizer of \( \mathcal{E} \).

If \( t \) is a tensor product of 1-forms on \( \mathcal{M} \) (or a sum of such), \( t \in \Lambda^\ell(\mathcal{M}) \), then the condition \( t|_S = 0 \) means that
\[ t(X_1, \ldots, X_\ell) \in \mathcal{E}, \ \forall X_1, \ldots, X_\ell \in \text{Norm}(\mathcal{E}), \]  
(A4.1.5a)

which is the same as
\[ [t(X_1, \ldots, X_\ell)] = 0, \ \forall X_1, \ldots, X_\ell \in \text{Norm}(\mathcal{E}). \]  
(A4.1.5b)

If \( t \) is a vector field on \( \mathcal{M} \), \( t \in \mathcal{D}(\mathcal{M}) \), then the object \( t|_S \) is not defined unless the vector field \( t \) is tangent to \( S \). The latter condition, \( t \) is tangent to \( S \), may be expressed in terms of \( \mathcal{D}(\mathcal{M}) \) as
\[ [t] = 0, \]  
(A4.1.6)

which \([t]\) is the class of the element \( t \in \mathcal{D}(\mathcal{M}) \) in the factor-space \( \mathcal{D}(\mathcal{M})/\text{Norm}(\mathcal{E}) \). (\( \text{Norm}(\mathcal{E}) \) is not an ideal in the Lie algebra \( \mathcal{D}(\mathcal{M}) \), so the factor-space \( \mathcal{D}(\mathcal{M})/\text{Norm}(\mathcal{E}) \) is not a Lie algebra either.)

If \( t \) is a tensor product of vector fields on \( \mathcal{M} \) (or a sum of such), \( t \in \mathcal{D}(\mathcal{M})^\otimes L \), then we define the condition \( t|_S = 0 \) by the equality
\[ [t] = 0, \]  
(A4.1.7)

where \([t]\) is the class of the element \( t \) in the factor-space \( (\mathcal{D}(\mathcal{M})/\text{Norm}(\mathcal{E}))^\otimes L \).
Finally, if $t$ is a tensor of the $(\ell, L)$-type on $\mathcal{M}$, we define the condition $t|_{S} = 0$ by the equation

$$[t(X_{1}, \ldots, X_{\ell})] = 0, \quad \forall X_{1}, \ldots, X_{\ell} \in \text{Norm}(\mathcal{E}).$$  \hfill (A4.1.8)

Now, let $M$ and $\mathcal{N}$ be two manifolds, and let $T$ and $\mathcal{T}$ be two tensor fields of the $(\ell, L)$-type defined on $M$ and $\mathcal{N}$ respectively. Let $\varphi : M \to N$ be a (smooth) map. With respect to this map, the tensor fields $T$ and $\mathcal{T}$ are compatible if, for each point $m \in M$,

$$\varphi_{*}(T|_{m}(\xi_{1}, \ldots, \xi_{\ell})) = \mathcal{T}|_{\varphi(m)}(\varphi_{*}(\xi_{1}), \ldots, \varphi_{*}(\xi_{\ell})), \quad \forall \xi_{1}, \ldots, \xi_{\ell} \in T_{m}(M).$$  \hfill (A4.1.9a)

Here we think of $(\ell, L)$-tensors as operators $D^{\otimes \ell} \to D^{\otimes L}$. When $\ell = 0$, the condition (A4.1.9a) takes the $\xi$-less form

$$\varphi_{*}(T|_{m}) = \mathcal{T}|_{\varphi(m)}, \quad \forall m \in M.$$  \hfill (A4.1.9b)

[When $L = 0$, the equation $\varphi_{*}(\cdot) = (\cdot)$ is to be understood as $(\cdot) = \varphi^{*}(\cdot)$.]  

Our first aim is to reformulate the compatibility condition (A4.1.9) in terms of the graph $\Gamma_{\varphi}$ considered as a submanifold in the manifold $M \times \mathcal{N}$. Recall that

$$\Gamma_{\varphi} = \{(m, \varphi(m)) \mid m \in M\}.$$  \hfill (A4.1.10)

Consider, on the product-manifold $M \times \mathcal{N}$, the $(\ell, L)$-type tensor

$$t = T - \mathcal{T}.$$  \hfill (A4.1.11)

This tensor is well-defined: since

$$T_{(m, n)}(M \times \mathcal{N}) = T_{m}(M) \oplus T_{n}(\mathcal{N}),$$  \hfill (A4.1.12)

$$t(\xi_{1} \oplus \eta_{1}, \ldots, \xi_{\ell} \oplus \eta_{\ell}) = T(\xi_{1}, \ldots, \xi_{\ell}) - T(\eta_{1}, \ldots, \eta_{\ell}), \quad \xi_{i} \in T_{m}(M), \; \eta_{i} \in T_{n}(\mathcal{N}).$$  \hfill (A4.1.13)

**Proposition A4.1.14.** The tensors $T$ and $\mathcal{T}$ are $\varphi$-compatible iff

$$(T - \mathcal{T})\bigg|_{\Gamma_{\varphi}} = 0.$$  \hfill (A4.1.15)

**Proof.** Let $(x^{i})$ and $(y^{\alpha})$ be local coordinates around $m \in M$ and $\varphi(m) \in \mathcal{N}$, respectively. In the induced local coordinates $(x^{i}, y^{\alpha})$ on $M \times \mathcal{N}$, the graph $\Gamma_{\varphi}$ is given by the equations

$$\{y^{\alpha} = \varphi^{\alpha}(x), \; \alpha = 1, \ldots, \text{dim}(\mathcal{N})\};$$  \hfill (A4.1.16)

the ideal $\mathcal{E} = \mathcal{E}(\Gamma_{\varphi})$ of functions on $M \times \mathcal{N}$ which vanish on $\Gamma_{\varphi}$ is generated by the elements

$$\theta^{\alpha} := y^{\alpha} - \varphi^{\alpha}(x), \; \alpha = 1, \ldots, \text{dim}(\mathcal{N}).$$  \hfill (A4.1.17)
The case $(\ell, L) = (0, 0)$: If $T$ and $\mathcal{T}$ are functions then

\[
(T - \mathcal{T})|_{\Gamma_{\varphi}} (m, \varphi(m)) = T(m) - \mathcal{T}(\varphi(m)) = (T - \varphi^*(\mathcal{T}))(m).
\]  

(A4.1.18)

Thus,

\[
\{(T - \mathcal{T})|_{\Gamma_{\varphi}} = 0\} \iff \{T = \varphi^*(\mathcal{T})\};
\]  

(A4.1.19)

**Lemma A4.1.20.** A tangent vector $(\xi \oplus \eta) \in T_{m, \varphi(m)}(M \times N)$ is tangent to the submanifold $\Gamma_{\varphi}$ at the point $(m, \varphi(m))$ iff

\[
\eta = \varphi_*(\xi).
\]  

(A4.1.21)

**Proof.** Let

\[
\xi = \sum_i \xi^i \frac{\partial}{\partial x^i}, \quad \eta = \sum_\alpha \eta^\alpha \frac{\partial}{\partial y^\alpha}.
\]  

(A4.1.22)

The vector $\xi \oplus \eta$ is tangent to $\Gamma_{\varphi}$ iff

\[
(\xi \oplus \eta)(y^\alpha - \varphi^\alpha(x)) = 0, \quad \alpha = 1, \ldots, \text{dim}(N).
\]  

(A4.1.23)

But

\[
(\xi \oplus \eta)(y^\alpha - \varphi^\alpha(x)) = \eta^\alpha - \sum_i \xi^i \frac{\partial \varphi^\alpha(x)}{\partial x^i} \bigg|_{x=x(m)}.
\]  

(A4.1.24)

This vanishes iff

\[
\eta = \sum_\alpha \eta^\alpha \frac{\partial}{\partial y^\alpha} = \sum_i \xi^i \frac{\partial \varphi^\alpha(x)}{\partial x^i} \bigg|_{x=x(m)} \frac{\partial}{\partial y^\alpha},
\]  

(A4.1.25)

and the R.H.S. is precisely $\varphi_*(\xi)$:

\[
[\varphi_*(\xi)]y^\alpha = \xi(\varphi_*(y^\alpha)) = \xi(\varphi^\alpha(x)) = \sum_i \xi^i \frac{\partial \varphi^\alpha(x)}{\partial x^i} \bigg|_{x=x(m)}.
\]  

(A4.1.26)

The case $(\ell, L) = (\ell, 0)$: Suppose

\[
T = \sum T(i)\, dx(i), \quad \mathcal{T} = \sum T(\alpha)\, dy(\alpha),
\]  

(A4.1.27a)

where

\[
dx(i) = dx^{i(1)} \otimes \ldots \otimes dx^{i(\ell)}, \quad dy(\alpha) = dy^{\alpha(1)} \otimes \ldots \otimes dy^{\alpha(\ell)}.
\]  

(A4.1.27b)

Then

\[
(T - \mathcal{T})(\xi_1 \oplus \eta_1, \ldots, \xi_\ell \oplus \eta_\ell) = T(\xi_1, \ldots, \xi_\ell) - \mathcal{T}(\eta_1, \ldots, \eta_\ell).
\]  

(A4.1.28)

This vanishes on $\Gamma_{\varphi}$ iff, by Lemma A4.1.20, the R.H.S. of formula (A4.1.28) vanishes when every $\eta_s = \varphi_*(\xi_s), \ s = 1, \ldots, \ell$:

\[
T(\xi_1, \ldots, \xi_\ell) = T(\varphi_*(\xi_1), \ldots, \varphi_*(\xi_\ell)).
\]  

(A4.1.29)
But this is the local form of the familiar condition \( T = \varphi^*(T) \);
The case \((\ell, L)\): Suppose
\[
T = \sum T^{(i)}_i \frac{\partial}{\partial x^{(i)}} \otimes dx^{(i)}, \quad T = \sum T^{(A)}_A \frac{\partial}{\partial y^{(A)}} \otimes dy^{(A)},
\]
(A4.1.30a)
where
\[
\frac{\partial}{\partial x^{(i)}} = \frac{\partial}{\partial x^{(i)(1)}} \otimes \cdots \otimes \frac{\partial}{\partial x^{(i)(L)}}, \quad \frac{\partial}{\partial y^{(A)}} = \frac{\partial}{\partial y^{(A)(1)}} \otimes \cdots \otimes \frac{\partial}{\partial y^{(A)(L)}}.
\]
(A4.1.30b)

By formulae (A4.1.5,21), \((T - T)|_{\Gamma_\varphi} = 0\) iff, for any \(\xi_1, \ldots, \xi_\ell \in T_m(M)\),
\[
\left[ \sum T^{(i)}_i \xi^{(i)} \frac{\partial}{\partial x^{(i)}} - \sum T^{(A)}_A (\varphi_*(\xi))^{(A)} \frac{\partial}{\partial y^{(A)}} \right] = 0.
\]
(A4.1.31)
Now,
\[
[\xi \oplus \eta] = [0 \oplus (\eta - \varphi_*(\xi))] = \eta - \varphi_*(\xi).
\]
(A4.1.32)
Hence, the equality (A4.1.31) can be rewritten as
\[
\sum T^{(i)}_i \xi^{(i)} \varphi_*(\frac{\partial}{\partial x^{(i)}}) = \sum T^{(A)}_A (\varphi_*(\xi))^{(A)} \frac{\partial}{\partial y^{(A)}}.
\]
(A4.1.33)
But this is exactly the \(\varphi\)–compatibility condition (A4.1.9a).

**Remark A4.1.34.** Suppose \(L = 0, \ell = 2\), so that
\[
T = T_{ij} dx^i \otimes dx^j, \quad T = T_{\alpha\beta} dy^\alpha \otimes dy^\beta.
\]
(A4.1.35)

If \(T\) and \(\mathcal{T}\) are skewsymmetric closed and nondegenerate, Proposition A4.1.14 reduces to the well-known criterion: a map \(\varphi\) between two symplectic manifolds \(M\) and \(\mathcal{N}\) is symplectic iff the graph \(\Gamma_\varphi\) is a Lagrangian submanifold in \(M \times \mathcal{N}\) w.r.t the symplectic structure \(T - \mathcal{T}\). If \(T\) and \(\mathcal{T}\) are symmetric, we get a criterion for a map to be a Riemannian map between two Riemannian manifolds. If \(L = 0, \ell = 1\), we can similarly treat contact maps, their symplectizations, etc. The same philosophy applies to holomorphic, etc, maps. We won’t pursue this avenue here since our main interest is in the Poisson case of \(L = 2, \ell = 0\).

**Exercise A4.1.36.** Suppose \(L = \ell, T = \mathcal{T} = \{\text{the identity map } D^{\otimes \ell} \to D^{\otimes \ell}\}\).
Show that the tensors \(T\) and \(\mathcal{T}\) are \(\varphi\)–related no matter what \(\varphi\) is.

From now on, \(L = 2\) and \(\ell = 0:\)
\[
T = \sum T^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}, \quad T = \sum T^{\alpha\beta} \frac{\partial}{\partial y^\alpha} \otimes \frac{\partial}{\partial y^\beta}.
\]
(A4.1.37)

We are going to translate Proposition A4.1.14 into an algebraic language. This amounts to the replacements: 1) \(M\) and \(\mathcal{N}\) by commutative rings \(R_M\) and \(R_\mathcal{N}\), respectively; 2) \(M \times \mathcal{N}\) by \(R_M \otimes R_\mathcal{N}\); 3) \(T\) and \(\mathcal{T}\) by the maps
\[
\{,\}_i : R_i \otimes R_i \to R_i, \quad i = M \text{ or } \mathcal{N},
\]
(A4.1.38)
with the map \{,\} being a derivation w.r.t. each of its arguments. Equivalently, instead of the Poisson brackets \{,\}_i we can consider the derivations

\[ \gamma_i : R_i \to \mathcal{D}(R_i), \quad \mathcal{D}(R_i) = \{\text{derivations of } R_i\}; \quad (A4.1.39) \]

4) \( \varphi : M \to N \) by a homomorphism \( \Phi : R_N \to R_M \); 5) The \( \varphi \)-compatibility condition of \( T \) and \( T \) by the compatibility of the Poisson brackets:

\[ \{\Phi(f), \Phi(g)\}_M = \Phi(\{f,g\}_N), \quad \forall f, g \in R_N. \quad (A4.1.40) \]

This can be equivalently stated as

\[ \gamma_M(\Phi(f)) \circ \Phi = \Phi \gamma_N(f), \quad \forall f \in R_N. \quad (A4.1.41) \]

We are going to reformulate the compatibility condition (A4.1.40) in terms of the ideal \( \mathcal{E} \subset R_M \otimes R_N \). This ideal is generated by the elements

\[ \theta_f = 1 \otimes f - \Phi(f) \otimes 1, \quad f \in R_N; \quad (A4.1.42) \]

this is an algebraic analog of formula (A4.1.17). Let us define the map \( \gamma : R_M \otimes R_N \to \mathcal{D}(R_M \otimes R_N) \) by the rule

\[ \gamma(a \otimes b) = a \otimes \gamma_N(b) - \gamma_M(a) \otimes b. \quad (A4.1.43) \]

This formula means that

\[ (\gamma(a \otimes b))(a' \otimes b') = aa' \otimes \{b,b'\} - \{a,a'\} \otimes bb'. \quad (A4.1.44) \]

Thus, \( \gamma \) is an algebraic version of \( T - T \).

**Lemma A4.1.45** (i) The map \( \gamma \) is a derivation;
(ii) If \( \gamma_M \) and \( \gamma_N \) are skew then so is \( \gamma \).

**Proof.** (i) We have

\[ \gamma((a \otimes b)(c \otimes d)) = \gamma(ac \otimes bd) = ac \otimes \gamma_N(bd) - \gamma_M(ac) \otimes bd = \]

\[ = ac \otimes (b\gamma_N(d) + d\gamma_N(b)) - (a\gamma_M(c) + c\gamma_M(a)) \otimes bd, \quad (A4.1.46) \]

while

\[ (a \otimes b)\gamma(c \otimes d) + (c \otimes d)\gamma(a \otimes b) = (a \otimes b)(c \otimes \gamma_N(d) - \gamma_M(c) \otimes d) + \]

\[ + (c \otimes d)(a \otimes \gamma_N(b) - \gamma_M(a) \otimes b) = ac \otimes b\gamma_N(d) - a\gamma_M(c) \otimes bd + \]

\[ + ca \otimes d\gamma_N(b) - c\gamma_M(a) \otimes db, \]

which is the same as (A4.1.46);
(ii) If \( \gamma_M \) and \( \gamma_N \) are skew then

\[ \{a,a'\} = -\{a',a\}, \quad \{b,b'\} = -\{b',b\}, \quad a,a' \in R_M, \ b,b' \in R_N. \]

Therefore, by formula (A4.1.44),

\[ \{a \otimes b, a' \otimes b'\} = aa' \otimes \{b,b'\} - \{a,a'\} \otimes bb' = -\{a' \otimes b', a \otimes b\}. \]
**Proposition A4.1.47.** The Poisson brackets in $R_M$ and $R_N$ are $\Phi$-compatible iff the ideal $\mathcal{E}$ generated by the elements $\{\theta_f = 1 \otimes f - \Phi(f) \otimes 1, \ f \in R_N\}$, is Poisson-closed:

$$\{\mathcal{E}, \mathcal{E}\} \subseteq \mathcal{E}. \quad (A4.1.48)$$

**Proof.** We have:

$$\{\theta_f, \theta_g\} = \{1 \otimes f - \Phi(f) \otimes 1, 1 \otimes g - \Phi(g) \otimes 1\} \quad [\text{by } (A4.1.44)] =$$

$$= (1 \otimes \{f, .\} + \{\Phi(f), .\} \otimes 1)(1 \otimes g - \Phi(g) \otimes 1) = 1 \otimes \{f,g\} - \{\Phi(f), \Phi(g)\} \otimes 1 =$$

$$= 1 \otimes \{f,g\} - \Phi(\{f,g\}) \otimes 1 + \{\Phi(\{f,g\}) \} \otimes 1. \quad (A4.1.49a)$$

$$+ (\Phi(\{f,g\}) - \{\Phi(f), \Phi(g)\}) \otimes 1. \quad (A4.1.49b)$$

The expression $(A4.1.49a)$ is $\theta_{\{f,g\}}$ and thus belongs to $\mathcal{E}$. From the expression $(A4.1.49b)$ we see that $\{\theta_f, \theta_g\}$ belongs to $\mathcal{E}$ if the equality $(A4.1.40)$ is satisfied. 

Formula $(A4.1.48)$ is the result we have aimed at in this Section: while in geometry we graduate from graphs to arbitrary submanifolds on which tensors of the form $T - T$ vanish, in algebra we simply consider *Poisson ideals*, - whether these ideals are generated by elements of the type $\theta_f$ or not is no longer important. The functional case is almost equally simple. This will be done in the next Section. As a warm-up, consider

**Exercise A4.1.50.** Suppose that the rings $R_M$ and $R_N$ are noncommutative.

(i) Can one still assume that the maps $\varphi_i : R_i \to Der(R_i), \ i = M, N$ are derivations?

(ii) Is formula $(A4.1.43)$ still justified?

§A4.2. THE INFINITE-DIMENSIONAL CASE

Chi Wan thought thrice, and then acted. When Confucius was informed of it, he said, 'Twice may do.'

In this Section we reformulate, in the most general differential-difference framework, the condition for a homomorphism to be Hamiltonian. This reformulation encourages the corresponding ideal to be Poisson-closed. As an example, a Bäcklund transformation of the KdV equation is considered.

In the notation of Chapter 11, let $C_q = R_q^{(\nu)}$ and let $B^q \in Mat(...) (O_{p_0} (C_q)^{[\bar{g}, \bar{\sigma}]}$ be a (formally Hamiltonian) matrix defining the (formally Hamiltonian) map

$$\gamma_q : C_q \to D^{\nu}(C_q), \ H \mapsto X_H, \quad (A4.2.1a)$$

$$X_H = X_H(q) = B^q \left( \frac{\delta H}{\delta q} \right). \quad (A4.2.1b)$$

Until we reach the application-to-examples stage, we do *not* assume that the matrix $B^q$ (and similar matrices to appear below) is Hamiltonian or even skewsymmetric.
Similarly, let \( C_u = R < u^{(g|v)}_\eta \) be another such ring, with its own matrix \( B^u \) defining the map \( \gamma_u: C_u \to D^{e^v}(C_u) \). The maps \( \gamma_q \) and \( \gamma_u \) define the corresponding “Poisson brackets”

\[
\{ H, F \}_s = (\gamma_s(H))(F), \quad s = q, u, \quad H, F \in C_s.
\] (A4.2.2)

Given a homomorphism \( \Phi: C_q \to C_u \), the Poisson brackets in \( C_q \) and \( C_u \) are compatible w.r.t. \( \Phi \) iff

\[
\Phi(\{ H, F \}_q) = \{ \Phi(H), \Phi(F) \}_u, \quad \forall H, F \in C_q.
\] (A4.2.3)

As in Exercise 11.2.8, this condition can be recast as the equation

\[
\Phi(B^q) = D(\Phi)B^u D(\Phi)^\dagger.
\] (A4.2.4)

Our aim is to reformulate this compatibility condition in terms of the ring \( C_{q,u} = R < q^{(g|v)}_i, u^{(g|v)}_\eta > \) and the two-sided ideal \( \mathcal{E} \) generated by the elements \( e^{(g|v)}_i \),

\[
e_i = q_i - \Phi_i, \quad \Phi_i = \Phi(q_i).
\] (A4.2.5)

As a matter of fact, we are going to take this reformulated condition in the form (A4.1.48) already established in the finite-dimensional nonfunctional case:

\[
\{ \mathcal{E}, \mathcal{E} \} \subset \mathcal{E}.
\] (A4.2.6)

The Poisson bracket in the ring \( C_{q,u} \) we define by the \((T-T)\)-rule of the preceding Section:

\[
X_{\mathcal{R}} \left( \begin{array}{c} \mathbf{q} \\ \mathbf{u} \end{array} \right) = \left( \begin{array}{cc} B^q & 0 \\ 0 & -B^u \end{array} \right) \left( \begin{array}{c} \delta H/\delta \mathbf{q} \\ \delta H/\delta \mathbf{u} \end{array} \right), \quad \forall H \in C_{q,u}.
\] (A4.2.7)

**Proposition A4.2.8.** The Poisson—closedness condition (A4.2.6) and the \( \Phi \)-compatibility condition (A4.2.4) are stably equivalent. (Recall that “stability” means that we are allowed to consider arbitrary extensions \( R' \supseteq R \).

**Proof.** We are going to transform the Poisson closedness condition (A4.2.6). Since the Poisson bracket is a derivation w.r.t. the second argument, we have:

\[
\{ E, ae_j^{(g|v)} b \} = \{ E, a \} e_j^{(g|v)} b + a e_j^{(g|v)} \{ E, b \} + a \{ E, e_j^{(g|v)} b \}.
\] (A4.2.9)

Hence, the condition \( \{ \mathcal{E}, \mathcal{E} \} \subset \mathcal{E} \) is equivalent to the condition

\[
\{ \mathcal{E}, e_j \} \subset \mathcal{E}, \quad \text{all } j.
\] (A4.2.10)

Next, since

\[
a e_j^{(g|v)} b \approx e_j \hat{g}^{-1}(-\partial)^\nu (ba),
\] (A4.2.11)

and, by formula (A4.2.7),

\[
\{ \text{trivial, } \forall \} = 0,
\]

the condition \( \{ \mathcal{E}, e_j \} \subset \mathcal{E} \) (A4.2.10) is equivalent to the condition

\[
\{ e_i h, e_j \} \subset \mathcal{E}, \quad \text{all } i, j, \quad \forall h \in C_{q,u}.
\] (A4.2.12)
Further, by formula (A4.2.7),
\[
\mathbf{X}_{e_i h} = \left( B^q(\delta(e_i h)/\delta q) \right) \left( -B^u(\delta(e_i h)/\delta u) \right), \tag{A4.2.13}
\]
while by formula (10.1.43),
\[
\frac{\delta(\cdot)}{\delta(\cdot)} = \sum \hat{g}^{-1}(-\partial)\nu \left( \frac{\partial^+ (\cdot)}{\partial (\cdot) (\nu \nu)} \right). \tag{A4.2.14}
\]
Therefore, each time the derivative
\[
\frac{\partial^+ (e_i h)}{\partial (\cdot)}
\]
is spent on \( h \) rather than on \( e_i \) in the calculation of the variational derivatives \( \delta(e_i h)/\delta(\cdot) \) entering formula (A4.2.13), we have the unmolested element \( e_i \in \mathcal{E} \) entering the corresponding component of \( \mathbf{X}_{e_i h} \) and thence of \( \{e_i h, \ldots\} \). The moral is that in handling the condition (A4.2.12) we can consider \( h \) to be \( q \)- and \( u \)-independent; equivalently, \( h \in R' \) for arbitrary extension \( R' \supset R \):
\[
\{e_i h, e_j\} \subset \mathcal{E}, \ \forall i, j, \ \forall h \in R'. \tag{A4.2.15}
\]
Finally, we can replace this condition by the condition
\[
\{\sum_i e_i h_i, e_j\} \subset \mathcal{E}, \ \forall j, \ \forall h_i \in R'. \tag{A4.2.16}
\]
Now, by formula (A4.2.5),
\[
\frac{\delta(\sum e_i h_i)}{\delta q} = \frac{\delta}{\delta q} (q^i h - \Phi^i h) = h, \tag{A4.2.17a}
\]
\[
= -\sum_i \left( \frac{D\Phi_i}{Du} \right)^\dagger (h_i). \tag{A4.2.17b}
\]
Therefore, by formula (A4.2.13),
\[
X\sum e_i h_i (q) = B^q (h), \quad X\sum e_i h_i (u) = \sum_i B^u \left( \frac{D\Phi_i}{Du} \right)^\dagger (h_i). \tag{A4.2.18}
\]
Hence,
\[
\{\sum e_i h_i, e_j\} = X\sum e_i h_i (q_j - \Phi_j) = [B^q (h)]_j - \sum_i \frac{D\Phi_j}{Du} B^u \left( \frac{D\Phi_i}{Du} \right)^\dagger (h_i). \tag{A4.2.19}
\]
This must belong to \( \mathcal{E} \). Since
\[
[B^q(\mathbf{h})]_j = [\Phi(B^q)(\mathbf{h})]_j \quad (\text{mod } \mathcal{E}),
\]  
formula (A4.2.19) turns the condition (A4.2.16) into the equality
\[
\{[(\Phi(B^q) - \frac{\partial \Phi}{\partial u} B^u \left( \frac{\partial \Phi}{\partial u} \right)^\dagger](\mathbf{h})]_j = 0 \quad (\text{mod } \mathcal{E}).
\]  
Since there are no \( q \)-dependencies left in this formula, it can be rewritten as
\[
[\Phi(B^q) - \frac{\partial \Phi}{\partial u} B^u \left( \frac{\partial \Phi}{\partial u} \right)^\dagger](\mathbf{h}) = 0.
\]  
Since \( \mathbf{h} \) is arbitrary, this becomes the \( \Phi \)-compatibility criterion (A4.2.4).

**Exercise A4.2.23.** Show that if, instead of formula (A4.2.7) corresponding to the \((T - T)\)-rule, we take the \((T + T)\)-rule
\[
X_H(q) = B^q(\delta H/\delta q), \quad X_H(u) = B^u(\delta H/\delta u),
\]  
the Poisson–closedness condition \( \{\mathcal{E}, \mathcal{E}\} \subset \mathcal{E} \) becomes equivalent to the property of the map \( \Phi \) being *anti*Hamiltonian:
\[
\Phi(\{H, F\}_q) = -\{\Phi(H), \Phi(F)\}_u, \quad \forall H, F \in C_q.
\]  

**Exercise A4.2.26.** All variables below are *commuting*.

(A) Let \( C_u = C[u^{(n)}], \quad C_v = C[v^{(n)}], \quad n \in \mathbb{Z}_+, \quad B^u = \partial, \quad B^v = \partial, \quad B^{u,v} = (\begin{smallmatrix} \partial & 0 \\ 0 & -\partial \end{smallmatrix}) \).

Let \( \mathcal{E} \) be the ideal in \( C_{u,v} \) generated by the element
\[
u - v - f(u + v)(u^{(1)} + v^{(1)}), \quad \text{some } f.
\]  

(i) Show that \( \{\mathcal{E}, \mathcal{E}\} \subset \mathcal{E} \) no matter what \( f = f(u + v) \) is. (\( f \) can be taken from an algebraic extension of the field \( C(u). \));

(ii) Show that
\[
\begin{aligned}
\frac{u^2}{2} &\equiv \frac{v^2}{2} \quad (\text{mod } \mathcal{E}); \\
\end{aligned}
\]  

(iii) Set
\[
H_u = u^3 - \frac{1}{2} u^{(1)2}, \quad H_v = v^2 - \frac{1}{2} v^{(1)2}.
\]  

Show that
\[
H_u \sim H_v \quad (\text{mod } \mathcal{E})
\]  

iff
\[
\frac{df}{d\eta} = f^3, \quad \eta = u + v;
\]  

(B) With \( C_u, C_v \) and \( \mathcal{E} \) as above, let \( B^u = \partial^3 + 2(\partial u + u\partial), \quad B^v = \partial^3 + 2(\partial v + v\partial), \quad B^{u,v} = \left( \begin{smallmatrix} B^u & 0 \\ 0 & -B^v \end{smallmatrix} \right). \)

(i') Show that \( \{\mathcal{E}, \mathcal{E}\} \subset \mathcal{E} \) iff the equation (A4.2.31) is satisfied;

(C) Let \( C_u = C[u_i^{(n)}], \quad C_v = C[v_i^{(n)}], \quad i = 1, \ldots, \ell, \quad n \in \mathbb{Z}_+, \quad B^v = S\partial, \quad B^v = S\partial, \)
\[ S \in \text{Mat}_\ell(C), \quad S^t = S. \] Fix \( \ell \) functions \( F_1, \ldots, F_\ell \):

\[ F_i = F_i(u + v), \quad (A4.2.32) \]

and consider the ideal \( \mathcal{E} \) in \( C_{u,v} \) generated by the elements

\[ u_i - v_i - \partial(F_i), \quad i = 1, \ldots, \ell. \quad (A4.2.33) \]

(i) Show that \( \{ \mathcal{E}, \mathcal{E} \} \subset \mathcal{E} \) iff

\[ (J^F S)^t = J^F S, \quad (J^F)_{ij} = \frac{\partial F_i}{\partial y_j}, \quad \eta = u + v; \quad (A4.2.34) \]

(ii) Assume that the matrix \( S \) is invertible. Show that if the equation (A4.2.34) is satisfied then

\[ \frac{u^t S^{-1} u}{2} \sim \frac{v^t S^{-1} v}{2} \quad (\text{mod} \ \mathcal{E}). \quad (A4.2.35) \]

**Remark A4.2.36.** If \( \mathcal{E} \) is a Poisson-closed ideal in \( C \) then \( X_H(\mathcal{E}) \subset \mathcal{E}, \forall H \in \mathcal{E} \). In geometric terms, the vector field \( X_H \) is tangent to the “submanifold \( \{ \mathcal{E} = 0 \} \).” In practice one meets this situation in the following guise, sometimes called a Bäcklund transformation. Suppose we have a ring \( C_q \) with a Hamiltonian matrix \( B^q \). Consider another copy of these two objects, call them \( C_u \) and \( B^u \). Let \( \mathcal{E} \) be an ideal in \( C_{q,u} \) Poisson-closed w.r.t. the Hamiltonian matrix \( B^{q,u} = (B^q, 0_u) \). Let \( H = H(q) \) be an element in \( C_q \) such that

\[ \mathcal{H} = H(q) - H(u) \approx 0 \quad (\text{mod} \ \mathcal{E}). \quad (A4.2.37) \]

Then the motion equations

\[ \partial_t(q) = B^q \left( \frac{\delta H(q)}{\delta q} \right), \quad \partial_t(u) = B^u \left( \frac{\delta H(u)}{\delta u} \right) \quad (A4.2.38) \]

preserve \( \mathcal{E} \). This is so because these motion equations are Hamiltonian in the ring \( C_{q,u} \) with the Hamiltonian \( \mathcal{H} \) (A4.2.37); and since

\[ \mathcal{H} \in \mathcal{E} + \{ \text{trivial in } C_{q,u} \}, \]

the conclusion \( X_H(\mathcal{E}) \subset \mathcal{E} \) follows from the condition \( \{ \mathcal{E}, \mathcal{E} \} \subset \mathcal{E} \). For example, from the statements (i'), (i), (ii) in the Exercise A4.2.26, it follows that formula (A4.2.27) defines a Bäcklund transformation for the KdV equation

\[ \partial_t(u) = [\partial^3 + 2(\partial u + u\partial)] \frac{\delta}{\delta u} \left( \frac{u^2}{2} \right) = \partial_x \left( u^3 - \frac{1}{2} u^{(1)2} \right) = 6uu^{(1)} + u^{(3)} \quad (A4.2.39) \]

whenever \( f \) satisfies the equation (A4.2.31). An additional example of a Bäcklund transformation will be found at the end of §A4.4. Of the main result of this Section, Proposition A4.2.8, one-half can be considered as the absolute case \( \Psi = id \) of the following relative result:

**Proposition A4.2.40.** Suppose, in addition to the rings \( C_q \) and \( C_u \) we are given a third ring \( C_v = R < v^{(g)}_v > \) and corresponding matrix \( B^v \) over the ring \( C_v \).
Let $\Psi : C_q \to C_v$ be a homomorphism. In the ring $C_{v,u}$, consider the matrix $B^{v,u} = \begin{pmatrix} B^v & 0 \\ 0 & -B^u \end{pmatrix}$ and the two-sided ideal $\mathcal{E}$ generated by the elements

$$E_i = \Psi_i - \Phi_i, \quad \Psi_i = \Psi(q_i).$$

(A4.2.41)

Then the Poisson-closedness condition $\{\mathcal{E}, \mathcal{E}\} \subseteq \mathcal{E}$ follows from the property of both homomorphisms $\Psi$ and $\Phi$ providing compatibility between the matrices $B^q, B^v$ and $B^u$.

**Proof.** Like in the Proof of the absolute Proposition A4.2.8, we find that

$$\left\{ \sum_i E_i h_i, E_j \right\} = \sum_i \left[ \frac{D\Psi_j}{Du} B^v \left( \frac{D\Psi_i}{Du} \right)^\dagger - \frac{D\Phi_j}{Du} B^u \left( \frac{D\Phi_i}{Du} \right)^\dagger \right](h_i).$$

(A4.2.42)

Since both $\Psi$ and $\Phi$ are compatible maps, the R.H.S. of this formula is

$$\{(\Psi(B^q) - \Phi(B^q))(h)\}_j = \sum_i [(\Psi - \Phi)(B^q)](h_i) \subseteq \mathcal{E}.$$  \hspace{1cm} (A4.2.43)

§A4.3. CLOSED 1-FORMS AS LAGRANGIAN SUBMANIFOLDS, THE VARIATIONAL VERSION

Let $\pi : T^*M \to M$ be the cotangent bundle of a manifold $M$. Identifying 1-forms $\omega$ on $M$ with their graphs $\Gamma_\omega \subset T^*M$, one can characterize closed forms as those whose graphs are Lagrangian submanifolds in the natural symplectic structure of $T^*M$. In this Section we establish, in the Poisson (rather than symplectic) language, an infinite-dimensional version of this well-known characterization.

We start with a ring $C_q = R < q^{(g)}_i >$, extend it to the ring $C_{p,q} = R < q^{(g)}_i, p^{(g)}_i >$, and fix the canonical Hamiltonian matrix $B^{p,q} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Choose a 1-form $\omega \in \Omega_0^1(C_q)$ (10.1.29b):

$$\omega = \sum_i dq_i F_i, \quad F_i \in C_q.$$  \hspace{1cm} (A4.3.1)

With such a form we associate a two-sided ideal $\mathcal{E} = \mathcal{E}_\omega$ in $C_{p,q}$, generated by the elements

$$e_i = p_i - F_i, \quad i \in I.$$  \hspace{1cm} (A4.3.2)

the components of the form $dq^i p - \omega$.

**Proposition A4.3.3.** The ideal $\mathcal{E}_\omega$ is Poisson-closed iff the form $\omega$ is exact, i.e., there exists $L \in C_q$ such that

$$\omega = \delta(L).$$  \hspace{1cm} (A4.3.4)

**Proof.** Calculating as in the preceding Section, we have

$$\frac{\delta}{\partial p} \left( \sum e_i h_i \right) = \frac{\delta}{\partial p} (p^i h - F^i h) = h,$$

(A4.3.5a)

$$\frac{\delta}{\partial q} \left( \sum e_i h_i \right) = - \sum \left( \frac{DF_i}{Dq} \right)^\dagger (h_i),$$

(A4.3.5b)
\[ X \sum e_i h_i \left( \frac{p}{q} \right) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \left( \sum \frac{h}{\frac{DF_i}{Dq_i}} (h_i) \right) = -\left( \sum \left( \frac{DF_i}{Dq_i} \right)^\dagger (h_i) \right), \quad (A4.3.6) \]

\[ \{ \sum e_i h_i, e_j \} = -\sum \left( \frac{DF_i}{Dq_j} \right)^\dagger (h_i) + \sum \frac{DF_j}{Dq_i} (h_i). \quad (A4.3.7) \]

Since the R.H.S. of the last expression is \( p \)-independent, the condition \( \{ \sum e_i h_i, e_j \} \subset \mathcal{E} \) becomes \( \{ \sum e_i h_i, e_j \} = 0 \), and this happens iff

\[ \sum_j \left[ \left( \frac{DF_i}{Dq_j} \right)^\dagger - \frac{DF_j}{Dq_i} \right] (h_i) = 0. \quad (A4.3.8) \]

Since \( h \) is arbitrary, the last equality is equivalent to the equation

\[ \left( \frac{DF_i}{Dq_j} \right)^\dagger = \frac{DF_j}{Dq_i}, \quad \text{all } i, j, \quad (A4.3.9) \]

and this is the component form of the equality

\[ \left( \frac{DF}{Dq} \right)^\dagger = \frac{DF}{Dq}. \quad (A4.3.10) \]

By Theorems 10.3.13,24 this equality is necessary and sufficient for the form \( \omega = dq^i F \) to be exact.

The shortest-lived mistakes
are always the best.
Moliere

\section*{§A4.4. GENERATING FUNCTIONS OF SYMPLECTIC MAPS AND THEIR GENERALIZATIONS}

Leibniz never married; he had considered it at the age of fifty; but the person he had in mind asked for time to reflect. This gave Leibniz time to reflect, too, and so he never married.

Bernard Le Bovier (1657 - 1757)

In this Section we demystify into a banality the formalism of generating functions in symplectic mechanics and field theory.

In classical mechanics, the generating-functions formulae arise via the following construction. Suppose \( (p, q) \leftrightarrow (\dot{p}, \dot{q}) \) is a symplectic map. This means that

\[ dp \wedge dq = d\dot{p} \wedge d\dot{q}. \quad (A4.4.1) \]
Since locally every closed 1-form is exact, there exists locally a function $S$ such that
\[ pdq - \dot{p}dq = dS. \] (A4.4.2)

If $S$ is taken to be a function of $q$ and $\dot{q}$ then formula (A4.4.2) is equivalent to the pair of formulae
\[ p = \frac{\partial S}{\partial q}, \quad \dot{p} = -\frac{\partial S}{\partial \dot{q}}. \] (A4.4.3)

All other versions of these formulae follow when combined with the canonical automorphisms
\[ (p_i, q_i) \mapsto (\epsilon_i q_i, -\epsilon_i p_i), \quad \epsilon_i = \pm 1, \] (A4.4.4)
for some of the $i$'s. For example, rewriting the relation (A4.4.2) as
\[ -q\dot{p} + \dot{q}\dot{p} = d(S - p'q + \dot{p}'\dot{q}) \] (A4.4.5)
and considering
\[ S' = S - p'q + \dot{p}'\dot{q} \] (A4.4.6)
as a function of $p$ and $\dot{p}$, we find
\[ q = -\frac{\partial S'}{\partial p}, \quad \dot{q} = \frac{\partial S'}{\partial \dot{p}}. \] (A4.4.7)

This is all, of course, well-known since the dawn of civilization. Implicit in this knowledge is the assumption that the resulting formulae can be resolved in favour of $\dot{p}$ and $\dot{q}$. Since this assumption is not necessarily true, what we have, at best, is a symplectic correspondence, i.e., a submanifold, in the $(p, q)$ space, whose associated ideal is Poisson-closed w.r.t. to the Hamiltonian structure
\[ B = \begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \] (A4.4.8)
in geometric terms, a Lagrangian submanifold.

We generalize this formalism in two directions. First, we move from classical mechanics to field theory, replacing partial derivatives by the variational ones. Second, instead of the canonical Hamiltonian matrix $b = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ in the $(p, q)$-space we take a more general Hamiltonian matrix
\[ b = \frac{p}{q} \begin{pmatrix} p & q \\ 0 & \mathcal{O} \end{pmatrix}, \] (A4.4.9)
where $\mathcal{O}$ is an arbitrary $(p, q)$-independent operator:
\[ \mathcal{O} \in \text{Mat}_{(\cdot)}(Op_0(R)[\hat{g}, \partial'])]. \] (A4.4.10)

**Proposition A4.4.11.** In the ring $C = C_{p,q,\tilde{p},\tilde{q}}$, consider the Hamiltonian matrix
\[ B = \begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}, \quad b = \begin{pmatrix} 0 & \mathcal{O} \\ -\mathcal{O}' & 0 \end{pmatrix}. \] (A4.4.12)
Pick an arbitrary element \( S \in C_{q, \hat{q}} \), and consider the two-sided ideal \( \mathcal{E} = \mathcal{E}_S \) in \( C \) generated by the elements

\[
e_i = \left[ p - \mathcal{O} \left( \frac{\delta S}{\delta q} \right) \right]_i, \quad \tilde{e}_i = \left[ \hat{p} + \mathcal{O} \left( \frac{\delta S}{\delta q} \right) \right]_i, \quad i = 1, \ldots \quad (A4.4.13)
\]

Then \( \mathcal{E} \) is Poisson-closed: \( \{ \mathcal{E}, \mathcal{E} \} \subset \mathcal{E} \).

**Proof.** Calculating as in §A4.2, take

\[
H = \sum e_i h_i = e^i h, \quad h_i \in R'.
\]

Writing \( S_q \) instead of \( \delta S/\delta q \), etc., we find:

\[
H = p^i h - \mathcal{O}(S_q)^i h \approx p^i h - S_q^i \mathcal{O}^i (h) \quad \Rightarrow \quad (A4.4.15)
\]

\[
H_p = h, \quad H_q = -D_q(S_q)^i \mathcal{O}^i (h), \quad H_{\hat{p}} = 0, \quad H_{\hat{q}} = -D_{\hat{q}}(S_{\hat{q}})^i \mathcal{O}^i (h) \quad \Rightarrow \quad (A4.4.16)
\]

\[
X_H(p) = \mathcal{O}(H_q) = -\mathcal{O}D_q(S_q)^i \mathcal{O}^i (h) = \mathcal{O}D_q(S_q)^i (\tilde{h}),
\]

\[
\tilde{h} := -\mathcal{O}^i (h),
\]

\[
X_H(q) = -\mathcal{O}^i (H_p) = -\mathcal{O}^i (h) = \tilde{h},
\]

\[
X_H(\hat{p}) = -\mathcal{O}(H_{\hat{q}}) = -\mathcal{O}(-D_{\hat{q}}(S_{\hat{q}})^i \mathcal{O}^i (h)) = -\mathcal{O}D_{\hat{q}}(S_{\hat{q}})^i (\tilde{h}),
\]

\[
X_H(\hat{q}) = \mathcal{O}^i (H_{\hat{p}}) = 0 \quad \Rightarrow \quad (A4.4.17e)
\]

\[
\{ H, e \} = X_H(e) = X_H(p - \mathcal{O}(S_q)) = \mathcal{O}D_q(S_q)^i (\tilde{h}) - \mathcal{O}[D_q(S_q)(X_H(q))] +
\]

\[
+D_q(S_q)(X_H(\hat{q})) = \mathcal{O}D_q(S_q)^i (\tilde{h}) - \mathcal{O}D_q(S_q)(\tilde{h}) = \mathcal{O}[D_q(S_q)^i - D_q(S_q)](\tilde{h}).
\]

By formula (10.3.23), the operator \( D(\cdot) \left( \frac{\delta S}{\delta (\cdot)} \right) \) is self-adjoint:

\[
\begin{pmatrix}
D_q(S_q) & D_{\hat{q}}(S_{\hat{q}})
\end{pmatrix}
\begin{pmatrix}
D_q(S_q) & D_{\hat{q}}(S_{\hat{q}})
\end{pmatrix}^\dagger = \begin{pmatrix}
D_q(S_q)^\dagger & D_{\hat{q}}(S_{\hat{q}})^\dagger
\end{pmatrix}
\Rightarrow \quad (A4.4.19)
\]

\[
D_q(S_q)^\dagger = D_q(S_q),
\]

\[
D_{\hat{q}}(S_{\hat{q}})^\dagger = D_{\hat{q}}(S_{\hat{q}}),
\]

By formula (A4.4.20), the expression (A4.4.18) vanishes:

\[
\{ \sum e_i h_i, e_j \} = 0.
\]

Similarly,

\[
\{ H, \tilde{e} \} = X_H(\tilde{e}) = X_H(\hat{p} + \mathcal{O}(S_{\hat{q}})) = -\mathcal{O}D_{\hat{q}}(S_{\hat{q}})^i (\tilde{h}) + \mathcal{O}[D_{\hat{q}}(S_{\hat{q}})(X_H(q)) +
\]

\[
+D_{\hat{q}}(S_{\hat{q}})(X_H(\hat{q})) = -\mathcal{O}D_{\hat{q}}(S_{\hat{q}})^i (\tilde{h}) + -\mathcal{O}D_{\hat{q}}(S_{\hat{q}})(\tilde{h}) = -\mathcal{O}[D_{\hat{q}}(S_{\hat{q}})^i - D_{\hat{q}}(S_{\hat{q}})](\tilde{h}),
\]

and this vanishes by formula (A4.4.21). Thus,

\[
\{ \sum e_i h_i, \tilde{e}_j \} = 0.
\]

\[
(A4.4.23)
\]
Exchanging tilded and untilded variables in formulae (A4.4.13,12,9) amounts to replacing $\mathcal{O}$ by $-\mathcal{O}$. Therefore, the already proven formulae (A4.4.22,23) yield their tilded version:

$$\{\sum \hat{e}_i h_i, e_j\} = \{\sum \hat{e}_i h_i, \hat{e}_j\} = 0. \quad (A4.4.24)$$

As an example, consider the NLS hierarchy of §3.5:

$$\partial_t \mathbf{p} = (L^n)^+ \mathbf{p}, \quad (A4.4.25a)$$
$$\partial_t \mathbf{q} = -[(L^n)^+ \mathbf{q}], \quad n \in \mathbb{N}, \quad (A4.4.25b)$$
$$L = \xi + p^t \xi^{-1} q. \quad (A4.4.25c)$$

For $n = 2$, this is the system (3.5.9,10):

$$\partial_t \mathbf{p} = p^{(2)} + 2(p^t q)p, \quad (A4.4.26a)$$
$$\partial_t \mathbf{q} = -q^{(2)} - 2q(p^t q). \quad (A4.4.26b)$$

The whole NLS hierarchy is Hamiltonian, with the Hamiltonian matrix being the canonical one: $b = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Now, consider the generating function

$$S = \mathbf{p}^{(1)t} \mathbf{q} - \frac{1}{2}(p^t q)^2. \quad (A4.4.27)$$

Then the formulae

$$\mathbf{p} = \frac{\delta S}{\delta \mathbf{q}} = \mathbf{p}^{(1)t} - (\hat{p}^t \mathbf{q}) \hat{\mathbf{p}}, \quad (A4.4.28a)$$
$$\mathbf{q} = \frac{\delta S}{\delta \mathbf{p}} = -q^{(1)} - \mathbf{q}(\hat{p}^t \mathbf{q}), \quad (A4.4.28b)$$

provide a symplectic correspondence in the $(\mathbf{p}, \mathbf{q}, \hat{\mathbf{p}}, \hat{\mathbf{q}})$-space. The two copies of the NLS Lax operator $L$ (A4.4.25c), in the notation

$$a = \hat{\mathbf{p}}, \quad b = \mathbf{q}, \quad (A4.4.29)$$

become

$$L_1 = \xi + [a^{(1)} - (a^t b)a^t] \xi^{-1} b, \quad (A4.4.30a)$$
$$L_2 = \xi - a^t \xi^{-1} [b^{(1)} + b(a^t b)]. \quad (A4.4.30b)$$

**Exercise A4.4.31.** Show that

$$\text{Res}(L_1^n) \approx \text{Res}(L_2^n) \quad (A4.4.32)$$

for $n = 1, 2$.

**Conjecture A4.4.33.** Formula (A4.4.32) holds true for all $n \in \mathbb{N}$.

By the Remark A4.2.36, Exercise A4.4.31 implies that formulae (A4.4.28) define a Bäcklund transformation of the NLS system (A4.4.26). If conjecture A4.4.33 were true, the same conclusion would apply to the whole NLS hierarchy (A4.4.25).
Doubt creeps in everywhere, with, however, a
signal exception: there is no skeptical music.
Cioran

APPENDIX A5. COVARIANT ASPECTS OF THE
HAMILTONIAN FORMALISM

The Hamiltonian formalism is not a covariant theory: it requires an explicit
separation of the time variable from all the other space-time variables. In this
Appendix we examine various problems which appear when one tries to extend a
given Hamiltonian system into a covariant form.

§A5.1. $GL_{m+1}$-GENERALITIES AND A $GL_2$-EXAMPLE:
The KdV Equation

In this Section we discuss obstructions to extending the Hamiltonian formalism
from a noncovariant into a covariant theory. As an example, we examine the KdV
and similar Hamiltonian systems.

A system of partial differential equations with $m + 1$ independent variables
$y = (y_1, \ldots, y_{m+1})$,

$$f(y, u, \frac{\partial u}{\partial y}, \ldots) = 0,$$

sometimes, very rarely, can be resolved in favor of $\frac{\partial u}{\partial t}$, $y = (t, x)$:

$$u_t = F(t, x, u, u_x, \ldots).$$

(A5.1.2)

When $F$ is $t$-independent, we get an evolution system

$$u_t = F(x, u, u_x, \ldots).$$

(A5.1.3)

This latter system allows a purely algebraic treatment; in particular, such a system
may be the subject of the Hamiltonian formalism. The latter is definitely not a
covariant theory: it is applicable only to evolution equations of the form (A5.1.3).
The natural question is whether the property of being a Hamiltonian system is the
property of the object described by a system of differential equations or only the
property of a very particular form assumed by this system in a specially chosen
coordinates. The latter case would be unphysical and disagreeable. The former
case would be physical and aesthetically satisfying. The theory to decide which is
which doesn’t exist. As shall be seen presently, the question is technically difficult.

A simplified version of this question restricts the allowable changes of independ-
ent variables to the linear ones:

$$y_{new} = M y_{old}, \quad M \in GL_{m+1}(\mathbb{R}).$$

(A5.1.4)

This simplified problem is also difficult and unsolved. The roots of the difficulty are
legion. One of them lies in the uncontrollable behavior of the variational derivations
$\delta H/\delta u$ under the $GL_{m+1}$ action (A5.1.4) applied to a Hamiltonian system
\[ \mathbf{u}_t = B \left( \frac{\delta H}{\delta \mathbf{u}} \right). \]  
(A5.1.5)

Another is connected with the number of \( x \)-derivatives entering the R.H.S. of the Hamiltonian system (A5.1.5): after a \( GL_{m+1} \)-action, these \( x \)-derivatives beget time-derivatives, of various orders; the resulting system is no longer of first-order in time, and to restore its evolution character one needs to introduce new dependent variables. Only for 1st-order systems of the type

\[ \mathbf{u}_t = F(\mathbf{u}, \mathbf{u}_x), \]  
(A5.1.6)

the enlargement of the original system under the \( GL_{m+1} \)-action is unnecessary.

To see what is involved, let us restrict ourselves to the simplest case of \( GL_2 \), rather than \( GL_{m+1} \), from now on. Let us work out an example of the KdV equation

\[ \partial_x(u) = \partial_\eta(3u^2 + u_{\eta\eta}) = \partial_\eta \left( \frac{\delta H}{\delta \mathbf{u}} \right), \quad H = u^3 - \frac{1}{2} \left( \frac{\partial \mathbf{u}}{\partial \eta} \right)^2. \]  
(A5.1.7)

We pick an arbitrary matrix

\[ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2, \]  
(A5.1.8)

generating the change of partial derivatives

\[ \partial_x = a \partial_t + b \partial_x, \]  
(A5.1.9)

\[ \partial_\eta = c \partial_t + d \partial_x. \]

If \( c = 0 \) then any Hamiltonian system (A5.1.5) under the \( GL_2 \) change (A5.1.9) becomes

\[ a\mathbf{u}_t + b \mathbf{u}_x = B^d \left( \frac{\delta H^d}{\delta \mathbf{u}} \right), \]  
(A5.1.10)

where the superscript ‘d’ denotes the rescaling of \( \eta \) by \( d \). If \( P = P(B) \) is the momentum Hamiltonian for the Hamiltonian matrix \( B \):

\[ B \left( \frac{\delta P}{\delta \mathbf{u}} \right) = \partial_\eta(\mathbf{u}), \]  
(A5.1.11)

then

\[ B^d \left( \frac{\delta P^d}{\delta \mathbf{u}} \right) = d \partial_x(\mathbf{u}), \]  
(A5.1.12)

so that formula (A5.1.10) can be rewritten as

\[ \mathbf{u}_t = a^{-1} B^d \frac{\delta}{\delta \mathbf{u}} \left( H^d - bd^{-1} P^d \right). \]  
(A5.1.13)

Thus, the case \( c = 0 \) is trivial. From now on,

\[ c \neq 0. \]  
(A5.1.14)
Denote
\[ \dot{\cdot} = \partial_t(\cdot), \quad (\cdot)' = \partial_x(\cdot). \] (A5.1.15)

The KdV equation (A5.1.7) under the $GL_2$-action (A5.1.9) becomes:
\[ a\dot{u} + bu' = 3u(c\dot{u} + du') + 3(c\dot{u} + du')u + (c_3 \ddot{u} + 3c^2d\dot{u}'' + 3cd^2\dot{u}''' + d^3u'''). \] (A5.1.16)

Introducing the new variables
\[ \dot{u} = \bar{u}_1, \quad \ddot{u}_1 = \bar{u}_2(=\ddot{u}), \] (A5.1.17)
we can recast the equation (A5.1.16) into a 1st-order-in-time system consisting of equations (A5.1.17) and
\begin{align*}
\dot{u}_2 &= c^{-3}[a\ddot{u}_1 + bu' - 3u(c\ddot{u}_1 + du') - 3(c\ddot{u}_1 + du')u - \\
&\quad - 3c^2d\dot{u}_2' - 3cd^2\dot{u}_1'' - d^3u'''].
\end{align*}
(A5.1.18)

This system is a bit unwieldy. Introducing the variables
\[ u_1 = \bar{u}_1 + c^{-1}du', \quad u_2 = \bar{u}_2 + 2c^{-1}d\ddot{u}_1' + (c^{-1}d)^2u'', \] (A5.1.19)
so that
\[ \bar{u}_1 = u_1 - c^{-1}du', \quad \bar{u}_2 = u_2 - 2c^{-1}du_1' + (c^{-1}d)^2u'', \] (A5.1.20)
we can rewrite the system (A5.1.17,18) in the form
\[ \begin{pmatrix} u \\ u_1 \\ u_2 \end{pmatrix}_t = \begin{pmatrix} u_1 - c^{-1}du' \\ u_2 - c^{-1}du_1' \\ -(c^{-4}\Delta u + c^{-1}du_2)' + c^{-3}au_1 - 3c^{-2}(uu_1 + u_1u) \end{pmatrix}, \] (A5.1.21a)
\[ \Delta = ad - bc. \] (A5.1.21b)

**Exercise A5.1.22.** Verify these formulae.

The system (A5.1.21) turns out to be Hamiltonian:

**Proposition A5.1.23.** In the ring $C = C_{u,u_1,u_2}$, consider the Hamiltonian
\[ H = \frac{1}{2}u_1^2 - c^{-1}du_1u_1 + \left(\frac{bc}{\Delta}u - \frac{3cd}{\Delta}u^2 - \frac{c^3d}{2\Delta}u_2\right)u_2 - \\
-\frac{c^{-2}ab}{2\Delta}u^2 + \frac{c^{-2}}{\Delta}(ad + 2bc)u^3 - \frac{9c^{-1}d}{2\Delta}u^4, \] (A5.1.24)
and the matrix
\[ B = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -J^t \\ 0 & J & \gamma\partial \end{pmatrix}, \quad J = \frac{Dh}{Du}, \quad h \in C_u, \quad \gamma = \text{const.} \] (A5.1.25)

Then:
(i) The matrix $B$ is Hamiltonian;
(ii) For
\[
\gamma = c^{-4} \Delta, \quad h = c^{-3} au - 3c^{-2} u^2, \quad (A5.1.26)
\]
the matrix $B$ and the Hamiltonian $H$ (A5.1.24) reproduce the motion equations (A5.1.21).

**Proof.** (i) The invertible change of variables
\[
u = u, \quad u_1 = u_1, \quad \tilde{u}_2 = u_2 - h(u), \quad (A5.1.27)
\]
brings the matrix $B$ (A5.1.25) into the constant-coefficient form
\[
B = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \gamma \partial \end{pmatrix}; \quad (A5.1.28)
\]
(ii) The vector of variational derivatives of the Hamiltonian $H$ (A5.1.24) is:
\[
\frac{\delta H}{\delta u} = c^{-1} du_1 + bcu_2/\Delta - 3cd(uu_2 + u_2 u)/\Delta - c^{-2} abu/\Delta +
\]
\[
+ 3c^{-2}(ad + 2bc)u^2/\Delta - 18c^{-1} du^3/\Delta, \quad (A5.1.29a)
\]
\[
\frac{\delta H}{\delta u_1} = u_1 - c^{-1} du', \quad (A5.1.29b)
\]
\[
\frac{\delta H}{\delta u_2} = (bcu - 3cdu^2 - c^3 du_2)/\Delta. \quad (A5.1.29c)
\]
For $h$ (A5.1.26), the entry $\frac{Dh}{Du}$ in the matrix $B$ (A5.1.25) is
\[
c^{-3} a - 3c^{-2}(\tilde{L}_u + \tilde{R}_u). \quad (A5.1.30)
\]
Applying this matrix $B$ to the vector $\frac{\delta H}{\delta (\ldots)}$, we recover the motion equations (A5.1.21).

Thus, the $GL_2$-orbit of the KdV equation consists of Hamiltonian systems. It is unlikely that the KdV equation is in any way special among the Hamiltonian systems of the form
\[
u_\tau = \partial_{\eta} \left( \frac{\delta H}{\delta u} \right). \quad (A5.1.31)
\]
But a Proof is lacking.

**Remark A5.1.32.** Suppose $\mathcal{H}$ is an integral of an evolution system
\[
u_\tau = F(\nu, \nu_\eta, \ldots): \quad (A5.1.33)
\]
\[
\mathcal{H}_\tau + (F\nu_\xi)_{\eta} \in \text{Com}, \quad (A5.1.34)
\]
where Com stands for the subspace of commutators in $C_u$. Applying the $GL_2$-action formulae (A5.1.8) to the evolution system (A5.1.33) and to the conservation law (A5.1.34), we get,
\[
(a\mathcal{H} + cF\nu_\xi)_t + (b\mathcal{H} + dF\nu_\xi)_x \in \text{Com}. \quad (A5.1.35)
\]
Thus the new integral, spread over the $GL_2$-orbit, is

$$\alpha \mathcal{H} + cF \ell u x. \quad (A5.1.36)$$

This formula points out another root of difficulty in attempting to construct a covariant Hamiltonian formalism: we now have to bring in fluxes whose avoidance is one of the salient features of the traditional Hamiltonian formalism.

However, the picture is not all of gloom. If dealing with a Lie group, in this case $GL_{m+1}$, is difficult, we may have a better luck dealing with the corresponding Lie algebra. Details in the next Section.

**Exercise A5.1.37.** Show that the $GL_2 - KdV$ system (A5.1.21) still has an infinite number of integrals.

**Exercise A5.1.38.** Instead of the $KdV$ equation (A5.1.7) consider a more general system

$$u_\tau = \partial_\eta (f + u_\eta \eta) = \frac{Df}{Du}(u_\eta) + u_{\eta \eta} = \partial_\eta \frac{\delta}{\delta u}(F - \frac{1}{2}(u_\eta)^2), \quad (A5.1.39a)$$

$$f = \frac{dF}{du}, \quad F \in C(u). \quad (A5.1.39b)$$

Show that formulae (A5.1.21, 24, 26) undergo the following changes:

$$\dot{u}_2 = -(c^{-4} \Delta u + c^{-1}du_2)' + c^{-3}au_1 - c^{-2}\frac{DF}{Du}(u_1), \quad (A5.1.40)$$

$$H = \frac{1}{2}u_2^2 - c^{-1}du_1'u_1 + (bcu - c^3 du_2/2 - cdf)u_2/\Delta -$$

$$-c^{-2}abu^2/2\Delta + c^{-2}(adF + bc\Phi)/\Delta - c^{-1}f^2/2\Delta, \quad (A5.1.41a)$$

$$\frac{d\Phi}{du} = u \frac{df}{du}, \quad (A5.1.41b)$$

$$h = c^{-3}au - c^{-2}f. \quad (A5.1.41c)$$

**Exercise A5.1.42.** The travelling wave solutions of an evolution equation and its $GL$-version, — are they in any way related?

**Definition A5.1.43.** In the case of $m$ independent variables, a Hamiltonian $H \in C_q$ is called a $s^{th}$ momentum of a Hamiltonian matrix $B$, denoted $H \in P^s(B)$, if

$$B \left( \frac{\delta H}{\delta q} \right) = \partial_s(q). \quad (A5.1.44)$$

**Exercise A5.1.45.** Suppose $B = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)$ is a canonical Hamiltonian matrix. Show that

$$\partial_s(p)^*q \in P^s(B). \quad (A5.1.46)$$
§A5.2. INFINITESIMAL GEOMETRIC PERTURBATIONS

In this Section we examine effects of infinitesimal \( \mathfrak{gl}_{m+1} \)-actions on Hamiltonian systems. 1-dimensional examples considered include the KdV and mKdV equations, and barothropic and adiabatic versions of the fluid dynamics.

For maximum lucidity, in this Section everything is allowed to commute.

Almost every matrix \( M \in GL_{m+1} \) has a triangular decomposition

\[
M = \mathcal{L} \mathcal{D} \mathcal{U},
\]

where \( \mathcal{D} \) is a diagonal matrix and \( \mathcal{L}(\mathcal{U}) \) is a lower-(upper)-triangular matrix with the 1’s on the diagonal. In its action on the vector of partial derivatives \( \nabla_y = (\partial_t, \nabla_x) \), the diagonal matrix \( \mathcal{D} \) acts as a rescaling, while the upper-triangular matrix \( \mathcal{U} \) mixes the \( x \)'s between themselves and leaves \( t \) alone. These are innocent actions, easy to accommodate to any fanciful scheme of things to come. For the lower-triangular matrix \( \mathcal{L} \), we have

\[
\mathcal{L} = EN,
\]

where

\[
\mathcal{L} = \begin{pmatrix}
1 & & \\
\alpha & \ddots & \\
& \ddots & 1
\end{pmatrix}, \quad \alpha \text{ is a column } m-\text{vector, } L \in Mat_m,
\]

\[
E = \begin{pmatrix}
1 & & \\
\alpha & \ddots & \\
& \ddots & 1
\end{pmatrix}, \quad N = \begin{pmatrix}
1 & & \\
0 & \ddots & \\
& 0 & 1
\end{pmatrix}
\]

The matrix \( N \) acts in a way similar to \( \mathcal{U} \). The matrix \( E \) acts as

\[
\partial_t \mapsto \partial_t; \quad \partial_{x_i} \mapsto \partial_{x_i} + \alpha_i \partial_t, \quad i = 1, \ldots, m.
\]

This is the matrix ultimately responsible for all the covariant Hamiltonian difficulties discussed in the preceding Section.

**Exercise A5.2.4.** Show that neither of the representations

\[
M = \mathcal{L} \mathcal{D} \mathcal{U} \quad \text{or} \quad M = \mathcal{U} \mathcal{D} \mathcal{L}
\]

covers the transposition \( (t,x) \mapsto (x,t), \quad M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \).

**Exercise A5.2.6.** Let \( m = 1, \quad a = a(u) \in C(u), \quad H = H(u) \in C(u) \).

(i) Show that the action (A5.2.3)

\[
\partial_t \mapsto \partial_t, \quad \partial_x \mapsto \partial_x + \alpha \partial_t,
\]

takes the Hamiltonian system

\[
u_t = (a \partial + \partial a) \left( \frac{\delta H}{\delta u} \right) = (a \partial + \partial a)(H_u)
\]

into the equation

\[
u_t = [1 - \alpha(2aH_{uu} + a_u H)]^{-1}(a \partial + \partial a)(H_u);
\]
(ii) Show that this equation is again a Hamiltonian system

\[ \dot{u}_t = B \frac{\delta \mathcal{H}}{\delta u}, \]  

\[ B = [1 - \alpha(2aH_{uu} + a_uH_u)]^{-1}(a\partial + \partial a)[1 - \alpha(2aH_{uu} + a_uH_u)]^{-1}, \]  

\[ \mathcal{H} = H - \alpha a(H_u)^2; \]  

(A5.2.10a)  

(A5.2.10b)  

(A5.2.10c)

(iii) Let \( H_0 = H_0(u) \) be such that

\[ dH_0/du = a^{-1/2}. \]  

Show that \( H_0 \in \text{Ker}(a\partial + \partial a) \), i.e.

\[ (a\partial + \partial a) \left( \frac{\delta H_0}{\delta u} \right) = 0. \]  

(A5.2.11)

(A5.2.12)

[Hint: \( a\partial + \partial a = 2\sqrt{a}\partial \sqrt{a} \);

(i^4) Show that equation (A5.2.8) implies

\[ H_{0,t} = 2\partial(\sqrt{a}H_u); \]  

(A5.2.13)

(i^5) Deduce from this that

\[ H_0 = H_0 - 2\alpha \sqrt{a}H_u \]  

is a conserved density of the equation (A5.2.9);

(i^6) Show that \( \mathcal{H}_0 \in \text{Ker}(B) \), with \( B \) given by formula (A5.2.10b);

(i^7) Show that

\[ \mathcal{H}_1 = \text{const}(\mathcal{H}_0)^2, \text{ const } = 1/4, \]  

(A5.2.15)

is a momentum for the Hamiltonian operator \( B \) (A5.2.10b):

\[ B \left( \frac{\delta \mathcal{H}_1}{\delta u} \right) = u_x; \]  

(A5.2.16)

(i^8) For \( \alpha = 0 \), formula (A5.2.15) yields the momentum \( H_1 = (H_0)^2/4 \) of the Hamiltonian operator \( B = (a\partial + \partial a) \). Show that

\[ H_{1,t} = \partial(H_0\sqrt{a}H_u - H); \]  

(A5.2.17)

(i^9) Deduce from this that

\[ \mathcal{H}_1 = H_1 - \alpha(H_0\sqrt{a}H_u - H) \]  

(A5.2.18)

is a conserved density of the perturbed equation (A5.2.9);

Show that

\[ \mathcal{H}_1 \neq \mathcal{H}_1 \text{ for } \alpha \neq 0. \]  

(A5.2.19)

Clearly, the more \( x \)-derivatives a Hamiltonian \( H \) contains the harder it is to come up with explicit formulae like (A5.2.10). As a general problem, it is probably hopeless. A temporary way out is to consider matrices \( M \) very close to the identity:

\[ M = 1 + \epsilon \tilde{M}, \quad \tilde{M} \in g\ell_{m+1}, \quad \epsilon^2 = 0. \]  

(A5.2.20)
In terms of the key matrix \( E \) \((A5.2.2c)\), the corresponding transformation \((A5.2.3)\) is

\[
\partial_t \mapsto \partial_t; \quad \partial_{x_i} \mapsto \partial_{x_i} + \epsilon \alpha_i \partial_t, \quad i = 1, \ldots, m, \quad \epsilon^2 = 0,
\]

so that

\[
\partial^\sigma \mapsto \partial^\sigma + \epsilon \sum_i \alpha_i \sigma_i \partial^{\sigma - 1} \partial_t, \quad \partial^\sigma = \partial^\sigma_{x_1} \ldots \partial^\sigma_{x_m}.
\]  

**Exercise A5.2.23.** Let \( m = 1 \). Show that under the action \((A5.2.21)\), the equation

\[
u_t = F(u, \ldots, u^{(n)}), \quad n \in \mathbb{N},
\]  

becomes

\[
u_t = (1 + \epsilon \sum_{i=0}^{n-1} i + 1) \frac{\partial F}{\partial u^{(i+1)}} (F).
\]  

The formula \((A5.2.25)\) is *quadratic* in \( F \). It is easy to see that this conclusion applies in general, for vector systems subject to \( gl_{m+1} \)-action. In the remainder of this Section we consider various Examples, arranged into Exercises, all in the \( gl_2 \)-case. Since in this case there is only one \( x \) present, we can put \( \alpha = 1 \) in formula \((A5.2.21)\) and consider the action

\[
\partial_t \mapsto \partial_t, \quad \partial_x \mapsto \partial_x + \epsilon \partial_t, \quad \epsilon^2 = 0 \quad \Rightarrow \quad \partial_x^n \mapsto \partial_x^n + \epsilon n \partial_x^{n-1} \partial_t.
\]  

**Exercise A5.2.27.** Let \( f = dF/du \in \mathcal{C}(u) \). Consider a KdV-like equation

\[
u_t = \partial(f - u_{xx}) = \partial \left( \frac{\delta H}{\delta u} \right), \quad H = F + \frac{1}{2} u_x^2.
\]  

(i) Show that the action \((A5.2.26)\) turns this equation into

\[
u_t = \left[ 1 + \epsilon (f_u - 3 \partial^2) \right] \partial(f - u_{xx});
\]  

(ii) Show that equation \((A5.2.28)\) implies

\[
H_t = \partial \left\{ \frac{1}{2} f^2 + u_x \partial(f) - u_{xx} f + \frac{1}{2} u_x^2 - u_x u_{xxx} \right\};
\]  

(iii) Deduce from this that

\[
\mathcal{H}_3 = F + \frac{1}{2} u_x^2 - \epsilon \left( \frac{1}{2} f^2 - u_{xx} f + \frac{1}{2} u_x^2 \right)
\]  

is a conserved density of the equation \((A5.2.29)\); 

(iii) Show that the Hamiltonian \(\mathcal{H}_3 \) \((A5.2.31)\) and the operator

\[
B^f = \partial + \epsilon (f_u \partial + \partial f_u - 4 \partial^3)
\]  

reproduce the perturbed equation \((A5.2.29)\); 

(i) Show that the operator \( B^f \) \((A5.2.32)\) is Hamiltonian over the ring \( C_u[\partial][\epsilon]/(\epsilon^2) \), \( C_u = C[u, u^{(1)}, \ldots] \);
(i^6) Consider another copy of the same equation (A5.2.28), with \( u \) renamed \( v \):

\[ v_t = \partial(2v^3 - v_{xx}), \quad (A5.2.33) \]

the mKdV equation. The Miura map

\[ u = v^2 + v_x, \quad (A5.2.34) \]

under the action (A5.2.25) becomes

\[ u = v^2 + v_x + \epsilon(6v^2v_x - v_{xxx}). \quad (A5.2.35) \]

Show that this map takes the Hamiltonian operator

\[ B^I = \partial + \epsilon(6v^2\partial + 6\partial v^2 - 4\partial^3) \]

into the Hamiltonian operator

\[ B^{II} = 2u\partial + 2\partial u - \partial^3 + \epsilon(6\partial^5 - 20(u\partial^3 + \partial^3 u) + 12(u'' + 2u^2)\partial + 12\partial(u'' + 2u^2)); \quad (A5.2.37) \]

(i^7) Show that equation (A5.2.28) has the conservation law

\[ \left( \frac{1}{2}u^2 \right)_t = (\Phi - uu'' + \frac{1}{2}u^2)_x, \quad \Phi_u = uf_u, \quad (A5.2.38) \]

(i^8) Deduce from this that the perturbed equation (A5.2.29) has the conserved density

\[ \mathcal{H}_2 = \frac{1}{2}u^2 - \epsilon(\Phi + \frac{3}{2}u^2); \quad (A5.2.39) \]

(i^9) Check that for the case

\[ f = 3u^2, \quad \Phi = 2u^3, \quad (A5.2.40) \]

the Hamiltonian \( \mathcal{H}_2 \) (A5.2.39) and the Hamiltonian operator \( B^{II} \) (A5.2.37) reproduce the perturbed KdV equation (A5.2.29);

(i^{10}) Let \( H_0 = u \in \text{Ker}(\partial) \);

\[ \partial \left( \frac{\delta H_0}{\delta u} \right) = 0. \quad (A5.2.41) \]

From the equation (A5.2.28) deduce that

\[ \mathcal{H}_0 = u - \epsilon(f - u'') \sim u - \epsilon f \]

is a conserved density of the perturbed equation (A5.2.29);

(i^{11}) Check that \( \mathcal{H}_0 \in \text{Ker}(B^I) \), with \( B^I \) being given by formula (A5.2.32);

(i^{12}) Check that

\[ \mathcal{H}_1 = \frac{u^2}{2} - \epsilon(uf - 2uu'') = \frac{1}{2}[u - \epsilon(f - 2u'')]^2 \quad (A5.2.43) \]

is a momentum of the Hamiltonian operator \( B^I \) (A5.2.32);
(i\textsuperscript{13}) Check that
\[ u(1 - 4\epsilon u) \]  
(A5.2.44)
is a momentum of the Hamiltonian operator \( B^{II} \) (A5.2.37)

**Exercise A5.2.45.** Consider the 1-dimensional barotropic fluid dynamics
\[ M_t = (M^2/\rho + P)_x, \quad \rho_t = M_x, \]  
(A5.2.46)
where the pressure \( P = P(\rho) \) is a given function of the density \( \rho \):
\[ P = \rho E_\rho - E, \quad P_\rho = \rho E_{\rho\rho}, \]  
(A5.2.47)

\( E = E(\rho) \) is the internal energy density, and \( M \) is the momentum density of the fluid. (The sign of time is antinomian). The system (A5.2.46) is Hamiltonian: it can be rewritten as

\[
\begin{pmatrix}
M \\
\rho
\end{pmatrix}_t = 
\begin{pmatrix}
M \partial + \partial M & \rho \partial \\
\partial \rho & 0
\end{pmatrix}
\begin{pmatrix}
\delta H \\
\delta M \\
\delta \rho
\end{pmatrix},
\]  
(A5.2.48)
\[ H = M^2/2\rho + E. \]  
(A5.2.49)

(i) Show that the action (A5.2.26) converts the system (A5.2.46) into the deformed system
\[ M_t = (M^2/\rho + P) + \epsilon[2M\rho^{-1}(M^2/\rho + P) + (P_\rho - M^2/\rho^2)M'], \]  
(A5.2.50a)
\[ \rho_t = M' + \epsilon(M^2/\rho + P)', \]  
(A5.2.50b)

(ii) Show that the system (A5.2.46) has the conservation lower
\[ H_t = (M^3/2\rho^2 + M E_\rho)_x; \]  
(A5.2.51)

(iii) Deduce from this that the perturbed system (A5.2.50) has the conserved density
\[ \mathcal{H} = M^2/2\rho + E - \epsilon(M^3/2\rho^2 + M E_\rho); \]  
(A5.2.52)

(iv) Show that in the variables \( u = M/\rho \) (the fluid velocity) and \( \rho \), the matrix
\[ B = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix} + \epsilon \begin{pmatrix} E_{\rho\rho} \partial + \partial E_{\rho\rho} & u \partial + \partial u \\ u \partial + \partial u & \rho \partial + \partial \rho \end{pmatrix}, \]  
(A5.2.53)

and the Hamiltonian \( \mathcal{H} \) (A5.2.52) generate the perturbed system (A5.2.50), now written in the form
\[ u_t = (1 + \epsilon u)(u^2/2 + E_\rho)' + \epsilon E_{\rho\rho}(\rho u)_t, \]  
(A5.2.54a)
\[ \rho_t = [\rho_u + \epsilon(\rho u^2 + P)]', \]  
(A5.2.54b)

(v) Show that the matrix \( B \) (A5.2.53) is Hamiltonian;
(i) Show that if $\epsilon$ is finite then the matrix $B$ (A5.2.53) is Hamiltonian iff

$$\rho E_{\rho \rho} = E_{\rho \rho};$$  \hspace{1cm} (A5.2.55)

(ii) Show that both

$$\mathcal{H}_0 = \rho(1 - \epsilon)u$$ and $\mathcal{H}_0 = u - \epsilon(\frac{u^2}{2} + E_\rho)$  \hspace{1cm} (A5.2.56)

belong to the kernel of the Hamiltonian matrix $B$ (A5.2.53);

(iii) Set

$$\mathcal{H}_1 = \mathcal{H}_0 \mathcal{H}_0 = \rho u - \epsilon(\frac{3}{2}\rho u^2 + \rho E_\rho).$$  \hspace{1cm} (A5.2.57)

Show that $\mathcal{H}_1$ is a momentum of the Hamiltonian matrix $B$ (A5.2.53). (Cf. formula (A5.2.15).)

**Exercise A5.2.58.** Consider the 1-dimensional adiabatic fluid dynamics:

$$M_t = (M^2 / \rho + P)_x, \quad \rho_t = M_x, \quad \eta_t = \rho^{-1} M \eta_x,$$  \hspace{1cm} (A5.2.59)

which differs from the barothropic system (A5.2.46) in two respects: first, we have a new variable $\eta$, the specific entropy, and second, the internal energy $E$ is a given function of $\rho$ and $\eta$: $E = E(\rho, \eta)$. The adiabatic fluid dynamics is a Hamiltonian system: it can be rewritten in the form

$$\begin{pmatrix} M \\ \rho \\ \eta \end{pmatrix}_t = \begin{pmatrix} \rho \partial + \partial M \\ \partial \rho \\ \eta_x \end{pmatrix} \begin{pmatrix} \rho \partial \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} \delta H/\delta M \\ \delta H/\delta \rho \\ \delta H/\delta \eta \end{pmatrix},$$  \hspace{1cm} (A5.2.60)

$$H = \frac{M^2}{2\rho} + E.$$  \hspace{1cm} (A5.2.61)

In the variables $u = M/\rho, \rho, \eta$, the Hamiltonian form (A5.2.60,61) becomes

$$\begin{pmatrix} u \\ \rho \\ \eta \end{pmatrix}_t = \begin{pmatrix} 0 \\ \partial \rho^{-1} \eta_x \\ \rho^{-1} \eta_x \end{pmatrix} \begin{pmatrix} \delta H/\delta u \\ \delta H/\delta \rho \\ \delta H/\delta \eta \end{pmatrix},$$  \hspace{1cm} (A5.2.62)

$$H = \rho u^2/2 + E.$$  \hspace{1cm} (A5.2.63)

The adiabatic motion equations in the $(u, \rho, \eta)$-variables,

$$u_t = uu_x + \rho^{-1} P_x, \quad \rho_t = (up)_x, \quad \eta_t = u\eta_x,$$  \hspace{1cm} (A5.2.64)

after the infinitesimal action (A5.2.26), become

$$\dot{u} = (1 + \epsilon u)(uu' + \rho^{-1} P') + \epsilon \rho^{-1}[P_u(u\rho)' + P_\eta u\eta'],$$  \hspace{1cm} (A5.2.65a)

$$\dot{\rho} = [u\rho + \epsilon(\rho u^2 + P)]',$$  \hspace{1cm} (A5.2.65b)

$$\dot{\eta} = u\eta'(1 + \epsilon u).$$  \hspace{1cm} (A5.2.65c)
(i) Show that the adiabatic motion equations (A5.2.64) imply
\[ H_t = (\rho u^3/2 + u \rho E_\rho)_x; \quad (A5.2.66) \]

(iii) Deduce that the perturbed system (A5.2.65) has the conserved density
\[ \mathcal{H} = \rho u^2/2 + E - \epsilon u(\rho u^2/2 + \rho E_\rho) = H - \epsilon \rho u H_\rho; \quad (A5.2.67) \]

(iii) Consider the matrix
\[ B = \begin{pmatrix} \epsilon(E_{\rho \rho} \partial + \partial E_{\rho \rho}) & \epsilon(u \partial + \partial u) + \partial & -\rho^{-1} \eta_x (1 + 2 \epsilon u) \\ \epsilon(u \partial + \partial u) + \partial & \epsilon(\rho \partial + \partial \rho) & -\epsilon \eta_x \\ \rho^{-1} \eta_x (1 + 2 \epsilon u) & \epsilon \eta_x & 0 \end{pmatrix}. \quad (A5.2.68) \]

Show that this matrix B and the Hamiltonian \( \mathcal{H} \) (A5.2.67) reproduce the perturbed motion equations (A5.2.65);

(iii) Show that the Hamiltonians
\[ \mathcal{H}_0 = \rho(1 - \epsilon u), \quad \tilde{\mathcal{H}}_0 = \rho \eta(1 - \epsilon u), \quad (A5.2.69) \]

belong to the Kernel of the Hamiltonian matrix B (A5.2.68);

(6) Set
\[ \mathcal{H}_1 = \rho u - \epsilon \rho(3/2 u^2 + E_\rho). \quad (A5.2.70) \]

Show that \( \mathcal{H}_1 \in P(B) \). (Cf. formula (A5.2.57).)

APPENDIX A6. NONCOMMUTATIVE SOLITONS

In commutative theory, multisolution solutions can be sometimes efficiently derived in the form of infinite series. Below we examine such a series-construction for the noncommutative KdV equation and observe, in this language, the precise place where an obstruction appears to the introduction of the \( \tau \)-function representations.

The series representation we are about to develop, follows along the lines of the commutative derivation of Wadati and Sawada in [WSa 1980a].

We start with the KdV equation
\[ u_t = (3u^2 + u_{xx})_x, \quad (A6.1) \]
and consider its potential version in the form
\[ w_t = 6\epsilon w_x^2 + w_{xxx}, \quad (A6.2) \]
\[ u = 2\epsilon w_x. \quad (A6.3) \]

Here we have introduced a formal parameter \( \epsilon \) commuting with everything. The purpose of such an introduction is to be able to look for solutions of the \( w \)-equation (A6.2) in the form of a series
\[ w = \sum_{i \geq 0} w_i \epsilon^i \quad (A6.4) \]
Substituting this representation into the $w$-equation (A6.2), we obtain the system

\[(\partial_x^3 - \partial_t)(w_0) = 0, \quad (A6.5a)\]

\[(\partial_x^3 - \partial_t)(w_{s+1}) = -6 \sum_{p+q=s} w_p' w_q', \quad s \in \mathbb{N}. \quad (A6.5b)\]

We start with the linear $w_0$ equation (A6.5a). Consider the objects

\[\varphi_\ell = A_\ell \exp[k(\ell)x + 4k(\ell)^3t], \quad \ell = 1, ..., N, \quad (A6.6)\]

where $A_\ell$'s are noncommutative constants, and $k(1), ..., k(n)$ are commutative constants, real or complex, subject to the condition

\[k(i) + k(j) \neq 0, \quad i, j = 1, ..., N. \quad (A6.7)\]

(In particular, none of the $k(i)$'s may vanish.) Set

\[\Phi_\ell = (\varphi_\ell)^2 = (A_\ell)^2 \exp[2k(\ell)x + 8k(\ell)^3t]. \quad (A6.8)\]

By definition, we have:

\[\partial_t(\Phi_\ell) = 8k(\ell)^3 \Phi_\ell, \quad (A6.9a)\]

\[\partial_x(\Phi_\ell) = 2k(\ell) \Phi_\ell \Rightarrow \quad (A6.9b)\]

\[\partial_t(\cap_{\ell \in S} \Phi_\ell) = 8(\sum_{\ell \in S} k(\ell)^3)(\cap_{\ell \in S} \Phi_\ell), \quad (A6.10a)\]

\[\partial_x(\cap_{\ell \in S} \Phi_\ell) = 2(\sum_{\ell \in S} k(\ell))(\cap_{\ell \in S} \Phi_\ell). \quad (A6.10b)\]

In particular,

\[(\partial_x^3 - \partial_t)(\Phi_\ell) = 0. \quad (A6.11)\]

We then set:

\[w_0 = \sum_{\ell=1}^{N} \Phi_\ell \quad (A6.12)\]

for some fixed $N \in \mathbb{N}$. The $w_0$-equation (A6.5a) is thereby satisfied.

Now set

\[w_1 = -\sum_{\ell m} \frac{1}{k(\ell) + k(m)} \Phi_\ell \Phi_m. \quad (A6.13)\]

By formulae (A6.10),

\[(\partial_x^3 - \partial_t)(w_1) = -\sum_{\ell m} \frac{1}{k(\ell) + k(m)} 8[(k(\ell) + k(m))^3 - k(\ell)^3 - k(m)^3] \Phi_\ell \Phi_m =\]

\[= -24 \sum_{\ell m} k(\ell)k(m) \Phi_\ell \Phi_m - 6 \sum_{\ell} \partial_x(\Phi_\ell) \sum_{m} \partial_x(\Phi_m) = -6(w'_0)^2. \quad (A6.14)\]

Thus, the $w_1$-equation, (A6.5b) for $s = 0$, is also satisfied.
To write down the general $w_n$-formula, we introduce the following notations:

If

$$\mathbf{\ell} = (\ell_1, ..., \ell_n), \quad 1 \leq \ell_1, ..., \ell_n \leq N$$

is a multiindex, we set

$$\Phi_{\ell} = \phi_{\ell_1} \cdots \phi_{\ell_n},$$

$$|\ell| = n,$$  \hspace{1cm} (A6.16)

$$\sigma(\ell) = \frac{1}{k(\ell_1) + k(\ell_2) \cdots k(\ell_{n-1}) + k(\ell_n)}, \quad |\ell| > 1,$$  \hspace{1cm} (A6.18a)

$$\sigma(\ell) = 1, \quad |\ell| = 1,$$  \hspace{1cm} (A6.18b)

$$f_\ell \equiv \sum_{j=1}^{|\ell|} k(\ell_j)^i, \quad i \in \mathbb{Z}_+.$$  \hspace{1cm} (A6.19)

Now set

$$w_i = (-1)^i \sum_{|\ell| = i+1} \sigma(\ell) \Phi_{\ell}, \quad i \in \mathbb{Z}_+.$$  \hspace{1cm} (A6.20)

**Proposition A6.21.** Formulae (A6.20) solve the $w$-system (A6.5).

**Proof.** Formula (A6.5a) has been already verified. To check formula (A6.5b), we have

$$(\partial_2^3 - \partial_i)(w_{s+1}) = (\partial_2^3 - \partial_i)(-1)^{s+1}(\sum_{|\ell| = s+2} \sigma(\ell) \Phi_{\ell}) \quad \text{[by (A6.10)]} =$$

$$= (-1)^{s+1} \sum \delta(f_\ell^3 - f_\ell)(\Phi_{\ell}).$$  \hspace{1cm} (A6.22)

**Exercise A6.23.** Show that

$$(z_1 + ... + z_{n+1})^3 - z_1^3 - ... - z_{n+1}^3 = 3 \sum_{j=1}^n (z_1 + ... + z_j)(z_j + z_{j+1})(z_{j+1} + ... + z_{n+1}).$$

From formula (A6.24) it follows that

$$[f_\ell(\ell^3 - f_\ell)] \sigma(\ell) = 3 \sum_{\ell - \ell_+ = \ell} f_\ell(f_\ell^3 - f_\ell)\sigma(\ell^3 - \ell)\sigma(\ell_+).$$  \hspace{1cm} (A6.25)

where $\ell_- = (\ell_1, ..., \ell_j), \ell_+ = (\ell_{j+1}, ..., \ell_\ell), \quad 1 \leq j \leq |\ell|$. Substituting formula (A6.25) into the expression (A6.22) we get:

$$24(-1)^{s+1} \sum_{|\ell| = s+2} f_\ell(f_\ell - f_\ell)\sigma(\ell)\sigma(\ell^3 - \ell)\sigma(\ell_+)\Phi_{\ell_-}\Phi_{\ell_+} =$$

$$= 6 \sum_{p+q = s} [\sum_{|\ell_-| = p+1} (-1)^p \partial_x(\sigma(\ell_-)\Phi_{\ell_-})][\sum_{|\ell_+| = q+1} (-1)^q \partial_x(\sigma(\ell_+))\Phi_{\ell_+}] =$$

$$= -6 \sum_{p+q = s} w'_p w'_q.$$  \hspace{1cm} (A6.26)
and this is the R.H.S of the equation (A6.5b).

The \( w \)-formulae (A6.20) can be recast into a more compact form. Introduce two matrices, \( \mathcal{M} \) and \( \mathcal{D} \):

\[
\mathcal{M}_{\alpha \beta} = \frac{1}{k(\alpha) + k(\beta)} \varphi_{\alpha} \varphi_{\beta}, \quad \alpha, \beta = 1, \ldots, N, \quad (A6.27)
\]

\[
\mathcal{D} = \text{diag} (\varphi_1, \ldots, \varphi_N). \quad (A6.28)
\]

Assume for the moment that all the noncommutative constants \( A_i \)'s in formula (A6.6) are invertible, so that the matrix \( \mathcal{D}^{-1} \) exists.

**Lemma A6.29.** Formulae (A6.20) can be recast as

\[
w_i = (-1)^i \text{Tr} (\mathcal{D}^{-1} \mathcal{M}_x \mathcal{M}^i \mathcal{D}), \quad i \in \mathbb{Z}_+. \quad (A6.30)
\]

**Proof.** Since

\[
\partial_x (\varphi_{\alpha} \varphi_{\beta}) = [k(\alpha) + k(\beta)] \varphi_{\alpha} \varphi_{\beta}, \quad (A6.31)
\]

we find that

\[
(\mathcal{M}_x)_{\alpha \beta} = \varphi_{\alpha} \varphi_{\beta}. \quad (A6.32)
\]

Hence, for \( i = 0 \), formula (A6.30) yields

\[
(D^{-1} \mathcal{M}_x \mathcal{D})_{\alpha \beta} = \varphi_{\beta}^2, \quad (A6.33)
\]

so that

\[
w_0 = \text{Tr} (D^{-1} \mathcal{M}_x \mathcal{D}) = \text{Tr} (\mathcal{M}_x). \quad (A6.34)
\]

For \( i > 0 \), we have

\[
(\mathcal{M}_x \mathcal{M}^i)_{\alpha \beta} = \sum_{\gamma(1) \ldots \gamma(i)} \varphi_{\alpha} \varphi_{\gamma(1)} \varphi_{\gamma(2)} \ldots \varphi_{\gamma(i)} \varphi_{\beta} \Rightarrow \ usefulness
\]

\[
(D^{-1} \mathcal{M}_x \mathcal{M}^i \mathcal{D})_{\alpha \beta} = \sum_{\gamma} \varphi_{\gamma(1)}^2 \ldots \varphi_{\gamma(i)}^2 \varphi_{\beta}^2 \sigma(\gamma, \beta), \quad (A6.35)
\]

and taking the Trace of this identity we get back the definition (A6.20).

By formulae (A6.430),

\[
w = \sum_i w_i \epsilon^i = \sum \text{Tr} [D^{-1} \mathcal{M}_x (-\epsilon \mathcal{M})^i \mathcal{D}] = \text{Tr} [D^{-1} \mathcal{M}_x (1 + \epsilon \mathcal{M})^{-1} \mathcal{D}]. \quad (A6.36)
\]

*Only when everything commutes*, can we further simplify this formula into

\[
w = \epsilon^{-1} \frac{\partial}{\partial x} \text{Tr} \log (1 + \epsilon \mathcal{M}) \Rightarrow \quad (A6.37)
\]

\[
u = 2 \epsilon w_x = \frac{\partial^2}{\partial x^2} 2 \text{Tr} \log (1 + \epsilon \mathcal{M}). \quad (A6.38)
\]
In general, no simplification of the noncommutative formula (A6.36) is possible unless \( N = 1 \); in the latter case we find

\[
u = 4k \frac{\partial}{\partial x} \frac{e\Phi}{2k - e\Phi} = -8k^2 \frac{\partial}{\partial x} \frac{1}{e\Phi - 2k}, \tag{A6.39a}\]

where

\[
\Phi = A^2 e^{\exp(2kx + 8k^3 t)} \tag{A6.39b}
\]

is the 1-solution solution.

**APPENDIX A7. THE NONCOMMUTATIVE KP EQUATION**

My father used to say that there was no message between human beings which could not be communicated on the back of a postcard. 
Auberon Waugh

The traditional (commutative) KP equation arises as the compatibility condition between first two nontrivial flows of the KP hierarchy. Below we perform similar compatibility operation on the noncommutative KP hierarchy.

We start with the KP Lax operator

\[
L = \xi + \sum_{i \geq 0} A_i \xi^{-i-1}, \tag{A7.1}\]

denote \( t_2 = y, \ t_3 = t \), and consider the KP flows \#2 and \#3:

\[
\partial_y(L) = [(L^2)_+/2, L], \tag{A7.2}
\]
\[
\partial_t(L) = [(L^3)_+/3, L], \tag{A7.3}\]

Since

\[
(L^2)_+/2 = \xi^2/2 + A_0, \tag{A7.4}
\]
\[
(L^3)_+/3 = \xi^3/3 + A_0 \xi + A_1 + A_0^{(1)}, \tag{A7.5}
\]

we find:

\[
\partial_y(L) = \partial_y(A_0)\xi^{-1} + \partial_y(A_1)\xi^{-2} + ... = [\xi^{2}/2 + A_0, \ \xi + A_0\xi^{-1} + A_1\xi^{-2} + A_2\xi^{-3} + ...]_+ =
\]
\[
= \{A_0^{(2)}/2 + A_1^{(1)}\}\xi^{-1} + \{A_1^{(2)}/2 + A_2^{(1)} + A_0 A_0^{(1)} + [A_0, A_1]\}\xi^{-2} + ... \Rightarrow \tag{A7.6}
\]
\[
\partial_y(A_0) = A_0^{(2)}/2 + A_1^{(1)}, \tag{A7.7}
\]
\[
\partial_y(A_1) = A_1^{(2)}/2 + A_2^{(1)} + A_0 A_0^{(1)} + [A_0, A_1], \tag{A7.8}
\]
\[
\partial_t(L) = \partial_t(A_0)\xi^{-1} + ... = [\xi^{3}/3 + A_0 \xi + A_1 + A_0^{(1)}, \ \xi + A_0\xi^{-1} + A_1\xi^{-2} + A_2\xi^{-3} + ...]_+ =
\]
\[
= \{A_0^{(3)}/3 + A_1^{(2)} + A_2^{(1)} + (A_0^{(1)})^{(1)}\}\xi^{-1} + ... \Rightarrow \tag{A7.9}
\]
\[ \partial_t(A_0) = A_0^{(3)}/3 + A_1^{(2)} + A_2^{(1)} + (A_0^{(3)})^{(1)}. \] (A7.10)

From the three equations (A7.7,8,10), we aim to eliminate \( A_1 \) and \( A_2 \); the remaining object will be the desired KP equation. From the equation (A7.7) we find

\[ A_1 = -A_0^{(1)}/2 + \partial^{-1}\partial_y(A_0). \] (A7.11)

From the equation (A7.8) we obtain

\[ A_1^{(2)} + A_2^{(1)} = \partial_y(A_1) + A_1^{(2)}/2 - [A_0, A_1] - A_0 A_0^{(1)}. \] (A7.12)

Substituting formulae (A7.11,12) into the equation (A7.10), we get:

\[
\partial_t(A_0) = A_0^{(3)}/3 + \left\{ \left\{ -\partial\partial_y(A_0)/2 + \partial^{-1}\partial_y^2(A_0) \right\} + \left\{ -A_0^{(3)}/2 + \partial\partial_y(A_0) \right\} \right\}/2 - \\
- [A_0, -A_0^{(1)}/2 + \partial^{-1}\partial_y(A_0) - A_0 A_0^{(1)} + (A_0^{(3)})^{(1)}] \Rightarrow \\
\partial_t(A_0) = A_0^{(3)}/12 + (A_0^{(2)}/2)^{(1)} + (\partial_y - adA_0)[\partial^{-1}\partial_y(A_0)].
\] (A7.13)

This is the noncommutative KP equation. It is nonlocal in \( A_0 \)-terms, but is local in \( w = \partial^{-1}(A_0) \)-terms.

Similar calculations yield the mKP and Potential mKP equations, and the corresponding Miura map. This is left to the reader as an Exercise.

In ancient times, men learned with a view of their own improvement. Nowadays, men learn with a view to the approbation of others.

Confucius (C.553 B.C. - 479 B.C.)

**APPENDIX A8. A LIST OF SCALAR EQUATIONS**

In this Appendix, most of the scalar equations from the book are collected together.

We start with continuous-space equations from Part A.

The Burgers hierarchy (2.5.12) of commuting flows

\[ \partial_t^n(u) = \text{const}_n(ad_u + \partial)(\hat{R}_u - \partial)^n, \quad n \in \mathbb{N}, \] (A8.1)

is linearized by the map \( \Phi \) (2.5.17)

\[ \Phi(u) = -U^{(1)}U^{-1}. \] (A8.2)

into the hierarchy (2.5.18)

\[ \partial_t^n(U) = \text{const}_n(-1)^{n+1}U^{(n)}. \] (A8.3)

Alternatively ("the mirror symmetry"), the first two of these formulae could be taken as (2.5.28,29):

\[ \partial_t^n(u) = \text{const}_n(\partial - ad_u)(\hat{L}_u - \partial)^n, \] (A8.4)

\[ \Phi(u) = -U^{-1}U^{(1)}. \] (A8.5)

In the zero-dispersion limit, equations (A8.1,4) become (2.5.31,34):

\[ \partial_t^n(u) = \text{const}_n n u_x u^{n-1}, \] (A8.6)
\[ \partial_t u = const_n nu^{n-1}u_x. \quad (A8.7) \]

One more version of the latter equations is (2.5.36):

\[ \partial_t u = const_n \partial(u^{n+1}). \quad (A8.8) \]

The KdV hierarchy (2.6.2,3):

\[ \partial_{t_{2n+1}}(h) = const_{2n+1} \partial(p_{-1}(2n + 1)), \quad n \in \mathbb{Z}_+, \quad (A8.9a) \]

\[ (\xi + \hbar \xi^{-1})^n = \sum_k \xi^k p_k(n), \quad \xi = d/dx. \quad (A8.9b) \]

For \( n = 1 \), we get (3.9.51)

\[ \partial_t h = (3\hbar^2 + h_{xx})_x. \quad (A8.10) \]

The quasiclassical limit of the KdV hierarchy (A8.9) is (3.9.195):

\[ \partial_{t_{2n+1}}(h) = const_n \partial(h^{n+1}). \quad (A8.11) \]

The mKdV equation of the first type (3.9.52) is

\[ \partial_t v = -3(v_x v^2 + v^2 v_x) + v_{xxx}, \quad (A8.12) \]

while the second-type version of this equation, (3.9.91), is

\[ \partial_t w = -6ww_x w + 3[w, w_{xx}] + w_{xxx}. \quad (A8.13) \]

The potential form (3.9.95)

\[ Pot(w) = -W_x W^{-1} \quad (A8.14) \]

of the equation (A8.13), (3.9.106), is

\[ \partial_t W = -3W_x W^{-1}W_{xx} + W_{xxx}; \quad (A8.15) \]

it is related to the KdV equation (A8.10) by the map (3.9.97):

\[ \Phi_2(h) = -W^{-1}W_{xx}. \quad (A8.16) \]

The mKdV equation (A8.12) has the following potential forms (3.9.111-117):

\[ \partial_t U = -3U_{xx} U^{-1}U_x + U_{xxx}, \quad (A8.17) \]

\[ Pot_+(v) = U_x U^{-1}, \quad (A8.18) \]

\[ \partial_t V = -3V_x V^{-1}V_x + V_{xxx}, \quad (A8.19) \]

\[ Pot_-(v) = -V^{-1}V_x. \quad (A8.20) \]
The quasiclassical limit of the mKdV hierarchy of which (A8.12) is the first non-trivial member, is (3.9.196)

\[ \partial_{t_{2n+1}}(v) = \text{const}_n \sum_{k=0}^{n} v^{2k} v_x v^{2(n-k)}. \]  

(A8.21)

The remaining equations in this Appendix are living in discrete spaces.

For each \( r \in \mathbb{N} \), the first flow (9.8.32) of an infinite hierarchy

\[ \partial_t(q) = [q + \ldots + q^{(r)}]q - q[q + \ldots + q^{(-r)}], \quad t = t_1, \]

(A8.22)

has the Bogoyavlensky limit (9.8.35)

\[ \frac{\partial q(x, t)}{\partial t} = \left[ \int_x^{x+c} q(\theta, t) d\theta \right] q(x, t) - q(x, t) \left[ \int_x^{x-c} q(\theta, t) d\theta \right]. \]  

(A8.23)

For \( r = 1 \), the flows (A8.22) and others in the hierarchy have alternative representations (14.3.25) and (14.6.34):

\[ \partial_t(q) = \text{Res}((L^n)_{+} - q\zeta^{-N-1}(L^n)_{+}\zeta^{N+1}), \quad L = \zeta(1 + q\zeta^{-1})^{-1}, \quad N = 1. \]  

(A8.24b)

The hierarchy (14.4.45) has the form

\[ \partial_t(H) = \text{Res}(\zeta^{-1}[(L^n)_{>0}, L]), \quad L = \zeta(1 - H\zeta^{-1})^{-1} H. \]  

(A8.25a)

Its quasiclassical limit (14.8.19) is

\[ \partial_t(H) = \text{const}_n H \partial_x(H^n) H. \]  

(A8.25b)

Formulae (14.5.14-18) yield

\[ \partial_t(H) = H[H^{(1)} - H^{(-1)}] H, \]  

(A8.26)

\[ \partial_t(V) = -VV^{(1)} - V^{(-1)}, \]  

(A8.27a)

\[ \partial_t(u) = u^{(1)} u - uu^{(-1)}, \]  

(A8.27b)

\[ u = V^{(2)} - 1 V. \]  

(A8.27c)

If the flow (A8.26) is rewritten as (14.5.20):

\[ \partial_t(H^{-1}) = (\Delta^{-1} - \Delta)(H), \]  

(A8.28a)

then the next flow (14.5.23) is

\[ \partial_t(H^{-1}) = (\Delta^{-1} - \Delta)(H[H^{(1)} + H^{(-1)}] H). \]  

(A8.28b)

The hierarchy (14.8.5) is

\[ \partial_t(u) = (u^{(n)} - u)u^{(n-1)} \ldots u, \quad n \in \mathbb{N}. \]  

(A8.29a)
Its zero-dispersion limit (14.8.6) is
\[ \partial_t(u) = nu_x u^{n-1}. \] (A8.29b)

The hierarchy (14.8.10,11)
\[ \partial_t(u) = (\Delta^2 - \Delta)[\text{Res}(L^n \zeta)], \quad L = (1 - \zeta^{-1}u)^{-1} \zeta, \quad n \in \mathbb{N}, \] (A8.30a)
has the quasiclassical limit (14.8.12,14)
\[ \partial_t(u) = \text{const}_n \partial(u^{n+1}). \] (A8.30b)

From the Relativistic Toda lattice comes the equation (15.2.13)
\[ \partial_t(q) = \epsilon q(1 - \Delta^{-1})(q), \] (48.31)
while from the shadow Relativistic Toda lattice comes the equation (15.3.30)
\[ \partial_t(q) = \epsilon^{-3}(q_1^{(1)-1}q_1 - 1). \] (A8.32)

From the constraint (15.6.42) of the deformed mKP hierarchy (15.6.3) comes the flow (15.6.69)
\[ \partial_t(Q) = \epsilon(1 - \Delta^{-1})(QQ^{(1)}). \] (48.33)

In the rest of this Appendix, the time is also discrete. Recall that \( \tilde{f} \) stands for \( f(t + \Delta t) \) when \( f \) stands for \( f(t) \), whatever \( f \) is.
Equation (17.3.57) (or (17.6.41) or (18.6.75)), discretizing (A8.27b), is
\[ (1 + h\tilde{a}^{(1)})\tilde{a} = \alpha(1 + h\alpha^{(-1)}). \] (A8.34)

The exponential function \( \partial_t(a) = a \) is discretized as (18.5.19,16)
\[ \tilde{a} = \lambda a, \quad \lambda(1 + \lambda h) = 1. \] (A8.35)

Equation (18.6.53,54), discretizing (A8.26), is
\[ (1 - h\tilde{\beta}^{(1)})^{-1}\tilde{\beta} = (1 - h\beta^{(-1)})^{-1}\beta. \] (A8.36)

Equation (18.6.69)
\[ \tilde{V} = V(1 + h\tilde{V}^{(1)-1}V^{(-1)}) \] (A8.37a)
discretizes the equation
\[ \partial_t(V) = VV^{(1)-1}V^{(-1)}. \] (A8.37b)

Equation (18.7.9,10), a discrete version of (A8.26) for \( v = H^{-1} \), is
\[ \tilde{v} = \beta^{-1}v^{(-1)}\beta^{(-1)}, \quad v = \beta^{-1} - h\beta^{(1)}. \] (A8.38)

A discrete version of (A8.33) is (19.3.30):
\[ \tilde{\alpha}(1 - h\epsilon\tilde{\alpha})^{-1}(1 - h\epsilon\alpha^{(-1)})^{-1} = (1 - h\epsilon\alpha^{(1)})^{-1}(1 - h\epsilon\alpha)^{-1}. \] (A8.39)
A discrete version of (A8.31) is (19.3.36):
\[ \hat{\alpha}(1 - \hat{h}\epsilon\hat{\alpha}^{-1})^{-1} = \alpha(1 - h\epsilon\alpha)^{-1}. \] (A8.40)

Equation (19.4.27c)
\[ (1 + h\hat{\alpha}(1))(1 - \epsilon\hat{\alpha})^{-1} = (1 + h\alpha)(1 - \epsilon\alpha)^{-1} \] (A8.41a)
discretizes the equation
\[ \partial_t(\alpha) = (\alpha - \epsilon^{-1})(\alpha^{(1)} - \alpha). \] (A8.41b)

Formula (10.11.59b)
\[ \hat{a} - a = -h\epsilon a \] (A8.42)
provides an exact time-discretization of its continuous-time counterpart
\[ \partial_t(a) = -a^2 \] (19.11.59a).

A few more discrete-time scalar systems, each one involving an intermediary variable, can be found in Part C.

APPENDIX A9. OPEN PROBLEMS AND CONJECTURES

When all questions are answered, it will be time to telephone the undertaker.
Elbert Hubbard

Open problems and Conjectures scattered throughout the book are collected in this Appendix.

It is not known officially whether the motion equations in various dressing spaces: (1.5.7), (2.7.9), (9.2.2), (9.4.2), (19,12.4,9), can be made Hamiltonian. (The reason the Hamiltonian property disappears is that there exist no nontrivial integrals, in the dressing spaces, of the dressing motion: all the Lax-space integrals become trivial when lifted into the dressing space:
\[ \text{Res}(K\xi^nK^{-1}) \approx \text{Res}(K^{-1}K\xi^n) = \text{Res}(\xi^n) = 0, \] (A9.0)
and one can show that no new integrals arise.)

**Conjecture 3.5.28.** If
\[ X_n^t\xi^{-1}q + p^t\xi^{-1}Y_n = 0, \quad \xi = d/dx, \] (A9.1a)
for some \( X_n, Y_n \in R_{pq} \), then
\[ X_n = A^1p + A^2q, \quad Y_n = A^2p - A^1q \] (A9.1b)
where \( A^1, A^2, A^3 \) are constant matrices, with \( A^2 \) and \( A^3 \) being skewsymmetric.

**Remark 14.1.29.** A noncommutative analog of the 2 + 1-dimensional dynamical system (14.1.5) and of the moments map (14.1.7) (where the \( y \)-dependence is not neglected) is not known, and neither is the dispersive analog of these 2+1-d objects
for the commutative KP hierarchy. This is one of the greatest mysteries of the whole theory of integrable systems.

**Conjecture 14.8.15.** The hierarchies (14.8.10)

\[ \partial_t(u) = \Delta^2 - \Delta [\text{Res}(L^n \xi)], \quad L = (1 - \xi^{-1}u)\xi, \quad \in \mathbb{N}, \quad (A9.2a) \]

and (9.8.11-13)

\[ \partial_t(q) = (1 - \Delta^{-N})(p_{-1}(n)), \quad (\xi^N + \xi^{-r}q)^{N+r}) = \sum_s \xi^{(N+r)s}p_s(n), \quad (A9.2b) \]

are isomorphic for all \( n \).

**Conjecture 15.2.9.** The remaining motion equations for \( q_1,..., \) resulting upon the imposition of the constraint \( q_0 = -\epsilon^{-1} \) on the quasirelativistic KP hierarchy (15.1.4), are regular in \( \epsilon \).

**Remark 15.2.10.** What are the reasons underlying the existence of the constraint \( \{q_0 = -\epsilon^{-1}\} \)? Quite likely, there exists at least one more similar constraint.

(From Exercise 15.2.11(ii)): What is the analog of the Relativistic Toda lattice constraint (15.2.12)

\[ q_0 = -\epsilon^{-1} - \epsilon q \quad (A9.3) \]

for the whole Relativistic KP hierarchy (15.1.4)?

What is the meaning of the shadow Relativistic Toda system (15.3.23) and the shadow Relativistic KP flow \# (15.3.24)?

\[ \partial_t(q_0) = \frac{1}{1 + \epsilon q_0^{(1)}} q_1 - q_1^{(-1)} \frac{1}{1 + \epsilon q_0^{(-1)}} (1 - \Delta^{-1})(q_1) + \mathcal{O}(\epsilon), \quad (A9.4) \]

\[ \partial_t(q_i) = \frac{1}{(1 - \epsilon q_0)^{(i+1)}} q_i + 1 - \left( q_i + 1 - \frac{1}{1 + \epsilon q_0} \right)^{(-1)} + \left( \frac{q_0}{1 + \epsilon q_0} \right)^{(i)} q_i - q_i \frac{q_0}{1 - \epsilon q_0} = \]

\[ = (1 - \Delta^{-1})(q_i - 1) + q_0^{(i)} q_i - q_i q_0 + \mathcal{O}(\epsilon), \quad i \in \mathbb{Z}_+. \quad (A9.5) \]

What is the nature of the Lax representation (15.3.26) for the flow (A9 5)?

\[ \partial_t(L) = [Sh, L], L = (1 - \epsilon \zeta)^{-1}(\zeta + \sum_{i \geq 0} \zeta^{-1} q_i) \]

\[ Sh = \frac{1}{1 + \epsilon q_0} (\zeta + q_0) = \zeta + \frac{q_0}{1 + \epsilon q_0} (1 - \epsilon \zeta). \quad (A9.6) \]

What is the analog of the constraint (A9.3), allowed by the shadow Relativistic Toda lattice (A9.4) (see 15.3.30) for the shadow KP flow (A9.5)?

For the most of the discrete-time evolutions of Part C, Hamiltonian property is known; neither it is known w.r.t. the additional Hamiltonian structures, such as the 2\(^{nd}\), which reappear when all the variables commute.

The Volterra system can be thought of either as a pair of Toda lattices or as an invariant sublattice of the 2\(^{nd}\) Toda flow. (See §19.8) No such property is known
for the Relativistic Toda lattice or its shadow version. Similar puzzle persists, more generally, for the whole quasi-relativistic KP hierarchy.

Let δ be the annihilator of $\text{Im}(\partial + \epsilon \text{ad}_u)$ of §A3.3:

$$\frac{\delta H}{\delta u} = \epsilon \text{ad}_H + \sum (-\partial - \epsilon \text{ad}_u)^{n+1} \left( \frac{\partial H}{\partial u^{(n)}} \right) + \epsilon \left[ \sum (-\partial - \epsilon \text{ad}_u)^n \left( \frac{\partial H}{\partial u^{(n)}} \right) \right] \text{ad}_u,$$

(A9.7)

Find $\text{Im}\delta$.

**Conjecture A4.4.33.** For all $n \in \mathbb{N}$,

$$\text{Res}(L_1^n) \approx \text{Res}(L_2^n),$$

(A9.8a)

$$L_1 = \xi + [a^{(1)} - (a^t b)a] \xi^{-1} b, \quad L_2 = \xi - a^t \xi^{-1} [b^{(1)} + b(a^t b)].$$

(A9.8b)

When Alexander surveyed his vast domain, he wept, for there were no more worlds to conquer.
NOTES AND COMMENTS

In gratiam corum, qui Latinam linguam non callent.

(Boastful remarks below should be taken with a grain of face value.)

Preface: This book’s results have been summarized in the author’s 1994 Los Alamos talk [Kup 1995b]. Gel’fand and Smirnov consider the zero-dimensional $\mathbb{Z}_2$-graded case in [GSm 1994].

Various noncommutative integrable systems had appeared sporadically in print. For example, the $KdV$ equation for $u(x,t)$ a matrix, was derived by Wadati and Kamijo in [WKa 1974]. More references appear below.

Kontsevich’s results appeared in his paper on noncommutative Symplectic Mechanics [Kon 1993].

Chapter 1: The algebraic scheme of integrability and dressing, followed in this and other Chapters, was distilled by Wilson [Wil 1979] following earlier works by Gel’fand and Dickey, and Manin. A modern exposition of these and related topics can be found in Dickey’s monograph [Dic 1991].

Chapter 2: is modelled on commutative treatment in [Kup 1985b].

Chapter 3: §3.1-§3.4 follow commutative treatment in [Kup 1985b, 1995a]. The “Gibbons form” in §3.5,6 and in many other places in the book is extracted from Gibbons’ paper [Gib 1981].

Chapters 4,5: are modelled on commutative treatment in [Kup 1992], where further references can be found.

Chapter 6: §6.2 is a noncommutative version of [Kup 1995b]. §6.4: Clebsch representations, an object of Lie-algebraic nature, underwrite many constructions of ideal Fluid Mechanics and Plasma Physics; the book [Kup 1992] develops these representations in considerable detail; for finite-dimensional Lie algebras, there exist also quantum Clebsch representations, they can be found in issues of the “Journal of Nonlinear Mathematical Physics”, the most ethical scientific journal in existence.

Chapter 7: §7.1: Quasirelativistic Lax equations were defined in [GKu 1990]. §7.2: Hamiltonian operations in this § are modelled on [Kup 1993].

Chapter 8: originates in works of Cherednik, Verdier, Wilson, and Flaschka; the presentation follows the path described in [Kup 1985a].

Chapter 14: originates in works of Hayles [Hay 1992].

Part B: Lattice Lax equations were introduced in [Kup 1982] and developed in [Kup 1985a], all in the commutative case of course. Equation (9.8.32), called by Narita in [Nar 1982] the “generalized Volterra equation”, was found by him to have 1- and 2-soliton solutions. This generalized Volterra system was extensively analysed in [Kup 1985a] as a part of a general theory. Bogoyavlensky studied the generalized Volterra equation in [Bog 1991]; this explains the name “Bogoyavlensky lattices” occasionally attached to this equation, see e.g. [Sur 1996b].

Chapter 15: Relativistic Toda lattice was discovered by Ruijsenaars and Schneider [RSc 1986], [Rui 1987, 1990a,b], and further studied by Alber [Alb 1990], Bruschi, Calogero, and Ragnisco [BCa 1987], [BRa 1988a,b, 1989a,b], and many others.
General quasirelativistic ansatz (15.1.3-4) was defined in [GKu 1990] (in the commutative case.)

**Part C:** Time-discretization as conjugation for lattice Lax equations, the approach taken in Part C, was proposed by Gibbons and Kupershmidt in [GKu 1992]. Different approaches can be found in works of Taha and Ablowitz [TAb 1984], Veselov [Ves 1988, 1991a,b], Moser and Veselov [MVe 1991], Suris [Sur 1990, 1991], Deift, Li, and Tomei [DLT 1992], and others. The KP case of Chapter 17 is modelled on the commutative treatment in [GKu 1992]. The Volterra system in §19.6-8 is treated similar to the commutative case in [Kup 1996]. Many systems inside Part C, and many outside it, have been worked out in the commutative realm by Suris, who has advanced the subject considerably; see: [Sur 1996a,b, 1997a,b], his many e-prints on solv-int e-board, and his papers in the *Journal of Nonlinear Mathematical Physics*. In particular, the main result of this Part, time-discretization as a factorization, arrived at empirically in §19.11, is used by Suris as a part of established commutative theory based on group-geometric considerations.

**Appendix A2:** Finite asymptotic expansions are essentially generalized linearizations; the latter are expansions of the type

\[ q = u_0 + \epsilon u_1, \quad \epsilon^2 = 0, \]

and the theory of such expansions has been developed in [Kup 1992].

**Appendix A3:** was the original motivation for this book, to develop a platform from which to attack quantum mathematical puzzles such as quantum integrable systems and Quantum Groups. In Quantum realm commutators can not be ignored, in contrast to the noncommutative approach in the rest of this book.

**Appendix A5:** follows the commutative presentation in [Kup 1991]. Fluid dynamical systems of §A5.2 are extensively treated in [Kup 1992] (and many other places.)

**Appendix A6:** Another useful reference on series solution is Rosales’ paper [Ros 1978]. Solutions of various noncommutative equations can be found in papers of Etingof, Gel’fand, and Retakh [EGR 1997a, b], [GRe 1991].

**Appendix A7:** The noncommutative KP equation (A7.13) has already appeared, as equation (44), in the paper [DFo 1992] of Dorfman and Fokas, derived there from a different point of view.
BIBLIOGRAPHY

We never disclose our sources.
Torquemada, on being asked whether it was ethical for the Inquisition's interrogators to keep secret the identity of the accusers.


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Lucky Job, who was not obliged
to annotate his lamentations!
Cioran

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