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Groups and Geometric Analysis

Integral Geometry, Invariant Differential Operators, and Spherical Functions

Sigurdur Helgason

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To Thor and Annie

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PREFACE

This volume is intended as an introduction to group-theoretic methods in analysis on spaces that possess certain amounts of mobility and symmetry.

The role of group theory in elementary classical analysis is a rather subdued one; the motion group of $\boldsymbol{R}^3$ enters rather implicitly in standard vector analysis, and the conformal groups of the sphere and of the unit disk become important tools primarily after the Riemann mapping theorem has been established.

In contrast, our point of view here is to place a natural transformation group of a given space in the foreground. We use this group as a guide for the principal concepts (like that of an invariant differential operator) and as motivation for the leading problems in analysis on the space. For examples of problems that arise naturally in such a framework, we call attention here to the Eigenfunction Problems (A, B, and C in Introduction §1, No. 1, p. 2), the Radon-inversion and Orbital Integral Problems (A–D in Chapter I, §3, p. 147), and the Invariant Differential Operator Problems (A–E in Chapter II, §4, p. 277).

The present volume is intended as a textbook and a reference work on the three topics in the subtitle. Its length is caused by a rather leisurely style with many applications and digressions, sometimes in the form of exercises (with solutions).

The introductory chapter deals with the two-dimensional case, requiring only elementary methods and no Lie group theory. While this example has considerable interest in itself, it serves here to connect our main topics to classical analysis. Its primary purpose, however, is to introduce techniques and theories that generalize to semisimple Lie groups and symmetric spaces, including Harish-Chandra's c-functions (treated in generality in Chapter IV), the author's work on harmonic analysis on symmetric spaces (to be treated in generality in another volume), and group-theoretic analysis of the eigenfunctions of the Laplacian, including classical boundary value properties of harmonic functions.

In Chapter I we give an exposition of invariant integration on homogeneous spaces and relate this to the structure theory of semisimple Lie groups. We formulate general analytic integral–geometric problems (Radon transforms and orbital integrals) for a double fibration of a homogeneous space and develop their solutions in an elementary fashion for some simple classes of homogeneous spaces.

In Chapter II we discuss the effect on differential operators of a group action on a manifold. This gives rise to a separation of variables for differential operators, projections, transversal parts, orbital parts, and radial parts of differential operators; all these are useful for problems with built-in invariance conditions.

Chapter III deals with linear group action on a vector space and with the corresponding invariant polynomials and harmonic polynomials. The results, together with a description of the orbits, find analytic applications in the case of the linear isotropy representation of a symmetric space.

In Chapter IV we study the (zonal) spherical functions, that is, the K-invariant eigenfunctions of the G-invariant differential operators on the Riemannian homogeneous space G/K. Their theory, as well as that of the corresponding spherical transform, is worked out in considerable detail for the case of symmetric space (and its tangent space). Here the connection with representation theory is particularly simple and important.

In the investigation of the spherical function's behavior at ∞, Harish-Chandra's beautiful c-function emerges. Eventually this function is expressed by the formula

$$c(\lambda) = c_0 \prod_{\alpha \in \Sigma_0^+} \frac{2^{-\langle i\lambda, \alpha_0\rangle}\Gamma(\langle i\lambda, \alpha_0\rangle)}{\Gamma(\frac{1}{2}(\frac{1}{2}m_\alpha + 1 + \langle i\lambda, \alpha_0\rangle))\Gamma(\frac{1}{2}(\frac{1}{2}m_\alpha + m_{2\alpha} + \langle i\lambda, \alpha_0\rangle))},$$

the notation being explained at the end of §6. As indicated there, each detail in this formula has its own significance. In particular, the location of the singularities of $c(\lambda)$ is crucial for the proof of the Paley–Wiener-type theorem for the spherical transform, which in turn enters into the proof of the corresponding inversion and Plancherel formula.

Chapter V deals with harmonic analysis on compact homogeneous spaces U/K. Because of the intimate connection with finite-dimensional representation theory for U, the chapter begins with a detailed exposition of the weight theory and the character theory for U.

Since this book is in part intended as a textbook, we now give some description of its level. The first third of the book is introductory and has on occasion been used as a textbook for first-year graduate students without background in Lie group theory. The remainder of the book requires some standard functional analysis results. The Lie-theoretic tools needed can for example be found in my book "Differential Geometry, Lie Groups, and Symmetric Spaces" (abbreviated [DS]), of which the present book can be considered an "analytic continuation." Thus, our aim has been to provide complete proofs of all the results in the book. Although this process of unification and consolidation has at times led to some simplifications of proofs, we have in the exposition been more concerned with clarity than brevity.

Each chapter begins with a short summary and ends with historical notes giving references to source material: an effort has been made to give appropriate credit to authors of individual results. At the same time we have tried to make these notes reflect the fact that the logical order of the exposition often differs drastically from the order of the historical development.

Much of the material in this book has been the subject of lectures at the Massachusetts Institute of Technology in recent years; some of the content of my lecture notes [1980c, 1981] has been incorporated in the Introduction and in Chapter I. Parts of the manuscript have been read and commented on by M. Baum, M. Cowling, M. Flensted-Jensen, F. Gonzales, A. G. Helmink, G. F. Helmink, B. Hoogenboom, A. Korányi, M. Mazzarello, J. Orloff, F. Richter, H. Schlichtkrull, and G. Travaglini. I am particularly indebted to T. Koornwinder for his numerous useful suggestions.

PREFACE TO THE 2000 PRINTING

The manuscript for this book was completed in late 1982 and was published by Academic Press in 1984. The book has been unavailable for a long time and I am happy to have this new printing published by the American Mathematical Society.

The book is the forerunner to my book "Geometric Analysis on Symmetric Spaces" published in 1994 by the American Mathematical Society as Vol. 39 in the Series "Mathematical Surveys and Monographs".

In this new printing I have taken the opportunity to correct a few inaccuracies and also to add a number of footnotes which are collected towards the end of the book in a section entitled "Some Details". The place of the footnotes is indicated by a dagger †. Some of these footnotes furnish alternative, more elementary proofs and some are clarifications of occasional tough spots in the exposition. In addition, an errata list appears after the Index.

Many of these corrections and footnotes are based on suggestions by readers and I would like to express my appreciation to A. Korányi, A. Mattuck, F. Rouvière, H. Schlichtkrull, G. Shimura, H. Stetkaer, and particularly P. Kuchment who translated the book into Russian.

SUGGESTIONS TO THE READER

Since this book is intended for readers with varied backgrounds and since it is written with several objectives in mind, we give some suggestions for its use.

The primary purpose has been to provide readers having a modest Lie-theoretic background with a self-contained account of the three topics in the subtitle. However, the various chapters are largely independent of each other and could individually serve as textbooks for one- or two-semester courses.

(i) The introductory chapter is so elementary that it could be used for an advanced undergraduate course.

(ii) Chapter I is an independent account of group-invariant integration and analytic integral geometry. A reader wishing only an elementary treatment of the Radon transform with some natural generalizations could read this chapter and skip §1 and §5.

(iii) Chapter II (together with §1 and §5 of Chapter I) provides an independent treatment of invariant differential operators. Some of the most natural problems for these operators are discussed in the introduction to §4, Chapter II.

(iv) Chapter III deals with invariants, particularly Weyl group invariants, harmonic polynomials, and orbit decomposition for the complex isotropy representation associated with a symmetric space. This chapter is almost entirely independent of the previous ones.

(v) Chapter IV, when preceded by Chapter I (§1 and §5) and Chapter II, provides an independent account of the theory of spherical functions with a certain degree of completeness.

(vi) Chapter V (together with Chapter IV, §1, No. 2) gives an independent exposition of analysis on compact homogeneous spaces with emphasis on the symmetric ones. The chapter starts with a brief account of the basic representation theory of compact Lie groups.

Exercises and Further Results. Each chapter ends with a few exercises; some of these are routine applications and others develop the general theory further. Stating such results with no hint of proof or reference I consider counterproductive; hence solutions are provided at the end of the book. These solutions are often fairly concise and some of them (indicated with a star) rely on other references. Hopefully such exercises can furnish suitable topics for student seminars.

A SEQUEL TO THE PRESENT VOLUME

GEOMETRIC ANALYSIS ON SYMMETRIC SPACES

was published in 1994 as

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Contents:

Chapter V. Differential Equations on Symmetric Spaces

1. Solvability. 2. Mean Value Theorems. 3. Harmonic Functions on Symmetric Spaces. 4. Harmonic Functions on Bounded Symmetric Domains. 5. The Wave Equation on Symmetric Spaces. 6. Eigenfunctions and Hyperfunctions. Exercises and Further Results.

Chapter VI. Eigenspace Representations

1. Generalities. 2. Irreducibility Criteria for a Symmetric Space. 3. Eigenspace Representations for the Horocycle Space G/MN. 4. Eigenspace Representations for the Complex Space G/N. 5. Two Models of the Spherical Representations. Exercises and Further Results.

Solutions to Exercises. Bibliography.

SOLUTIONS TO EXERCISES

INTRODUCTION

A. The Spaces R^n and S^n.

A.1. (i) The space spanned by the translates of f can be decomposed into subspaces on which $O(n)$ acts irreducibly. Each of these will contain a function ϕ_k for some k, so by the irreducibility of the spaces E_k these subspaces are among the E_k.

(ii) The group $O(1, 1)$ is generated by the transformations

$$a_t\colon \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \begin{pmatrix} \operatorname{ch} t & \operatorname{sh} t \\ \operatorname{sh} t & \operatorname{ch} t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \begin{pmatrix} \pm x_1 \\ -x_2 \end{pmatrix}.$$

Consider the function f on $x_1^2 - x_2^2 = -1$ defined by

$$f(x_1, x_2) = \sinh^{-1}(x_1).$$

Then $f^{a_t}(x_1, x_2) = f(x_1, x_2) - t$, so f is $O(1, 1)$ finite, yet is not the restriction of a polynomial.

A.2. Consider for $\lambda \neq 0$ the transform $F \to f$ given by

$$f(x) = \int_{S^{n-1}} e^{i\lambda(x,\omega)} F(\omega)\, d\omega, \qquad F \in L^2(S^{n-1}).$$

Generalizing Lemma 2.7, we see that the map $F \to f$ is one-to-one. Using Theorem 2.7 of Chapter II, we let S_λ be a sphere in R^n with center 0 such that each $f \in \mathscr{E}_\lambda(R^n)$ is determined by its restriction $f \mid S_\lambda$ to S_λ. Because of Theorem 3.1, the space $E_k = E_k(S^{n-1})$ is exactly the space of functions $f \in C(S^{n-1})$ which are $O(n)$-finite and for which the representation of $O(n)$ on the space of translates is equivalent to δ (in Prop. 3.2). The space $E_k(S_\lambda)$ is similarly characterized. The maps

$$F \to f \mid S_\lambda, \qquad F \to f, \qquad F \in L^2(S^{n-1}),$$

are, by the above, one-to-one and commute with the action of $O(n)$. By the characterization of E_k indicated the first map sends $E_k(S^{n-1})$ injectively into $E_k(S_\lambda)$, hence surjectively. Hence each $O(n)$-finite $f \in \mathscr{E}_\lambda(R^n)$

has the form stated. The proof of the irreducibility criterion for T_λ now proceeds as for Theorem 2.6.

A.3. There is a natural bijection of $\mathscr{H}_k/\mathscr{H}_{k-1}$ onto E_k commuting with the $\boldsymbol{O}(n)$ action. Thus we see that already $\boldsymbol{O}(n)$ acts irreducibly on $\mathscr{H}_k/\mathscr{H}_{k-1}$.

A.4. The Lie algebra $\mathfrak{sl}(2, \boldsymbol{C})$ of $\boldsymbol{SL}(2, \boldsymbol{C})$ consists of the complex 2×2 matrices of trace 0 viewed as a real Lie algebra. Given $X \in \mathfrak{sl}(2, C)$, $u \in \mathscr{E}(\boldsymbol{R}^2)$ we put

$$(X^*u)(z) = \left(\frac{d}{dt} u(\exp(-tX) \cdot z)\right)_{t=0}, \qquad z \in \boldsymbol{R}^2.$$

By the quasi-invariance of L, X^*u is harmonic if u is harmonic. The statement to be proved is that if $A: \mathscr{E}_0(\boldsymbol{R}^2) \to \mathscr{E}_0(\boldsymbol{R}^2)$ is a continuous linear mapping such that $AX^* = X^*A$ for all $X \in \mathfrak{su}(2, \boldsymbol{C})$ then A is a scalar. Taking X as

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}$$

we find that A commutes with $\partial/\partial x$ and $\partial/\partial y$, and therefore maps the subspaces $\mathscr{H}$ and $\overline{\mathscr{H}}$ (of holomorphic and antiholomorphic functions, respectively) into themselves. Taking X as

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

we find that A commutes with the operator $x\partial/\partial x + y\partial/\partial y$, which on $\mathscr{H}$ coincides with $z\partial/\partial z$. This implies easily that A is a scalar on $\mathscr{H}$ and similarly on $\overline{\mathscr{H}}$, hence on $\mathscr{E}_0(\boldsymbol{R}^2)$.

A.5. (i) Let $D_k(n) = \dim P_k$ and $d_k(n) = \dim H_k$. Writing a polynomial in $(x_1, \ldots, x_n)$ as a polynomial in x_n with coefficients in $(x_1, \ldots, x_{n-1})$ we see that

$$D_k(n) = \sum_{p=0}^{k} D_{k-p}(n-1) = D_k(n-1) + D_{k-1}(n),$$

so by induction on $k + n$ we have

$$D_k(n) = \frac{n(n+1)\cdots(n+k-1)}{k!} = \binom{n+k-1}{k}.$$

Since by Eq. (4) of §3 $d_k(n) = D_k(n) - D_{k-2}(n) = D_k(n-1) + D_{k-1}(n-1)$, part (i) follows.

Part (ii) follows from Eq. (3) of §3. For (iii) we have by the mean-value theorem for harmonic functions

$$\int_{\mathbf{R}^n} h(x+y)f(x)\,dx = \left(\int_{\mathbf{R}^n} f(x)\,dx\right)h(y)$$

if f is any rapidly decreasing radial function. Here we can take $f(x) = e^{-(1/2)|x|^2}$ and since h is a polynomial we can replace y by any complex $z \in \mathbf{C}^n$. This gives

$$\int_{\mathbf{R}^n} h(x-iy)e^{-(1/2)(x,x)}\,dx = (-i)^k(2\pi)^{(1/2)n}h(y).$$

Using Cauchy's theorem on the holomorphic function

$$z \to h(z-iy)e^{-(1/2)(z,z)}$$

we can shift the integral on the left to $\mathbf{R}^n + iy$. Then (iii) follows. (The result is given in Bochner [1955] with a proof involving Bessel functions.)

(iv) Let $\phi \in E_k$ and let h_ϕ denote its unique extension to an element of H_k. Note that $\partial(h_\phi)(|x|^{2-n})$ is a harmonic function of homogeneity $2-n-k$. Writing it as $|x|^{2-n-k}\psi(x/|x|)$ we see using Eq. (3) of §3 that ψ is an eigenfunction of L with eigenvalue

$$-(2-n-k)(2-n-k+n-2),$$

so $\psi \in E_k$. The mapping

$$\phi \to \partial(h_\phi)(|x|^{2-n})|_{S^{n-1}} = \psi$$

thus maps E_k into itself. Since it clearly commutes with the rotations, Schur's lemma and Theorem 3.1(ii) show that the mapping is a scalar multiple of the identity. This proves (iv) except for the value of the constant $c_{k,n}$, which can easily be calculated by using h_{k+1} from part (ii).

A computational proof (using induction on k) can easily be given for h of the form $(a, x)^k$ $a \in \mathbf{C}^n$ isotropic.

B. The Hyperbolic Plane

B.1. In fact, if $g \in G$ the point $g \cdot o$ can be written $na \cdot o$ $(n \in N, a \in A)$ so $g^{-1}na \in K$.

B.2. Part (i) follows from [DS] (p. 548), which also shows that $\mathbf{SU}(1, 1)$ and the conjugation $z \to \bar{z}$ generate the group of isometries of D. Since the conjugation $z \to \bar{z}$ of D corresponds to the map $w \to 1/\bar{w}$ of the upper half plane part (ii) follows.

(iii) Since c maps Euclidean circles into circles it is clear that as a set $S_r(i)$ is a Euclidean circle. It passes through the points $(0, e^{\pm 2r})$ so the statement is clear.

B.3. This follows from (24) and (32) combined with the classical Paley–Wiener theorem for $\boldsymbol{R}$. Another proof can be given by means of Abel's integral equation

$$\phi(u) = \int_{\boldsymbol{R}} F(u + y^2)\, dy,$$

appearing in the proof of Theorem 4.6.

B.4. (i) Clearly $\phi_{-\lambda}(z)$ is real for all z if and only if $\tilde{f}(\lambda)$ is real for all real $f \in \mathscr{D}^{\natural}(D)$, which by B3 is equivalent to

$$\int_{-\infty}^{\infty} e^{-i\lambda t}\phi(t)\, dt$$

being real for every real $\phi \in \mathscr{D}^{\natural}(\boldsymbol{R})$. This in turn is equivalent to $\cos(\lambda t) = \cos(\bar{\lambda} t)$ and (i) follows.

(ii) Let $\lambda = \xi + i\eta$ where $\xi,\ \eta \in \boldsymbol{R}$. Assume first $|\eta| \leq 1$. Then if $f \in L^1(D)$ and is radial

$$\begin{aligned}\int |f(z)\phi_{-\lambda}(z)|\, dz &\leq \int |f(z)|\phi_{i\eta}(z)\, dz \\ &= \int_{\boldsymbol{R}} e^{\eta t}F_{|f|}(t)\, dt = \int_0^{\infty} (e^{\eta t} + e^{-\eta t})F_{|f|}(t)\, dt \\ &\leq 2\int_0^{\infty} e^{t}F_{|f|}(t)\, dt < \infty\end{aligned}$$

because we have

$$\int_D |f(z)|dz = \int_{\boldsymbol{R}} e^{-t}F_{|f|}(t)\, dt < \infty$$

and $F_{|f|}$ is even. Thus ϕ_λ has a finite integral against any L^1-function and so is bounded.

On the other hand, suppose ϕ_λ bounded. If $\eta = 0$ there is nothing to prove and since $\phi_\lambda = \phi_{-\lambda}$ we can assume $\eta < 0$. Then Theorem 4.5 applies and $e^{(-i\lambda + 1)r}\phi_\lambda(a_r \cdot 0)$ has a nonzero limit as $r \to +\infty$. But the boundedness of ϕ_λ then implies $\mathrm{Re}(-i\lambda + 1) \geq 0$ and so $\eta \geq -1$ as desired.

B.5. The calculation is the same as that of Lemma 4.9 (cf. Helgason [1976], p. 203).

B.6. This follows by approximating f by functions $f_n \in \mathscr{D}^{\natural}(D)$ and using (32).

B.7. As in text expand the function $f \in \mathscr{D}(D)$,

$$f(z) = \sum_{m \in \boldsymbol{Z}} f_m(z),$$

where $f_m(e^{i\theta}z) = e^{im\theta}f_m(z)$. Then $\tilde{f}_m$ and $\hat{f}_m$ have the form

$$\tilde{f}_m(\lambda, e^{i\theta}) = e^{im\theta}\tilde{f}_m(\lambda, 1),$$
$$\hat{f}_m(e^{i\theta}, t) = e^{im\theta}\hat{f}_m(1, t),$$

and

$$\tilde{f}_m(\lambda, 1) = \int_{\boldsymbol{R}} e^{-i\lambda t}e^t\hat{f}_m(1, t)\, dt.$$

Also, by (58),

$$\tilde{f}_m(\lambda, 1) = \frac{p_m(-i\lambda)}{p_m(i\lambda)}\, \tilde{f}_m(-\lambda, 1)$$

and since $p_m(i\lambda)$ and $p_m(-i\lambda)$ are relatively prime this implies

$$\tilde{f}_m(\lambda, 1) = p_m(-i\lambda)h(\lambda),$$

where $h(\lambda)$ even and of exponential type. Now the result follows from the Fourier inversion

$$\hat{f}_m(1, t)e^t = (2\pi)^{-1} \int_{\boldsymbol{R}} \tilde{f}_m(\lambda, 1)e^{i\lambda t}\, d\lambda.$$

B.8. These are the circular arcs connecting -1 and 1 inside D. They are the curves $x_t\colon r \to x(r, t)$ in Exercise B9 and consist of points of fixed non-Euclidean distance from the geodesic $A \cdot o$.

B.9. We have $x(r, t) = a_r \cdot x(0, t) = a_r \cdot (\tanh t\ i)$ where $a_r \in \boldsymbol{SU}(1, 1)$ is as in §4. This gives the formula for $x(r, t)$. Next substitute $\zeta = x(r, t)$ into the formula

$$dx = \tfrac{1}{2}i\,\frac{d\zeta \wedge d\bar{\zeta}}{(1 - |\zeta|^2)^2}.$$

B.10. We have by (24)–(27)

$$f(\tanh s) = F((\mathrm{ch} s)^2)$$

if $F(x) = x^{-(1/2)(i\lambda+1)}$. It follows that

$$\begin{aligned} F_f(t) &= \int_{\boldsymbol{R}} F((\mathrm{ch} t)^2 + y^2)\, dy \\ &= (\mathrm{ch} t)^{-i\lambda} \int_{\boldsymbol{R}} (1 + z^2)^{-(1/2)(i\lambda+1)}\, dz \\ &= (\mathrm{ch} t)^{-i\lambda}\pi c(\lambda). \end{aligned}$$

C. Fourier Analysis on the Sphere

(i) Immediate from §3. No. 1.

(ii) The first relation and the second for $(s, a) > 0$ are clear from the uniqueness of ϕ_k in Prop. 3.2. For $(s, a) < 0$ the second relation holds at least up to a constant factor which however is 1 because by the first relation $C_k^n(-1) = (-1)^k$

(iii) Denote the integral by $F(s, s')$. Since $F(-s, -s') = F(s, s')$ we may assume $(s', a) > 0$. Next observe $F(us, us')$ if u is a rotation of B. Let u_ϕ be a rotation of angle ϕ in the $x_n x_{n+1}$ plane. If we can prove

$$F(\mu_\phi s, u_\phi s') = F(s, s')$$

for all ϕ with $(u_\phi s', a) > 0$ then (iii) is reduced to the case $s' = a$ and then (ii) applies. Thus it suffices to prove $\partial F(u_\phi s, u_\phi s')/\partial \phi = 0$.

Each $b \in B$ can be written

$$b = (c_1 \cos\theta, c_2 \cos\theta, \ldots, c_{n-1} \cos\theta, \sin\theta, 0), \qquad c \in \boldsymbol{S}^{n-2}.$$

Putting

$$g(c, \theta, \phi, s)$$

$$= i \cos\theta \sum_{j=1}^{n-1} c_j s_j + i \sin\theta (s_n \cos\phi - s_{n+1} \sin\phi) + s_n \sin\phi + s_{n+1} \cos\phi$$

we have

$$e_{b,k}(u_\phi s) = [g(c, \theta, \phi, s)]^k,$$

$$f_{b,k}(u_\phi s') = [g(c, \theta, \phi, s')]^{-k-n+1}.$$

For each $m \in \boldsymbol{Z}$ we have the differential equation

$$i \frac{\partial g^m}{\partial \phi} - \cos\theta \frac{\partial g^m}{\partial \theta} = m g^m \sin\theta$$

and the volume element decomposition

$$db = (\text{const}) \cos^{n-2}\theta \, d\theta \, dc \qquad (-\pi/2 \le \theta \le \pi/2).$$

Using these facts we get after simple computation

$$\frac{\partial F(u_\phi s, u_\phi s')}{\partial \phi}$$

$$= -i \int_{c \in S^{n-2}} \int_{-\pi/2}^{\pi/2} \frac{\partial}{\partial \theta} [g(c, \theta, \phi, s)^k g(c, \theta, \phi, s')^{-k-n+1} \cos^{n-1}\theta] \, d\theta = 0.$$

(The result is from Sherman [1975], the proof from J.-G. Yang [1983].)

(iv) Let $\phi_k(s) = C_k^n((s, a))$ ($k \in \mathbf{Z}^+$) so by Lemma 3.5

$$f = \sum_{k=0}^{\infty} d(k) f * \phi_k.$$

Then

$$\begin{aligned}\int_B e_{b,k}(s')\tilde{f}(b, k)\, db &= \int_{S^n} f(s)\left(\int_B e_{b,n}(s') f_{b,n}(s)\, db\right) ds \\ &= \int_{S^n} f(s) C_k^n((s, s'))\, ds \\ &= \int_{O(n+1)} f(u \cdot a) C_k^n((u \cdot a, s'))\, du\end{aligned}$$

so we have the compact analog of Lemma 4.7.

$$f * \phi_k(s) = \int_B e_{b,k}(s)\tilde{f}(b, k)\, db$$

and the result follows using the expansion above. (With a suitable definition of $\tilde{f}$ the result holds for all $f \in \mathscr{E}(S^n)$; cf. Sherman [1975].)

CHAPTER I

A. Invariant Measures

A.1. (i) If H is compact, $|\det(\mathrm{Ad}_G(H))|$ and $|\det(\mathrm{Ad}_H(H))|$ are compact subgroups of the multiplicative groups of the positive reals, hence identically 1.

(ii) G/H has an invariant measure so $|\det \mathrm{Ad}_H(h)| = |\det \mathrm{Ad}_G(h)|$, which by unimodularity of G equals 1.

(iii) Let $G_0 = \{g \in G : |\det \mathrm{Ad}_G(g)| = 1\}$. Then G_0 is a normal subgroup of G containing H. Since $\mu(G/H) < \infty$, Prop. 1.13 shows that the group G/G_0 has finite Haar measure, and hence is compact. Thus the image $|\det \mathrm{Ad}_G(G)|$ is a compact subgroup of the group of positive reals, and hence consists of 1 alone.

A.2. The element $H = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ spans the Lie algebra $\mathfrak{o}(2)$ and $\exp \mathrm{Ad}(g)tH = g \exp tHg^{-1} = \exp(-tH)$.

A.3. We have $\det \mathrm{Ad}(\exp X) = \det(e^{\mathrm{ad}\, X}) = e^{\mathrm{Tr}(\mathrm{ad}\, X)}$, so (i) follows. For (ii) we know that G/H has an invariant measure if and only if

$$\exp(\mathrm{Tr}(\mathrm{ad}_{\mathfrak{g}}\, T)) = \exp(\mathrm{Tr}(\mathrm{ad}_{\mathfrak{h}}\, T)), \quad T \in \mathfrak{h}.$$

Put $T = tX_i$ ($r < i \leq n$), $t \in \mathbf{R}$, and differentiate with respect to t. Then the desired relations follow.

A.4. To each $g \in \boldsymbol{M}(n)$ we associate the translation T_x by the vector $x = g \cdot o$ and the rotation k given by $g = T_x k$. Then $kT_x k^{-1} = T_{k \cdot x}$, so

$$g_1 g_2 = T_{x_1} k_1 T_{x_2} k_2 = T_{x_1 + k_1 \cdot x_2} k_1 k_2.$$

Since $g_{k_1, x_1} \cdot g_{k_2, x_2} = g_{k_1 k_2, x_1 + k_1 \cdot x_2}$ this shows that the mapping $g \to g_{k,x}$ is an isomorphism. Also

$$\int f(g_{k_0, x_0} g_{k,x})\, dk\, dx = \int f(g_{k_0 k,\, x_0 + k_0 \cdot x})\, dk\, dx$$

$$= \int f(g_{k,x})\, dk\, dx$$

since dx is invariant under $x \to x_0 + k_0 \cdot x$.

A.5. By [DS], Chapter II, §7, the entries ω_{ij} in the matrix $\Omega = X^{-1}\, dX$ constitute a basis of the Maurer–Cartan forms (the left invariant 1-forms) on $\boldsymbol{GL}(n, \boldsymbol{R})$. Writing $dX = X\Omega$ we obtain from [DS] (Chapter I, §2, No. 3) for the exterior products

$$\prod_{i,j} dx_{ij} = (\det X)^n \prod_{i,j} \omega_{ij},$$

so $|\det X|^{-n} \prod_{i,j} dx_{ij}$ is indeed a left invariant measure. The same result would be obtained from the right invariant matrix $(dX)X^{-1}$ so the unimodularity follows.

A.6. Let the subset $G' \subset G$ be determined by the condition $\det X_{11} \neq 0$ and define a measure $d\mu$ on G' by

$$d\mu = |\det X_{11}|^{-1} \prod_{(i,j) \neq (1,1)} dx_{ij}.$$

If dg is a bi-invariant Haar measure on G we have (since $G - G'$ is a null set)

$$\int_G f(g)\, dg = \int_{G'} f(g)\, dg = \int_{G'} f(g) J(g)\, d\mu,$$

where J is a function on G'. Let T be a diagonal matrix with $\det T = 1$ and $t_1, \ldots, t_n$ its diagonal entries. Under the map $X \to TX$ the product $\prod_{(i,j)\neq(1,1)} dx_{ij}$ is multiplied by $t_1^{n-1} t_2^n \cdots t_n^n$ and $|\det X_{11}|$ is multiplied by $t_2 t_3 \cdots t_n$. Since $\det T = 1$, these factors are equal, so the set G' and the measure μ are preserved by the map $X \to TX$. If A is a supertriangular matrix with diagonal 1, the mapping $X \to AX$ is supertriangular with diagonal 1 if the elements x_{ij} are ordered lexicographically. Thus $\prod_{(i,j)\neq(1,1)} dx_{ij}$ is unchanged and a simple inspection shows[†] $\det((AX)_{11}) = \det(X_{11})$. It follows that G' and $d\mu$ are invariant under

[†]See "Some Details," p. 611.

each map $X \to UX$ where U is a supertriangular matrix in G. By transposition, G' and $d\mu$ are invariant under the map $X \to XV$ where V is a lower triangular matrix in G. The integral formulas above therefore show that $J(UXV) \equiv J(X)$. Since the products UV form a dense subset of G ([DS], Chapter IX, Exercise A2) μ is a constant multiple of dg. For another, more down-to-earth proof see Gelfand and Naimark [1957] (Chapter I, §4).

A.7. A simple computation shows that the measures are invariant under multiplication by diagonal matrices as well as by unipotent matrices; hence they are invariant under $\boldsymbol{T}(n, \boldsymbol{R})$; cf. Gelfand and Naimark [1957] (Chapter I).

A.8. G leaves invariant the metric $y^{-2}(dx^2 + dy^2)$ on H ([DS], Chapter X, Exercise G1) and the Riemannian measure is $y^{-2}\,dx\,dy$, so D has finite area. Since $G = \Gamma\pi^{-1}(D)$ where $\pi: G \to H$ is the natural projection $g \to g \cdot i$, $\mu(G/\Gamma) < \infty$.

Remarks. (i) D is actually a fundamental domain for Γ in the sense that each Γ-orbit in H meets D and if two distinct points of D are in the same Γ-orbit they lie on the boundary of D. Also, T and S generate Γ (see, e.g., Serre [1970]).

(ii) The finiteness $\mu(G/\Gamma) < \infty$ for $G = \boldsymbol{SL}(n, \boldsymbol{R})$, $\Gamma = \boldsymbol{SL}(n, \boldsymbol{Z})$ is classical (Hermite, Siegel; for an elementary proof, see Bourbaki [1963], *Intégration*, Chapter 7, Exercise 7, §3). For an extension to semisimple groups see Borel and Harish-Chandra [1962].

A.9. Part (i) is straightforward. For Part (ii) we just indicate the proof in the case of $\boldsymbol{G}'_{2,4}$. The invariant measure is of the form

$$D(a, b, c, d)\, da\, db\, dc\, dd$$

where the density D is to be determined. Requiring invariance under rotations in the (f_1, f_2)-plane and in the (e_1, e_2)-plane gives, respectively,

$$(1) \quad D(a\cos\phi - b\sin\phi, a\sin\phi + b\cos\phi, c\cos\phi - d\sin\phi, c\sin\phi + d\cos\phi) = D(a, b, c, d),$$

$$(2) \quad D(a\cos\phi - c\sin\phi, b\cos\phi - d\sin\phi, a\sin\phi + c\cos\phi, b\sin\phi + d\cos\phi) = D(a, b, c, d).$$

Invariance under rotations in the (e_1, f_1)-plane gives

$$(3) \qquad D(a', b', c', d') = (a\sin\phi - \cos\phi)^4 D(a, b, c, d)$$

where

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = (\cos\phi - a\sin\phi)^{-1}$$

$$\times \begin{pmatrix} a\cos\phi + \sin\phi & b \\ c & d\cos\phi + (bc - ad)\sin\phi \end{pmatrix}.$$

From (1) we obtain with $r = (a^2 + b^2)^{1/2}$,

$$(4) \qquad D(r, 0, r^{-1}(ac + bd), r^{-1}(ad - bc)) = D(a, b, c, d),$$

and from (3) we derive

$$(5) \qquad D(0, 0, -(a^2 + 1)^{-1}c, d) = (a^2 + 1)^2 D(a, 0, c, d).$$

Combining (4) and (5) we get

$$(6) \quad D(a, b, c, d) = (r^2 + 1)^{-2} D(0, 0, -(r^2 + 1)^{-1/2} r^{-1}(ac + bd), r^{-1}(ad - bc)).$$

Also, by (1) and (5),

$$D(0, 0, c, d) = D(0, 0, 0, (c^2 + d^2)^{1/2},$$

$$D(a, 0, 0, d) = (a^2 + 1)^{-2} D(0, 0, 0, d),$$

whence

$$D(a, 0, 0, 0) = (a^2 + 1)^{-2} D(0, 0, 0, 0).$$

But (1) and (2) imply

$$D(a, 0, 0, 0) = D(0, a, 0, 0) = D(0, 0, a, 0) = D(0, 0, 0, a).$$

The formula for D now follows easily.

A.10. (i) This comes from the usual formula for surface area in $\boldsymbol{R}^n$.

(ii) The invariant measure can be written

$$f(x_1, \dots, x_{p+q+1})\, dx_1 \cdots dx_{p+q}$$

letting $x_1, \dots, x_{p+q}$ serve as coordinates. Since $\boldsymbol{O}(p, q)$ acting on $x_1, \dots, x_{p+q}$ leaves the measure invariant f is a function of

$$x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$$

and x_{p+q+1} only, hence of x_{p+q+1} only. Now use the invariance under the group

$$(x_1, x_2, \dots, x_{p+q+1}) \to (x_1 \operatorname{ch} t + x_{p+q+1} \operatorname{sh} t, x_2, \dots, x_1 \operatorname{sh} t + x_{p+q+1} \operatorname{ch} t).$$

This shows that

$$f(x_{p+q+1})|x_{p+q+1}| = \text{const.}$$

A.11. (i)–(ii) follow by routine computation. For (iv) it is sufficient just to verify that given α, $\beta \in \boldsymbol{C}$, $\alpha\beta \neq 0$, $|\alpha|^2 + |\beta|^2 = 1$ there exist unique t, θ, ϕ such that

$$0 \leq t < \pi, \qquad 0 \leq \theta < 2\pi, \qquad -2\pi \leq \phi < 2\pi,$$

and such that (1) holds. For further information concerning this exercise see, e.g., Vilenkin [1968].

B. Radon Transforms

B.1. Let $\phi \in \mathscr{D}_H(\boldsymbol{P}^n)$ and put

$$\Phi(\lambda, \omega) = \int_{\boldsymbol{R}} \phi(\omega, p) e^{-ip\lambda}\, dp.$$

Then Φ is holomorphic in λ. Let $a \in \boldsymbol{C}^n$ be isotropic and consider the expression

$$\left(\frac{d^l}{d\lambda^l} \int_{S^{n-1}} \Phi(\lambda, \omega)(a, \omega)^k\, d\omega\right)_{\lambda=0},$$

where l, $k \in \boldsymbol{Z}^+$, $l < k$. Using $\phi \in \mathscr{D}_H(\boldsymbol{P}^n)$ we see this expression vanish. Now apply Theorem 2.10 of the Introduction to Φ.

B.2. (i, ii) As in Introduction, §3, let S_{km} $[1 \leq m \leq d(k)]$ be an orthonormal basis of the eigenspace E_k of $L = L_{S^{n-1}}$. Then we have the following result.

Theorem. *The range $\mathscr{D}(\boldsymbol{R}^n)^\wedge$ consists of the functions $\psi \in \mathscr{D}(\boldsymbol{P}^n)$ which when expanded,*

$$\psi(\omega, p) = \sum_{k \geq 0,\ 1 \leq m \leq d(k)} \psi_{km}(p) S_{km}(\omega),$$

have the property that, for each $k \in \boldsymbol{Z}^+$, $1 \leq m \leq d(k)$,

$$\psi_{km}(p) = \frac{d^k}{dp^k} \phi_{km}(p)$$

where $\phi_{km} \in \mathscr{D}(\boldsymbol{R})$ is even.

By Theorem 2.10, $\mathscr{D}(\boldsymbol{R}^n)^\wedge$ is characterized by

$$\int_{\boldsymbol{R}} \psi_{km}(p) p^l\, dp = 0 \qquad \text{for} \quad l < k,$$

which (by Fourier transform) is equivalent to the description above. [The evenness of ϕ_{km} comes from the property $\psi(-\omega, -p) = \psi(\omega, p)$.]

B.3. Let $T \in \mathscr{E}'(\boldsymbol{P}^n)$ such that $\check{T} = 0$. Then $T(\hat{f}) = 0$ for each $f \in \mathscr{D}(\boldsymbol{R}^n)$. With notation from B2 define $T_{km} \in \mathscr{E}'(\boldsymbol{R})$ by

$$T_{km}(\phi) = \int \phi(p) S_{km}(\omega)\, dT(p, \omega).$$

Since the expansion for ψ (in B2) converges in $\mathscr{E}(\boldsymbol{P}^n)$ (Introduction, Theorem 3.4 or Chapter V, Corollary 3.4),

$$T(\psi) = \sum_{k,m} T_{km}(\psi_{km}).$$

Thus by B2, $d^k/dp^k(T_{km})$ annihilates all even functions in $\mathscr{D}(\boldsymbol{R})$. On the other hand, it annihilates all odd functions because

$$\int \phi(-p)\, dT_{km}(p) = (-1)^k \int \phi(p)\, dT_{km}(p).$$

Thus $d^k/dp^k(T_{km}) = 0$, so, since T_{km} has compact support, $T_{km} = 0$.

B.4. Because of Eq. (46) of §2 we have

$$\mathscr{N} = \left\{\phi \in \mathscr{E}(\boldsymbol{P}^n) : \int \hat{f}(\xi)\phi(\xi)\, d\xi = 0 \qquad \text{for} \quad f \in \mathscr{D}(\boldsymbol{R}^n)\right\};$$

in other words, $\mathscr{N}$ equals the annihilator $(\mathscr{D}^\wedge)^\perp$ of $\mathscr{D}(\boldsymbol{R}^n)^\wedge$ in $\mathscr{E}(\boldsymbol{P}^n)$. But then (since $d\xi = d\omega\, dp$) Theorem 2.10 implies that $\mathscr{N}$ contains each space $E_k \otimes p^l$ if $k - l > 0$ and even. Denoting now

$$\langle \phi, \psi \rangle = \int_{\boldsymbol{P}^n} \phi(\xi)\psi(\xi)\, d\xi,$$

we shall prove for $\phi \in \mathscr{D}(\boldsymbol{P}^n)$

$$(*) \quad \phi \in \mathscr{D}(\boldsymbol{R}^n)^\wedge \Leftrightarrow \langle \phi, E_k \otimes p^l \rangle = 0 \qquad \text{for} \quad k - l > 0, \text{ even}.$$

It suffices to verify the implication $\Leftarrow$. For this we expand

$$\phi(\omega, p) = \sum_{r \geq 0} \sum_{1 \leq m \leq d(r)} \phi_{rm}(p) S_{rm}(\omega).$$

The condition $\langle \phi, E_k \otimes p^l \rangle = 0$ then implies

$$\int_{\boldsymbol{R}} \phi_{km}(p) p^l\, dp = 0, \qquad k - l > 0, \text{ even}.$$

But since $\phi_{km}(-p) = (-1)^k \phi_{km}(p)$ this integral vanishes for $k - l$ odd. Hence $\phi \in \mathscr{D}(\boldsymbol{R}^n)^\wedge$ as stated (cf. B2).

Now suppose $\psi \in \mathscr{N}$ and decompose

$$(**) \qquad \psi = \sum_{k \geq 0} \psi_k$$

where (cf. B2)

$$\psi_k(\omega, p) = \sum_{1 \leq m \leq d(k)} \psi_{km}(p) S_{km}(\omega).$$

The decomposition $\psi = \sum \psi_k$ is the one given by Corollary 3.4 of Chapter V, so the various ψ_k transform under inequivalent representations of $\boldsymbol{O}(n)$. Thus the condition $\check{\psi} = 0$ implies $\check{\psi}_k = 0$ for each k.

In analogy with (**) we decompose each $f \in \mathscr{D}(\boldsymbol{R}^n)$

$$f = \sum_{k \geq 0} f_k$$

[Corollary 3.4, Chapter V or Introduction, Eq. (19), §3.] Then by (*) we have for each k and each $\phi \in \mathscr{D}(\boldsymbol{P}^n)_k$

$$\phi \in (\mathscr{D}(\boldsymbol{R}^n)_k)\widehat{\ } \Leftrightarrow \langle \phi, E_k \otimes p^l \rangle = 0, \qquad k - l > 0, \text{ even}.$$

For a fixed k the spaces $E_k \otimes p^l$ ($l < k$, $k - l$ even) span a finite-dimensional space $\tilde{E}_k$ whose annihilator $(\tilde{E}_k)^\perp$ in $\mathscr{D}(\boldsymbol{P}^n)_k$ is $(\mathscr{D}(\boldsymbol{R}^n)_k)\widehat{\ }$. But $\check{\psi}_k = 0$ implies that ψ_k lies in the double annihilator $((\tilde{E}_k)^\perp)^\perp$, which by the finite dimensionality equals $\tilde{E}_k$. Since (**) converges in the topology of $\mathscr{E}(\boldsymbol{P}^n)$ (Corollary 3.4, Chapter V), ψ lies in the closed subspace generated by the spaces $E_k \otimes p^l$.

B.5. (Sketch) It is easy to prove by approximation that each Σ in the image $(\mathscr{E}')\widehat{\ }$ has the property stated. Conversely, suppose $\Sigma \in \mathscr{E}'(\boldsymbol{P}^n)$ has the stated property. Because of the Remark following Theorem 2.17 we define $S \in \mathscr{D}'(\boldsymbol{R}^n)$ by

$$S(f) = \Sigma(\Lambda \hat{f}).$$

Let dg be a Haar measure on the isometry group $G = \boldsymbol{M}(n)$. If $g \in G$ let S^g denote the image of S under g [i.e., $S^g(f) = \int f(g \cdot x)\, dS(x)$] and define Σ^g similarly. If $F \in \mathscr{D}(G)$ the distribution

$$f \to \int_G F(g) S^g(f) dg, \qquad f \in \mathscr{D}(\boldsymbol{R}^n),$$

is given by a function s on $\boldsymbol{R}^n$. Also, the distribution

$$\phi \to \int_G F(g) \Sigma^g(\phi)\, dg, \qquad \phi \in \mathscr{D}(\boldsymbol{P}^n),$$

is given by a compactly supported function σ on $\boldsymbol{P}^n$. Since Σ vanishes on the kernel $\mathscr{N}$ from B4 we have $\sigma(\mathscr{N}) = 0$ so by Theorem 2.10 $\sigma = \hat{h}$ for some $h \in \mathscr{D}(\boldsymbol{R}^n)$. The Remark following Theorem 2.17 now shows $s(x) = c\, h(x)$ so $s \in \mathscr{D}(\boldsymbol{R}^n)$. Hence, by Exercise E2, Chapter II, $S \in \mathscr{E}'(\boldsymbol{R}^n)$. Since $\hat{s} = c\, \sigma$ a suitable limit argument shows $\hat{S} = c\, \Sigma$ as desired.

A related result has been proved by Ambrose (unpublished) and a generalization by Hertle [1983].

C. Spaces of Constant Curvature

C.1. Let $I: Q_{-1} \to \boldsymbol{R}^{n+1}$ denote the identity mapping. We shall prove

$$I^*\Phi^*(ds^2) = dx_1^2 + \cdots + dx_n^2 - dx_{n+1}^2,$$
$$I^*\Psi^*(d\sigma^2) = dx_1^2 + \cdots + dx_n^2 - dx_{n+1}^2,$$

which is the substance of C1. But by [DS] (Chapter I, §3. No. 3)

$$\Phi^*(dy_i) = (x_{n+1} + 1)^{-1}dx_i - x_i(x_{n+1} + 1)^{-2}\, dx_{n+1},$$

so

$$\Phi^*(ds^2) = \sum_{i=1}^{n} dx_i^2 + (x_{n+1} + 1)^{-2} \sum_1^n x_i^2\, dx_{n+1}^2 - 2(x_{n+1} + 1)^{-1}\left(\sum_1^n x_i\, dx_i\right) dx_{n+1}.$$

But on Q_{-1} we have

$$\sum_1^n x_i^2 = x_{n+1}^2 - 1, \qquad \sum_1^n x_i\, dx_i = x_{n+1}\, dx_{n+1},$$

so

$$I^*(\Phi^*(ds^2)) = \sum_1^n dx_i^2 + \frac{x_{n+1} - 1}{x_{n+1} + 1}\, dx_{n+1}^2 - \frac{2x_{n+1}}{x_{n+1} + 1}\, dx_{n+1}^2 = \sum_1^n dx_i^2 - dx_{n+1}^2.$$

Also

$$\Psi^*(dz_i) = (x_{n+1} - x_n)^{-1}\, dx_i - x_i(x_{n+1} - x_n)^{-2}(dx_{n+1} - dx_n) \qquad (1 \le i < n),$$
$$\Psi^*(dz_n) = -(x_{n+1} - x_n)^{-2}(dx_{n+1} - dx_n).$$

But since

$$I^*\left(\sum_1^{n-1} x_i\, dx_i + x_n\, dx_n - x_{n+1}\, dx_{n+1}\right) = 0$$

a computation similar to that made before gives

$$I^*(\Psi^*(d\sigma^2)) = dx_1^2 + \cdots + dx_n^2 - dx_{n+1}^2.$$

C.2. We may use as the hyperbolic plane the unit disk $|z| < 1$ with the metric

$$ds^2 = 4\,\frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2},$$

which gives curvature -1 and the distance

$$d(z_1, z_2) = \log \frac{1 + \delta(z_1, z_2)}{1 - \delta(z_1, z_2)}, \qquad \delta(z_1, z_2) = \frac{|z_2 - z_1|}{|1 - \bar{z}_1 z_2|},$$

[Introduction, §1, Eq. (6)]. This means that

$$\delta = \tanh(d/2)$$

It suffices to verify (2) for a triangle $ABC = oz_1z_2$ with z_1 on the x axis. Then $A = o$, $B = \tanh(c/2)$, $C = \tanh(b/2)e^{iA}$, so

$$\tanh \frac{a}{2} = \delta(B, C) = \frac{|\tanh(b/2)e^{iA} - \tanh(c/2)|}{|1 - \tanh(b/2)e^{iA} \tanh(c/2)|}.$$

Now (2) follows by a simple computation using the formulas

$$\cosh a = \frac{1 + \tanh(a/2)}{1 - \tanh^2(a/2)}, \qquad \sinh a = \frac{2 \tanh(a/2)}{1 - \tanh^2(a/2)}.$$

Formula (1) can be derived directly from (2).

For curvature $-\varepsilon^2$ we use the model D_ε:

$$ds^2 = 4 \frac{dx^2 + dy^2}{(1 - \varepsilon^2(x^2 + y^2))^2} \qquad |z| < \varepsilon^{-1}.$$

Then we find

$$\varepsilon |z| = \tanh[\tfrac{1}{2}\varepsilon d(o, z)].$$

We consider again a triangle $ABC = oz_1z_2$ with z_1 on the x axis. Then

$$A = o, \qquad B = \frac{1}{\varepsilon} \tanh\left(\frac{\varepsilon c}{2}\right), \qquad C = \frac{1}{\varepsilon} \tanh\left(\frac{\varepsilon b}{2}\right) e^{iA}.$$

The mapping

$$z \to (z - z_1)/(1 - \varepsilon^2 \bar{z}_1 z)$$

maps the disk $|z| < \varepsilon^{-1}$ isometrically onto itself so

$$a = d(z_1, z_2) = d\left(o, \frac{z_2 - z_1}{1 - \varepsilon^2 \bar{z}_1 z_2}\right)$$

and

$$\varepsilon(|z_2 - z_1|)/|1 - \varepsilon^2 \bar{z}_1 z_2| = \tanh(\varepsilon a/2).$$

This equation can be written

$$\tanh\left(\frac{\varepsilon a}{2}\right) = \frac{|\tanh(\varepsilon b/2)e^{iA} - \tanh(\varepsilon c/2)|}{|1 - \tanh(\varepsilon b/2)e^{iA} \tanh(\varepsilon c/2)|}.$$

Thus (1) and (2) hold for D_ε if we replace a, b, and c by εa, εb, and εc. Letting $\varepsilon \to 0$, D_ε formally tends to the flat plane and formulas (1) and (2) converge to the standard trigonometric formulas

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},$$

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

D. Some Results in Analysis

D.1. As in Introduction §3 consider the bilinear form

$$\langle p, q\rangle = \left[p\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)q\right](0)$$

on the space P_k of homogeneous polynomials of degree k. It is nondegenerate and the annihilator of the set of polynomials $x \to (x, \omega)^k$ as ω runs through S^{n-1} is 0.

D.3. It is obvious that

$$\lim_{r=0} r \int_0^\infty F(re^t, re^{-t})e^{-t}\, dt = 0$$

and if $r > 0$

$$r \int_0^\infty F(re^t, re^{-t})e^t\, dt = \int_0^\infty F(\tau, r^2\tau^{-1})\, d\tau \to \int_0^\infty F(\tau, 0)\, d\tau.$$

D.4. Let $\alpha \to 0$ in the formula $L(\alpha^{-1}(r^\alpha - 1)) = \alpha r^{\alpha-2}$.
Note, however, that while

$$L((\log r)r^\alpha) = 2\alpha r^{\alpha-2} + \alpha^2(\log r)r^{\alpha-2},$$

multiplication by $\log r$ is not continuous in $\mathscr{S}'$ so we cannot state $\lim_{\alpha\to 0} \alpha^2(\log r)r^{\alpha-2} = 0$ (remark by R. Melrose).

D.5. Let $a = \limsup_{t\to+\infty} |f(t)|$; if $\varepsilon > 0$ then $|f(t)| \le a + \varepsilon$ for all sufficiently large t so

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T |f(t)|^2\, dt \le (a + \varepsilon)^2.$$

Now $(*)$ follows. Next let $l_j = \operatorname{Re} k_j$. If $l_j < 0$ then $\lim_{t\to+\infty} p_j(t)e^{k_j t} = 0$. Suppose $l_j \ge 0$ for $1 \le j \le s$, $l_j < 0$ for $j > s$. Then

$$\limsup_{t\to+\infty} \left| p_0(t) + \sum_{j=1}^{s} p_j(t)e^{k_j t}\right| \le a.$$

Thus we may assume $s = r$. We first claim $l_1 = l_2 = \cdots = l_r = 0$. Otherwise let $l = \max(l_1, \ldots, l_r) > 0$ and suppose $l_1 = \cdots = l_i = l$, $l_j < l$ for $j > i$. Let n be the maximal degree among $p_1, \ldots, p_i$ and put

$$c'_j = \lim_{t \to +\infty} p_j(t)/t^n.$$

Then

$$\begin{aligned} 0 &= \limsup_{t \to +\infty} t^{-n} e^{-lt} |p_0(t) + p_1(t)e^{k_1 t} + \cdots + p_r(t)e^{k_r t}| \\ &= \limsup_{t \to +\infty} |c'_1 e^{(k_1 - l)t} + \cdots + c'_i e^{(k_i - l)t}|, \end{aligned}$$

contradicting (*). Thus $l = 0$.

Next let m be the maximal degree among $p_0, \ldots, p_r$ and put $c_j = \lim_{t \to +\infty} p_j(t)/t^m$. If $m > 0$ our assumption implies

$$\lim_{t \to +\infty} t^{-m} \left| p_0(t) + \sum_{1 \le j \le r} p_j(t) e^{k_j t} \right| = 0,$$

whence

$$\lim_{t \to +\infty} \left| c_0 + \sum_{1 \le j \le r} c_j e^{k_j t} \right| = 0,$$

contradicting (*). Hence $m = 0$, so each p_j is a constant and (*) implies $|p_0| \le a$.

CHAPTER II

A. The Laplace-Beltrami Operator

A.1. The differential equations for geodesics {[DS], Chapter I, §5, Eq. (3)} show that $\Gamma^k_{ij}(p) = 0$. Now use Prop. 2.6.

A.2. The results are local so we may assume M has an orthonormal basis $X_1, \ldots, X_n$ of the vector fields. Let $\omega^1, \ldots, \omega^n$ be the dual basis of 1-forms. Then $\omega^k = \omega_{X_k}$ and $\omega^1 \wedge \cdots \wedge \omega^n$ is the volume element. The Lie derivative $\theta(X_k)$ satisfies

$$\theta(X_k)\omega^i = \sum_j (\Gamma^i_{jk} - \Gamma^i_{kj})\omega^j$$

(use [DS], pp. 540, 44–45), whence by Lemma 2.2,

$$\operatorname{div}(X_k) = \sum_i (\Gamma^i_{ik} - \Gamma^i_{ki}).$$

On the other hand, $*\omega^1 = \omega^2 \wedge \cdots \wedge \omega^n$, etc., and $\delta\omega^k$ can be calculated by means of the first structural equation ([DS], Chapter I, Theorem 8.1) in terms of the Γ^i_{jk}. One finds $\delta\omega^k = -\operatorname{div} X_k$ and now $\delta(\omega_X) = -\operatorname{div} X$ follows by using Eq. (14) of §2.

Finally, if $f \in \mathscr{E}(M)$ we have

$$\operatorname{grad} f = \sum_i (X_i f) X_i, \qquad df = \sum_i (X_i f)\omega_i,$$

$$\operatorname{div}(\operatorname{grad} f) = \sum_i X_i^2 f + \sum_i (X_i f) \operatorname{div} X_i,$$

$$\begin{aligned}
\Delta f &= -\delta\, df = -\delta\Big(\sum_i (X_i f)\omega^i\Big) \\
&= *\, d\left[\Big(\sum_i X_i f\Big) * \omega^i\right] \\
&= \sum_i (X_i f) * d * \omega^i + * d\Big(\sum_i X_i f\Big) \wedge * \omega^i \\
&= \sum_i (X_i f) \operatorname{div} X_i + * \sum_{i,j} (X_i X_j f)\omega^j \wedge * \omega^i \\
&= \operatorname{div} \operatorname{grad} f.
\end{aligned}$$

A.3. The space G/H is in each case symmetric, hence reductive, so $D(G/H)$ is determined by the algebra $I(\mathfrak{m})$ of $\mathrm{Ad}_G(H)$-invariants on $\mathfrak{m}$ (cf. Theorem 6.1 in Chapter I). The space G/H being isotropic, each $q \in I(\mathfrak{m})$ is constant on each "sphere" $\{X \neq 0 \text{ in } \mathfrak{m} : g_e(X, X) = c\}$ in $\mathfrak{m}$; hence $q(X) = F(g_e(X, X))$ where F is a function. But then F is a polynomial; now use Corollary 4.10.

A.4. (i) We have $g^{ij} = g^{ji}$ so by Theorem 4.3

$$\begin{aligned}
&((g^{ij}\tilde{X}_i\tilde{X}_j + g^{ji}\tilde{X}_j\tilde{X}_i)f)(e) \\
&\quad = 2(g^{ij}\lambda(X_iX_j)f)(e) = 2g^{ij}\left(\frac{\partial^2}{\partial t_i\,\partial t_j} f(\exp(t_iX_i + t_jX_j))\right)_{t=0}
\end{aligned}$$

For the pseudo-Riemannian structure given by B the one-parameter subgroups are the geodesics through e; the Christoffel symbols therefore vanish at e for the canonical coordinates. Thus by Proposition 2.6, Ω coincides with the Laplace-Beltrami operator L_G at e, hence everywhere by their invariance. Now $\Omega \in \boldsymbol{Z}(G)$ by Corollary 4.5.

(ii) Choosing the basis (X_i) compatible with the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the argument above shows that $\mu(\Omega)$ and $L_{G/K}$ coincide at the origin $\{K\}$, hence everywhere by the invariance.

A.5. Using notation from A2 we have $g = \sum_i \omega^i \otimes \omega^i$ so (i) follows from the proof of A2. Formula (1) follows from $L^{\exp tX} = h(\exp tX)L$ by differentiation.

(iii) Let M have pseudo-Riemannian structure g and put $h = \tau^2 g$. Then $h_{ij} = \tau^2 g_{ij}$, $h^{ij} = \tau^{-2} g^{ij}$, $\bar{h} = \tau^{2n}\bar{g}$, and by (22) §2 the new Christoffel symbols are

$$(1) \qquad \Gamma_{ij}^{*k} = \Gamma_{ij}^k + \delta_i^k \partial_j(\log \tau) + \delta_j^k \partial_i(\log \tau) - g_{ij} \sum_i g^{lk} \partial_l(\log \tau)$$

Thus by Proposition 2.6 the new Laplacian L^* is given by

$$L^* f = \tau^{-2}[Lf + (n-2)(\operatorname{grad} f)(\log \tau)],$$

which by (17) §2 can also be written

$$(2) \qquad L^* f = \tau^{-n/2-1}[L(\tau^{n/2-1} f) - L(\tau^{n/2-1}) f].$$

Now we need the identity

$$(3) \qquad -L(\tau^{n/2-1}) = c_n(K^* \tau^{n/2+1} - K\tau^{n/2-1}),$$

where K^* is the scalar curvature for the Riemannian structure h (see Yamabe [1960], where the scalar curvature has the same sign). The desired result now follows from (2) and (3) because, L and K being invariant under isometries, we can take T as the identity mapping.

This quasi-invariance of L occurs several places in the literatures; see, e.g., Friedlander [1975], Ørsted [1981b], and also Kosman [1975]. To a large extent, it goes back to Cotton [1900].

A.7. Let U be an arbitrary coordinate neighborhood on M and let μ be given on U by

$$d\mu = \delta(x)\, dx_1 \cdots dx_n = \delta(x)\, dx.$$

Then the equations

$$\int (Xf)(x) g(x) \delta(x)\, dx = \int f(x)(Xg)(x)\delta(x)\, dx, \qquad f \in \mathscr{D}(U), \quad g \in \mathscr{E}(U),$$

$$\int [(Xf)g + f(Xg)]\delta(x)\, dx = \int X(fg) 1 \delta(x)\, dx = \int fg \cdot 0 \delta(x)\, dx = 0$$

imply

$$\int (Xf)(x) g(x) \delta(x)\, dx = 0$$

for all f, g, whence $X = 0$.

A.8. (Harish-Chandra [1960]) During the proof of Proposition 5.23 we saw

$$Y_\beta = (\coth c)Z_\beta - (\sinh c)^{-1}Z_\beta^{a^{-1}}, \qquad c = \beta(H),$$

so

$$X_\beta = (\sinh c)^{-1}\{e^c Z_\beta - Z_\beta^{a^{-1}}\}.$$

Hence

$$-X_\beta X_{-\beta} = (\sinh c)^{-2}\{Z_\beta Z_{-\beta} + (Z_\beta Z_{-\beta})^{a^{-1}} - e^c Z_\beta Z_{-\beta}^{a^{-1}} - e^{-c}Z_\beta^{a^{-1}}Z_{-\beta}\}.$$

Using

$$\begin{aligned} Z_\beta Z_{-\beta}^{a^{-1}} &= Z_{-\beta}^{a^{-1}}Z_\beta + [Z_\beta, Z_{-\beta}^{a^{-1}}] \\ &= Z_{-\beta}^{a^{-1}}Z_\beta + (\cosh c)[Z_\beta, Z_{-\beta}] + (\sinh c)[Z_\beta, Y_{-\beta}] \end{aligned}$$

we obtain

$$\begin{aligned} -X_\beta X_{-\beta} = {}&(\sinh c)^{-2}\{Z_\beta Z_{-\beta} + (Z_\beta Z_{-\beta})^{a^{-1}} - e^c Z_{-\beta}^{a^{-1}}Z_\beta - e^{-c}Z_\beta^{a^{-1}}Z_{-\beta} \\ &-\tfrac{1}{2}(e^{2c}+1)[Z_\beta, Z_{-\beta}] - \tfrac{1}{2}(e^{2c}-1)[Z_\beta, Y_{-\beta}]\} \end{aligned}$$

Here we apply θ and note that $\theta(Z_\beta^{a^{-1}}) = Z_{\beta}^{a}$, etc. This gives a formula for $\theta(X_\beta X_{-\beta})$ in which we replace β by $-\beta$ and a^{-1} by a. Adding, we obtain

$$\begin{aligned} &-(X_\beta X_{-\beta} + \theta(X_{-\beta}X_\beta)) \\ &\quad = (\sinh c)^{-2}\{Z_\beta Z_{-\beta} + Z_{-\beta}Z_\beta + (Z_\beta Z_{-\beta} + Z_{-\beta}Z_\beta)^{a^{-1}} \\ &\quad\quad -2e^c Z_{-\beta}^{a^{-1}}Z_\beta - 2e^{-c}Z_\beta^{a^{-1}}Z_{-\beta}\} \\ &\quad\quad + e^c(\sinh c)^{-1}\{[Z_{-\beta}, Y_\beta] - [Z_\beta, Y_{-\beta}]\}. \end{aligned}$$

Put $\Omega_+ = \Sigma_{\beta\in P_+}(X_\beta X_{-\beta} + X_{-\beta}X_\beta)$. Then $\theta\Omega_+ = \Omega_+$ so that $\Omega_+ = \frac{1}{2}(\Omega_+ + \theta\Omega_+)$. Also, $[Z_\beta, Y_{-\beta}] + [Y_\beta, Z_{-\beta}] = A_\beta$ so that the formula above gives

$$\begin{aligned} \Omega_+ = {}&\sum_{\beta\in P_+} \coth c A_\beta \\ &- \sum_{\beta\in P_+} (\sinh c)^{-2}\{Z_\beta Z_{-\beta} + Z_{-\beta}Z_\beta + (Z_\beta Z_{-\beta} + Z_{-\beta}Z_\beta)^{a^{-1}} \\ &- 2e^c Z_{-\beta}^{a^{-1}}Z_\beta - 2e^{-c}Z_\beta^{a^{-1}}Z_{-\beta}\} \end{aligned}$$

Since the mapping $\beta \to -\beta^\theta = \beta^*$ is a permutation of P_+ we have $X_{\beta^*} = c_\beta \theta X_{-\beta}$, $X_{-\beta^*} = c_{-\beta}\theta X_\beta$ where $c_\beta \in \mathbf{C}$. Since θ preserves B and since $B(X_\beta, X_{-\beta}) = 1$, we have $c_\beta c_{-\beta} = 1$. Hence

$$Z_{\beta^*}Z_{-\beta^*} = c_\beta c_{-\beta} Z_{-\beta}Z_\beta = Z_{-\beta}Z_\beta.$$

This gives the desired formula. See also Anderson [1979].

B. The Radial Part

B.1. Let $v_0 \in V$ be such that the orbit $H \cdot v_0$ has maximal dimension. Then $\dim(H \cdot v) = \dim(H \cdot v_0)$ for v in a neighborhood N_0 of v_0 in V. According to Chevalley [1946] (Chapter III, §7, Theorem 1) there exists a coordinate system $\{x_1, \ldots, x_n\}$ near v_0 in V such that

$$x_i(v_0) = 0 \quad (1 \leq i \leq n)$$

and such that each slice $x_{r+1} = a_{r+1}, \ldots, x_n = a_n$ is a part of an orbit $H \cdot v$. Then the submanifold $x_1 = 0, \ldots, x_r = 0$ can serve as W (remark by R. Palais).

B.2. This is clear from (33′) because here $\delta(r) = A(r)$.

B.3. By (i) the orbit $H \cdot w$ is locally compact and thus homeomorphic to H/H^w. By (ii), H^v is compact for each $v \in V^*$ so no change is required in the proof of Theorem 3.7.

B.4. The right-hand side of the formula is independent of the choice of basis (H_i). But with (H_i) orthonormal the computation is the same as in the proof of Proposition 3.9.

C. Invariant Differential Operators

C.1. The group $G = \boldsymbol{GL}(n, \boldsymbol{R})$ acts on the space $\boldsymbol{P}_n$ of $n \times n$ symmetric positive definite matrices by

$$p \to gp{}^tg, \qquad \text{if} \quad p \in \boldsymbol{P}_n, \quad g \in G, \quad {}^tg = \text{transpose of } g,$$

and the isotropy group of I is $K = \boldsymbol{O}(n)$. For the Lie algebras we have $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and $\mathfrak{p}$ is the space of all $n \times n$ real symmetric matrices X. If $k \in K$, then $\mathrm{Ad}_G(k)|\mathfrak{p}$ is given by $X \to kXk^{-1}$. The algebra $I(\mathfrak{p})$ of invariant polynomial functions is generated by

$$\mathrm{Tr}(X), \ \mathrm{Tr}(XX), \ \ldots, \ \mathrm{Tr}(X^n).$$

This boils down to the fact that the symmetric polynomial in n variables $\lambda_1, \ldots, \lambda_n$ (the eigenvalues of X) is a polynomial in

$$\lambda_1 + \cdots + \lambda_n, \ldots, \lambda_1^n + \cdots + \lambda_n^n.$$

By Corollary 4.10 the algebra $\boldsymbol{D}(G/K)$ is generated by the corresponding operators D_i $(1 \leq i \leq n)$. With the inner product $\mathrm{Tr}(XY)$ on $\mathfrak{p}$ the matrices $e_i = E_{ii}$, $1 \leq i \leq n$, $e_{ij} = 2^{-1/2}(E_{ij} + E_{ji})$ $(i > j)$ form an orthonormal basis. Writing the matrix $X = (x_{ij})$ in the form $X = \sum \xi_{ii} e_i + \sum \xi_{ij} e_{ij}$ we have $\xi_{ii} = x_{ii}$, $\xi_{ij} = 2^{1/2} x_{ij}$. If

$$\mathrm{Tr}(X^k) = P_k(\xi_{pq})$$

we have to determine $P_k(\partial/\partial\xi_{pq})$. But for $a = 2^{-1/2}$

$$X = \begin{pmatrix} \xi_{11}, & \dots, & a\xi_{n1} \\ \vdots & & \vdots \\ a\xi_{n1}, & \dots, & \xi_{nn} \end{pmatrix}$$

and we have

$$\begin{pmatrix} \dfrac{\partial}{\partial\xi_{11}}, & \dots, & a\dfrac{\partial}{\partial\xi_{n1}} \\ \vdots & & \vdots \\ a\dfrac{\partial}{\partial\xi_{n1}}, & \dots, & \dfrac{\partial}{\partial\xi_{nn}} \end{pmatrix} = \frac{\partial}{\partial X}$$

the symmetric matrix with entries $\frac{1}{2}(1 + \delta_{ij})\,\partial/\partial x_{ij}$. Since $\mathrm{Tr}(X^k) = P_k(\xi_{pq})$ we have

$$\mathrm{Tr}\left(\left(\frac{\partial}{\partial X}\right)^k\right) = P_k\left(\frac{\partial}{\partial\xi_{pq}}\right).$$

Thus the operators

$$(D_k f)(g \cdot 0) = \left[\mathrm{Tr}\left(\left(\frac{\partial}{\partial X}\right)^k\right) f(g \exp X \cdot o)\right]_{X=0}$$

generate $\boldsymbol{D}(G/K)$.

A different description of $\boldsymbol{D}(G/K)$ is given Maass [1955] and Selberg [1956].

C.3. (i) Since $\mathrm{Ad}_G(m)$ fixes H each $P \in I(\mathfrak{a} + \mathfrak{l} + \mathfrak{q})$ can be written

$$P(H, T, X) = \sum_k H^k P_k(T, X).$$

But each $P_k(T, X)$ is invariant under the action

$$(T, X) \to (mT, mX), \qquad m \in M = \boldsymbol{SO}(n-1).$$

By a standard result about invariants (cf. Weyl [1939], p. 31) P_k is a polynomial in $|T|^2$, $T \cdot X$, and $|X|^2$. This proves (i).

(ii) We have the commutation relations $X_i = [T_i, H]$, $[X_i, T_j] = \delta_{ij}H$, $[X_i, H] = T_i$ {[DS], Eq. (17) p. 407}. The symmetrization λ satisfies

$$\lambda(H|T|^2) = \frac{1}{6}\sum_{i=1}^{n-1} 2(T_i T_i H + T_i H T_i + H T_i T_i).$$

Using the commutation relations this can be written

$$\sum_{i=1}^{n-1} T_i T_i H - \frac{1}{2}\sum_{i=1}^{n-1}(X_i T_i + T_i X_i) + \frac{n-1}{6} H$$

and also as

$$\sum_{i=1}^{n-1} H T_i T_i + \frac{1}{2}\sum_{i=1}^{n-1}(X_i T_i + T_i X_i) + \frac{n-1}{6} H.$$

By subtraction, $[D_H, D_{|T|^2}] = -2D_{T \cdot X}$; the other formulas from (ii) follow similarly.

C.4. In the notation of Theorem 5.16 we have

$$D\phi_\nu = \gamma(D)(i\nu)\phi_\nu \qquad D \in \boldsymbol{D}_K(G).$$

Let $s \in W$, let $m_s \in M'$ be a representative and put $N^s = m_s N m_s^{-1}$. Then

$$g = n \exp A^s(g) k, \qquad k \in K,\ A^s(g) \in \mathfrak{a},\ n \in N^s.$$

Let $\gamma^s : \boldsymbol{D}_K(G) \to \boldsymbol{D}_W(A)$ be the homomorphism defined by using the Iwasawa decomposition $G = N^s AK$ in place of $G = NAK$ and put

$$\phi_\nu^s(g) = \int_K e^{(i\nu + s\rho)(A^s(kg))}\, dk.$$

Then by Eq. (38′) of §5,

$$D\phi_\nu^s = \gamma^s(D)(i\nu)\phi_\nu^s.$$

Comparing the two Iwasawa decompositions we see that

$$A^s(g) = sA(m_s^{-1} g).$$

As a result $\phi_\nu^s = \phi_\nu$, so $\gamma^s = \gamma$.

The result is given in Schiffmann [1979] with a different proof.

D. Restriction Theorems

D.4. For $R > 0$ select C_R such that

$$|\phi(H) - \phi(H')| \leq C_R |H - H'|$$

for all H, H' in the ball $|H| < R$ in $\mathfrak{a}$. The K-invariant extension Φ satisfies such an inequality on each maximal abelian subspace $\mathfrak{b}$ of $\mathfrak{p}$. Given $X, Y \in \mathfrak{p}$ the distance from X to the orbit $\mathrm{Ad}(K)Y$ is minimized at a point $\mathrm{Ad}(k_0)Y$ for which X and $\mathrm{Ad}(k_0)Y$ commute (Prop. 5.18, Chapter I) and so lie in the same $\mathfrak{b}$. Thus $|\Phi(X) - \Phi(Y)| \leq C_R |X - Y|$ if both $|X|$, $|Y| < R$. (cf. Helgason [1980a], Proposition 2.4).

E. Distributions

E.1. By definition

$$\int_G (f \times T)(g \cdot o)F(g \cdot 0)\, dg = (\tilde{f} * \tilde{T})(\tilde{F})$$

$$= \int_G \int_G \tilde{F}(gh)\tilde{f}(g)\, dg\, d\tilde{T}(h)$$

$$= \int_G \tilde{F}(g)\left(\int_G f(gh^{-1} \cdot o)\, d\tilde{T}(h)\right) dg,$$

proving the first relation. The second follows in the same way.

E.3. If $F \in \mathscr{D}(G)$ then

$$(s * (D^*)^{\vee} t)(F) = \int\left(\int F(xy)\, ds(x)\right) d((D^*)^{\vee} t)(y)$$

$$= \iint F(xy^{-1})\, ds(x)\, d(D^*\check{t})(y) = \int (\check{F} * s)(y)\, d(D^*\check{t})(y)$$

$$= \check{t}(\check{F} * Ds) = \iint \check{F}(yx^{-1})\, d(Ds)(x)\, d\check{t}(y)$$

$$= \iint F(xy)\, d(Ds)(x)\, dt(y) = (Ds * t)(F).$$

Use this result for $s = \tilde{S}$, $t = \tilde{T}$, $D \in \lambda(I(\mathfrak{p}))$. Then, using (13) §5 and the bi-invariance of $t^{\natural}$,

$$D(s * t) = D(s * \delta * t) = D(s * t^{\natural}) = s * Dt^{\natural}$$
$$= s * (D^*)^{0} t^{\natural} = s * (D^*)^{\vee} t^{\natural} = Ds * t^{\natural} = Ds * t.$$

F. The Wave Equation

F.1. By Eq. (63) of §5 we have for $x_0 \in \boldsymbol{R}^n$, $r \geq 0$,

$$\int_{S^r(x_0)} v(x, 0)\, d\omega(x) = \int_{S^r(0)} v(x_0, y)\, d\omega(y).$$

The function $v(x_0, y)$ is constant in y on hyperplanes perpendicular to the y_1 axis. Proceeding as with Lemma 2.30 in Chapter I we therefore obtain

$$(1) \qquad \Omega_n r^{n-1}(M^r u_0)(x) = \Omega_{n-1} r \int_{-r}^{r} (r^2 - t^2)^{(1/2)(n-3)} u(x_0, t)\, dt.$$

Case I. n odd. Divide Eq. (1) by r, differentiate $\frac{1}{2}(n-3)$ times with respect to r^2, and then apply $\partial/\partial r$. This gives

$$(2) \qquad u(x_0, r) + u(x_0, -r) = \frac{\Omega_n}{[\frac{1}{2}(n-3)!\Omega_{n-1}} \frac{\partial}{\partial r}\left(\frac{\partial}{\partial(r^2)}\right)^{(1/2)(n-3)} [r^{n-2}(M^r u_0)(x_0)].$$

Now the function $w(x, t) = u_t(x, t)$ satisfies the wave equation with initial data $w_0 = u_1$, $w_1 = u_{tt}(x, 0) = L_{\boldsymbol{R}^n} u_0$. We can replace u by w, u_0 by $w_0 = u_1$ in the last formula and get by integration

$$(3) \qquad u(x_0, r) - u(x_0, -r) = \frac{\Omega_n}{[\frac{1}{2}(n-3)]!\Omega_{n-1}} \left(\frac{\partial}{\partial(r^2)}\right)^{(1/2)(n-3)} [r^{n-2}(M^r u_1)(x_0)].$$

Adding (2) and (3) an explicit solution formula for n odd is obtained.

Case II. n even. Again divide (1) by r and differentiate $\frac{1}{2}(n-2)$ times with respect to r^2. This gives

$$\frac{n-3}{2}\frac{n-5}{2}\cdots\frac{1}{2}\int_0^r (r^2 - t^2)^{-1/2}[u(x_0, t) + u(x_0, -t)]\, dt = \frac{\Omega_n}{\Omega_{n-1}}\left(\frac{\partial}{\partial(r^2)}\right)^{(1/2)(n\ \ 2)} \{r^{n-2}(M^r u_0)(x_0)\}.$$

The integral on the left is inverted as in Eq. (19) of §2 in Chapter I. Again we can use the identity replacing u by u_t and u_0 by u_1. Here one gets the solution formula

$$u(x_0, \pm r) = \frac{1}{[\frac{1}{2}(n-2)]!}\left[\frac{\partial}{\partial r}\int_0^r \frac{t}{(r^2 - t^2)^{1/2}}\left(\frac{\partial}{\partial(t^2)}\right)^{(1/2)(n-2)} [t^{n-2}(M^t u_0)(x_0)]\, dt \pm \int_0^r \frac{t}{(r^2 - t^2)^{1/2}}\left(\frac{\partial}{\partial(t^2)}\right)^{(1/2)(n-2)} [t^{n-2} M^t u_1(x_0)]\, dt\right].$$

F.2. In accordance with Chapter I, Exercise C1, we use the half-space model for $\boldsymbol{H}^n$ and have

$$ds^2 = y_n^{-2}(dy_1^2 + \cdots + dy_n^2), \qquad y_n > 0.$$

Then the Laplace–Beltrami operator is given by

$$L = y_n^2\left(\frac{\partial^2}{\partial y_1^2} + \cdots + \frac{\partial^2}{\partial y_{n-1}^2}\right) + y_n^2 \frac{\partial^2}{\partial y_n^2} - (n-2)y_n \frac{\partial}{\partial y_n},$$

which by the substitution $y_n = e^s$ can be written

$$L = e^{2s}\left(\frac{\partial^2}{\partial y_1^2} + \cdots + \frac{\partial^2}{\partial y_{n-1}^2}\right) + e^{(1/2)(n-1)s}\left[\frac{\partial^2}{\partial s^2} - \left(\frac{n-1}{2}\right)^2\right] \circ e^{-(1/2)(n-1)s}.$$

With $y_n = e^s$ put

$$v(x, y) = v(x_1, \ldots, x_n, y_1, \ldots, y_n) = e^{(1/2)(n-1)s}u(x_1, \ldots, x_n, s).$$

Then, using the differential equation for u,

$$L_x(v(x, y)) = e^{(1/2)(n-1)s}\left(\frac{\partial^2 u}{\partial s^2} - \left(\frac{n-1}{2}\right)^2 u\right)$$
$$= L_y(v(x, y)).$$

Now use Theorem 5.28 for the point $(x_0, y_0) = (x, o)$, where o is the origin $(0, \ldots, 0, 1)$ in $\boldsymbol{H}^n$. Since $v(x, o) = u(x, 0)$ we obtain

$$\int_{S_r(x)} u(z, 0)\, d\omega(z) = \int_{S_r(o)} v(x, y)\, d\omega(y).$$

The non-Euclidean sphere $S_r(o)$ is a Euclidean sphere with center $(0, \ldots, 0, \operatorname{ch} r)$ and radius $\operatorname{sh} r$. On the plane $y_n = e^s$ the metric is

$$e^{-2s}(dy_1^2 + \cdots + dy_{n-1}^2)$$

so the intersection with the sphere $S_r(o)$ is a non-Euclidean sphere of Euclidean radius $[2e^s(\operatorname{ch} r - \operatorname{ch} s)]^{1/2}$; its non-Euclidean $(n-2)$-dimensional area is

$$\Omega_{n-1}[2e^{-s}(\operatorname{ch} r - \operatorname{ch} s)]^{(1/2)(n-2)}$$

and the function $y \to v(x, y)$ is constant on it. The arc-element on the circle

$$y_1 = (2e^s(\operatorname{ch} r - \operatorname{ch} s))^{1/2}, \qquad y_2 = \cdots = y_{n-1} = 0, \qquad y_n = e^s$$

is

$$y_n^{-1}(dy_1^2 + dy_n^2)^{1/2} = \operatorname{sh} r[2e^s(\operatorname{ch} r - \operatorname{ch} s)]^{-\frac{1}{2}} ds.$$

The integral relation above thus becomes

(4)
$$\Omega_n \operatorname{sh}^{n-1} r(M^r u_0)(x) = \Omega_{n-1} \operatorname{sh} r \int_{-r}^{r} u(x, s)(2 \operatorname{ch} r - 2 \operatorname{ch} s)^{(1/2)(n-3)}\, ds.$$

The function $w(x, s) = u_s(x, s)$ satisfies the same differential equation as $u(x, s)$. This we can in the last formula replace $u(x, s)$ by $u_s(x, s)$ and $u_0(x)$ by $u_1(x)$. Using now the method of the last problem we obtain:

Case I. n odd. The solution $u(x, s)$ is given by

$$(5)\quad u(x,s) = \frac{1}{2(\frac{1}{2}(n-3))!}\frac{\Omega_n}{\Omega_{n-1}}\left[\frac{\partial}{\partial s}\left(\frac{\partial}{\partial(2\,\mathrm{ch}\, s)}\right)^{(n-3)/2}[\mathrm{sh}^{n-2}s(M^s u_0)(x)] + \left(\frac{\partial}{\partial(2\,\mathrm{ch}\, s)}\right)^{(n-3)/2}[\mathrm{sh}^{n-2}s(M^s u_1)(x)]\right].$$

In particular, Huygens's principle holds.

Case II. n even. From (4) we obtain

$$\frac{n-3}{2}\cdot\frac{n-5}{2}\cdots\frac{1}{2}\int_0^s(2\,\mathrm{ch}\, s - 2\,\mathrm{ch}\, r)^{-1/2}[u(x,r) + u(x,-r)]\,dr = \frac{\Omega_n}{\Omega_{n-1}}\left(\frac{\partial}{\partial(2\,\mathrm{ch}\, s)}\right)^{(1/2)(n-2)}[\mathrm{sh}^{n-2}s(M^s u_0)(x)].$$

This Abel-type integral equation can be solved for $u(x, s) + u(x, -s)$. Using (4) on u_s we get an analogous equation for $u(x, s) - u(x, -s)$, so a solution formula for $u(x, s)$ follows.

For relevant references see Hölder [1938] (for the pure equation $Lu - u_{ss} = 0$ with Huygens's principle absent), Günther [1957, p. 23] (giving (5) for $n = 3$, using a different method), Helgason [1964a, 1977c, 1984], Lax and Phillips [1978], Ørsted [1981b], and Kiprijanov and Ivanov [1981].

CHAPTER III

1. Since the invariants of $\boldsymbol{O}(p, q)$ are the polynomials in Q (compare Exercise A3, Chapter II) this follows from Theorem 1.2 [for $(p, q) \neq (1, 1)$]. For $(p, q) = (1, 1)$ use substitution $y_1 = x_1$, $y_2 = ix_2$ and observe that a polynomial identity $P(\cosh\zeta, \sinh\zeta) = 0$ implies $P(\cos\theta, i\sin\theta) = 0$.

2. (i) If λ is singular then K_λ has a Lie algebra larger than that of M so $K_\lambda \neq M$ and $s_\alpha\lambda = \lambda$ for a certain Weyl reflection $s_\alpha \neq e$ in W. If λ is regular then K_λ and M have the same Lie algebra. If $k \in K_\lambda$ and $\lambda = \xi + i\eta$ $(\xi, \eta \in \mathfrak{a}^*)$ then $k\xi = \xi$, $k\eta = \eta$. For each restricted root α either $\alpha(A_\xi)$ or $\alpha(A_\eta)$ is nonzero. Thus the centralizers $\mathfrak{z}_\xi$ and $\mathfrak{z}_\eta$ of ξ and η, respectively, in $\mathfrak{p}$ have intersection $\mathfrak{a}$. Since k leaves each of these centralizers invariant we have $k \in M'$, the normalizer of $\mathfrak{a}$ in K. Then the Weyl group element $s = \mathrm{Ad}(k)|\mathfrak{a}$ fixes the set $\boldsymbol{R}A_\xi + \boldsymbol{R}A_\eta$ pointwise. But

$$|\alpha(A_\xi)| + |\alpha(A_\eta)| > 0$$

for each restricted root α so this set contains a regular element. Hence $s = e$ so $k \in M$.

(ii) The statement $f \in \mathscr{E}_\lambda(\mathfrak{p})$ is proved just as was Lemma 3.14.

(iii) This follows from (ii) because each M-harmonic polynomial on $\mathfrak{p}$ is K_λ-harmonic.

3. Let $h_1, \ldots, h_w$ be a homogeneous basis of H. As in Exercise D1 of Chapter II consider the normal field extension

$$\boldsymbol{C}(S)/\boldsymbol{C}(I) \qquad (S = S(\mathfrak{a}^{\boldsymbol{C}}),\ I = I(\mathfrak{a}^{\boldsymbol{C}})),$$

which has Galois group W. Because of Theorem 3.4 the h_i $(1 \leq i \leq r)$ are linearly independent over $\boldsymbol{C}(I)$. Each element of $\boldsymbol{C}(S)$ can be written as a quotient of a polynomial and an invariant polynomial. Thus the h_i form a basis of $\boldsymbol{C}(S)$ over $\boldsymbol{C}(I)$ so the representation $\sigma \to (a_{ij}(\sigma))$ of W given by

$$\sigma \cdot h_j = \sum_i a_{ij}(\sigma) h_i$$

is equivalent to the regular representation.

7. Let $\alpha_1, \ldots, \alpha_l$ be the simple restricted roots; $H_1, \ldots, H_l$ the dual basis of $\mathfrak{a}_0$; $a = \exp(\sum_i t_i H_i)$ any member of A. It is now easy to see that $a^2 = e \Leftrightarrow t_j \in \pi i \boldsymbol{Z}$ for each j. Thus $\{0, 1\}^l$ is in a one-to-one correspondence with F.

8. Because of the results quoted it suffices to prove $a \cdot j = j$ for $a \in F$ and $j \in I(\mathfrak{a})$. Since a acts trivially on $\mathfrak{a}$ this is obvious.

CHAPTER IV

A. Representations

A.1. (i) The proof of Corollary 1.4 gives the result.

A.2. Let π, π' be two unitary representations of G on Hilbert spaces V and V', respectively. Assume the linear homeomorphism $A: V \to V'$ satisfies $A\pi(x) \equiv \pi'(x)A$. Passing to adjoints one finds

$$A^*A\pi(x) = \pi(x)A^*A.$$

Let P be a positive bounded self-adjoint operator such that $P^{-2} = A^*A$. Then AP sets up the desired unitary equivalence (cf. Borel [1972], §5).

A.3. (i) Let $v \to \hat{v}$ be the canonical mapping of V into $(V')'$. Consider the functional $X_t : \lambda \to (d/dt)\langle x_t, \lambda\rangle$ on V', which does belong to $(V')'$ (Dunford and Schwartz [1958], Theorem II 1.18). The function $t \to X_t(\lambda)$ is C^1 for each $\lambda \in V'$. Fix $t \in I$ and an $\varepsilon > 0$ such that $t + s \in I$ for $|s| \leq \varepsilon$.

Let

$$V'_n = \{\lambda \in V' : |(1/s)\langle X_{t+s} - X_t, \lambda\rangle| \le n \text{ for } |s| \le \varepsilon\}.$$

Then V'_n is closed for $n = 1, 2,$ and $V' = \bigcup_n V'_n$ so by the Baire Category Theorem there exists an $m \in \mathbf{Z}^+$ such that V'_m contains a ball $B_\delta(\lambda_0)$. Thus

$$\|\lambda\| < \delta \Rightarrow |(1/s)\langle X_{t+s} - X_t, \lambda + \lambda_0\rangle| \le m \qquad \text{for} \quad |s| \le \varepsilon,$$

which implies that the mapping $t \to X_t$ is continuous from I to $(V')'$. Since

$$\begin{aligned}\frac{1}{h}\langle \hat{x}_{t+h} - \hat{x}_t, \lambda\rangle &= \frac{1}{h}\int_t^{t+h} \frac{d}{ds}\langle \hat{x}_s, \lambda\rangle\, ds \\ &= \frac{1}{h}\int_t^{t+h} \langle X_s, \lambda\rangle\, ds = \left\langle \frac{1}{h}\int_t^{t+h} X_s\, ds, \lambda\right\rangle\end{aligned}$$

we see that

$$\frac{1}{h}(\hat{x}_{t+h} - \hat{x}_t) - X_t = \frac{1}{h}\int_t^{t+h} (X_s - X_t)\, ds \to 0$$

in the norm on $(V')'$ as $h \to 0$. But $\hat{V}$ is closed in $(V')'$, so $X_t \in \hat{V}$. Since the injection $v \to \hat{v}$ of V in $(V')'$ is isometric there is a unique continuous mapping $t \to \xi_t$ of I into V such that $\hat{\xi}_t = X_t$. This shows that $t \to x_t$ is of class C^1.

(ii) Let $X_1, \ldots, X_n$ be a basis for the Lie algebra $\mathfrak{g}$ of G. If N_e is a suitable neighborhood of e in G the inverse of the mapping $(t_1, \ldots, t_n) \to \exp(t_1X_1)\cdots\exp(t_nX_n)$ is a local coordinate system on N_e. Assuming $x \to \langle\pi(x)v, \lambda\rangle$ C^∞ on G for each $\lambda \in V'$ we see from (i) that the map $(t_1, \ldots, t_n) \to \pi(\exp(t_1X_1)\cdots\exp(t_nX_n))v$ of $\mathbf{R}^n$ into V has continuous partial derivatives of any order.

A.5. (i) As an orbit $\pi(G)\cdot v$ the image $I(G/K)$ is a submanifold of V and the injectivity is clear from the maximality of K ([DS], Chapter VI, Exercise A3).

(ii) Let ϕ be a G-finite function on G/K, V_ϕ the space of complex linear combinations of translates $\phi^{\tau(g)}$, T the natural representation of G on V_ϕ, and $T^{\mathbf{C}}$ the extension of T to a representation of $\mathfrak{g}^{\mathbf{C}}$ and $G^{\mathbf{C}}$ on V_ϕ. We have $T(g)\psi = \psi^{\tau(g)}$, $\psi \in V_\phi$. Define $\phi^{\mathbf{C}}$ on $G^{\mathbf{C}}$ by $\phi^{\mathbf{C}}(x) = (T^{\mathbf{C}}(x^{-1})\phi)(o)$ where o is the origin in G/K. Then $\phi^{\mathbf{C}}(g) = \phi(gK)$ for $g \in G$. Also $\phi^{\mathbf{C}}$ is a $G^{\mathbf{C}}$-finite function on $G^{\mathbf{C}}$. [If $\phi = e_1, \ldots, e_n$ is a basis of V_ϕ and $T^{\mathbf{C}}(x)e_j = \sum_i T_{ij}(x)e_i$ then $\phi^{\mathbf{C}}(x) = \sum_i T_{i1}(x^{-1})e_i(o)$.].

Let $k \in K$. Then if $x \in G^{\mathbf{C}}$,

$$\phi^{\mathbf{C}}(xk) = (T^{\mathbf{C}}(k^{-1}x^{-1})\phi)(o) = (T(k^{-1})T^{\mathbf{C}}(x^{-1})\phi)(o),$$

so $\phi^{C}(xk) = \phi^{C}(x)$. In particular, we can define ϕ_0 on U/K by $\phi_0(uk) = \phi^{C}(u)$. Then ϕ_0 is K-finite. Denoting by I_0 the injection $uK \to \pi(u)v$ of U/K into V_0 we have by Introduction, Exercise A1 $\phi_0 = p_0 \circ I_0$ where p_0 is a polynomial function on V_0. If p is the canonical extension of p_0 to a (holomorphic) polynomial on V put $\psi = p \circ I$. Since ψ is a G-finite function on G/K we can repeat the above process, replacing ϕ by ψ. Then $\phi_0 = \psi_0$ so $\phi^C = \psi^C$ by holomorphic continuation.

A.6. If $D \in \boldsymbol{D}(K)$ let as in Chapter II, §5 ε_D denote the distribution with support $\{e\}$ defined by $\varepsilon_D(f) = (D\check{f})(e)$. The mapping $D \to \varepsilon_D$ then identifies $\boldsymbol{D}(K)$ with a subspace of $\mathscr{D}'(K)$. Each element ϕ in the space $\mathscr{A}(K)$ of analytic functions on K gives a linear functional $\tau_\phi \colon T \to T(\phi)$ on $\mathscr{D}'(K)$; we assign to $\mathscr{D}'(K)$ the topology $\sigma(\mathscr{D}'(K), \mathscr{A}(K))$, i.e., the weakest topology making all these functionals continuous. From general theory it is then known that the dual of $\mathscr{D}'(K)$ for this topology consists just of the functionals τ_ϕ ($\phi \in \mathscr{A}(K)$). Since no such functional annihilates the subspace $\boldsymbol{D}(K) \subset \mathscr{D}'(K)$ (by Taylor's formula [DS], Chapter II, (6) §1) we conclude that $\boldsymbol{D}(K)$ is a dense subspace of $\mathscr{D}'(K)$. Second, by the definition of the topology $\sigma(\mathscr{D}'(K), \mathscr{A}(K))$, the convolution $T \to T * \chi_\delta$ is a continuous mapping of $\mathscr{D}'(K)$ into itself. Thus $\boldsymbol{D}(K)\chi_\delta$ is dense in $\mathscr{D}'(K) * \chi_{\check{\delta}}$ which, however, by Corollary 1.8, equals the finite-dimensional space H_δ.

B. Spherical Functions

B.1. We have by Lemma 6.5 $\rho(H(ak)) \leq \rho(\log a)$ for $a \in A^+$ so $e^{-\rho(\log a)} \leq \phi_0(a)$. Next put $\psi(a) = e^{\rho(\log a)}\phi_0(a)$ ($a \in A$). Let $b \in A$. Then using Lemma 5.19 of Chapter I we find [since $H(abk) = H(ak(bk)) + H(bk)$]

$$\phi_0(a) = \int_K e^{-\rho(H(ak))}\, dk = \int_K e^{-\rho(H(abk))} e^{-\rho(H(bk))}\, dk$$

so

$$\psi(a) \leq \psi(ab) \qquad \text{if} \quad a \in A, \quad b \in \overline{A^+}. \tag{1}$$

Select $H_0 \in \mathfrak{a}^+$ such that $\alpha_i(H_0) = 1$ ($1 \leq i \leq l$) and put for $t > 0$

$$F(t) = \psi(\exp tH_0) = ce^{\rho(tH_0)}\{\partial(\pi)_\lambda(\pi(\lambda)\phi_\lambda(\exp tH_0))\}_{\lambda=0}.$$

Then

$$F(t) = \sum_{\mu \in \Lambda} p_\mu(t) e^{-t\mu(H_0)},$$

where p_μ is a polynomial of degree $\leq d$. Using the uniform convergence for $1 \leq t < \infty$; this implies $|F(t)| \leq a_0(1 + t)^d$ for $t \geq 0$, a_0 being a constant.

Let $H \in \mathfrak{a}^+$ and put $\beta(H) = \max_i \alpha_i(H)$. Then $\beta(H)H_0 - H \in \overline{\mathfrak{a}^+}$, so inequality (1) above implies

$$\psi(\exp H) \le F(\beta(H)) \le a_0(1 + \beta(H))^d \le a(1 + |H|)^d$$

as desired.

B.2. If $f \in \mathscr{D}(\mathfrak{a})$ we can write it as a Fourier transform

$$f(H) = \int_{\mathfrak{a}^*} g(\lambda)e^{-i\lambda(H)}\, d\lambda$$

and then our formula comes from Eq. (38) of §6 if we ignore problems of convergence. The problem is that the integral for $c_s(\lambda)$ is not absolutely convergent for $\lambda \in \mathfrak{a}^*$. One way to overcome the difficulty is to retrace the steps in the proof of the product formula (33). A simpler way is to shift the integration in the formula for f and write

$$f(H) = \int_{\mathfrak{a}^* - i\rho} g(\lambda)e^{-i\lambda(H)}\, d\lambda,$$

because then the absolute convergence of (36) is guaranteed by (37).

B.3. We know from Eq. (34) of §6 that $\rho(A_\alpha) = \rho_\alpha(A_\alpha)$ if α is one of the simple roots $\alpha_1, \dots, \alpha_l$. For $\beta \in \Sigma_0^+$ put as in §5, $\beta = \Sigma_1^l\, m_i \alpha_i$ and $m(\beta) = \Sigma_i\, m_i$. Assume now β not simple and select an i such that $\langle \beta, \alpha_i \rangle > 0$. Then the root $\gamma = s_{\alpha_i}\beta = \beta - c_i\alpha_i$ $(c_i \in \boldsymbol{Z}^+)$ satisfies $m(\gamma) < m(\beta)$ and $\gamma \in \Sigma_0^+$. We have $\rho(A_\beta) \ge \rho(A_\gamma)$. After finitely many steps we get $\rho(A_\beta) \ge \rho(A_{s\beta})$ where $s \in W$ and $s\beta$ is simple. By the first part of the proof, $\rho(A_{s\beta}) = \rho_{s\beta}(A_{s\beta}) = \rho_\beta(A_\beta)$ (cf. Kostant [1975a]).

B.4. Using B2 for $\overline{N}_s = \overline{N}$, $p = d$, $f(H) = [1 + \rho(H)]^{d+\varepsilon}$ we have by B3,

$$1 + \rho(H(\bar{n}_1) + \cdots + H(\bar{n}_d)) \ge 1 + \rho_1(H(\bar{n}_1)) + \cdots + \rho_d(H(\bar{n}_d))$$

so

$$[1 + \rho(H(\bar{n}_1) + \cdots + H(\bar{n}_d))]^{d+\varepsilon} \ge \{[1 + \rho_1(H(\bar{n}_1))] \cdots [1 + \rho_d(H(\bar{n}_d))]\}^{1+\varepsilon/d},$$

which reduces to result to the rank-one case.

B.5. A corollary of Theorem 4.3.

B.7. Writing $\eta = -i\lambda$ we have

$$\phi_\lambda(g) = \int_K e^{\rho(A(kg))}e^{-\eta(A(kg))}\, dk.$$

Here we can replace $e^{-\eta(A(kg))}$ by the average $w^{-1}\sum_{s\in W} e^{-s\eta(A(kg))}$. However, if $H \in \mathfrak{a}$, $\sum_{s\in W} s\cdot H$ is fixed under each s_α and hence is 0. By convexity of e^x, $\sum_{i=1}^n e^{a_i} \geq n$ whenever $\sum_i a_i = 0$. Thus

$$\sum_{s\in W} e^{-s\eta(A(kg))} \geq w,$$

giving the desired result.

B.8. We have $2\rho = (m_\alpha + 2m_{2\alpha})\alpha$ and $\langle\alpha, \alpha\rangle = \frac{1}{2}(m_\alpha + 4m_{2\alpha})^{-1}$. From Eq. (8) of §5 we have

$$2(m_\alpha + 4m_{2\alpha})\Delta(L_X) = \frac{d^2}{dt^2} + \{m_\alpha \coth t + 2m_{2\alpha}\coth 2t\}\frac{d}{dt}.$$

Also

$$\langle\lambda, \lambda\rangle + \langle\rho, \rho\rangle = \tfrac{1}{2}[1/(m_\alpha + 4m_{2\alpha})][\langle\lambda, \alpha_0\rangle^2 + (\tfrac{1}{2}m_\alpha + m_{2\alpha})^2]$$

and the rest follows by direct computation.

B.9. (i) Let $f^*(g) = \overline{f(g^{-1})}$. Then since $\phi_{-\lambda}(g) \equiv \phi_\lambda(g^{-1})$ we have $(f^*)^{\sim}(\lambda) = \overline{\tilde{f}(\lambda)}$ for $\lambda \in \mathfrak{a}^*$. Thus if $\lambda \in \mathfrak{a}^*$

$$\int_G\int_G \phi_\lambda(h^{-1}g)f(h)\overline{f(g)}\,dh\,dg = \int_G \phi_{-\lambda}(g)(f^* * f)(g)\,dg = |\tilde{f}(\lambda)|^2$$

so ϕ_λ is positive definite.

B.10. Considering the group $\mathrm{Ad}_{MN}(M)$ acting on $\mathfrak{n}$ we see from Chapter II, Theorem 4.9, that with the identification $MN/M = N$ the algebra $\boldsymbol{D}(MN/M)$ is generated by the two commuting operators $\tilde{X}^2 + \tilde{Y}^2$ and $\tilde{Z}$ ($\tilde{V}$ being the left invariant vector field on N generated by $V \in \mathfrak{n}$). Putting for $F \in \mathscr{E}(N)$,

$$f(x, y, z) = F(\exp(xX + yY + zZ))$$

we have, by [DS], Chapter II, Lemma 1.8, since

$$[X, Y] = 2Z, \qquad [X, Z] = [Y, Z] = 0,$$

$$(\tilde{X}F)(n) = \frac{\partial f}{\partial x} - y\frac{\partial f}{\partial z}, \qquad (\tilde{Y}F)(n) = \frac{\partial f}{\partial y} + x\frac{\partial f}{\partial z}, \qquad (\tilde{Z}F)(n) = \frac{\partial f}{\partial z},$$

whence

$$(\tilde{X}^2 + \tilde{Y}^2)F(n) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + (x^2 + y^2)\frac{\partial^2 f}{\partial z^2} + 2x\frac{\partial^2 f}{\partial y\,\partial z} - 2y\frac{\partial^2 f}{\partial x\,\partial z}.$$

Thus the spherical functions on $N = MN/M$ are given by

$$f(x, y, z) = \phi(x^2 + y^2)e^{\alpha z},$$

where $\alpha \in \boldsymbol{C}$ and ϕ satisfies the differential equation

$$\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} + r^2\alpha^2\phi = -\lambda^2\phi$$

for some $\lambda \in \boldsymbol{C}$.

This example and its generalization to G of real rank one are given in Korányi [1980]; see also Kaplan and Putz [1977].

B.12. Let $\lambda_0 = \xi_0 + i\eta_0$ $(\xi_0, \eta_0 \in \mathfrak{a}^*)$ be such that $-A_{\eta_0} \in \overline{\mathfrak{a}^+}$ and ϕ_{λ_0} is bounded. We have to prove $\eta_0 + \rho$ nonnegative on $\mathfrak{a}^+$. Let $U \subset V \subset W$ be the subgroups of the Weyl group W leaving fixed λ_0 and η_0, respectively. We can assume that for the lexicographic ordering of $\mathfrak{a}^*$ defined by means of the simple roots $\alpha_1, \ldots, \alpha_l$ we have $\xi_0 \geq s\xi_0$ for $s \in V$. In particular, $\alpha(A_{\xi_0}) \geq 0$ for $\alpha \in \Sigma^+$ satisfying $\alpha(A_{\eta_0}) = 0$.

Lemma. *The subgroup U of W leaving λ_0 fixed is generated by the refections s_{α_i} where α_i is a simple root vanishing at A_{λ_0}.*

For this let U' denote the subgroup of U generated by these s_{α_i}. The group U is generated by s_α as α runs through the roots, vanishing at A_{λ_0} ([DS], Chapter VII, Theorem 2.15). For each such $\alpha > 0$ we shall prove $\alpha = s\alpha_p$ where $s \in U'$ and α_p is a simple root vanishing at A_{λ_0}. We prove this by induction on $\sum m_i$ if $\alpha = \sum m_i\alpha_i$ $(m_i \in \boldsymbol{Z}^+ - \{0\})$. The statement is clear if $\sum m_i = 1$ so assume $\sum m_i > 1$. Since $\langle \alpha, \alpha \rangle > 0$ we have $\langle \alpha, \alpha_k \rangle > 0$ for some k among the indices i above. Then $s_{\alpha_k}\alpha \in \Sigma^+$ so $s_{\alpha_k}\alpha = \sum m'_j\alpha_j$ $(m'_j \in \boldsymbol{Z}^+)$ and $\sum m'_j < \sum m_i$. Now $\alpha(A_{\lambda_0}) = 0$ so $\alpha_i(A_{\eta_0}) = 0$ for each i above, whence $\alpha_i(A_{\xi_0}) \geq 0$ by our convention. But then $\alpha(A_{\lambda_0}) = 0$ implies $\alpha_i(A_{\xi_0}) = 0$. In particular, $s_{\alpha_k} \in U$. Thus the induction assumption applies to $s_{\alpha_k}\alpha$, giving a $s' \in U'$ such that $s_{\alpha_k}\alpha = s'\alpha_p$. Hence $\alpha = s\alpha_p$ with $s \in U'$. But then $s_\alpha = ss_{\alpha_p}s^{-1}$, proving the lemma.

Let π' denote the product of the positive roots vanishing on λ_0, π'' the product of the remaining positive roots. We multiply the formula (Theorem 5.7) for $\phi_\lambda(a)$ by $\pi'(\lambda)$, then apply the differential operator $\partial(\pi')$ on $\mathfrak{a}_C^*$ and evaluate at $\lambda = \lambda_0$. This gives the formula

$$\sum_{s \in w} (\det s)e^{s\rho(\log a)}\phi_{\lambda_0}(a) = c \sum_{s \in W} p_s(\log a)e^{is\lambda_0(\log a)},$$

with $c \in \boldsymbol{C}$, the p_s being certain polynomials. Now write the sum on the right in the form

$$e^{-\eta_0(\log a)}\left(\sum_{V/U} e^{is\xi_0(\log a)} \sum_{t \in U} p_{st}(\log a)\right) + \sum_{s \in W - V} p_s(\log a)e^{is\lambda_0(\log a)}.$$

Using the lemma above we see that $\pi'(sH) = (\det s)\pi'(H)$ for $s \in U$. Examining the definition of p_s, i.e.,

$$p_s(\log a) = \{\partial(\pi')_\lambda[(\det s)\pi''(\lambda)^{-1}e^{is\lambda(\log a)}]\}_{\lambda=\lambda_0}e^{-is\lambda_0(\log a)},$$

we see that the highest-degree term in $\sum_{s \in U} p_s(\log a)$ is a nonzero constant multiple of $\pi'(\log a)$. Hence

$$\sum_{t \in U} p_t(\log a) \not\equiv 0.$$

Thus the expression $q(a)$ inside the large parentheses above is nonzero (cf. Chapter I, Exercise D5). We have

$$\left|\sum_{s \in W} p_s(\log a)e^{is\lambda_0(\log a)}\right|$$

$$\geq e^{-\eta_0(\log a)}\left| |q(a)| - \sum_{s \in W-V} |p_s(\log a)|e^{(\eta_0 - s\eta_0)(\log a)}\right|$$

and

$$\left|\sum_{s \in W} (\det s)e^{s\rho(\log a)}\right| \leq e^{\rho(\log a)}\left(1 + \sum_{s \neq e} e^{(s\rho-\rho)(\log a)}\right).$$

Now $(\eta_0 - s\eta_0)(H) < 0$ for $H \in \mathfrak{a}^+$, and $s \in W - V$, and of course $s\rho - \rho < 0$ on $\mathfrak{a}^+$ if $s \neq e$. Fix a unit vector $H \in \mathfrak{a}^+$ such that $(s\xi_0)(H) \neq \xi_0(H)$ for $s \in V - U$ and write

$$q(\exp tH) = \sum_j q_j(t)e^{ir_jt},$$

where the r_j are different real numbers and the q_j are polynomials. The q_j which corresponds to $r_j = \xi_0(H)$ equals $\sum_{s \in U} p_s(\exp tH)$ whose leading term is $\pi'(tH)$ (up to a constant factor). By Chapter I, Exercise D5, the function $t \to q(\exp tH)$ is unbounded. The inequalities above, together with the boundedness of ϕ_{λ_0}, thus imply that $e^{-(\eta_0+\rho)(tH)}$ is bounded for $t \geq 0$; thus $\eta_0 + \rho \geq 0$ on a dense subset of $\mathfrak{a}^+$, hence on all of $\mathfrak{a}^+$.

B.13. For $x \in G$, consider the mean value operator M^x given by

$$(M^xf)(g) = \int_K f(gkx)\,dk$$

for f in the space

$$C_K(G) = \{f \in C(G) : f(gk) = f(g) \quad \text{for} \quad g \in G, k \in K\}.$$

Let $\mathfrak{g}$ be the Lie algebra of G. For $X \in \mathfrak{g}$, $m \in \boldsymbol{Z}^+$, let D_m^X denote the operator

$$D_m^X = \int_K \mathrm{Ad}(k)\tilde{X}^m\,dk \in \boldsymbol{D}_K(G).$$

Let $f \in C_K(G)$ be analytic. Then if $X \in \mathfrak{g}$ is sufficiently small, the formula

$$M^{\exp X} f = \sum_0^\infty \frac{1}{m!} D_m^X f$$

holds on a neighborhood of e in G (cf. Helgason [1957a, 1962a], Chapter X). Assuming (i) we thus derive the relation

$$(M^x M^y f)(e) = (M^y M^x f)(e)$$

for x, y near e, hence for all $x, y \in G$ by analyticity. Using approximation by analytic functions (loc. cit. Chapter X, Lemma 7.9) the relation is seen to hold for all $f \in C_K(G)$. It follows that if $f \in C_c^\natural(G)$, then

$$\int_K f(xky^{-1})\, dk = \int_K f(y^{-1}kx)\, dk.$$

Integrating this relation against $g(y)$ ($g \in C_c^\natural(G)$) and using the unimodularity we get $f * g = g * f$, proving (ii).

Conversely, assume (ii) holds and let $D, E \in \boldsymbol{D}_K(G)$, $f, g \in \mathscr{D}^\natural(G)$. Then since D is left invariant, $D(f * g) = f * Dg = Dg * f$, so

$$DE(f * g) = D(Eg * f) = Eg * Df = Df * Eg,$$

$$ED(f * g) = ED(g * f) = Eg * Df.$$

Thus E and D commute on the subspace $\mathscr{D}^\natural * \mathscr{D}^\natural$ of $\mathscr{D}^\natural$, hence, by density, on all of $\mathscr{D}^\natural$.

Now let $F \in \mathscr{D}(G) \cap C_K(G)$ and consider

$$F^*(x, y) = (M^x F)(y).$$

Let subscript 1 denote differentiation with respect to the first variable. Since F^* is bi-invariant under K in the first variable, we have

$$(D_1 F^*)(x, y) = \int_K (DF)(ykx)\, dk = (DF)^*(x, y),$$

so

$$(DF)(y) = (D_1 F^*)(e, y).$$

Hence

$$\begin{aligned}(DEF)(y) &= (D_1(EF)^*)(e, y) = (D_1 E_1 F^*)(e, y)\\ &= (E_1 D_1 F^*)(e, y) = (EDF)(y),\end{aligned}$$

so (i) holds. Part (ii) $\Rightarrow$ (i) is from Helgason [1979]; part (i) $\Rightarrow$ (ii) was noticed independently by Thomas [1984].

B.14. The adjoint representation of U is irreducible and the multiplicity of the 0 weight (Chapter V, §1) equals dim T. Now the result follows from B.13 and Theorem 3.5, Chapter V.

B.16. Assuming $L\colon C_c^\natural(G) \to C$ to be a continuous surjective homomorphism we have, by the definition of a measure, a measure μ on G, bi-invariant under K such that

$$L(f) = \int_G f(x)\, d\mu(x), \qquad f \in C_c^\natural(G).$$

Take $g \in C_c^\natural(G)$ such that $L(g) = 1$. Then

$$L(f) = L(f)L(g) = L(f * g) = \int_G f * g(x)\, d\mu(x)$$
$$= \int_G f(x)\mu * \check{g}(x)\, dx$$

so $\mu = \mu * \check{g}$ is a continuous function which by Lemma 3.2 is spherical.

B.17. Let r be the maximal rank of the matrices $(\Phi_i(x_j))_{i,j}$ for all sets of $x_j \in S$, and choose $x_1, \ldots, x_r$ such that $(\Phi_i(x_j))_{i,j}$ has rank r. If $r < w$ there exists a nonzero w-tuple $(\lambda_1, \ldots, \lambda_w)$ depending on $x_1, \ldots, x_r$ such that if $v_j = (\Phi_1(x_j), \ldots, \Phi_w(x_j))$ then

$$\sum_{i=1}^{w} \lambda_i \Phi_i(x_j) = 0, \qquad 1 \le j \le r.$$

Let $x \in S$ be arbitrary and put $v = (\Phi_1(x), \ldots, \Phi_w(x))$. Since the matrix formed by the column vectors $v_1, \ldots, v_r, v$ has rank r (by the maximality) there exist numbers $a_j(x)$ such that $v = \sum_1^r a_j(x)v_j$. Then, by the above,

$$\sum_{i=1}^{w} \lambda_i \Phi_i(x) = \sum_{i=1}^{w} \lambda_i \sum_{j=1}^{r} a_j(x)\Phi_i(x_j) = 0.$$

Since $x \in S$ is arbitrary, this is a contradiction.

C. Spherical Transforms

C.1. (i) Let $\lambda = \xi + i\eta$ ($\xi, \eta \in \mathfrak{a}^*$) be such that ϕ_λ is bounded. For $F \in L^\natural(G)$ we have

$$\infty > \int_G |F(g)|\, dg = \int_{KAN} |F(kan)| e^{2\rho(\log a)}\, dk\, da\, dn$$
$$= \int_A e^{2\rho(\log a)} \left(\int_N |F(an)|\, dn \right) da,$$

so $\int_N F(an)\, dn$ exists for almost all $a \in A$. Also $\phi_{i\eta}$ is bounded by Theorem 8.1, so

$$\int_K \int_G |F(g) e^{(i\lambda - \rho)(H(gk))}|\, dg\, dk = \int_G |F(g)| \phi_{i\eta}(g)\, dg < \infty,$$

whence by Fubini, whenever ϕ_λ is bounded,

$$(*)\qquad \int_G F(g)\phi_\lambda(g)\,dg = \int_G F(g)e^{(i\lambda-\rho)(H(g))}\,dg = \int_A \phi_F(a)e^{i\lambda(\log a)}\,da,$$

where

$$\phi_F(a) = e^{\rho(\log a)}\int_N F(an)\,dn.$$

Replacing F by $|F|$ and taking $\lambda = 0$ in $(*)$ we conclude that $\phi_{|F|} \in L^1(A)$; hence $\phi_F \in L^1(A)$. The assumption $\tilde{F}(\lambda) = 0$ $(\lambda \in \mathfrak{a}^*)$ and the injectivity of the Fourier transform on $L^1(A)$ then imply $\phi_F \equiv 0$ almost everywhere on A. But then $(*)$ together with Prop. 3.8 implies that $F \equiv 0$ almost everywhere on G. The argument also gives part (ii).

C.2. Clearly $\tilde{T}$ is W-invariant. Also since T is K-invariant

$$\tilde{T}(\lambda) = T^*(A_\lambda)$$

where

$$T^*(Y) = \int_{\mathfrak{p}} e^{-iB(Y,Z)}\,dT(Z).$$

By the Paley–Wiener theorem for $\mathscr{E}'(\mathfrak{p})$ (cf. Hörmander [1963]) T^* and therefore $\tilde{T}$ satisfies the desired inequality.

Conversely, suppose Φ satisfies the stated conditions. Consider the K-invariant distribution T on $\mathfrak{p}$ defined by (cf. (2) §9)

$$T(f) = \int_{\mathfrak{a}^*} \tilde{f^\natural}(\lambda)\Phi(\lambda)\delta(\lambda)\,d\lambda, \qquad f \in \mathscr{D}(\mathfrak{p}),$$

where $f^\natural(Z) = \int f(k\cdot Z)\,dk$. Let $\eta_\varepsilon \in \mathscr{D}_K(\mathfrak{p})$ have support in $B_\varepsilon(0)$. Then $\Phi(\lambda)\tilde{\eta}_\varepsilon(\lambda)$ is entire of exponential type $\leq R + \varepsilon$ so by Theorem 9.2, $T * \eta_\varepsilon$ has support in $B_{R+\varepsilon}$. Letting $\varepsilon \to 0$, we see that $T \in \mathscr{E}'_K(\mathfrak{p})$.

C.3. (i) Writing $a_r = \exp rH$, $\alpha(H) = 1$, we have as in [DS], p. 417, $\bar{n}a_r = u_1 \exp A^+(\bar{n}a_r)u_2$ $(u_1, u_2 \in K)$, whence

$$a_r\theta(\bar{n})^{-1}\bar{n}a_r = u_2^{-1}\exp 2A^+(\bar{n}a_r)u_2,$$

giving

$$\operatorname{Tr}(a_r\theta(\bar{n})^{-1}\bar{n}a_r) = 2\operatorname{ch}(2\alpha[A^+(\bar{n}a_r)]) + 1,$$

and (i) follows by direct computation.

(ii) If α is the root in Σ^+ the formula reads, replacing $\operatorname{ch} x$ by $2\operatorname{ch}^2(\frac{1}{2}x) - 1$,

$$\operatorname{ch}^2(\tfrac{1}{2}\alpha[A^+(\bar{n}a)]) = \operatorname{ch}^2(\tfrac{1}{2}\alpha(\log \alpha)) + (1/16m_\alpha)e^{\alpha(\log a)}|X|^2.$$

If 2α is the root in Σ^+ the formula reads, writing $\beta = 2\alpha$,

$$\operatorname{ch}^2(\tfrac{1}{2}\beta[A^+(\bar{n}a)]) = \operatorname{ch}^2(\tfrac{1}{2}\beta(\log a)) + (1/16m_{2\alpha})e^{\beta(\log a)}|Y|^2.$$

which is the same formula.

(iii) See solution to B8.

(iv) The change of variables $X \to e^{(1/2)r}X$, $Y \to e^r Y$ cancels out the factor $e^{\rho(\log a)}$, so $(*)$ follows.

(v) Putting

$$\Phi(u) = \int_{\mathfrak{g}_{-2\alpha}} \phi(u^2 + \|Y\|^2)\,dY,$$

we have

$$\psi(v) = \int_{\mathfrak{g}_{-\alpha}} \Phi(v^2 + \|X\|^2)\,dX.$$

Introducing polar coordinates in $\mathfrak{g}_{-2\alpha}$ and then putting $t^2 = u^2 + \|Y\|^2$, we find

$$\Phi(u) = c_1 \int_u^\infty t\phi(t^2)(t^2 - u^2)^{(1/2)(m_{2\alpha}-2)}\,dt,$$

and similarly

$$\psi(v) = c_2 \int_v^\infty s\Phi(s^2)(s^2 - v^2)^{(1/2)(m_\alpha - 2)}\,ds,$$

c_1 and c_2 being constants.

(vi) These two integral transforms can be inverted by the method used in the proof of Theorem 2.6 of Chapter I. This gives the conclusions

$$\begin{aligned} \psi(v) &= 0 \qquad \text{for} \quad v > A, \\ \Rightarrow \Phi(s^2) &= 0 \qquad \text{for } s > A, \\ \Rightarrow \phi(t^2) &= 0 \qquad \text{for} \quad t > A^2. \end{aligned}$$

Now assume $F_f(a_r) = 0$ for $\|\log a_r\| > R$. Since $\|\log a_r\| = r$ we have $\psi(\sqrt{\operatorname{ch} r}) = 0$ for $r > R$, i.e., $\sqrt{\operatorname{ch} r} > \sqrt{\operatorname{ch} R}$. By the above, $\phi(t^2) = 0$ for $t > \operatorname{ch} R$, that is, $f(g) = 0$ if $\delta(o, g \cdot o) > R$.

C.4. Select $f \in \mathscr{D}^\natural(G)$ such that the function $H \to f(\exp H)$ on $\mathfrak{a}^+$ has support inside the set

$$\mathfrak{a}^+(M) = \{H \in \mathfrak{a}^+ : \alpha(H) > M \text{ for } \alpha \in \Sigma^+\},$$

$M > 0$ being fixed.

With $\delta(H) = \prod_{\alpha > 0} 2 \sinh \alpha(H)$, consider the function $f_\varepsilon \in \mathscr{D}^\natural(G)$ given by

$$f_\varepsilon(H)\delta^{1/2}(H) = \varepsilon^{(1/2)l} f(\varepsilon H)\delta^{1/2}(\varepsilon H), \qquad H \in \mathfrak{a}^+.$$

Then

$$\int_G |f_\varepsilon(g)|^2\, dg = \int_G |f(g)|^2\, dg.$$

Put $\chi(s, \lambda) = \mathbf{c}(-s\lambda)/\mathbf{c}(-\lambda)$ and

$$F_\varepsilon(\lambda) = \varepsilon^{-(1/2)l} \sum_{s\in W} \chi(s, \lambda) \int_{\mathfrak{a}^+} f(H)\delta^{1/2}(H)e^{-is\lambda(H)/\varepsilon}\, dH.$$

By the Euclidean Plancherel formula,

$$(1) \qquad \int_{\mathfrak{a}^*} |F_\varepsilon(\lambda)|^2\, d\lambda \to |W| \int_G |f(g)|^2\, dg \qquad \text{as} \quad \varepsilon \to 0.$$

On the other hand,

$$\tilde{f}_\varepsilon(\lambda) = \int_{\mathfrak{a}^+} f_\varepsilon(H)\phi_{-\lambda}(H)\, \delta(H)\, dH,$$

and, using the expansion for $\phi_{-\lambda}$ and the Euclidean Paley–Wiener theorem on $\mathfrak{a}$, it can be proved that

$$(2) \qquad \int_{\mathfrak{a}^*} |\tilde{f}_\varepsilon(\lambda)\mathbf{c}(-\lambda)^{-1} - F_\varepsilon(\lambda)|^2\, d\lambda \to 0 \qquad \text{as} \quad \varepsilon \to 0.$$

Since

$$c \int_G |f(g)|^2\, dg = \int_{\mathfrak{a}^*} |\tilde{f}_\varepsilon(\lambda)\mathbf{c}(\lambda)^{-1}|^2\, d\lambda$$

(1) and (2) would imply $c = |W|$.

This proof, with further details, was given by Rosenberg [1977]; in Harish-Chandra [1958b] (§15) the measures $d\lambda$ and da are regularly normalized, the measure dg is normalized by

$$\int f(g)\, dg = \int_{KAN} f(kan)e^{2\rho(\log a)}\, dk\, da\, dn,$$

and the Plancherel formula (Theorem 7.5) is proved with the same constant $c = |W|$. In particular, the two normalizations of dg considered above must agree. See also Mneimné [1983].

C.5. From the inversion formula we have

$$f(\exp H) = \frac{1}{|W|} \int_{\mathfrak{a}^*} \tilde{f}(\lambda)\phi_\lambda(\exp H)|\mathbf{c}(\lambda)|^{-2}\, d\lambda.$$

Using Theorem 5.7 we get for $H \in \mathfrak{a}$, $\varepsilon(s) = \det(s)$,

$$i^{-d}\pi(\rho) \sum_{s\in W} \varepsilon(s)e^{s\rho(H)}f(\exp H) = \int_{\mathfrak{a}^*} \tilde{f}(\lambda)\pi(\lambda)e^{i\lambda(H)}\, d\lambda,$$

and in this last integral we can shift the integration to the extremes of the tube $\mathfrak{a}^* + iC(\rho)$ where $C(\rho)$ is the convex hull of the orbit $W \cdot \rho$. This gives

$$i^{-d}\pi(\rho) \sum_{s \in W} \varepsilon(s) e^{s\rho(H)}(\Delta_0 f)(\exp H) = \int_{\mathfrak{a}^*} \psi^*(\lambda)\, \pi(\lambda) e^{i\lambda(H)}\, d\lambda$$

where

$$\psi^*(\lambda) = \pi(\lambda)^{-1} \sum_{s \in W} \tilde{f}(\lambda + is\rho)\pi(\lambda + is\rho).$$

Reintroducing the formula for ϕ_λ this gives the desired formula.

CHAPTER V

A. Representations

A.1. Apply Eq. (16) of §1 to the representation $\pi = \mathrm{Ad}_U$; then $\pi(L_U) = cI$, $\lambda = \delta$ and $\mathrm{Tr}(\pi(L_u)) = -c \dim \mathfrak{u}$ so $c = -1$.

A.2. (i) Use Proposition 6.10, Chapter IV. For (ii) suppose $\lambda \in \Lambda$ and $|\lambda| = |\rho|$. By Chapter I, Proposition 5.15(ii)

$$\sum_{s \in W} (\det s) e^{s\lambda} = \left(\sum_{s \in W} (\det s) e^{s\rho} \right) \sum_{\mu} a_\mu e^\mu.$$

By the W-invariance we have $a_\mu \neq 0$ for some μ with $H_\mu \in \overline{\mathfrak{t}^+}$. But then if λ_0 is the highest among $s\lambda$ ($s \in W$) we have $\lambda_0 = \rho + \mu$. Then $\langle \mu, \rho \rangle \geq 0$, so $|\lambda_0|^2 \geq |\rho|^2 + |\mu|^2$, which implies $\mu = 0$.

A.3. While the formula for $\dim(\pi)$ (Theorem 1.8) is obtained by comparing the lowest-degree terms in x, the present formula is obtained by considering the next power of x.

A.4. Clear from A1 and A3 if we also recall

$$\langle \mu, \nu \rangle = \sum_{\alpha \in \Delta} \langle \mu, \alpha \rangle \langle \nu, \alpha \rangle$$

([DS], Chapter III, Exercise C4).

A.5. As in the proof of Eq. (25), Chapter IV, §5, we have

$$\sum_{\mu \in \Lambda} \mathscr{P}(\mu) e^{-\mu} = \prod_{\alpha \in \Delta^+} (1 + e^{-\alpha} + e^{-2\alpha} + \cdots)$$
$$= \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{-1} = e^\rho \left(\sum_s (\det s) e^{s\rho} \right)^{-1}.$$

Hence

$$\chi_\pi = \sum_{\mu \in \Lambda(\pi)} m_\mu e^\mu = \sum_s (\det s) e^{s(\lambda+\rho)} \left(\sum_s (\det s) e^{s\rho} \right)^{-1}$$
$$= \sum_s (\det s) e^{s(\lambda+\rho)} \sum_{\nu \in \Lambda} \mathscr{P}(\nu) e^{-\nu-\rho},$$

so (i) follows by putting $\nu = s(\lambda + \rho) - (\mu + \rho)$ (cf. Cartier [1961]).

For (ii) note that by Eq. (16) of §1 and Prop. 3.12 of Chapter II we have

$$(1) \qquad \Delta(L_U)\left(\sum_{\mu \in \Lambda(\pi)} m_\mu e^\mu \right) = (\langle \rho, \rho \rangle - \langle \rho + \lambda, \rho + \lambda \rangle) \sum_{\mu \in \Lambda(\pi)} m_\mu e^\mu.$$

By Theorem 3.7 of Chapter II

$$\Delta(L_U) = L_T + \operatorname{grad}(\log \delta)$$

and

$$\delta = e^{2\rho} \prod_{\alpha \in \Lambda^+} (1 - e^{-\alpha})^2.$$

For $\nu \in \Lambda$ let H_ν be determined by $\langle H, H_\nu \rangle = \nu(H)$. Then as a differential operator on T, $H_\nu e^\mu = \langle \mu, \nu \rangle e^\mu$. Let $H_1, \ldots, H_l$ be a basis of $\mathfrak{t}$ such that $-\langle H_i, H_j \rangle = \delta_{ij}$. Then

$$\begin{aligned} \operatorname{grad}(\log \delta) &= \sum_i H_i(\log \delta) H_i \\ &= 2\rho(H_i)H_i + 2 \sum_i \sum_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{-1} e^{-\alpha} \alpha(H_i) H_i \\ &= -H_{2\rho} - 2 \sum_{\alpha \in \Delta^+} e^{-\alpha}(1 - e^{-\alpha})^{-1} H_\alpha, \end{aligned}$$

so

$$\Delta(L_U) = L_T - H_{2\rho} - 2\left(\sum_{\alpha \in \Delta^+, k \geq 1} e^{-k\alpha} \right) H_\alpha.$$

Now we have

$$\begin{aligned} L_T\left(\sum_\mu m_\mu e^\mu \right) &= -\sum_\mu m_\mu \langle \mu, \mu \rangle e^\mu, \\ -H_{2\rho}\left(\sum_\mu m_\mu e^\mu \right) &= -\sum m_\mu \langle \mu, 2\rho \rangle e^\mu, \\ -2 \sum_{\alpha, k} e^{-k\alpha} H_\alpha \left(\sum_\mu m_\mu e^\mu \right) &= -2 \sum_\mu \left(\sum_{\nu - k\alpha = \mu} m_\nu \langle \alpha, \nu \rangle \right) e^\mu. \end{aligned}$$

Substituting this into (1) we obtain the desired formula (ii) in analogy with Eq. (12), Chapter IV, §5.

For the last statement put $\lambda = 0$ in (i).

A.6. (i) Consider the mapping A from the proof of Theorem 1.8. Let $l_\nu \in \boldsymbol{Z}^+$ be determined by $\chi_\lambda \chi_\mu = \sum_{\nu \in \Lambda(+)} l_\nu \chi_\nu$. Then multiplying by $A(e^\rho)$ we get

$$\begin{aligned}\sum_{\nu \in \Lambda(+)} l_\nu A(e^{\nu+\rho}) &= \chi_\lambda A(e^{\mu+\rho})\\ &= \left(\sum_{\xi \in \Delta} m_\xi e^\xi\right) \sum_{s \in W} (\det s) e^{s(\mu+\rho)}\\ &= \sum_{\xi \in \Lambda} \left(\sum_{s \in W} (\det s) m_{\xi+\rho-s(\mu+\rho)}\right) e^{\xi+\rho}.\end{aligned}$$

Now $H_{\nu+\rho} \in \mathfrak{t}^+$ whereas $H_{s(\nu+\rho)} \notin \mathfrak{t}^+$ for $s \neq e$ in W. Thus l_ν is the coefficient to $e^{\nu+\rho}$ on the left. Hence

$$l_\nu = \sum_{s \in W} (\det s) m_{\nu+\rho-s(\mu+\rho)}$$

and now the result follows from A.5(i).

(ii) We have $\chi_\lambda = \sum_{\nu \in \Lambda(\pi_\lambda)} m_\nu e^\nu$ and a similar formula for χ_μ. Since $m_\lambda = m_\mu = 1$ the result follows by multiplication.

(iii) Let π denote the restriction of π_λ to the algebra $\mathfrak{g}^\alpha + \mathfrak{g}^{-\alpha} + \boldsymbol{C}H_\alpha$, which is isomorphic to $\mathfrak{sl}(2, \boldsymbol{C})$. Now apply Lemma 1.2 of the Appendix.

A.8. In fact $\chi_{J \otimes J} = (\chi_J)^2$ so by Lemma 2.6 for $A = I$,

$$\int_{\boldsymbol{U}(n)} |\chi_{J \otimes J}(u)|^2 = 2$$

and the statement follows.

A.9. The simple roots α_i $(1 \leq i \leq n)$ are given by (cf. [DS], Chapter III, §8)

$$\alpha_i(H) = e_i(H) - e_{i+1}(H),$$

where H is a diagonal matrix of trace 0 with diagonal elements $e_i(H)$. The fundamental weights ω_j satisfy $\langle \omega_j, \check{\alpha}_i \rangle = \delta_{ij}$ so since $\langle e_i, e_i \rangle = \frac{1}{2}(n+1)^{-1}$ we obtain $\omega_j = e_1 + \cdots + e_j$. If $v_1, \ldots, v_{n+1}$ is the standard basis of $V = \boldsymbol{C}^{n+1}$ the elements $v_{i_1} \wedge \cdots \wedge v_{i_r}$ $(i_1 < \cdots < i_r)$ form a basis of $\bigwedge^r V$. For $t \in T$ [maximal torus formed by the diagonal elements in $\boldsymbol{SU}(n+1)$] we have for $t = \exp H$

$$(\textstyle\bigwedge^r J)(t)(v_{i_1} \wedge \cdots \wedge v_{i_r}) = (e_{i_1} + \cdots + e_{i_r})(H) v_{i_1} \wedge \cdots \wedge v_{i_r}.$$

Thus $e_{i_1} + \cdots + e_{i_r}$ is a weight of $\bigwedge^r J$ and $\omega_r = e_1 + \cdots + e_r$ is the highest one. The reflection s_{α_i} interchanges e_i and e_{i+1} so the Weyl group W consists of all permutations of the e_i. Thus W permutes transitively the weights of $\bigwedge^r J$, so it must be irreducible.

The construction of the fundamental representations of the simple Lie algebras goes back to Cartan [1913]. For $\boldsymbol{Sp}(n)$ the results are similar; one has to replace $\bigwedge^r V$ by the subspace of primitive elements (in the sense of Corollary 2.2 of Chapter III); for $\boldsymbol{Spin}(2n+1)$ [the double covering of $\boldsymbol{SO}(2n+1)$] one needs in addition to the exterior powers $\bigwedge^r J$, the *spin-representation*; for $\boldsymbol{Spin}(2n)$ one needs in addition to the exterior powers $\bigwedge^r J$, the two *semi-spin-representations.* For detailed description see Weyl [1939]; Boerner [1955]; Bourbaki, *Groupes et algèbres de Lie*, Chapter 8.

A.10. With notation from §1 let $V = \sum V_\mu$, where μ runs through $\Lambda(\pi)$. If $v' \in V'$ is such that $v'(V_\mu) = 0$ for all $\mu \in \Lambda(\pi)$ except μ_0 then $\check{\pi}(H)v' = -\mu_0(H)v'$. Thus $-\mu_0 \in \Lambda(\check{\pi})$ and $\dim V'_{-\mu_0} = \dim V_{\mu_0}$. This proves (i). Part (ii) follows from Theorem 1.3. Part (iv) follows since $-s$ permutes the "co-roots" $\check{\alpha}_j$.

A.11. For $G = \boldsymbol{SL}(2, \boldsymbol{R})$ we have $\mathrm{Ad}(g)X = gXg^{-1}$. Since $M = \pm I$, $N = \left(\begin{smallmatrix} 1 & \boldsymbol{R} \\ 0 & 1 \end{smallmatrix}\right)$ we find quickly

$$K = \{g \in G : \mathrm{Ad}(g)X_1 = X_1\},$$
$$MN = \{g \in G : \mathrm{Ad}(g)(X_1 + X_3) = X_1 + X_3\},$$

so the two maps are injections. If

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then

$$g^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

so

$$\begin{aligned} \mathrm{Ad}(g)X_1 &= \tfrac{1}{2}(a^2 + b^2 + c^2 + d^2)X_1 - (bd + ac)X_2 \\ &\quad + \tfrac{1}{2}(a^2 + b^2 - c^2 - d^2)X_3, \\ \mathrm{Ad}(g)\tfrac{1}{2}(X_1 + X_3) &= \tfrac{1}{2}(a^2 + c^2)X_1 - acX_2 + \tfrac{1}{2}(a^2 - c^2)X_3, \end{aligned}$$

so (i) and (ii) follow easily. For (iii) consider the horocycle $N \cdot X_1 = \mathrm{Ad}(N)(X_1)$, which by the above is the section of G/K with the plane $x_1 = x_3 + 1$. Writing $Z_0 = \frac{1}{8}(X_1 + X_3)$ this has the form $B(X, Z_0) = -1$. Since G permutes the horocycles transitively, (iii) follows.

B. Fourier Series

B.1. Part (i) is clear from Lemmas 2.4 and 2.4′ and then (ii) follows from Theorem 2.9.

B.2. Let A be an operator on $\mathscr{H}_\lambda$. Then $A = PU$ where U is unitary and P is the unique nonnegative square root of AA^*. Then $P = \sum_1^d c_i P_i$ where $c_i \geq 0$; the c_i^2 are the eigenvalues of AA^* and the P_i are mutually orthogonal projections, $\sum_i P_i = I$. Let $2c = \max c_i$. Then we can write $c_i = \alpha_i + \beta_i$ $(1 \leq i \leq d)$ where $|\alpha_i| = |\beta_i| = c$ for all i (cf. Example to Theorem 2.3). Put

$$V = c^{-1}\left(\sum_i \alpha_i P_i\right)U, \qquad W = c^{-1}\left(\sum_i \beta_i P_i\right)U.$$

Then V and W are unitary and since $\|A\| = \|AA^*\|^{1/2} = 2c$ we have

$$A = \tfrac{1}{2}\|A\|(V + W).$$

This shows that the Fourier series of f can be multiplied on the left by any hyperfunction B, bounded in the sense of Eq. (44) of §2, giving a function $f_B \in L^1$. By the closed-graph theorem the map $B \to f_B$ is continuous, i.e., $\|f_B\| \leq C \sup_\lambda \|B_\lambda\|$ where C is a constant. Restricting B to Fourier coefficients of functions $g \in L^1$ we see that the mapping $g \to f * g$ is spectrally continuous so by Theorem 2.3, $f \in L^2(K)$.

Remark. This corollary of Theorem 2.3 (Helgason [1957]) has been extended by Figa-Talamanca and Rider [1967] so that the assumption is only required for a set of unitary hyperfunctions Γ of positive measure in the product group $\prod_{\lambda \in K} U(d_\lambda)$. They also proved [1966] a certain "converse" to Exercise B2, namely that if $f \in L^2(K)$ then for any $p < \infty$ there exists a unitary hyperfunction Γ such that $f_\Gamma \in L^p(K)$. For the circle group such results are classical (Littlewood [1924]; Zygmund [1959], Vol. II, §8.14).

B.4. (Sketch) Since $L = L_U$ satisfies

$$L(U_\lambda) = (\langle \rho, \rho \rangle - \langle \lambda + \rho, \lambda + \rho \rangle)U_\lambda$$

[because of Lemma 1.6 and (16)] we see from (3) §2 that the Fourier coefficients of $f \in \mathscr{E}(U)$ form a rapidly decreasing hyperfunction. Conversely, let $\{A_\lambda\}_{\lambda \in \Lambda(\pi)}$ be a rapidly decreasing hyperfunction and define $f \in C(U)$ by

$$f(u) = \sum_{\lambda \in \hat{u}} d_\lambda \operatorname{Tr}(A_\lambda U_\lambda(u)),$$

the series being clearly uniformly convergent. Let dU_λ be the differential of U_λ. If $H \in \mathfrak{t}$ then $dU_\lambda(H)$ is a diagonal matrix with diagonal formed by $\mu(H)$ as μ runs over the weights of U_λ. Thus by Theorem 1.3(iv)

$$\mathrm{Tr}(dU_\lambda(H)(dU_\lambda(H))^*) \leq d_\lambda |\lambda|^2 |H|^2,$$

and by the conjugacy theorem for $\mathfrak{u}$ this holds for H replaced by any $X \in \mathfrak{u}$. Since d_λ is majorized by a polynomial in λ (Theorem 1.8) this inequality implies quickly that the series for f remains uniformly convergent after repeated differentiation. Hence $f \in \mathscr{E}(U)$.

B.5. (i) By Exercise B1 we know that $f_S = f * \mu_S$, where μ_S is a measure with Fourier–Stieltjes series

$$\mu_S \sim \sum_{\lambda \in S} d_\lambda \,\mathrm{Tr}(U_\lambda(k))$$

to μ_S is a central idempotent. The converse is obvious. Under the assumption in (ii) μ_S is a positive measure so by a theorem of Wendel [1954], $\mu_S * \mu_S = \mu_S$ implies that μ_S is the Haar measure on a compact subgroup $H \subset K$. The Fourier–Stieltjes series above then shows that S equals the annihilator $H^\perp$ so by Proposition 2.11 we are finished, (cf. Helgason [1958]).

The central idempotent measures for abelian groups were determined (in increasing generality) by Helson [1953], Rudin [1959], and Cohen [1960]; for classes of nonabelian compact groups by Rider [1970, 1971] and Ragozin [1972] (this latter for all simple compact Lie groups).

APPENDIX

In this Appendix we collect some scattered results which are used in the text but which are nevertheless somewhat extraneous to the main themes. In §§1–2 we have primarily followed the treatment in Dixmier [1974], but we have also used some proofs from Chevalley [1955b], Jacobson [1953], and Iwahori [1959]. Proposition 2.12 is due to Jacobson [1951]. In §3 we follow to some extent the treatment in Zariski and Samuel [1958, 1960]; the proof of Theorem 3.5(ii) was kindly communicated to the author by Arthur Mattuck.

§1. The Finite-Dimensional Representations of $\mathfrak{sl}(2, C)$

In this section we write down for reference the finite-dimensional representations of $\mathfrak{sl}(2, C)$; these are familiar from most books on Lie algebras.

The elements

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

form a basis of $\mathfrak{sl}(2, C)$ satisfying

$$(1) \qquad [H, X] = 2X, \qquad [H, Y] = -2Y, \qquad [X, Y] = H.$$

Let $k \in Z^+$. Consider the linear mapping

$$\pi_k : \mathfrak{sl}(2, C) \to \mathfrak{gl}(k + 1, C)$$

given by

$$\pi_k(H) = \begin{pmatrix} k & & & \\ & k-2 & & \\ & & \ddots & \\ & & & -k \end{pmatrix} \qquad \text{(diagonal matrix)}$$

$$\pi_k(X) = \begin{pmatrix} 0 & \lambda_1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_2 & \cdots & 0 \\ & & & & \\ 0 & 0 & \cdots & 0 & \lambda_k \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \qquad \pi_k(Y) = \begin{pmatrix} 0 & 0 & \cdots & & 0 \\ 1 & 0 & \cdots & & 0 \\ & & & & \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

where $\lambda_j = j(k - j + 1)$, $1 \leq j \leq k$.

Proposition 1.1. *The mapping π_k is an irreducible representation of $\mathfrak{sl}(2, \boldsymbol{C})$ on $\boldsymbol{C}^{k+1}$.*

Proof. To see that π_k is a representation one just verifies directly that $\pi_k(H)$, $\pi_k(X)$, and $\pi_k(Y)$ satisfy the bracket relations in (1). For the irreducibility we can take $k > 0$. Suppose $0 \neq V \subset \boldsymbol{C}^{k+1}$ is an invariant subspace. Let $v \in V - \{0\}$. If $e_1, \ldots, e_{k+1}$ is the standard basis of $\boldsymbol{C}^{k+1}$ we have

$$\pi_k(H)e_i = (k - 2i + 2)e_i \qquad (1 \leq i \leq k + 1), \tag{2}$$

$$\pi_k(Y)e_i = e_{i+1} \quad (1 \leq i \leq k), \qquad \pi_k(Y)e_{k+1} = 0, \tag{3}$$

$$\pi_k(X)e_1 = 0, \qquad \pi_k(X)e_{i+1} = \lambda_i e_i \quad (1 \leq i \leq k). \tag{4}$$

Thus, if $n \in \boldsymbol{Z}^+$ is suitably chosen, $\pi_k(Y)^n v$ is a nonzero multiple of e_{k+1}, whence $e_{k+1} \in V$. Using (4) successively, we see that each $e_i \in V$, so $V = \boldsymbol{C}^{k+1}$.

Lemma 1.2. *Let π be a representation of $\mathfrak{sl}(2, \boldsymbol{C})$ on a finite-dimensional vector space V. Let $v_0 \neq 0$ be an eigenvector of $\pi(H)$, $\pi(H)v_0 = \lambda v_0$, $\lambda \in \boldsymbol{C}$. Then if $k \in \boldsymbol{Z}^+$,*

(i) $\pi(H)\pi(X)^k v_0 = (\lambda + 2k)\pi(x)^k v_0$,
$\pi(H)\pi(Y)^k v_0 = (\lambda - 2k)\pi(Y)^k v_0$.

(ii) *Suppose $\pi(X)v_0 = 0$. Putting*

$$v_i = \pi(Y)^i v_0, \quad i = 0, 1, 2, \ldots,$$

we have

$$\pi(X)v_i = i(\lambda - i + 1)v_{i-1} \quad (i \geq 1),$$

so if $v_{i_0} = 0$, $v_{i_0 - 1} \neq 0$, we have $\lambda = i_0 - 1$.

Proof. (i)

$$\pi(H)\pi(X)v_0 = \pi(X)\pi(H)v_0 + \pi([H, X])v_0 = (\lambda + 2)\pi(X)v_0$$

and the first formula follows by iteration; the other is proved in the same way. For (ii) we use induction on i. For $i = 1$, $\pi(X)\pi(Y)v_0 = \pi(Y)\pi(X)v_0 + \pi(H)v_0 = \lambda v_0$; assuming formula for i we have by the first part

$$\begin{aligned}\pi(X)v_{i+1} &= \pi(X)\pi(Y)v_i = \pi(Y)\pi(X)v_i + \pi(H)v_i \\ &= i(\lambda - i + 1)v_i + (\lambda - 2i)v_i = (i + 1)(\lambda - i)v_i.\end{aligned}$$

Theorem 1.3. *Let π be an irreducible representation of $\mathfrak{sl}(2, \mathbf{C})$ on a complex vector space V of dimension $k + 1$. Then π is equivalent to π_k; that is, $\pi_k(X)Av = A\pi(X)v$ ($v \in V$, $X \in \mathfrak{sl}(2, \mathbf{C})$) for a suitable linear bijection $A: V \to \mathbf{C}^{k+1}$.*

Proof. The endomorphism $\pi(H)$ of the complex vector space V has an eigenvector $v \neq 0$, say $\pi(H)v = \mu v$ ($\mu \in \mathbf{C}$). Then by Lemma 1.2(i)

$$\pi(H)\pi(X)^k v = (\mu + 2k)\pi(X)^k v$$

and since $\pi(H)$ has at most finitely many eigenvalues there exists a $k_0 \in \mathbf{Z}^+$ such that $\pi(X)^{k_0}v \neq 0$, $\pi(X)^{k_0+1}v = 0$. Put $v_0 = \pi(X)^{k_0}v$, $\lambda = \mu + 2k_0$, so

$$\pi(H)v_0 = \lambda v_0, \qquad \pi(X)v_0 = 0.$$

Put $v_i = \pi(Y)^i v_0$ ($i \geq 0$) and let v_l be the last nonzero v_i. The vectors $v_0, \ldots, v_l$ have different eigenvalues for $\pi(H)$ and the space $\sum_0^l \mathbf{C}v_i$ is invariant, hence equals V. Thus $l = k$.

Since $(k + 1)(\lambda - k)v_k = \pi(X)v_{k+1} = 0$ we conclude $\lambda = k$. Finally, using (2)–(4) we see that the linear mapping given by $v_0 \to e_1$, $v_1 \to e_2$, $\ldots, v_k \to e_{k+1}$ sets up an equivalence between π and π_k.

Corollary 1.4. *Let π be any finite-dimensional representation of $\mathfrak{sl}(2, \mathbf{C})$. Then*

(5) $\quad$ Range $\pi(X) \cap$ Kernel $\pi(Y) = 0 =$ Range $\pi(Y) \cap$ Kernel $\pi(X)$.

For the representation $\pi = \pi_k$ this is obvious from (3) and (4). Since the representation π is semisimple ([DS], Chapter III, Exercise B3) it decomposes into irreducibles which by Theorem 1.3 have the form π_k. This proves (5) in general.

Corollary 1.5. *Let π be any finite-dimensional representation of $\mathfrak{sl}(2, \mathbf{C})$ on a vector space V. Then*

(i) *The endomorphism $\pi(H)$ is semisimple and its eigenvalues are all integers. Let V_r be the eigenspace of $\pi(H)$ for the eigenvalue r.*

(ii) Kernel $\pi(X) \cap$ Range $\pi(X) \subset \sum_{r>0} V_r \subset$ Range $\pi(X)$.
(iii) Kernel $\pi(Y) \cap$ Range $\pi(Y) \subset \sum_{r<0} V_r \subset$ Range $\pi(Y)$.

Again it suffices to verify this in the case $\pi = \pi_k$ and there it is immediate from (2)–(4).

§2. Representations and Reductive Lie Algebras

1. Semisimple Representations

Let k be a field, $\mathfrak{g}$ a Lie algebra over k and V a vector space over k. *We assume $k = \mathbf{R}$ or $\mathbf{C}$ and that* $\dim \mathfrak{g}$ *and* $\dim V$ *are both finite*. We shall now recall some notions and results about *representations* of $\mathfrak{g}$ on V, i.e., homomorphisms of $\mathfrak{g}$ into $\mathfrak{gl}(V)$. An injective representation is usually called a *faithful representation.*

Let π_1 and π_2 be representations of $\mathfrak{g}$ on V_1 and V_2, respectively; π_1 and π_2 are said to be *equivalent* if there exists a linear bijection $A: V_1 \to V_2$ such that $\pi_2(X)Av = A\pi_1(X)v$ for $v \in V_1$, $X \in \mathfrak{g}$. If we form the direct sum $V = V_1 \oplus V_2$ we can define a representation π of $\mathfrak{g}$ on V by $\pi(X)(v_1 + v_2) = \pi_1(X)v_1 + \pi_2(X)v_2$; π is then called the *direct sum* of π_1 and π_2. This can obviously be extended to several representations π_i of $\mathfrak{g}$.

Given the representations $\pi_1: \mathfrak{g} \to \mathfrak{gl}(V_1)$, $\pi_2: \mathfrak{g} \to \mathfrak{gl}(V_2)$, the *tensor product* $\pi_1 \otimes \pi_2$ is the representation of $\mathfrak{g}$ on $V_1 \otimes V_2$ given by

$$\pi_1 \otimes \pi_2(X)(v_1 \otimes v_2) = (\pi_1(X)v_1) \otimes v_2 + v_1 \otimes (\pi_2(X)v_2),$$

$X \in \mathfrak{g}$, $v_1 \in V_1$, $v_2 \in V_2$. If K is a Lie group with Lie algebra $\mathfrak{k}$ and ρ_i a representation of K on V_i $(i = 1, 2)$ then the representation $\rho_1 \otimes \rho_2$ as defined in Chapter IV, §1, No. 2 has differential

$$d(\rho_1 \otimes \rho_2) = d\rho_1 \otimes d\rho_2.$$

Let π be a representation of $\mathfrak{g}$ on V; π is said to be *irreducible* if the only $\pi(\mathfrak{g})$-invariant subspaces of V are 0 and V; π is said to be *semisimple* (or *completely reducible*) if for each $\pi(\mathfrak{g})$-invariant subspace $W \subset V$ there exists a complementary invariant subspace W', i.e.,

$$V = W \oplus W', \qquad \pi(\mathfrak{g})W' \subset W'. \tag{1}$$

Proposition 2.1. *The representation π is semisimple if and only if it is the direct sum of irreducible representations.*

Proof. Suppose π is a semisimple representation of $\mathfrak{g}$ on V. If $W \subset V$ is an invariant subspace the *subrepresentation* of $\mathfrak{g}$ on W given by $X \to$

$\pi(X)|W$ is also semisimple. Using this remark on W' in (1) and iterating we obtain a direct decomposition $V = \bigoplus_{i=1}^{r} V_i$ where for each i the representation $X \to \pi(X)|V_i$ is irreducible.

Conversely, suppose we have such a direct decomposition of V. Suppose $W \subset V$ is any invariant subspace. Then either $W = V$ or some V_i, say V_{i_1}, is not contained in W. Put $W_1 = W + V_{i_1}$. By the irreducibility, $W \cap V_{i_1} = 0$ so we have the direct decomposition $W_1 = W \oplus V_{i_1}$. Repeating this argument with W_1 in place of W we obtain if $V_{i_2} \not\subset W_1$, and if we put $W_2 = W_1 + V_{i_2}$,

$$W_2 = W \oplus V_{i_1} \oplus V_{i_2}.$$

The process stops when each V_i belongs to a certain W_j, in which case

$$V = W_j = W \oplus V_{i_1} \oplus V_{i_2} \oplus \cdots \oplus V_{i_j}.$$

Then the sum $W' = \sum_{p=1}^{j} V_{i_p}$ satisfies (1).

Proposition 2.2. *Let π be a semisimple representation of $\mathfrak{g}$ on V. Let $V^{\mathfrak{g}}$ denote the joint null space, i.e.,*

$$V^{\mathfrak{g}} = \{v : \pi(X)v = 0 \quad \text{for all } X \in \mathfrak{g}\},$$

and let $[\pi(\mathfrak{g})V]$ denote the subspace of V generated by the vectors $\pi(X)v$, $X \in \mathfrak{g}$, $v \in V$. Then

$$V = V^{\mathfrak{g}} \oplus [\pi(\mathfrak{g})V]$$

and the subspace $[\pi(\mathfrak{g})V]$ is the only $\pi(\mathfrak{g})$-invariant subspace W complementary to $V^{\mathfrak{g}}$.

Proof. If $V = V^{\mathfrak{g}} \oplus W$, $\pi(\mathfrak{g})W \subset W$, we have for each $v \in V$, $X \in \mathfrak{g}$, $v = v_0 + w$ ($v_0 \in V^{\mathfrak{g}}$, $w \in W$), $\pi(X)v = \pi(X)w \in W$. Thus $[\pi(\mathfrak{g})V] \subset W$. Let $W_1 \subset W$ be a $\pi(\mathfrak{g})$-invariant subspace complementary to $[\pi(\mathfrak{g})V]$. If $w_1 \in W_1$, then $\pi(X)w_1 \in [\pi(\mathfrak{g})V] \cap W_1 = 0$ so $w_1 \in V^{\mathfrak{g}}$ whence $w_1 = 0$. Thus $W_1 = 0$ and $W = [\pi(\mathfrak{g})V]$.

Proposition 2.3. *Let π be a representation of the real Lie algebra $\mathfrak{g}$ on the real vector space V. Let $\pi^{\mathbf{C}}$ denote the induced representation of the complexification $\mathfrak{g}^{\mathbf{C}}$ on the complexified vector space $V^{\mathbf{C}}$, i.e.,*

$$\pi^{\mathbf{C}}(X_1 + iX_2)(v_1 + iv_2) = \pi(X_1)v_1 - \pi(X_2)v_2 + i[\pi(X_2)v_1 + \pi(X_1)v_2],$$

$X_j \in \mathfrak{g}$, $v_j \in V$. Then

(i) *π is semisimple $\Leftrightarrow \pi^{\mathbf{C}}$ is semisimple.*

(ii) *If π is irreducible then either $\pi^{\mathbf{C}}$ is irreducible or is the direct sum of two irreducible components.*

Proof. (i) $\Leftarrow$. Let $\sigma: V^{\mathbf{C}} \to V^{\mathbf{C}}$ denote the conjugation of $V^{\mathbf{C}}$ with respect to V. Let $W \subset V$ be a $\pi(\mathfrak{g})$-invariant subspace. Then $W^{\mathbf{C}} = W + iW$ is a $\pi^{\mathbf{C}}(\mathfrak{g}^{\mathbf{C}})$-invariant subspace of $V^{\mathbf{C}}$ so $W^{\mathbf{C}}$ has an invariant complement $Z' \subset V^{\mathbf{C}}$. Then the space

$$Z = (1 + \sigma)[Z' \cap (1 - \sigma)^{-1}(iW)]$$

satisfies

$$V = W \oplus Z, \qquad \pi(\mathfrak{g})Z \subset Z$$

(cf. [DS], Chapter III, solution to Exercise B3), so π is semisimple.

(i) $\Rightarrow$. For this we may assume, by Proposition 2.1, that π is irreducible and $\pi^{\mathbf{C}}$ not irreducible. Let $Z \subset V^{\mathbf{C}}$ be any $\pi^{\mathbf{C}}(\mathfrak{g}^{\mathbf{C}})$-invariant subspace, $Z \neq \{0\}$, $Z \neq V^{\mathbf{C}}$. Then

$$V^{\mathbf{C}} = Z + \sigma Z, \qquad Z \cap (\sigma Z) = \{0\}. \tag{2}$$

In fact, since $\sigma(Z + \sigma Z) = Z + \sigma Z$ we have $Z + \sigma Z = Y + iY$ where $Y = (Z + \sigma Z) \cap V$. Then $Y \neq 0$ is a $\pi(\mathfrak{g})$-invariant subspace of V so $Y = V$ and $V^{\mathbf{C}} = Z + \sigma Z$. Similarly, $Z \cap (\sigma Z) = \{0\}$ so (2) holds. Also σZ is a complex subspace of $V^{\mathbf{C}}$ since $\alpha\sigma(v) = \sigma(\bar{\alpha}v)$ for $\alpha \in \mathbf{C}$, $v \in V^{\mathbf{C}}$. Finally,

$$\pi^{\mathbf{C}}(X_1 + iX_2)\sigma v = \sigma\pi^{\mathbf{C}}(X_1 - iX_2)v \tag{3}$$

so σZ is $\pi^{\mathbf{C}}(\mathfrak{g}^{\mathbf{C}})$-invariant. Thus (2) implies that $\pi^{\mathbf{C}}$ is semisimple.

(ii) It suffices to show that the space Z in (2) above is irreducible under $\pi^{\mathbf{C}}(\mathfrak{g}^{\mathbf{C}})$. But if $U \subset Z$ is an invariant subspace, $U \neq 0$, $U \neq Z$, then (2) holds for it, which is a contradiction.

2. Nilpotent and Semisimple Elements

Generalizing the notion of nilpotent and semisimple linear transformations we shall now define such notions in a semisimple Lie algebra.

Lemma 2.4. *Let V be a vector space over k and $\mathfrak{g} = \mathfrak{gl}(V)$. Let $X \in \mathfrak{g}$ be nilpotent. Then $ad_{\mathfrak{g}}\, X$ is nilpotent.*

Proof. Since $(\mathrm{ad}_{\mathfrak{g}}\, X)Y = XY - YX$ we have

$$(\mathrm{ad}_{\mathfrak{g}}\, X)^p(Y) = \sum_{i=0}^{p} (-1)^i \binom{p}{i} X^{p-i}\, Y X^i,$$

so the lemma follows.

Lemma 2.5. *Let* $\mathfrak{g} = \mathfrak{gl}(n, k)$ *and let* $A \in \mathfrak{g}$ *be a diagonal matrix*

$$A = d_1 E_{11} + \cdots + d_n E_{nn}.$$

Then

$$\mathrm{ad}_{\mathfrak{g}} A(E_{ij}) = (d_i - d_j)E_{ij}$$

so in the basis (E_{ij}) *of* $\mathfrak{g}$, $ad_{\mathfrak{g}}\, A$ *is given by a diagonal matrix.*

Lemma 2.6. *Let* V *be a vector space over* k, A *an endomorphism of* V, *and* $A = S + N$ *its additive Jordan decomposition,* S *semisimple,* N *nilpotent, and* $SN = NS$. *Let* $\mathfrak{g} = \mathfrak{gl}(V)$. *Then*

$$\mathrm{ad}_{\mathfrak{g}}\, A = \mathrm{ad}_{\mathfrak{g}}\, S + \mathrm{ad}_{\mathfrak{g}}\, N \tag{4}$$

is the additive Jordan decomposition of $ad_{\mathfrak{g}}\, A$.

Proof. Passing to the complexifications in the case $k = \boldsymbol{R}$ we may assume $k = \boldsymbol{C}$. Then S is diagonalizable and so is $\mathrm{ad}_{\mathfrak{g}}\, S$ by Lemma 2.5. Also Lemma 2.4 implies that $\mathrm{ad}_{\mathfrak{g}}\, N$ is nilpotent. Since in addition

$$[\mathrm{ad}_{\mathfrak{g}}, S, \mathrm{ad}_{\mathfrak{g}}\, N] = \mathrm{ad}_{\mathfrak{g}}([S, N]) = 0$$

(4) is indeed the Jordan decomposition.

Proposition 2.7.* *Let* $\mathfrak{g}$ *be a semisimple subalgebra of* $\mathfrak{gl}(V)$. *Let* $A \in \mathfrak{g}$ *be arbitrary and* $A = S + N$ *is additive Jordan decomposition. Then* $S, N \in \mathfrak{g}$.

Proof. In the case $k \in \boldsymbol{R}$ we pass to the complexification so we may assume $k = \boldsymbol{C}$. Let $\mathscr{V}$ be the set of subspaces of V invariant under $\mathfrak{g}$. For each $W \in \mathscr{V}$ let

$$\mathfrak{g}_W = \{X \in \mathfrak{gl}(V) : X(W) \subset W, \mathrm{Tr}(X \mid W) = 0\}.$$

Then $\mathfrak{g}_W$ is a subalgebra of $\mathfrak{gl}(V)$; it contains $\mathfrak{g}$ because, by semisimplicity, $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. Next consider the normalizer

$$\mathfrak{n} = \{A \in \mathfrak{gl}(V) : [A, \mathfrak{g}] \subset \mathfrak{g}\}$$

and put

$$\mathfrak{g}_* = \mathfrak{n} \cap \left(\bigcap_{W \in \mathscr{V}} \mathfrak{g}_W \right).$$

Let $A \in \mathfrak{g}_*$, put $\tilde{A} = \mathrm{ad}_{\mathfrak{gl}(V)}(A)$, and let

$$A = S + N, \qquad \tilde{A} = \tilde{S} + \tilde{N}$$

* Cf. [DS], Chapter IX, Exercise A6.

be their additive Jordan decompositions. By (4), $\tilde{S} = \mathrm{ad}_{\mathfrak{gl}(V)}(S)$, $\tilde{N} = \mathrm{ad}_{\mathfrak{gl}(V)}(N)$. Since $\tilde{S}$ and $\tilde{N}$ are polynomials in $\tilde{A}$ we have $S, N \in \mathfrak{n}$. Since S and N are polynomials in A they leave each $W \in \mathscr{V}$ invariant; also $\mathrm{Tr}(N|W) = 0$ and $\mathrm{Tr}(S|W) = \mathrm{Tr}(A|W) = 0$ so $S, N \in \mathfrak{g}_W$. Thus S and N belong to $\mathfrak{g}_*$.

It suffices now to prove $\mathfrak{g}_* = \mathfrak{g}$.

Since $\mathfrak{g}$ is an ideal in $\mathfrak{g}_*$ the Killing form B of $\mathfrak{g}$ is the restriction of the Killing form B_* of $\mathfrak{g}_*$. Hence B_* is nondegenerate on $\mathfrak{g}$ so we have the direct decomposition $\mathfrak{g}_* = \mathfrak{g} \oplus \mathfrak{g}^\perp$ where the orthogonal complement $\mathfrak{g}^\perp$ is an ideal in $\mathfrak{g}_*$. Now since the representation of $\mathfrak{g}$ on V is semisimple (Weyl's theorem) we have $V = \bigoplus_i V_i$ where each V_i is $\mathfrak{g}$-irreducible. Each $X \in \mathfrak{g}^\perp$ commutes elementwise with $\mathfrak{g}$ so by Schur's lemma it acts on V_i by a scalar multiplication. Since $\mathrm{Tr}(X|V_i) = 0$ we conclude $X|V_i = 0$ so $X = 0$ and $\mathfrak{g} = \mathfrak{g}_*$.

Corollary 2.8. *With the notation of Proposition 2.7, $A \in \mathfrak{g}$ is semisimple (nilpotent) if and only if $\mathrm{ad}_{\mathfrak{g}}(A)$ is semisimple (nilpotent).*

Let $A = S + N$ be the additive Jordan decomposition. Since $\mathrm{ad}_{\mathfrak{g}}(A) = \mathrm{ad}_{\mathfrak{gl}(V)}(A)|\mathfrak{g}$ we deduce from Lemma 2.6 that $\mathrm{ad}_{\mathfrak{g}}(A) = \mathrm{ad}_{\mathfrak{g}}(S) + \mathrm{ad}_{\mathfrak{g}}(N)$ is the additive Jordan decomposition of $\mathrm{ad}_{\mathfrak{g}}(A)$. Now $A = S \Leftrightarrow \mathrm{ad}_{\mathfrak{g}}(A) = \mathrm{ad}_{\mathfrak{g}}(S)$, so the corollary is verified.

Corollary 2.9. *Let $\mathfrak{g}$ be a semisimple Lie algebra, $A \in \mathfrak{g}$. The following conditions are equivalent:*

(i) $\mathrm{ad}_{\mathfrak{g}}(A)$ *is semisimple (resp. nilpotent).*

(ii) *There exists a faithful representation π of $\mathfrak{g}$ such that $\pi(A)$ is semisimple (resp. nilpotent).*

(iii) *For each representation π of $\mathfrak{g}$ $\pi(A)$ is semisimple (resp. nilpotent).*

Proof. (iii) $\Rightarrow$ (ii) is obvious; (ii) $\Rightarrow$ (i) by Corollary 2.8. Finally, (i) $\Rightarrow$ (iii) as follows: assuming $\mathrm{ad}_{\mathfrak{g}}(A)$ semisimple, consider a representation π of $\mathfrak{g}$ on V. Let $\mathfrak{g}' = \mathfrak{g}/\mathrm{Kernel}(\pi)$, view $\mathfrak{g}'$ as the subalgebra $\pi(\mathfrak{g}) \subset \mathfrak{gl}(V)$, and put $A' = \rho(A)$ where $\rho: \mathfrak{g} \to \mathfrak{g}'$ is the natural mapping. If $\mathfrak{v} \subset \mathfrak{g}'$ satisfies $[A', \mathfrak{v}] \subset \mathfrak{v}$ then $[A, \rho^{-1}\mathfrak{v}] \subset \rho^{-1}\mathfrak{v}$ so there exists a $\mathfrak{u} \subset \mathfrak{g}$ such that $\rho^{-1}\mathfrak{v} \oplus \mathfrak{u} = \mathfrak{g}$, $[A, \mathfrak{u}] \subset \mathfrak{u}$. But then $\rho(\mathfrak{u})$ satisfies $\mathfrak{v} \oplus \rho(\mathfrak{u}) = \mathfrak{g}'$ and $[A', \rho(\mathfrak{u})] \subset \rho(\mathfrak{u})$. Thus $\mathrm{ad}_{\mathfrak{g}'}(A')$ is semisimple. By Corollary 2.8 used on $\pi(\mathfrak{g}) = \mathfrak{g}' \subset \mathfrak{gl}(V)$ we conclude that $A' = \pi(A)$ is semisimple as desired. The nilpotent case of (i) $\Rightarrow$ (iii) is proved in a similar way.

Definition. Let $\mathfrak{g}$ be a semisimple Lie algebra. An element $A \in \mathfrak{g}$ verifying the conditions of Corollary 2.9 is said to be *semisimple* (resp. *nilpotent*).

Corollary 2.10.

(i) *Let* $\mathfrak{g}$ *be a semisimple Lie algebra. Then each* $A \in \mathfrak{g}$ *has a unique decomposition*

$$A = S + N \tag{5}$$

where $S \in \mathfrak{g}$ *is semisimple,* $N \in \mathfrak{g}$ *is nilpotent.*

(ii) *Let* $\mathfrak{g}_1 \subset \mathfrak{g}$ *be a semisimple subalgebra. If* $A \in \mathfrak{g}_1$, *then* $S, N \in \mathfrak{g}_1$. *An element* $A \in \mathfrak{g}_1$ *is semisimple (resp. nilpotent) in* $\mathfrak{g}_1$ *if and only if it is semisimple (resp. nilpotent) in* $\mathfrak{g}$.

This follows directly from Proposition 2.7.

Definition. Decomposition (5) is called the *additive Jordan decomposition of A.*

3. Reductive Lie Algebras

Reductive Lie algebras are natural generalizations of semisimple Lie algebras; in this subsection we prove for them a few results which are useful primarily in Chapter III, §4. A Lie algebra will mean a Lie algebra over $\boldsymbol{R}$ or $\boldsymbol{C}$ and a representation will mean a finite-dimensional representation.

Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a subalgebra.

Definition. The subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is said to be *reductive in* $\mathfrak{g}$ if the representation $H \to \mathrm{ad}_{\mathfrak{g}}(H)$ of $\mathfrak{h}$ on $\mathfrak{g}$ is completely reducible (i.e., semisimple). A Lie algebra $\mathfrak{g}$ is said to be *reductive* if it is reductive in itself. Obviously, if $\mathfrak{h}$ is reductive in $\mathfrak{g}$ then $\mathfrak{h}$ is reductive.

Proposition 2.11. *A Lie algebra* $\mathfrak{g}$ *is reductive if and only if it is the direct sum*

$$\mathfrak{g} = \mathfrak{a} + \mathfrak{s} \tag{6}$$

of an abelian ideal $\mathfrak{a}$ *and a semisimple ideal* $\mathfrak{s}$. *In that case,*

$$\mathfrak{a} = \mathrm{center}(\mathfrak{g}), \qquad \mathfrak{s} = [\mathfrak{g}, \mathfrak{g}]. \tag{7}$$

Proof. Since the representation $\mathrm{ad}_{\mathfrak{g}}$ is semisimple $\mathfrak{g}$ has a decomposition $\mathfrak{g} = \sum \mathfrak{g}_i$ into subspaces on which $\mathrm{ad}_{\mathfrak{g}}$ acts irreducibly. In particular, each $\mathfrak{g}_i$ is an ideal of $\mathfrak{g}$. Each ideal of $\mathfrak{g}_i$ is an ideal in $\mathfrak{g}$ so ([DS], Chapter II, §6; Chapter III, Exercise B8) $\mathfrak{g}_i$ is either a simple Lie algebra or one-dimensional. Decomposition (6) now follows by taking $\mathfrak{s}$ as the sum of the simple $\mathfrak{g}_i$ and $\mathfrak{a}$ the sum of the 1-dimensional $\mathfrak{g}_i$.

The converse is clear since by the semisimplicity of $\mathfrak{s}$, $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{g}) = \mathrm{ad}_{\mathfrak{g}}(\mathfrak{s})$ is semisimple.

Proposition 2.12. *Let $\mathfrak{g}$ be a reductive Lie algebra (over $\boldsymbol{R}$ or $\boldsymbol{C}$) and π a representation of $\mathfrak{g}$ on a finite-dimensional vector space V. Then π is semisimple if and only if the restriction $\pi|$center $(\mathfrak{g})$ is semisimple.*

Proof. We shall use the familiar fact from linear algebra that a commutative family of endomorphisms is semisimple if and only if each member of the family is semisimple.

Assume first π is semisimple and consider the decomposition (6). Fix $H \in \mathfrak{a}$ and write $\pi(H) = N + S$ where N and S are the nilpotent and semisimple parts of $\pi(H)$, respectively. Let W be the null space of N. Since N is a polynomial in $\pi(H)$ ([DS], Chapter III, §1) N commutes with each element in $\pi(\mathfrak{g})$. Hence $\pi(\mathfrak{g})$ leaves the subspace $W \subset V$ invariant. Let $W' \subset V$ be a complementary invariant subspace. Since $\pi(H)W' \subset W'$ we have $NW' \subset W'$ and the restriction $N' = N|W'$ is nilpotent. But since $W \cap W' = 0$, N' has no null vector $x \neq 0$ in W'. But, N' being nilpotent, this implies $N' = 0$ ([DS], Chapter III, §1). Thus $N = 0$, so $\pi(H)$ is semisimple, whence by the initial remark, $\pi|\mathfrak{a}$ is semisimple.

For the converse we may, by Proposition 2.3, assume that $\mathfrak{g}$ is a Lie algebra over $\boldsymbol{C}$. The commutative family $\pi(\mathfrak{a})$ now assumed semisimple, let $V = \bigoplus_i V_i$ be the direct decomposition into joint eigenspaces of $\pi(\mathfrak{a})$. Then each V_i is $\pi(\mathfrak{g})$-invariant and $\pi(\mathfrak{g})|V_i = \pi(\mathfrak{s})|V_i$. Since $\mathfrak{s}$ is semisimple this last family of operators is semisimple. Thus the representation $\pi_i\colon X \to \pi(X)|V_i$ of $\mathfrak{g}$ on V_i is semisimple. Decomposing this into irreducibles we conclude from Proposition 2.1 that π is semisimple.

Proposition 2.13. *Let $\mathfrak{g}$ be a reductive Lie algebra, π_1 and π_2 two semisimple representations of $\mathfrak{g}$. Then $\pi_1 \otimes \pi_2$ is semisimple.*

Proof. Because of Propositions 2.1 and 2.3 we may assume $\mathfrak{g}$ complex, π_1 and π_2 irreducible. Using (6) we see that $\pi_1(\mathfrak{a})$ commutes elementwise with $\pi_1(\mathfrak{g})$ so by Schur's lemma consists of scalar multiples of the identity. The same holds for $\pi_2(\mathfrak{a})$ and therefore also for $(\pi_1 \otimes \pi_2)(\mathfrak{a})$, whence, by Proposition 2.12, $\pi_1 \otimes \pi_2$ is semisimple.

Remark. The result holds also for $\mathfrak{g}$ nonreductive (see, e.g., Dixmier [1974], Chapter I).

Proposition 2.14. *Suppose the subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is reductive in $\mathfrak{g}$ and let π be a representation of $\mathfrak{g}$ on V.*

(i) *Let W be the sum of the subspaces W_i of V which are invariant and irreducible under $\pi(\mathfrak{h})$. Then $\pi(\mathfrak{g})W \subset W$.*

(ii) *If π is semisimple, so is $\pi|\mathfrak{h}$.*

Proof. Let $W_i \subset V$ be a subspace which is $\pi(\mathfrak{h})$-invariant and irreducible and consider the representation $\rho: X \to \operatorname{ad}_{\mathfrak{g}}(X)$ of $\mathfrak{h}$ on $\mathfrak{g}$. By Prop. 2.13 the representation $\rho \otimes (\pi|\mathfrak{h})$ of $\mathfrak{h}$ on $\mathfrak{g} \otimes W_i$ is semisimple. Consider now the linear map $A: \mathfrak{g} \otimes W_i \to V$ given by $A(X \otimes w) = \pi(X)w$. Then A commutes with the action of $\mathfrak{h}$; in fact if $Y \in \mathfrak{h}$

$$\begin{aligned} A((\rho \otimes (\pi|\mathfrak{h}))(Y)(X \otimes w)) &= A([Y, X] \otimes w + X \otimes \pi(Y)w) \\ &= \pi([Y, X])\omega + \pi(X)\pi(Y)w = \pi(Y)A(X \otimes w). \end{aligned}$$

Hence we have a semisimple representation of $\mathfrak{h}$ on $A(\mathfrak{g} \otimes W_i)$. Thus $\pi(\mathfrak{g})W_i = A(\mathfrak{g} \otimes W_i) \subset W$. This proves (i). For part (ii) we may assume π irreducible. Then by (i) $W = V$. Thus $V = \sum_i W_i$ and by irreducibility $W_i \cap W_j = 0$ unless $W_i = W_j$. Thus V is the direct sum of certain W_i so $\pi|\mathfrak{h}$ is semisimple.

Corollary 2.15. *If $\mathfrak{h}$ is reductive in $\mathfrak{g}$ and $\mathfrak{g}$ reductive in $\mathfrak{k}$, then $\mathfrak{h}$ is reductive in $\mathfrak{k}$.*

Proposition 2.16. *Let $\mathfrak{k}$ be semisimple and $\mathfrak{g}$ reductive in $\mathfrak{k}$. If $X \in \mathfrak{g}$ and $\operatorname{ad}_{\mathfrak{k}}(X)$ nilpotent, then $X \in [\mathfrak{g}, \mathfrak{g}]$.*

Proof. We may assume $\mathfrak{g}$ and $\mathfrak{k}$ complex. Let $X = C + A$ where $C \in$ center($\mathfrak{g}$) and $A \in [\mathfrak{g}, \mathfrak{g}]$ and let $A = S + N$ be the additive Jordan decomposition of A in $[\mathfrak{g}, \mathfrak{g}]$. Let $\pi = \operatorname{ad}_{\mathfrak{k}}|\mathfrak{g}$. Then π is a faithful semisimple representation of $\mathfrak{g}$ which has a decomposition $\pi = \bigoplus_i \pi_i$ into irreducibles π_i. By 2.9 for $[\mathfrak{g}, \mathfrak{g}]$, $\pi(N)$ is nilpotent, $\pi(S)$ semisimple. By Prop. 2.12, $\pi(C)$ is semisimple, so since $\pi(S)$ and $\pi(C)$ commute, $\pi(S + C)$ is semisimple. Since $\pi(X) = \pi(N) + \pi(S + C)$ is nilpotent and $\pi(N)$ and $\pi(S + C)$ commute it follows that $\pi(S + C) = 0$. For each i $\pi_i(C)$ commutes elementwise with the irreducible $\pi_i(\mathfrak{g})$, so is a scalar, and $\operatorname{Tr}(\pi_i(S)) = \operatorname{Tr}(\pi_i(A)) = 0$. Hence $\pi_i(C) = 0$, so $\pi(C) = 0$ and $C = 0$, as desired.

§3. Some Algebraic Tools

We recall that "field" always means commutative field of characteristic 0. In this section we only consider rings and algebras which are commutative, have an identity element, and no divisors of 0. A module M over a ring A is called *finite* if there exist finitely many elements $x_1, \ldots, x_m \in M$ such that $M = Ax_1 + \cdots + Ax_m$.

Definition. Let B be a ring and A a subring of B (with the same identity). An element $x \in B$ is called *integral* over A if it satisfies an equation

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0 \qquad (1)$$

with leading coefficient 1, where $a_i \in A$. If each $x \in B$ is integral over A, the ring B is said to be integral over A.

Lemma 3.1. *Let A be a subring of B and let $x \in B$. The following conditions are equivalent:*

(i) *x is integral over A.*

(ii) *The ring $A[x]$ (the subring of B generated by A and x) is a finite A-module.*

(iii) *There exists an intermediate ring R, $A[x] \subset R \subset B$, such that R is a finite A-module.*

Proof. (i)$\Rightarrow$(ii) There exists an integer $n > 0$ such that $x^n \in \sum_{i=0}^{n-1} Ax^i$. It follows that $x^{n+r} \in \sum_{i=0}^{n-1} Ax^{i+r}$ so by induction on r, $x^{n+r} \in \sum_{i=0}^{n-1} Ax^i$ for each $r > 0$.

(ii)$\Rightarrow$(iii) Take $R = A[x]$.

(iii)$\Rightarrow$(i) Select $y_1, \ldots, y_n \in R$ such that $R = \sum_{i=1}^n Ay_i$. Then we have for suitable elements $a_{ij} \in A$, $xy_i = \sum_{j=1}^n a_{ij}y_j$ $(1 \leq i \leq n)$. Writing this system of linear equations as

$$\sum_{j=1}^{n} (\delta_{ij}x - a_{ij})y_j = 0 \qquad (1 \leq i \leq n),$$

we conclude that the determinant $d = \det(\delta_{ij}x - a_{ij})$ satisfies $dy_i = 0$ for each i. Then $dR = 0$ so $d = 0$. But this is an equation of the form (1) so x is integral over A.

By induction we conclude from Lemma 3.1,

Lemma 3.2. *Let $x_1, \ldots, x_n$ be elements of a ring B which are integral over a subring A. Then the ring $A[x_1, \ldots, x_n]$ (the subring of B generated by A and $x_1, \ldots, x_n$) is a finite A-module.*

From this lemma and Lemma 3.1(iii) we obtain the following:

Corollary 3.3. *Let A be a subring of a ring B. The set of elements in B which are integral over A form a subring $\bar{A}$ of B containing A.*

Definition. The subring $\bar{A}$ is called the *integral closure* of A in B.

Definition. A ring A is called *integrally closed* if it coincides with its integral closure in the quotient field $C(A)$ of A.

Lemma 3.4. *A unique factorization domain A is integrally closed.*

Proof. Let $x \in C(A)$ be integral over A. Then $x = \alpha/\beta$ where we may assume that α and β are relatively prime. Now x satisfies an equation

$$x^n + a_1x^{n-1} + \cdots + a_n = 0$$

so

$$\alpha^n + a_1\alpha^{n-1}\beta + \cdots + \alpha_n\beta^n = 0.$$

If β has a prime factor γ, then γ must divide α^n and therefore γ divides α. This contradicts the fact that α and β are relatively prime. Hence $x \in A$.

In particularly, the symmetric algebra $S(V)$ over a finite-dimensional vector space V is integrally closed.

Theorem 3.5. *Let K be an algebraically closed field and A and B finitely generated algebras (with identity) over K. Suppose that $A \subset B$ and that B is integral over A. Then*

(i) *Each homomorphism $\varphi: A \to K$ extends to a homomorphism ψ of B into K.*

(ii) *If each homomorphism $\varphi: A \to K$ extends uniquely to a homomorphism of B into K then the quotient fields $C(A)$ and $C(B)$ coincide.*

Proof of (i). If $\varphi(A) = \{0\}$ we define ψ by $\psi(B) = \{0\}$. If $\varphi(A) \neq \{0\}$ then the kernel of φ is a maximal ideal $\mathfrak{m}$ of A. We first prove the existence of a maximal ideal $\mathfrak{n}$ of B such that $\mathfrak{n} \cap A = \mathfrak{m}$. For this purpose consider the set of all ideals $\mathfrak{p}$ of B satisfying $\mathfrak{p} \cap A \subset \mathfrak{m}$. This set is partially ordered under inclusion and every (totally) ordered subset has an upper bound. By Zorn's lemma there exists a maximal element $\mathfrak{n}$ of the set. If $\mathfrak{n} \cap A$ is a proper subset of $\mathfrak{m}$, let x be an element in $\mathfrak{m}$ which does not belong to $\mathfrak{n}$. Then $\mathfrak{n}$ is a proper subset of the ideal $\mathfrak{n} + Bx$; by the choice of $\mathfrak{n}$, $(\mathfrak{n} + Bx) \cap A$ is not a part of $\mathfrak{m}$. In other words, there exists an element $z \in B$ and an element $y \in A$, not in $\mathfrak{m}$, such that $zx - y \in \mathfrak{n}$. Now z satisfies an equation $z^n + a_1z^{n-1} + \cdots + a_n = 0$, $a_i \in A$, which after multiplication by x^n and use of the congruence $zx \equiv y \pmod{\mathfrak{n}}$ gives

$$y^n + a_1xy^{n-1} + \cdots + a_nx^n \equiv 0 \pmod{\mathfrak{m}}.$$

This contradicts $x \in \mathfrak{m}$, $y \notin \mathfrak{m}$. Hence $\mathfrak{n} \cap A = \mathfrak{m}$.

If $\mathfrak{n}$ were not a maximal ideal in B, suppose $\mathfrak{n}'$ is a maximal ideal in B satisfying the proper inclusions $\mathfrak{n} \subset \mathfrak{n}' \subset B$. Then $\mathfrak{n}' \cap A$ is an ideal in A properly containing $\mathfrak{m}$ so $A \subset \mathfrak{n}'$. Let $x \in B$. Since x is integral over $\mathfrak{n}'$ we have $x^l \in \mathfrak{n}'$ for some integer l. Since $B/\mathfrak{n}'$ is a field we conclude that $x \in \mathfrak{n}'$. Thus $\mathfrak{n}' = B$, which is a contradiction. This proves that $\mathfrak{n}$ is a maximal ideal in B so $B/\mathfrak{n}$ is a field. Let 1 denote the identity of A. Then the mapping $\alpha \to \alpha \cdot 1$ $(\alpha \in K)$ is an isomorphism of K into A and the mapping $\alpha \to \alpha \cdot 1 + \mathfrak{n}$ is an isomorphism of K into $B/\mathfrak{n}$. Now select $b_1, \ldots, b_n \in B$ such that $B = K[b_1, \ldots, b_n]$. Then $B = A[b_1, \ldots, b_n]$ so by Lemma 3.2, B is a finite A-module. Writing $B = Ax_1 + \cdots + Ax_m$ $(x_i \in B)$ and using $A = K + \mathfrak{m}$ we see that $B/\mathfrak{n}$ is a finite extension of K. Since K is algebraically closed we have $B/\mathfrak{n} = K$. The natural mapping $\psi: B \to K$ gives the desired extension of φ.

Proof of (ii). In order to prove (ii) we make use of the following theorem (the Noether normalization theorem, see, e.g., Zariski and Samuel [1958], p. 266).

Let $R = k[y_1, \ldots, y_n]$ be a finitely generated algebra over a field k and let d be the transcendence degree of the quotient field $k(y_1, \ldots, y_n)$ over k. There exist d linear combinations $z_1, \ldots, z_d$ of the y_i with coefficients in k, algebraically independent over k, such that R is integral over $k[z_1, \ldots, z_d]$.

Combining this theorem with (i) we see that if k is algebraically closed there exists a homomorphism of $k[y_1, \ldots, y_n]$ onto k.

Suppose now that the quotient fields $C(A)$ and $C(B)$ were different. Let $a_1, \ldots, a_m$ be a set of generators of A so $A = K[a_1, \ldots, a_m]$. Pick any element $\alpha \in B$ which does not belong to $C(A)$. We shall find a homomorphism of $K[a_1, \ldots, a_m]$ into K which has more than one extension to a homomorphism of $K[a_1, \ldots, a_m, \alpha]$ into K. Since B is integral over $K[a_1, \ldots, a_m, \alpha]$, (ii) will then follow from (i). Let

$$p_\alpha(x) = x^n + f_1(a)x^{n-1} + \cdots + f_n(a) = 0$$

be the polynomial with coefficients in the field $C(A) = K(a_1, \ldots, a_m)$ of lowest degree having α as a zero and leading coefficient 1. [Here a stands for the m-tuple $(a_1, \ldots, a_m)$]. Let $q(a)$ denote the product of all the denominators of all the $f_i(a)$ with the discriminant of the polynomial $p_\alpha(x)$. From the remark above we see that there exists a homomorphism φ of $K[a_1, \ldots, a_m, 1/q(a)]$ onto K. The image of the polynomial $p_\alpha(x)$ under φ will then be a polynomial with coefficients in K having n distinct roots, say $\alpha_1, \ldots, \alpha_n$. Fix one α_i. We wish to extend φ (or more precisely, the restriction of φ to $K[a_1, \ldots, a_m]$) to a homomorphism $\psi_i : K[a_1, \ldots, a_m, \alpha] \to K$ by putting $\psi_i(\alpha) = \alpha_i$. The condition for this being possible is that whenever α satisfies a polynomial equation $p(a, x) = 0$ with coefficients in $K[a_1, \ldots, a_m]$ then α_i satisfies the corresponding equation $p(\varphi(a), x) = 0$ with coefficients in K. Since the polynomial $p_\alpha(x)$ has minimum degree it divides $p(a, x)$:

$$p(a, x) = p_\alpha(x)q(a, x).$$

Here the polynomial $q(a, x)$ can be found by long division; its coefficients $g_i(a)$ are rational expressions in $a_1, \ldots, a_m$ whose denominators divide the product of the denominators of the $f_i(a)$. Since φ does not map this product into 0 it is clear that $p(\varphi(a), x)$ vanishes for $x = \alpha_i$. Now $n > 1$ and the homomorphisms ψ_i $(1 \leq i \leq n)$ are all different. This concludes the proof.

SOME DETAILS

As explained in the preface the present short section furnishes detailed explanations of a few statements in the text, the proof of which is only indicated there.

Page 9, line 11^-: As remarked by Stetkær one does not have to use the analyticity here. For a differential equation $y'' + f_1(r)y' + f_0(r)y = 0$ it is a classical result that if $y_1(x)$ is a solution then

$$y_2 = y_1 \int \tfrac{1}{y_1^2} e^{-\int f_1(r)\,dr}\, dr$$

is another solution. In our case formula (13a) shows that the solution $y_1(r) = \varphi_{\lambda,n}(r)$ is divisible by $r^{|n|}$. As a result, $y_2(r)$ is singular near $r = 0$, also for $n = 0$. This shows that all solutions which are smooth near $r = 0$ are proportional.

Page 19, line 12^+: The remark for Page 9 applies here to show that all smooth solutions are proportional.

Page 27, line 1^+: Combining this formula with Lemma 3.6 we obtain

$$\int_{\mathbf{S}^{n-1}} e^{i(x,w)} p(w)\, dw = \left(\frac{i}{2\pi}\right)^k p(x) \int_{\mathbf{S}^{n+2k-1}} e^{i(x,w)}\, dw$$

Page 28, line 16^-: Since $\Phi_{\lambda,\delta}(re_n) = \Phi_{r\lambda,\delta}(e_n)$ this constant is independent of λ (remark by Stetkær).

Page 30, line 1^+: In fact, putting $t = x(\tau)/|x|$ we get

$$L(\gamma) \geq \int_\alpha^\beta \frac{|x'(\tau)|}{1 - x(\tau)^2}\, d\tau \geq \int_\alpha^\beta \frac{x'(\tau)}{1 - x(\tau)^2}\, d\tau = \int_0^1 \frac{|x|}{1 - t^2 x^2}\, dt = L(\gamma_0)$$

Page 50, line 7^+: The remark for Page 9 shows that indeed all the smooth solutions of (45) are proportional.

Page 64, line 8^+, 6^-: The function $e^{(i\lambda+1)\langle z,\zeta\rangle}$ can be written

$$(1-z\overline{z})^\nu \overline{z}^{-\nu}\left(\frac{\zeta-z}{\zeta}\right)^{-\nu}\left(\frac{1}{\overline{z}}-\zeta\right)^{-\nu}.$$

If $|\zeta|=1$ then $\frac{\zeta-z}{\zeta}\in S_{|z|}(1)$ so $\log\left(\frac{\zeta-z}{\zeta}\right)$ is holomorphic for ζ in an annulus around $S_1(0)$. Also, $\log\left(\frac{1}{\overline{z}}-\zeta\right)$ is holomorphic for ζ in a slit plane from $\frac{1}{\overline{z}}$ to ∞.

Page 65, line 7^+: Here we give a more detailed proof of the formula

$$\int_K\left(\int_B f(k,b)\,dT(b)\right)dk=\int_B\left(\int_K f(k,b)\,dk\right)dT(b)\,, \tag{1}$$

where $f(k,b)=e^{(i\lambda+1)\langle gk\cdot z,b\rangle}$, $g\in G, z\in D$ fixed, since the text only gives and indication of a proof.

The function $b\to f(k,b)$ extends to a holomorphic function in an annulus U containing B (same for all k). This extension has a Laurent expansion

$$f(k,\zeta)=\sum_{-\infty}^{\infty}A_n(k)\zeta^n\,,\ A_n(k)=\tfrac{1}{2\pi i}\int_{|\zeta|=R}\frac{f(k,\zeta)}{\zeta^{n+1}}\,d\zeta\,, \tag{2}$$

converging uniformly on each compact subset $C\subset U$. We have

$$|A_n(k)|\leq\frac{M}{R^n} \tag{3}$$

whenever the circle $|\zeta|=R$ is contained in U, M being a constant. As $N\to\infty$ we have

$$\sum_{-N}^{N}A_n(k)\zeta^n\to f(k,\zeta)$$

uniformly on $K\times C$. After taking $\int_K dk$ we still have uniform convergence for $\zeta\in C$. Since the embedding $\mathcal{H}(U)\to\mathcal{A}(B)$ is continuous this implies

$$\int_B\left(\int_K\sum_{-N}^{N}A_n(k)\,dk\right)\chi_n(b)\,dT(b)\to\int_B\left(\int_K f(k,b)\,dk\right)dT(b)\,, \tag{4}$$

χ_n being as in Proposition 4.11.

On the other hand, the left hand side of (4) equals

$$\sum_{-N}^{N} \int_K \left(\int_B A_n(k)\chi_n(b)\, dT(b) \right) dk = 2\pi \sum_{-N}^{N} \int_K A_n(k) a_{-n}\, dk\,. \tag{5}$$

We shall prove that

$$\sum_{-\infty}^{\infty} \int_K |A_n(k) a_{-n}|\, dk < \infty\,. \tag{6}$$

Assuming this, the right hand side of (5) has (by the dominated convergence theorem) the limit

$$2\pi \sum_n \int_K A_n(k) a_{-n}\, dk = \int_K \sum_n A_n(k) \int_B \chi_n(b)\, dT(b)\, dk\,. \tag{7}$$

This last expression equals

$$\int_K \left(\int_B \sum_n A_n(k)\chi_n(b)\, dT(b) \right) dk\,, \tag{8}$$

because the uniform convergence of (2) implies that $\sum_n A_n(k)\chi_n(b)$ converges in $\mathcal{A}(B)$. On the other hand expression (8) equals

$$\int_K \left(\int_B f(k,b)\, dT(b) \right) dk$$

so (1) would follow from (4).

To verify (6) we consider the cases $n \geq 0$ and $n < 0$ separately. We know by Lemma 4.21 that

$$\sum_n |a_n| r^{|n|} < \infty \qquad 0 \leq r < 1\,. \tag{9}$$

(i) $n \geq 0$. Choose $R > 1$ such that (3) holds and then $r < 1$ such that $Rr > 1$. Then by (9), $|A_n(k) a_{-n}| \leq \text{const.}(Rr)^{-n}$.

(ii) $n < 0$. Choose $R < 1$ such that (3) holds and then $r < 1$ such that $R < r < 1$. Then by (9), $|A_n(k) a_{-n}| \leq \text{const.} \left(\frac{R}{r}\right)^{-n}$. Combining these cases (i) and (ii) relation (6) follows so (1) is proved.

Page 81, line 11^+: Here it is worth while remarking that each measure μ has the form $w - \nu$ where w and ν are positive measures. To see this define for $f \geq 0$, $w(f) = \sup_{0 \leq g \leq f} \mu(g)$. Then if f, f' are ≥ 0 it is easy to see that $w(f) + w(f') \leq w(f + f')$. On the other hand, if $h \geq 0$, $0 \leq h \leq f + f'$ then putting $g = \min(f, h)$, $g' = h - g$ we have $0 \leq g \leq f$, $0 \leq g' \leq f'$ and $h = g + g'$. Hence $\mu(h) = \mu(g) + \mu(g') \leq w(f) + w(f')$ whence $w(f + f') \leq w(f) + w(f')$. Thus $w(f + f') = w(f) + w(f')$ for $f, f' \geq 0$. If $k \in C_c(S)$ is written $k = f - g = f' - g'$ (f, g, f', g' all ≥ 0) then $f + g' = f' + g$ so $w(k)$ is well-defined as $w(f) - w(g)$. Thus w is a positive measure and so is $\nu = w - \mu$.

Page 92, line 6^-: The uniqueness can be proved as follows. If μ and μ' are two left invariant Haar measures, define ν by $\nu(f) = \mu'(\check{f})$ where $\check{f}(g) = f(g^{-1})$, $f \in C_c(G)$. For $g \in C_c(G)$ consider the function $s \to f(s) \int g(ts)\, d\nu(t)$ and integrate it with respect to μ. Since the ν-integral is constant in s the result is

$$\begin{aligned}
\mu(f)\nu(g) &= \int f(s) \int g(ts)\, d\nu(t)\, d\mu(s) = \int d\nu(t) \int f(s) g(ts)\, d\mu(s) \\
&= \int d\nu(t) \int f(t^{-1}\sigma) g(\sigma)\, d\mu(\sigma) = \int g(\sigma)\, d\mu(\sigma) \int f(t^{-1}\sigma)\, d\nu(t)\,.
\end{aligned}$$

Put $h(\sigma) = \int f(t^{-1}\sigma)\, d\nu(t)/\mu(f)$. Then the formula shows $\nu = h\mu$ (h independent of f). Taking $\sigma = e$ we deduce

$$h(e)\mu(f) = \nu(\check{f}) = \mu'(f)$$

so μ and μ' are proportional.

Page 115, line 1^+: More explicitly, we have for each $F \in \mathcal{S}(R)$,

$$\int_{\mathbf{R}} \operatorname{sgn} s\, e^{ist} \left(\int_{\mathbf{R}} F(p) e^{-isp}\, dp \right) ds = 2\pi (\mathcal{H}F)(t)\,.$$

In fact if we apply both sides to $\widetilde{\psi}$ ($\psi \in \mathcal{S}(\mathbf{R})$) the left-hand side is (by (40))

$$\begin{aligned}
&\int_{\mathbf{R}} \left(\int_{\mathbf{R}} \operatorname{sgn} s \widetilde{F}(s) e^{ist}\, ds \right) \widetilde{\psi}(t)\, dt \\
&\qquad = \int_{\mathbf{R}} (\mathcal{H}F)^{\sim}(s) 2\pi \psi(s)\, ds = 2\pi \int_{\mathbf{R}} (\mathcal{H}F)(t) (\widetilde{\psi})(t)\, dt
\end{aligned}$$

Page 176, line 3^+: To verify the surjectivity statement in full we note by Introduction, Theorem 3.4 that

$$f = \sum_{k,m} a_{km} S_{km} \qquad (k \text{ even})$$

and by (32)–(34)

$$\widehat{f} = \Omega_n \sum_{k,m} a_{km} \varphi_k \left(\frac{\pi}{2}\right) S_{km} \qquad (k \text{ even}).$$

But by p. 23

$$\varphi_{2k}\left(\frac{\pi}{2}\right) = \frac{\Omega_{n-1}}{\Omega_n}(-1)^k \int_0^\pi (\cos\varphi)^{2k}(\sin\varphi)^{n-2}\, d\varphi \sim k^{-\frac{n-1}{2}}$$

By the characterization in Introduction, Theorem 3.4(ii), the surjectivity of $f \to \widehat{f}$ follows.

Page 194, line 14^-: The factors $1 - e^{-\alpha}$ and $1 - e^{-\beta}$ are relatively prime.

Proof. Let $\omega_1, \dots \omega_\ell$ be as above, put $\Lambda = \widetilde{T}(R)$, so $\Lambda = \oplus_1^\ell \mathbf{Z}\omega_i \approx \mathbf{Z}^\ell$.

If $\alpha = \sum_1^\ell n_k \omega_k$ and d the highest common factor of the n_k then $-\alpha = du_1$ where $d > 0$, $u_1 \in \Lambda$, and coordinates $m_j = n_j/d$ of u_1 have no common factor. This being the case, we can extend the row $m_1, \dots, m_\ell$ to a matrix $(m_{ij}) \in \mathbf{SL}(\ell, \mathbf{Z})$ with $m_{1j} = m_j$. We extend u_1 to a basis $u_1, u_2', \dots, u_\ell'$ of Λ over $\mathbf{Z}$ by $u_k' = \sum_1^\ell m_{kj}\omega_j (k \geq 2)$. Let $\Lambda' = \oplus_2^\ell \mathbf{Z}u_k'$. We have

$$-\beta = nu_1 + \mu\,, \qquad \mu \neq 0 \text{ in } \Lambda'\,.$$

Then $\mu = mu_2$ where $u_2 \in \Lambda'$ has coordinates relative to (u_k') which have no common factor. We can then by the earlier argument for Λ extend u_2 to a basis (u_k) of Λ'. Then $\Lambda = \oplus_1^\ell \mathbf{Z}u_i$ and

$$-\alpha = du_1\,, \qquad -\beta = nu_1 + mu_2\,, \qquad m \neq 0\,.$$

The algebra F is isomorphic to the ring

$$A = \mathbf{Z}\left[X_1, \dots, X_\ell, X_1^{-1}, \dots, X_\ell^{-1}\right] \qquad X_i = e^{u_i}$$

and

$$\begin{aligned} 1 - e^{-\alpha} &= 1 - X_1^d\,, \\ 1 - e^{-\beta} &= 1 - X_1^n X_2^m\,. \end{aligned}$$

Since $1 - X_1^d = \prod_{\nu=0}^{d-1}(1 - \epsilon^\nu X_1)$ where $\epsilon = e^{\frac{2\pi}{d}}$, and since no factor $1 - \epsilon^\nu X_1$ divides $1 - X_1^n X_2^m$ (it does not vanish for $X_1 = e^{-\nu}$) the elements $1 - e^{-\alpha}$, $1 - e^{-\beta}$ are relatively prime.

Page 206, line 1^+: Since $H = \begin{pmatrix} 0(1,q) & 0 \\ 0 & 1 \end{pmatrix}$ is the isotropy group of $(0,\dots,0,1)$ it leaves each plane $y_{q+2} = $ const. invariant. Thus $\mathbf{S}_r(0)$ is a section of a plane with our quadric (or rather the part of this section which lies in $\mathbf{D}_0$).

Page 250, line 3^+: Proof of (31). Given $x' = (x_1,\dots,x_{n-1})$ put

$$I(x') = \{x_n : (x', x_n) \in B_R\}\ .$$

By Schwarz's inequality,

$$\begin{aligned} |\varphi(x_1,\dots,x_n)|^2 &\leq \int_{-R}^{x_n} dt \int_{I(x')} |(\partial_n\varphi)(x',t)|^2\,dt \\ &\leq 2R\int_{I(x')} |(\partial_n\varphi)(x',t)|^2\,dt\,. \end{aligned}$$

The right hand side is independent of x_n so

$$\begin{aligned} \int_{B_R} |\varphi(x)|^2\,dx &= \int_{|x'|\leq R} dx' \int_{I(x')} |\varphi(x',x_n)|^2\,dx_n \\ &\leq \int_{|x'|\leq R} dx' \int_{I(x')} \left(2R\int_{I(x')} |(\partial_n\varphi)(x',t)|^2\,dt\right) dx_n \\ &\leq 4R^2 \int_{|x'|<R} dx' \int_{I(x')} |(\partial_n\varphi)(x',t)|^2\,dt \\ &= 4R^2 \int_{B_R} |(\partial_n\varphi)(x)|^2\,dx\,, \end{aligned}$$

which proves (31) even in a stronger form.

Page 281, line 16^-: To prove (5) recall that

$$(\widetilde{X}^m f)(g) = \left\{\frac{d^m}{dt^m} f(g\exp tX)\right\}_{t=0}\,.$$

Put $F(x_1,\dots,x_n) = f(g\exp X)$. By induction

$$\frac{d^m}{dt^m}F(tx_1,\dots,tx_n) = \sum x_{i_1}\dots x_{i_m}(\partial_{i_1}\dots\partial_{i_m}F)(tx_1,\dots,tx_n)\,,$$

which for $t=0$ gives

$$(\widetilde{X}^m f)(g) = \sum x_{i_1}\dots x_{i_m}\lambda(X_{i_1}\dots X_{i_m})f(g) = (\lambda(X^m)f)(g)\,.$$

Page 281, line 8^+: That $\lambda(P)$ is a differential operator is even easier to see from (5) and the fact that the powers X^m ($X \in \mathfrak{g}\,, m \in \mathbf{Z}^+$) span $S(\mathfrak{g})$.

Page 444, line 7^-: Actually, Lemma 6.8 gives (with $\sigma = (s')^{-1}$)

$$N^\sigma = \exp(\mathfrak{n}^\sigma \cap \overline{\mathfrak{n}}) \exp(\mathfrak{n}^\sigma \cap \mathfrak{n}) .$$

If there were an element $x \in N^\sigma \cap \overline{N}$ not in $\exp(\mathfrak{n}^\sigma \cap \overline{\mathfrak{n}})$ then the formula gives $x = x_1 x_2$ and $x_2 = \overline{N} \cap N = e$. Hence $\exp(\mathfrak{n}^\sigma \cap \overline{\mathfrak{n}}) = N^\sigma \cap \overline{N}$, proving (28).

Page 558, line 2^-: Note in fact that $(AX)_{11} = A_{11} X_{11}$ if the x_{ij} are ordered by $x_{11}, \ldots, x_{1n}, x_{2n}, \ldots x_{2n}, \ldots x_{n1}, \ldots x_{nn}$.

BIBLIOGRAPHY

ADIMURTI, KUMARESAN S.
1979 On the singular support of distributions and Fourier transforms on symmetric spaces. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **6** (1979), 143–150.

AHLFORS, L. V.
1975 Invariant operators and integral representations in hyperbolic space. *Math. Scand.* **36** (1975), 27–43.
1981 "Möbius Transformations in Several Variables." Lecture Notes. Univ. of Minnesota, 1981.

ALLAMIGEON, A. C.
1961 Propriétés globales des espaces harmoniques. *C. R. Acad. Sci. Paris* **252** (1961), 1093–1095.

ANDERSON, M. F.
1979 A simple expression for the Casimir operator on a Lie Group. *Proc. Amer. Math. Soc.* **77** (1979), 415–420.

ARAKI, S. I.
1962 On root systems and an infinitesimal classification of irreducible symmetric spaces. *J. Math. Osaka City Univ.* **13** (1962), 1–34.

ÁSGEIRSSON, L.
1937 Über eine Mittelwertseigenschaft von Lösungen homogener linearer partieller Differentialgleichungen 2. Ordnung mit konstanten Koeffizienten. *Math. Ann.* **113** (1937), 321–346.

ASKEY, R.
1975 "Orthogonal Polynomials and Special Functions." Regional Conf. Ser. in Appl. Math. SIAM, Philadelphia, 1975.

ATIYAH, M. *et al.*
1979 "Representation Theory of Lie Groups." London Math. Soc. Lecture Notes 34. Cambridge Univ. Press, London and New York, 1979.

BAGCHI, SOMESH CHANDRA, and SITARAM, A.
1979 Spherical mean periodic functions on semisimple Lie groups. *Pacific J. Math.* **84** (1979), 241–250.

BARBASH, D.
1979 Fourier inversion for unipotent invariant integrals. *Trans. Amer. Math. Soc.* **249** (1979), 51–83.

BARBASH, D., and VOGAN, D.

1980 The local structure of characters. *J. Funct. Anal.* **37** (1980), 27–55.

1982 Primitive ideals and orbital integrals in complex classical groups. *Math. Ann.* **259** (1982), 153–199.

1983 Primitive ideals and orbital integrals in complex exceptional groups. *J. Algebra* **80** (1983), 350–382.

BARGMANN, V.

1947 Irreducible unitary representations of the Lorentz groups. *Ann. of Math.* **48** (1947), 568–640.

BARKER, W.

1975 The spherical Bochner theorem on semisimple Lie groups. *J. Funct. Anal.* **20** (1975), 179–207.

BARUT, A. D., and RACZKA, R.

1977 "Theory of Group Representations and Applications." Polish Scientific Publishers, Warsaw, 1977.

BELTRAMI, E.

1864 Ricerche di analisi applicata alla geometria. *Giornale di Mat.* **2, 3** (1864) (1865), Opera I, p. 107–198.

1869 Zur Theorie des Krümmungsmaasses. *Math. Ann.* **1** (1869), 575–582.

BEREZIN, F. A.

1957 Laplace operators on semi-simple Lie groups. *Trudy Moskov. Mat. Obšč.* **6** (1957), 371–463 [*English transl.*: *Amer. Math. Soc. Transl.* **21** (1962), 239–339].

BEREZIN, F. A., and GELFAND, I. M.

1956 Some remarks on the theory of spherical functions on symmetric Riemannian manifolds. *Trudy Moscov. Mat. Obšč.* **5** (1956), 311–351 [*English transl.: Amer. Math. Soc. Transl.* **21** (1962), 193–239].

BEREZIN, F. A., GELFAND I. M., GRAEV, M. A., and NAIMARK, M. A.

1960 Representations of groups. *Usp. Math. Nauk* **11** (1956), 13–40 [*English transl.*: *Amer. Math. Soc. Transl.* **16** (1960), 325–353].

BEREZIN, F. A., and KARPELEVIČ, F. I.

1958 Zonal spherical functions and Laplace operators on some symmetric spaces. *Dokl. Akad. Nauk. USSR* **118** (1958), 9–12.

BERG, C.

1973 Dirichlet forms on symmetric spaces. *Ann. Inst. Fourier* **23** (1973), 135–156.

BERLINE, N., and VERGNE, M.

1981 Équations de Hua et intégrales de Poisson. *C. R. Acad. Sci. Paris Ser. A* **290** (1980), 123–125. *In* "Non-Commutative Harmonic Analysis and Lie Groups," Lecture Notes in Math. No. 880, pp. 1–51. Springer-Verlag, New York, 1981.

BERNAT, P., CONZE, N. *et coll.*

1972 "Représentations des groupes de Lie résolubles." Dunod, Paris, 1972.

BESSE, A.

1978 Manifolds all of whose geodesics are closed. "Ergebnisse der Math.," Vol. 93. Springer-Verlag, New York, 1978.

BHANU MURTHY, T. S.

1960a Plancherel's measure for the factor space SL(n, **R**)/SO(n, **R**). *Dokl. Akad. Nauk. SSSR* **133** (1960), 503–506.

1960b The asymptotic behavior of zonal spherical functions on the Siegel upper half-plane. *Dokl. Akad. Nauk SSSR* **135** (1960), 1027–1029.

BOAS, R. P. Jr.

1954 "Entire Functions." Academic Press, New York, 1954.

BOCHNER, S.

1955 "Harmonic Analysis and the Theory of Probability." Univ. of California Press, Berkeley, California, 1955.

BOCKWINKEL, H. B. A.

1906 On the propagation of light in a biaxial crystal about a midpoint of oscillation. *Verh. Konink. Acad. V. Wet. Wis-en. Natur.* **14**, (1906), 636.

BOERNER, H.

1955 "Darstellungen von Gruppen mit Berücksichtigung der Bedürfnisse der Modernen Physik." Springer-Verlag, Berlin and New York, 1955; Zweite Auflage, 1967.

BOPP, N.

1981 Distributions de type K-positif sur l'espace tangent. *J. Funct. Anal.* **44** (1981), 348–358.

BOREL, A.

1972 "Représentations de Groupes Localement Compacts," Lecture Notes in Math. No. 276. Springer-Verlag, Berlin and New York, 1972.

BOREL, A., and HARISH-CHANDRA

1962 Arithmetic subgroups of algebraic groups. *Ann. of Math.* **75** (1962), 485–535.

BOREL, A., and WALLACH, N.

1980 "Continuous Cohomology, Discrete Subgroups, and Representations of Reductive Groups," Ann. Math. Studies. Princeton Univ. Press, Princeton, New Jersey, 1980.

BOROVIKOV, W. A.

1959 Fundamental solutions of linar partial differential equations with constant coefficients. *Trudy Moscov Mat. Obšč.* **8** (1959), 199–257.

BOTT, R.

1956 An application of the Morse theory to the topology of Lie groups. *Bull. Soc. Math. France* **84** (1956), 251–286.

BOURBAKI, N.

"Éléments de Mathématique," Vol. VI, Intégration, Chapters 1–8. Hermann, Paris, 1952–1963.

BOURBAKI, N.

Éléments de mathématique. "Groupes et algébres de Lie," Chapters I–VIII. Hermann, Paris, 1960–1975.

BOURBAKI, N.

"Éléments de Mathématique," Vol. V, Espaces vectoriels topologiques, Chapters I–V. Hermann, Paris, 1953–1955.

BRACEWELL, R. N., and RIDDLE, A. C.

1967 Inversion of fan beam scans in radio astronomy, *Astrophys. J.* **150** (1967), 427–434.

BRAUER, R.

1926 Über die Darstellung der Drehungsgruppe durch Gruppen linearer Substitutionen. Dissertation, Berlin, 1926.

1937 Sur la multiplication des caracteristiques des groupes continus et semi-simples. *C. R. Acad. Sci. Paris* **204** (1937), 1784–1786.

BRUHAT, F.

1956 Sur les représentations induites des groupes de Lie. *Bull. Soc. Math. France* **84** (1956), 97–205.

CALDERON A. P., and ZYGMUND, A.

1957 Singular differential operators and differential equations. *Amer. J. Math.* **79** (1957), 901–927.

CARTAN, É.

1896 Le principe de dualité et certaines intégrales multiples de l'espace tangential et de espace reglé. *Bull. Soc. Math. France* **24** (1896), 140–177.

1913 Les groupes projectifs qui ne laissent invariant aucune multiplicité plane. *Bull. Soc. Math. France* **41** (1913), 53–96.

1914 Les groupes projectifs continus réels qui ne laissent invariant aucune multiplicité plane. *J. Math. Pure Appl.* **10** (1914), 149–186.

1922 "Lecons sur les Invariant Integraux." Hermann, Paris, 1922. Reprinted 1958.

1927 Sur certaines formes riemanniennes remarquables des géometries a groupe fondamental simple. *Ann. Sci. École Norm. Sup.* **44** (1927), 345–467.

1929 Sur la detérmination d'un système orthogonal complet dans un espace de Riemann symétrique clos. *Rend. Circ. Mat. Palermo* **53** (1929), 217–252.

1950 "Leçons sur la Géometrie Projective Complexe," 2nd ed. Gauthier-Villars, Paris, 1950.

CARTAN, H.

1957 Fonctions Automorphes. Seminaire 1957–1958, Paris.

CARTER, R. W.

1972 "Simple Groups of Lie Type." Wiley, New York, 1972.

CARTER, R. W., and LUSZTIG, G.

1974 On the representations of the general linear and symmetric group *Math. Z.* **136** (1974), 193–242.

CARTIER, P.

1955 Articles in Séminaire Sophus Lie [1] (1955).

1961 On H. Weyl's character formula. *Bull. Amer. Math. Soc.* **67** (1961), 228–230.

CARTIER, P., and DIXMIER, J.

1958 Vecteurs analytiques dans les répresentations des groupes de Lie. *Amer. J. Math.* **80** (1958), 131–145.

CECCHINI, C.

1972 Lacunary Fourier series on compact Lie groups. *J. Funct. Anal.* **11** (1972), 191–203.

CERÈZO, A., and ROUVIÈRE, F.

1972 Sur certains opérateurs differentiels invariants du groupe hyperbolique. *Ann. Sci. Ecole Norm. Sup.* **5** (1972), 581–597.

1973 Opérateurs differentiels invariants sur un groupe de Lie. Séminaire Goulaouic-Schwartz 1972–1973. École Polytech., Paris, 1973.

CHAMPETIER, C. and DELORME, P.

1981 Sur les représentations des groupes de déplacements de Cartan. *J. Funct. Anal.* **43** (1981), 258–279.

CHANG, W.

1979a Global solvability of the Laplacians on pseudo-Riemannian symmetric spaces. *J. Funct. Anal.* **34** (1979), 481–492.

1979b Global solvability of bi-invariant differential operators on solvable Lie groups. "Non Commutative Harmonic Analysis," Lecture Notes in Math. No. 728, pp. 8–16. Springer-Verlag, Berlin and New York, 1979.

1982 Invariant differential operators and P-convexity of solvable Lie groups. *Adv. in Math.* **46** (1982), 284–304.

CHERN, S. S.

1942 On integral geometry in Klein spaces. *Ann. of Math.* **43** (1942), 178–189.

CHEVALLEY, C.

1946 "Theory of Lie Groups," Vol. I. Princeton Univ. Press, Princeton, New Jersey, 1946.

1952 The Betti numbers of the exceptional simple Lie groups. *Proc. Internat. Congr. Math. 1950* **2** (1952), 21–24.

1955a Invariants of finite groups generated by reflections, *Amer. J. Math.* **77** (1955), 778–782.

1955b "Théorie des groupes de Lie," Vol. III. Hermann, Paris, 1955.

CHÈBLI, H.

1974 Sur un théoreme de Paley-Wiener associé a la décomposition spectrale d'un opérateur de Sturm-Liouville sur $(0, \infty)$. *J. Funct. Anal.* **17** (1974), 447–461.

CLERC, J. L.

1972 Les sommes partielles de la decomposition en harmoniques sphérique ne convergent pas dans $L^p(p \neq 2)$. *C. R. Acad. Sci. Paris Ser. A–B* **274** (1972), A59–A61.

1976 Une formule de type Mehler–Heine pour les zonal d'un espace riemannien symétrique. *Studia Math.* **57** (1976), 27–32.

1979 Multipliers on symmetric spaces. *Proc. Symp. Pure Math.* **35**, Part 2, (1979), 345–353.

1980 Transformation de Fourier sphérique des espaces de Schwartz. *J. Funct. Anal.* **37** (1980), 182–202.

1981 Estimations à l'infini des fonctions des Bessel généraliseés. *C. R. Acad. Sci. Paris Math.* **292** (1981), 429–430.

CODDINGTON, E. A., and LEVINSON, N.

1955 "Theory of Ordinary Differential Equations." McGraw-Hill, New York, 1955.

COHEN, P.

1960 On a conjecture of Littlewood and idempotent measures. *Amer. J. Math.* **82** (1960), 191–212.

COIFMAN, R. R., and WEISS, G.

1971 "Analyse harmonique noncommutative sur certains espaces homogènes," Springer Lecture Notes No. 242. Springer Verlag, Berlin and New York, 1971.

CORMACK, A. M.

1963–1964 Representation of a function by its line integrals, with some radiological application I, II. *J. Appl. Phys.* **34** (1963), 2722–2727; **35** (1964), 2908–2912.

CORMACK, A. M., and QUINTO, T.

1980 A Radon transform on spheres through the origin in $\mathbb{R}^n$ and applications to the Darboux equation. *Trans. Amer. Math. Soc.* **260** (1980), 575–581.

COTTON, E.

1900 Sur les invariant differentiels.... *Ann. Éc. Norm. Sup.* **17** (1900), 211–244.

COURANT, R., and LAX, A.

1955 Remarks on Cauchy's problem for hyperbolic partial differential equations with constant coefficients in several independent variables. *Comm. Pure Appl. Math.* **8** (1955), 497–502.

COEXTER, H. S. M.

1934 Discrete groups generated by reflections. *Ann. of Math.* **35** (1934), 588–621.

1957 "Non-Euclidean Geometry." Univ. of Toronto Press, Toronto, 1957.

CROFTON, M. W.

1868 On the theory of local probability. *Phil. Trans. Roy. Soc. London* **158** (1868), 181–199.

1885 Probability. *Encyclopaedia Britannica*, 9th ed. (1885), 768–788.

DADOK, J.

1980 Solvability of invariant differential operators of principal type on certain Lie groups and symmetric spaces. *J. D'Analyse* **37** (1980), 118–127.

1982 On the C^∞ Chevalley theorem. *Adv. in Math.* **44** (1982), 121–131.

1983 Polar coordinates induced by actions of compact Lie groups (preprint).

DE GEORGE D. L., and WALLACH, N.

1978 Limit formulas for multiplicities in $L^2(G/\Gamma)$. *Ann. of Math.* **107** (1978), 133–150.

DELORME, P.

1981 Homomorphismes de Harish-Chandra et K-types minimaux de series principales des groupes de Lie semisimples reels. *In* "Non-Commutative Harmonic Analysis and Lie Groups." Lecture Notes in Math. No. 880. Springer-Verlag, Berlin and New York, 1981.

1982 Theorème de type Paley–Wiener pour les groupes de Lie semisimples reels avec une seule classe de conjugaison de sons groupes des Cartan. *J. Funct. Anal.* **47** (1982), 26–63.

DE MICHELE, L., and RICCI, F. (eds.)

1982 *Proc. Seminar Topics Mod. Harmonic Anal.* Turin and Milano, 1982.

DE RHAM, G.

1955 "Variétés différentiables." Hermann, Paris, 1955.

DIEUDONNÉ, J.
1980 "Special Functions and Linear Representations of Lie Groups," Conf. Board Math. Sci. Series. No. 42. American Mathematical Society, Providence, Rhode Island, 1980.
1975 "Éléments d'Analyse," Vols. V and VI. Gauthier-Villars, Paris, 1975.

DIXMIER, J.
1958 Sur les représentations unitaires des groupes de Lie nilpotents III. *Canad. J. Math.* **10** (1958), 321–348.
1959 Sur l'algèbre enveloppante d'une algèbre de Lie nilpotente. *Arch. Math.* **10** (1959), 321–326.
1967 Sur le centre de l'algèbre enveloppante d'une algèbre de Lie, *C. R. Acad. Sci. Paris Ser. A* **265** (1967), 408–410.
1974 "Algèbres Enveloppants." Gauthier-Villars, Paris, 1974.
1975 Sur les algèbres enveloppantes de $\mathfrak{sl}(n, C)$ et $\mathfrak{af}(n, C)$. *Bull. Sci. Math* **100** (1975), 57–95.
1981 Sur les invariants des formes binaires. *C. R. Acad. Sci. Paris* **292** (1981), 987–990.

DIXMIER, J., and MALLIAVIN, P.
1978 Factorisation de fonctions et de vecteurs indéfiniment différentiables. *Bull. Sci. Math.* **102** (1978), 305–330.

DOOLEY, A. H.
1979 Norms of characters and lacunarity for compact Lie groups. *J. Funct. Anal.* **32** (1979), 254–267.

DRESELER, B.
1977 "On Summation Processes of Fourier Expansions for Spherical Functions," Lecture Notes in Math. Vol. 571, pp. 65–84. Springer-Verlag, Berlin and New York, 1977.
1981 Norms of zonal spherical functions and Fourier series on compact symmetric spaces. *J. Funct. Anal.* **44** (1981), 74–86.

DRESELER, B., and SCHEMPP, W.
1975 On the convergence and divergence behavior of approximation processes in homogeneous Banach spaces. *Math. Z.* **143** (1975), 81–89.

DUFLO, M.
1970 Fundamental series representations of a semisimple Lie group. *Funct. Anal. Appl.* **4** (1970), 122–126.
1972 Charactère des représentations des groupes résolubles associées a une orbite entière. *In* Bernat *et al.* [1972].
1975 Représentations irréductibles des groupes semi-simples complexes. *In* "Analyse Harmonique sur les Groupes de Lie," Lecture Notes in Math. No. 497, pp. 26–88. Springer-Verlag, Berlin and New York, 1975.
1976 Sur l'analyse harmonique sur les groupes de Lie résolubles. *Ann. Sci. Ecole Norm. Sup.* **9** (1976), 107–144.
1977 Opérateurs differentiels bi-invariants sur un groupe de Lie. *Ann. Sci. Ecole Norm. Sup.* **10** (1977), 265–288.
1979a Opérateurs invariants sur un espace symétrique. *C. R. Acad Sci. Paris Ser. A* **289** (1979), 135–137.

1979b Représentations unitaires irréductibles des groupes simples complexes de rang deux. *Bull. Soc. Math. France* **107** (1979), 55–96.

DUFLO, M., and WIGNER, D.
1978 Sur la résolubilité des équations differentielles invariantes sur un groupe de Lie. *Sém. Goulaouic-Schwartz, 1977–1978*. École-Polytech., Palaisesu, 1978.

DUISTERMATT, J. J., KOLK, J. A. C., and VARADARAJAN, V. S.
1979 Spectra of compact locally symmetric manifolds of negative curvature. *Invent. Math.* **52** (1979), 27–93.
1983 Functions, flows, and oscillatory integrals on flag manifolds and conjugacy classes in real semisimple Lie groups. Composite Math. 49 (1983), 309–398.

DUNFORD, N., and SCHWARTZ, J. J.
1958 "Linear Operators," Part I. Wiley (Interscience), New York, 1958.

DUNKL, C. F.
1977 Spherical functions on compact groups and applications to special functions. *Symposia Mathematica* **22**, (1977), 145–161.

DUNKL, C. F., and RAMIREZ, D.
1971 "Topics in Harmonic Analysis." Appleton, New York, 1971.

DYM, H., and MCKEAN, H. P.
1972 "Fourier Series and Integrals." Academic Press, New York, 1972.

EDWARDS, R. E.
1972 "Integration and Harmonic Analysis on Compact Groups," London Math. Soc. Lecture Notes Series No. 8. Cambridge Univ. Press, London and New York, 1972.

EDWARDS, R. E., and ROSS, K. A.
1973 Helgason's number and lacunarity constants. *Bull. Austral. Math. Soc.* **9** (1973), 187–218.

EGUCHI, M., and OKAMOTO, K.
1977 The Fourier transform of the Schwartz space on a symmetric space. *Proc. Japan Acad.* **53** (1977), 237–241.

EHRENPREIS, L.
1954 Solutions of some problems of division I. *Amer. J. Math.* **76** (1954), 883–903.
1956 Some properties of distributions on Lie groups. *Pacific J. Math.* **6** (1956), 591–605.
1969 "Fourier Analysis in Several Complex Variables." Wiley, New York, 1969.

EHRENPREIS, L., and MAUTNER, F.
1955 Some properties of the Fourier transform on semi-simple Lie groups I. *Ann. of Math.* **61** (1955), 406–443; II, III. *Trans. Amer. Math. Soc.* **84** (1957), 1–55; **901** (1959), 431–484.

ERDÉLYI, A., MAGNUS, W., OBERHETTINGER, F., and TRICOMI, F. G.
"Higher Transcendental Functions" (Bateman Manuscript Project), Vols. I, II, and III. McGraw-Hill, New York, 1953, 1953, 1955.

EYMARD, P.
1977 Le noyau de Poisson et la théorie des groupes. *Symposia Mathematica* **22** (1977), 107–132.

FARAUT, J.
1979 Distributions sphériques sur les espaces hyperboliques. *J. Math. Pures Appl.* **58** (1979), 369–444.
1982 Un théoreme de Paley–Wiener pour la transformation de Fourier sur un espace Riemannien symétrique de rang un. *J. Funct. Anal.* **49** (1982), 230–268.

FARAUT, J., and HARZALLAH, K.
1972 Fonctions sphériques de type positif sur les espaces hyperboliques. *C. R. Acad. Sci. Paris* **274** (1972), A1396–A1398.

FATOU, P.
1906 Séries trigonométriques et séries de Taylor. *Acta Math.* **30** (1906), 335–400.

FIGA-TALAMANCA, A., and RIDER, D.
1966 A theorem of Littlewood and lacunary series for compact groups. *Pacific J. Math.* **16** (1966), 505–514.
1967 A theorem on random Fourier series on noncommutative groups. *Pacific J. Math.* **21** (1967), 487–492.

FLATTO, L.
1978 Invariants of finite reflection groups. *Enseign. Math.* **24** (1978), 237–292.

FLENSTED-JENSEN, M.
1972 Paley–Wiener theorems for a differential operator connected with symmetric spaces. *Ark. Mat.* **10** (1972), 143–162.
1975 "Spherical Functions and Discrete Series," Springer Lecture Notes No. 466, pp. 65–78. Springer-Verlag, Berlin and New York, 1975.
1977a Spherical functions on a simply connected semisimple Lie group. *Amer. J. Math.* **99** (1977), 341–361.
1977b Spherical functions on a simply connected semisimple Lie Group, II. *Math. Ann.* **228** (1977), 65–92.
1978 Spherical functions on a real semi-simple Lie group. A method of reduction to the complex case. *J. Funct. Anal.* **30** (1978), 106–146.
1980 Discrete series for semisimple symmetric spaces. *Ann. of Math.* **111** (1980), 253–311.
1981 K-finite joint eigenfunctions of $U(\mathfrak{g})^K$ on a non-Riemannian semisimple symmetric space G/H. *In* "Non-Commutative Harmonic Analysis and Lie Groups," Lecture Notes in Math. No. 880. Springer-Verlag, Berlin and New York, 1981.

FLENSTED-JENSEN, M., and KOORNWINDER, T.
1973 The convolution structure for Jacobi function expansions. *Ark. Mat.* **11** (1973), 245–262.
1979a Positive-definite spherical functions on a noncompact rank one symmetric space. *In* "Analyse Harmonique sur les Groupes de Lie II," Springer Lecture Notes No. 739. Springer-Verlag, Berlin and New York, 1979.
1979b Jacobi functions: the addition formula and the positivity of the dual convolution structure. *Ark. Math.* **17** (1979), 139–151.

FLENSTED-JENSEN, M., and RAGOZIN, D. L.
1973 Spherical functions are Fourier transforms of L_1-functions. *Ann. Sci. École. Norm. Sup.* **6** (1973), 457–458.

FOLLAND, G. B., and STEIN, E.
1974 Estimates for the ∂_b-complex and analysis on the Heisenberg group. *Comm. Pure Appl. Math.* **27** (1974), 429–522.

FREUDENTHAL, H.

1954 Zur Berechnung der Charaktere der halbeinfachen Lieschen Gruppen. I. *Indag. Math.* **16** (1954), 363–368.

1956 The existence of a vector of weight 0 in irreducible Lie groups without centre. *Proc. Amer. Math. Soc.* **7** (1956), 175–176.

FREUDENTHAL, H., and DE VRIES, H.

1969 "Linear Lie Groups." Academic Press, New York, 1969.

FRIEDLANDER, F. C.

1975 "The Wave Equation in Curved Space." Cambridge Univ. Press, London and New York, 1975.

FROTA-MATTOS, L. A.

1978 Analytic continuation of the Fourier series on connected compact Lie groups. *J. Funct. Anal.* **29** (1978), 1–15.

FUGLEDE, B.

1958 An integral formula. *Math. Scand.* **6** (1958), 207–212.

FUNK, P.

1916 Über eine geometrische Anwendung der Abelschen Integralgleichung. *Math. Ann.* **77** (1916), 129–135.

FURSTENBERG, H.

1963 A Poisson formula for semisimple Lie groups. *Ann. of Math.* **77** (1963), 335–386.

1965 Translation-invariant cones of functions on semi-simple Lie groups. *Bull. Amer. Math. Soc.* **71** (1965), 271–326.

FURSTENBERG, H., and TZKONI, I.

1971 Spherical functions and integral geometry. *Israel Math. J.* **10** (1971), 327–338.

GÅRDING, L.

1947 Note on continuous representations of Lie groups. *Proc. Nat. Acad. Sci. U.S.A.* **33** (1947), 331–332.

1960 Vecteurs analytiques dans les représentations des groups de Lie. *Bull. Soc. Math. France* **88** (1960), 73–93.

1961 Transformation de Fourier des distributions homogènes. *Bull. Soc. Math. France* **89** (1961), 381–428.

GANGOLLI, R.

1967 Positive definite kernels on homogeneous spaces and certain stochastic processes related to Levy's Brownian motion of several parameters. *Ann. Inst. H. Poincaré Sect. B.* **3** (1967), 121–226.

1968 Asymptotic behaviour of spectra of compact quotients of certain symmetric spaces. *Acta Math.* **12** (1968), 151–192.

1971 On the Plancherel formula and the Paley-Wiener theorem for spherical functions on semisimple Lie groups. *Ann. of Math.* **93** (1971), 150–165.

1972 "Spherical functions on semisimple Lie group. Short Courses, Washington Univ." Dekker, New York, 1972.

GELFAND, I. M.

1950a Spherical functions on symmetric spaces. *Dokl. Akad. Nauk USSR* **70** (1950), 5–8. *Amer. Math. Soc. Transl.* **37** (1964), 39–44.

1950b The center of an infinitesimal group algebra. *Mat. Sb.* **26** (1950), 103–112.
1960 Integral geometry and its relation to group representations. *Russ. Math. Surveys* **15** (1960), 143–151.

GELFAND, I. M., and GRAEV, M. I.
1955 Analogue of the Plancherel formula for the classical groups. *Trudy Moscow. Mat. Obšč.* **4** (1955), 375–404, (also in Gelfand and Naimark [1957]).
1968 Complexes of straight lines in the space $\mathbb{C}^n$. *Funct. Anal. Appl.* **2** (1968), 39–52.

GELFAND, I. M., GRAEV, M. I., and SHAPIRO, S. J.
1969 Differential forms and integral geometry. *Funct. Anal. Appl.* **3** (1969), 24–40.

GELFAND, I. M., GRAEV, M. I., and VILENKIN, N.
1966 "Generalized Functions," Vol. 5, Integral Geometry and Representation Theory. Academic Press, New York, 1966.

GELFAND, I. M., and NAIMARK, M. A.
1948 An analog of Plancherel's formula for the complex unimodular group. *Dokl. Akad. Nauk USSR* **63** (1948), 609–612.

GELFAND, I. M., and NAIMARK, M. R.
1952 Unitary representation of the unimodular group containing the identity representation of the unitary subgroup. *Trudy Moscov. Mat. Obšč.* **1** (1952), 423–475.
1957 "Unitäre Darstellungen der Klassischen Gruppen." Akademie Verlag, Berlin, 1957.

GELFAND, I. M., and RAIKOV, D. A.
1943 Irreducible unitary representations of locally compact groups. *Mat. Sb.* **13** (1943), 301–316.

GELFAND, I. M., and SCHILOV, G. E.
1960 "Verallgemeinerte Funktionen," Vol. I. German Transl. VEB, Berlin, 1960.

GELFAND, I. M., and SHAPIRO, S. J.
1955 Homogeneous functions and their applications, *Uspehi Mat. Nauk.* **10** (1955), 3–70.

GINDIKIN, S. G.
1967 Unitary representations of groups of automorphisms of Riemannian symmetric spaces of null curvature. *Funct. Anal. Appl.* **1** (1967), 28–32.
1975 Invariant generalized functions in homogeneous domains. *Funct. Anal. Appl.* **9** (1975), 50–52.

GINDIKIN, S. G., and KARPELEVIČ, F. I.
1962 Plancherel measure of Riemannian symmetric spaces of non-positive curvature, *Dokl. Akad. Nauk SSSR* **145** (1962), 252–255.
1969 On an integral connected with symmetric Riemannian spaces of negative curvature. *Amer. Math. Soc. Transl. (2)* **85** (1969), 249–258.

GIULINI, S., SOARDI, P. M., and TRAVAGLINI, G.
1982 Norms of characters and Fourier series on compact Lie groups. *J. Funct. Anal.* **46** (1982), 88–101.

GODEMENT, R.
1948 Les fonctions de type positif et la théorie des groupes. *Trans. Amer. Math. Soc.* **63** (1948), 1–84.

1951 Sur la théorie des représentations unitaires. *Ann. of Math.* **53** (1951), 68–124.

1952a Une généralisation du théorème de la moyenne pour les fonctions harmoniques, *C. R. Acad. Sci. Paris* **234** (1952), 2137–2139.

1952b A theory of spherical functions I. *Trans. Amer. Math. Soc.* **73** (1952), 496–556.

1957a Introduction aux travaux de A. Selberg. *Séminaire Bourbaki* No. 144 *Paris* (1957).

1957b Articles in H. Cartan, Fonctions automorphes, Séminaire, 1957–1958, Paris.

1966 The decomposition of $L^2(G/\Gamma)$ for $\Gamma = SL(2, Z)$, *Proc. Symp. Pure Math.* **9** (1966), 211–244.

GODIN, P.

1982 Hypoelliptic and Gevrey hypoelliptic invariant differential operators on certain symmetric spaces. *Ann. Scuola Norm. Pisa* **IX** (1982), 175–209.

GOODMAN, R.

1969a Analytic domination by fractional powers of a positive operator. *J. Funct. Anal.* **3** (1969), 55–76.

1969b Analytic and entire vectors for representations of Lie groups. *Trans. Amer. Math. Soc.* **143** (1969), 55–76.

GRAY, A., and WILLMORE, T.

1982 Mean-value theorems for Riemmanian manifolds. *Proc. Roy. Soc. Edinburgh Sect. A* **92A** (1982), 343–364.

GRINBERG, E. L.

1983 Spherical harmonics and integral geometry on projective spaces. *Trans. Amer. Math. Soc.* **279** (1983), 187–213.

GROSS, K. I.

1978 On the evolution of noncommutative harmonic analysis. *Amer. Math. Mon.* **85** (1978), 525–548.

GROSS, K. I., HOLMAN, W. J., and KUNZE, R.

1979 A new class of Bessel functions and applications in harmonic analysis. *Proc. Symp. Pure Math.* **35** (1979), Part 2, Harmonic Analysis in Euclidean Spaces. American Mathematical Society, Providence, Rhode Island, 1979.

GÜNTHER, P.

1957 Über einige specielle Probleme aus der Theorie der linearen partiellen Differentialgleichungen 2. Ordnung. *Ber. Verh. Sächs. Akad. Wiss. Leipzig* **102** (1957), 1–50.

GUILLEMIN, V.

1976 Radon Transform on Zoll surfaces. *Advances in Math.* **22** (1976), 85–199.

1979a A Szegö-type theorem for symmetric spaces. *Ann. Math. Stud.* **93** (1979), 63–78.

1979b Some micro-local aspects of analysis on compact symmetric spaces. *Ann. Math. Stud.* **93** (1979), 79–111.

GUILLEMIN, V., and POLLACK, A.

1974 "Differential Topology." Prentice Hall, Englewood Cliffs, New Jersey, 1974.

GUILLEMIN, V., and STERNBERG, S.

1977 "Geometric Asymtotics," Math. Surveys No. 14. American Mathematical Society, Providence, Rhode Island, 1977.

1979 Some problems in integral geometry and some related problems in micro-local analysis. *Amer. J. Math.* **101** (1979), 915–955.

GUTKIN, E. A.

1970 Representations of the principal series of a complex semisimple Lie group. *Funct. Anal.* **4** (1970), 117–121.

HAAR, A.

1933 Der Maassbegriff in der Theorie der Kontinuerlichen Gruppen. *Ann. of Math.* **34** (1933), 147–169.

HARISH-CHANDRA

1951a On some applications of the universal enveloping algebra of a semi-simple Lie algebra. *Trans. Amer. Math. Soc.* **70** (1951), 28–96.

1951b Plancherel formula for complex semisimple Lie groups. *Proc. Nat. Acad. Sci. U.S.A.* **37** (1951), 813–818.

1952 Plancherel formula for the 2 × 2 real unimodular group. *Proc. Nat. Acad. Sci. U.S.A.* **38** (1952), 337–342.

1953 Representations of semisimple Lie groups, I. *Trans. Amer. Math. Soc.* **75** (1953), 185–243.

1954a Representations of semisimple Lie groups, II. *Trans. Amer. Math. Soc.* **76** (1954), 26–65.

1954b Representations of semisimple Lie groups, III. *Trans. Amer. Math. Soc.* **76** (1954), 234–253.

1954c Plancherel formula for the right invariant functions on a semisimple Lie group. *Proc. Nat. Acad. Sci. U.S.A.* **4** (1954), 200–204.

1954d The Plancherel formula for complex semisimple Lie groups. *Trans. Amer. Math. Soc.* **76** (1954), 485–528.

1956a Representations of semisimple Lie groups, VI. *Amer. J. Math.* **78** (1956), 564–628.

1956b The characters of semisimple Lie groups. *Trans. Amer. Math. Soc.* **83** (1956), 98–163.

1957a Differential operators on a semi-simple Lie algebra, *Amer. J. Math.* **79** (1957), 87–120.

1957b A formula for semisimple Lie groups. *Amer. J. Math.* **79** (1957), 733–760.

1958a Spherical functions on a semi-simple Lie group I. *Amer. J. Math.* **80** (1958), pp. 241–310.

1958b Spherical functions on a semisimple Lie group, II. *Amer. J. Math.* **80** (1958), 553–613.

1959 Some results on differential equations and their applications, *Proc. Nat. Acad. Sci. U.S.A.* **45** (1959), 1763–1764.

1960 Differential equations and semisimple Lie groups (1960) (unpublished).

1964a Invariant distributions on Lie algebras. *Amer. J. Math.* **86** (1964), 271–309.

1964b Invariant differential operators and distributions on a semi-simple Lie algebra. *Amer. J. Math.* **86** (1964), 534–564.

1964c Some results on an invariant integral on a semi-simple Lie algebra. *Ann. of Math.* **80** (1964), 551–593.

1966 Discrete series for semisimple Lie groups II. *Acta Math.* **116** (1966), 1–111.

1970 Harmonic analysis on semisimple Lie groups. *Bull. Amer. Math. Soc.* **78** (1970), 529–551

HASHIZUME, M., KOWATA, A., MINEMURA, K., and OKAMOTO, K.

1972 An integral representation of an eigenfunction of the Laplacian on the Euclidean space. *Hiroshima Math. J.* **2** (1972), 535–545.

HECKMAN, G. J.

1980 Projections of orbits and asymtotic behaviour of multiplicities for compact Lie groups. Thesis, Leiden, 1980.

HEINE, H. E.

1878–1880 "Kugelfunktionen I, II." Berlin, 1878–1880.

HEJHAL, D. A.

1976 "The Selberg Trace Formula for PSL(2, **R**)," Lecture Notes in Mathematics No. 548. Springer-Verlag, Berlin and New York, 1976.

HELGASON, S.

1957a Topologies of group algebras and a theorem of Littlewood. *Trans. Amer. Math. Soc.* **86** (1957), 269–283.

1957b Partial differential equations on Lie groups. *Scand. Math. Congr., 13th, Helsinki, 1957*, 110–115.

1958 Lacunary Fourier series on non-commutative groups. *Proc. Amer. Math. Soc.* **9** (1958), 782–790.

1959 Differential operators on homogeneous spaces. *Acta Math.* **102** (1959), 239–299.

1961 Some remarks on the exponential mapping for an affine connection. *Math. Scand.* **9** (1961), 129–146.

1962a "Differential Geometry and Symmetric Spaces." Academic Press, New York, 1962.

1962b Some results in invariant theory. *Bull. Amer. Math. Soc.* **68** (1962), 367–371.

1963 Invariants and fundamental functions. *Acta Math.* **109** (1963), 241–258.

1964a Fundamental solutions of invariant differential operators on symmetric spaces. *Amer. J. Math.* **86** (1964), 565–601.

1964b A duality in integral geometry: some generalizations of the Radon transform. *Bull. Amer. Math. Soc.* **70** (1964), 435–446.

1965a The Radon transform on Euclidean spaces, compact two-point homogeneous spaces and Grassmann manifolds. *Acta Math.* **113** (1965), 153–180.

1965b Radon–Fourier transforms on symmetric spaces and related group representations. *Bull. Amer. Math. Soc.* **71** (1965), 757–763.

1966a A duality in integral geometry on symmetric spaces, *Proc. U.S.-Japan Seminar in Differential Geometry, Kyoto, 1965.* Nippon Hyronsha, Tokyo, 1966.

1966b An analogue of the Paley–Wiener theorem for the Fourier transform on certain symmetric spaces. *Math. Ann.* **165** (1966), 297–308.

1968 Lie groups and symmetric spaces. "Battelle Rencontres," pp. 1–71. Benjamin, New York, 1968.

1970a A duality for symmetric spaces with applications to group representations. *Advan. Math.* **5** (1970), 1–154.

1970b Group representations and symmetric spaces. *Actes Congr. Internat. Math.* **2** (1970), 313–319.

1972a "Analysis on Lie Groups and homogeneous Spaces," Conf. Board Math. Sci. Series, No. 14. American Mathematical Society, Providence, Rhode Island, 1972.

1972b A formula for the radial part of the Laplace–Beltrami operator. *J. Differential Geometry* **6** (1972), 411–419.

1973a The surjectivity of invariant differential operators on symmetric spaces. *Ann. of Math.* **98** (1973), 451–480.

1973b Paley–Wiener theorems and surjectivity of invariant differential operators on symmetric spaces and Lie groups. *Bull. Amer. Math. Soc.* **79** (1973), 129–132.

1974 Eigenspaces of the Laplacian; integral representations and irreducibility. *J. Functional Anal.* **17** (1974), 328–353.

1976 A duality for symmetric spaces with applications to group representations, II. Differential equations and eigenspace representations. *Advan. Math.* **22** (1976), 187–219.

1977a Some results on eigenfunctions on symmetric spaces and eigenspace representations. *Math. Scand.* **41** (1977), 79–89.

1977b Invariant differential equations on homogeneous manifolds. *Bull. Amer. Math. Soc.* **83** (1977), 751–774.

1977c Solvability questions for invariant differential operators. In "Colloquim on Group Theoretical Methods in Physics." Academic Press, New York, 1977.

1978 "Differential Geometry, Lie Groups and Symmetric Spaces." Academic Press, New York, 1978.

1979 Invariant differential operators and eigenspace representations, pp. 236–286 in Atiyah *et al.* [1].

1980a A duality for symmetric spaces with applications to group representations, III. Tangent space analysis. *Advan. Math.* **30** (1980), 297–323.

1980b Support of Radon transforms. *Advan. Math.* **38** (1980), 91–100.

1980c "The Radon Transform." Birkhäuser, Basel and Boston, Massachusetts, 1980.

1980d The X-ray transform on a symmetric space. *Proc. Conf. Diff. Geom. and Global Anal., Berlin, 1979,* Lecture Notes in Math. No. 838. Springer-Verlag, New York, 1980.

1981 "Topics in Harmonic Analysis on Homogeneous Spaces." Birkhäuser, Basel and Boston, Massachusetts, 1981.

1983a Ranges of Radon Transforms. AMS Short Course on Computerized Tomography, January, 1982. *Proc. Symp. Appl. Math.* Amer. Math. Soc. Providence, Rhode Island, 1983.

1983b The Range of the Radon transform on Symmetric Spaces. In *Proc. Conf. Representation Theory of Reductive Lie Groups, Utah, 1982* (P. Trombi, ed.), pp. 145–151. Birkhäuser, Basel and Boston, Massachusetts, 1983.

1984 Wave equations on homogeneous spaces. *In* "Lie Group Representations III." Lecture Notes in Math. No. 1077, Springer-Verlag, New York, 1984, 254–287.

HELGASON, S., and JOHNSON, K.

1969 The bounded spherical functions on symmetric spaces. *Adv. in Math.* **3** (1969), 586–593.

HELGASON, S., and KORÀNYI, A.

1968 A Fatou-type theorem for harmonic functions on symmetric spaces. *Bull. Amer. Math. Soc.* **74** (1968), 258-263.

HELSON, H.

1953 A note on harmonic functions. *Proc. Amer. Math. Soc.* **4** (1953), 686–691.

HERTLE, A.

1982 A characterization of Fourier and Radon transforms on Euclidean spaces. *Trans. Amer. Math. Soc.* **273** (1982), 595–608.

1983 Continuity of the Radon transform and its inverse on Euclidean space. *Math. Zeitschr.* **184** (1983), 165–192.

1984 On the range of the Radon transform and its dual. *Math. Ann.* **267** (1984), 91–99.

HERZ, C.

1970 Functions which are divergences. *Amer. J. Math.* **92** (1970), 641–656.

HESSELINK, W. H.

1979 Desingularization of varieties of null forms. *Invent. Math.* **55** (1979), 141–163.

HEWITT, E., and ZUCKERMAN, H.

1959 Some theorems on lacunary series with extensions to compact groups. *Trans. Amer. Math. Soc.* **93** (1959), 1–19.

HEWITT, W., and ROSS, K. A.

1970 "Abstract Harmonic Analysis, II." Springer-Verlag, Berlin and New York, 1970.

HIRAI, T.

1968 The characters of some induced representations of semisimple Lie groups. *J. Math. Kyoto Univ.* **8** (1968), 313–363.

HIRAOKA, K., and MATSUMOTO, S., and OKAMOTO, K.

1977 Eigenfunctions of the Laplacian on a real hyperboloid of one sheet. *Hiroshima Math. J.* **7** (1977), 855–864.

HOBSON, E. W.

1931 "The Theory of Spherical and Elliptical Harmonics." Cambridge Univ. Press, London and New York, 1931.

HOLDER, E.

1938 Poissonsche Wellenformel in nichtenklidischen Raumen. Ber. Verh. Sachs. *Akad. Wiss. Leipzig* **90** (1938), 55–66.

HOLE, A.

1974 Invariant differential operators and polynomials of Lie transformation groups. *Math. Scand.* **34** (1974), 109–123.

1975 Representations of the Heisenberg group of dimension $2n + 1$ on eigenspaces. *Math. Scand.* **37** (1975), 129–141.

HOOGENBOOM, B.

1982 Spherical functions and differential operators on complex Grassmann manifolds. *Ark. Mat.* **20** (1982), 69–85.

1983 "Intertwining Functions and Compact Lie Groups." Proefschrift, Amsterdam, 1983.

HORN, A.

1954a Doubly stochastic matrices and the diagonal of a rotation matrix. *Amer. J. Math.* **76** (1954), 620–630.

1954b On the eigenvalues of a matrix with prescribed singular values. *Proc. Amer. Math. Soc.* **5** (1954), 4–7.

HOWE, R.

1980 On the role of the Heisenberg group in harmonic analysis. *Bull. Amer. Math. Soc.* **3** (1980), 821–843.

HUA, LO-KENG

1981 "Starting with the Unit Circle; Background to Higher Analysis." Springer-Verlag, Berlin and New York, 1981.

HURWITZ, A.

1897 Über die Erzeugung der Invarianten durch Integration. *Gött. Nachr.* (1897), 71–90.

HÖRMANDER, L.

1963 "Linear Partial Differential Operators." Springer-Verlag, Berlin and New York, 1963.

IWAHORI, N.

1959 On real irreducible representations on Lie algebras. *Nagoya Math. J.* **14** (1959), 59–83.

1964 On the structure of a Hecke ring of a Chevalley group over a finite field. *J. Fac. Sci. Univ. Tokyo* **10** (1964), 215–236.

IWAHORI, N., and MATSUMOTO, H.

1965 On some Bruhat decompositions and the structure of the Hecke rings of p-adic Chevalley groups. *Inst. Hautes Etudes Sci. Publ. Math.* **25** (1965), 5–48.

JACOBSEN, J.

1982 Invariant differential operators on some homogeneous spaces for solvable Lie groups. Preprint No. 34, Aarhus Univ., 1982.

1983 Eigenspace representations of nilpotent Lie groups, II. *Math. Scand.* **52** (1983), 321–333.

JACOBSEN, J., and STETKAER, H.

1981 Eigenspace representations of nilpotent Lie groups. *Math. Scand.* **48** (1981), 41–55.

JACOBSON, N.

1951 Completely reducible Lie algebras of linear transformations. *Proc. Amer. Math. Soc.* **2** (1951), 105–133.

1953 "Lectures in Abstract Algebra," Vol. II. Van Nostrand Reinhold, New York, 1953.

1962 "Lie Algebras." Wiley (Interscience), New York, 1962.

JACQUET, H.

1967 Fonctions de Whittaker associées aux groupes de Chevalley. *Bull. Soc. Math. France* **95** (1967), 243–309.

JAKOBSEN, H. P.

Intertwining differential operators for $Mp(n, R)$ and $SU(n, n)$. *Trans. Amer. Math. Soc.* **246** (1978), 311–337.

JAKOBSEN, H. P., and VERGNE, M.

1977 Wave and Dirac operators and representations of the conformal group. *J. Funct. Anal.* **24** (1977), 52–66.

JAMES, A. T.

1961 Zonal polynomials of the real positive definite symmetric matrices. *Ann. of Math.* **74** (1961), 456–469.

JOHN, F.

1934 Bestimmung einer Funktion aus ihren Integralen über gevisse Mannigfaltigkeiten. *Math. Ann.* **109** (1934), 488–520.

1935 Anhängigheit zwischen den Flächenintegralen einer stetigen Funktion. *Math. Ann.* **111** (1935), 541–559.

1938 The ultrahyperbolic differential equation with 4 independent variables. *Duke Math. J.* **4** (1938), 300–322.

1955 "Plane Waves and Spherical Means." Wiley (Interscience), New York, 1955.

JOHNSON, K. D.

1980 On a ring of invariant polynomials on a Hermitian symmetric space. *J. of Algebra* **67** (1980), 72–81.

JOHNSON, K., and KORÁNYI, A.

1980 The Hua operators on bounded symmetric domains of tube type. *Ann. of Math.* **111** (1980), 589–608.

KAPLAN, A., and PUTZ, R.

1977 Boundary behaviour of harmonic forms on a rank one symmetric space. *Trans. Amer. Math. Soc.* **231** (1977), 369–384.

KARPELEVIČ, F. I.

1962 Orispherical radial parts of Laplace operators on symmetric spaces. *Sov. Math.* **3** (1962), 528–531.

1965 The geometry of geodesics and the eigenfunctions of the Beltrami–Laplace operator on symmetric spaces. *Trans. Moscow Math. Obšč.* **14** (1965), 48–185; *Trans. Moscow Math. Soc.* (1965), 51–199.

KASHIWARA, M., KOWATA, A., MINEMURA, K., OKAMOTO, K., OSHIMA, T., and TANAKA, M.

1978 Eigenfunctions of invariant differential operators on a symmetric space. *Ann. of Math.* **107** (1978), 1–39.

KASHIWARA, M., and VERGNE, M.

1976 Remarque sur la covariance de certains opérateurs differentiels. *In* "Non-Commutative Harmonic Analysis." Lecture Notes in Math. No. 587, 119–137. Springer-Verlag, New York, 1976.

1978a The Campbell–Hausdorff formula and invariant distributions. *Invent. Math.* **47** (1978), 249–272.

1978b On the Segal–Shale–Weil representation and harmonic polynomials. *Invent. Math.* **44** (1978), 1–47.

1979 *K*-types and singular spectrum. *In* "Non-Commutative Harmonic Analysis." Lecture Notes in Math. No. 728. Springer-Verlag, Berlin and New York, 1979.

KHALGUI, M. S.

1982 Sur les characteres des groupes de Lie. *J. Funct. Anal.* **47** (1982), 64–77.

KIPRIJANOV, I. A., and IVANOV, L. A.

1981 The Euler–Poisson–Darboux equation in a Riemannian space. *Soviet Math. Dokl.* **24** (1981), 331–335.

KIRILLOV, A.

1962 Unitary representations of nilpotent Lie groups. *Russian Math. Surveys* **17** (1962), 53–104.

1968 The characters of unitary representations of Lie groups. *Funct. Anal. Appl.* **2** (1968), 40–55; **3** (1969), 36–47.

KNAPP, A. W., and SPEH, B.

1981 Status of classification of irreducible unitary representations. *Proc. Conf. on Harmonic Anal., 1981* Lecture Notes in Math. Vol. 908. Springer, New York, 1982.

KNAPP, A. W., and WILLIAMSON, R. E.

1971 Poisson integrals and semisimple groups. *J. Anal. Math.* **24** (1971), 53–76.

KNAPP, A. W., and ZUCKERMAN, G. J.
1982 Classification of irreducible tempered representations of semisimple groups. *Ann. of Math.* **116** (1982), 389–455.

KOBAYASHI, S., and NOMIZU, K.
1963 "Foundations of Differential Geometry" Vols. I and II. Wiley (Interscience), New York, 1963, 1969.

KÖTHE, G.
1969 "Topological Vector Spaces." Springer-Verlag, New York, 1969.

KOLK, J. A. C.
1977 "The Selberg Trace Formula and Asymtotic Behaviour of Spectra." Proefschrift, Utrecht, 1977.

KOLMOGOROFF, A.
1928 Über die Summen durch den Zufall bestimmter unabhängiger Grössen. *Math. Ann.* **99** (1928), 309–319; **102** (1930), 484–488.

KOORNWINDER, T. H.
1973 The addition formula for Jacobi polynomials and spherical harmonics. *SIAM J. Appl. Math.* **25** (1973), 236–246.
1974 Jacobi polynomials II. An analytic proof of the product formula. *SIAM J. Math. Anal.* **5** (1974), 125–137.
1975 A new proof of a Paley–Wiener theorem for the Jacobi transform. *Ark. Mat.* **13** (1975), 145–159.
1981 "Invariant Differential Operators on Non-reductive Homogeneous Spaces." Publ. Math. Centrum, Amsterdam, Report ZW 153, 1981.
1982 The representation theory of $SL(2\boldsymbol{R})$. A non-infinitesimal approach. *L'Enseign. Math.* **28** (1982), 53–90.

KORÁNYI, A.
1963 On the boundary values of holomorphic functions in wedge domains. *Bull. Amer. Math. Soc.* **69** (1963), 475–480.
1970 Generalizations of Fatou's theorem to symmetric spaces. *Rice Univ. Stud.* **56** (1970), 127–136.
1979 A survey of harmonic functions on symmetric spaces. *Proc. Symp. Pure Math.* Vol. 35, Part 1, Harmonic Analysis in Euclidean Spaces. American Mathematical Society, Providence, Rhode Island, 1979.
1980 Some applications of Gelfand pairs in classical analysis. *In* "Harmonic Analysis and Group Representations." CIME, Cortona, 1980.
1982a On the injectivity of the Poisson transform. *J. Funct. Anal.* **45** (1982), 293–296.
1982b Kelvin transforms and harmonic polynomials on the Heisenberg group. *J. Funct. Anal.* **49** (1982), 177–185.
1982c Geometric properties of Heisenberg type groups. *Advan. Math.* (to appear).

KORÁNYI, A., and MALLIAVIN, P.
1975 Poisson formula and compound diffusion associated to an overdetermined elliptic system on the Siegel halfplane of rank two. *Acta Math.* **134** (1975), 185–209.

KOSMAN, Y.
1975 Sur les degrés conformes des opérateurs differentiels. *C. R. Acad. Sci. Paris Ser. A* 280 (1975), 229–232.

KOSTANT, B.

1959a The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group. *Amer. J. Math.* **81** (1959), 973–1032.

1959b A formula for the multiplicity of a weight. *Trans. Amer. Math. Soc.* **93** (1959), 53–73.

1963 Lie group representations on polynomial rings. *Amer. J. Math.* **85** (1963), 327–404.

1973 On convexity, the Weyl group and the Iwasawa decomposition. *Ann. Sci. École Norm. Sup.* **6** (1973), 413–455.

1975a On the existence and irreducibility of certain series of representations. *In* "Lie Groups and Their Representations" (I. M. Gelfand, ed.), pp. 231–329. Halsted, New York, 1975.

1975b Verma modules and the existence of quasiinvariant partial differential operators. *In* "Non-Commutative Harmonic Analysis." Lecture Notes in Math. No. 466. Springer-Verlag, New York, 1975.

KOSTANT, B., and RALLIS, S.

1971 Orbits and Lie group representations associated to symmetric spaces. *Amer. J. Math.* **93** (1971), 753–809.

KOWALSKI, O. and VANHECKE, L.

1983 Opérateurs différentiels invariants et symmétries géodesignes préservant le volume. *C. R. Acad. Sci. Paris.* **296** (1983), 1001–1004.

KOWATA, A., and OKAMOTO, K.

1974 Harmonic functions and the Borel–Weil theorem. *Hiroshima Math. J.* **4** (1974), 89–97.

KOWATA, A., and TANAKA, M.

1980 Global solvability of the Laplace operator on a non-compact affine symmetric space. *Hiroshima Math. J.* **10** (1980), 409–417.

KRÄMER, M.

1979 Some remarks suggesting an interesting theory of harmonic functions on $SU(2n+1)/Sp(n)$ and $SO(2n+1)/U(n)$. *Arch. Math.* **33** (1979/80), 76–79.

KUBOTA, T.

1973 "Elementary Theory of Eisenstein Series." Halsted, New York, 1973.

KUČMENT, P. A.

1981 Representations of solutions of invariant differential equations on certain symmetric spaces. *Sov. Math. Dokl.* **24** (1981), 104–106.

LANG, S.

1975 "$SL_2(R)$." Addison-Wesley, Reading, Massachusetts, 1975.

LANGLANDS, R. P.

1973 On the classification of irreducible representations of real algebraic groups. Preprint, Princeton, 1973.

LASALLE, M.

1978 Séries de Laurent des fonctions holomorphes dans la complexification d'un espace symétrique compact. *Ann. Sci. Ecole Norm. Sup.* **11** (1978), 167–210.

1982 Transformeés de Poisson, algèbres de Jordan et équations de Hua. *C. R. Acad. Sci. Paris* **294** (1982), 325–328.

LAX, P. D., and PHILLIPS, R. S.
1967 "Scattering Theory." Academic Press, New York, 1967.
1978 An example of Huygens' principle. *Comm. Pure Appl. Math.* **31** (1978), 415–423.
1979 Translation representations for the solution of the non-Euclidean wave equation. *Comm. Pure. Appl. Math.* **32** (1979), 617–667.
1982 A local Paley–Wiener theorem for the Radon transform of L^2 functions in a non-Euclidean setting. *Comm. Pure Appl. Math.* **35** (1982), 531–554.

LEPOWSKY, J.
1975 On the Harish-Chandra homomorphism. *Trans. Amer. Math. Soc.* **208** (1975), 193–218.
1976 Cartan subspaces of symmetric Lie algebras. *Trans. Amer. Math. Soc.* **216** (1976), 217–228.
1977 Generalized Verma modules, the Cartan–Helgason theorem and the Harish-Chandra homomorphism. *J. Algebra* **49** (1977), 470–495.

LEPOWSKY, J., and MCCOLLUM, G. W.
1976 Cartan subspaces of symmetric Lie algebras. *Trans. Amer. Math. Soc.* **216** (1976), 217–228.

LEWIS, J. B.
1970 Eisenstein series on the boundary of the disk. Thesis, MIT, Cambridge, Massachusetts, 1970.
1978 Eigenfunctions on symmetric spaces with distribution-valued boundary forms. *J. Funct. Anal.* **29** (1978), 287–307.

LEWY, H.
1957 An example of a smooth linear partial differential equation without solution. *Ann. of Math.* **66** (1957), 155–158.

LICHNÈROWICZ, A.
1963 Opèrateurs differentiels invariants sur un espace symétrique. *C. R. Acad. Sci. Paris* **257** (1963), 3548–3550.
1964 Opèrateurs différentiels invariants sur un espace homogène. *Ann. Sci. École Norm. Sup.* **81** (1964), 341–385.

LICHNÈROWICZ, A., and WALKER, A. G.
1945 Sur les espaces Riemanniens harmoniques de type hyperbolique normal. *C. R. Acad. Sci. Paris* **221** (1945), 397–396.

LICHTENSTEIN, W.
1979 Qualitative behavior of special functions on compact symmetric spaces. *J. Funct. Anal.* **34** (1979), 433–455.

LIE, S.
1897 Die Theorie der Integralinvarianten ist ein Corollar der Theorie der Differentialinvarianten. *Ber. Sächs. Gesellsch. Leipzig* (1897), 342–357.

LIMIC, N., NIDERLE, J., and RACZKA, R.
1967 Eigenfunction expansions associated with the second-order invariant operator of hyperboloids and cones, III. *J. Math. Phys.* **8** (1967), 1079–1093.

LITTLEWOOD, J. E.
1924 On the mean value of power series. *Proc. London Math. Soc.* **25** (1924), 328–337.

LOHOUÉ, N. and RYCHENER, T.
1982 Die Resolvente von Δ auf symmetrischen Räumen vom nichtkompakten. *Typ. Comment. Math. Helv.* **57** (1982), 445–468.

LOOMIS, L. H.
1953 "Abstract Harmonic Analysis." Van Nostrand Reinhold, New York, 1953.

LOWDENSLAGER, D. B.
1958 Potential theory in bounded symmetric homogeneous complex domains. *Ann. of Math.* **67** (1958), 467–484.

LUCQUIAND, J.-C.
1978 Expression sous forme covariante des fonctions sphériques zonales attachées aux groupes SO(N) et SH(N). *C. R. Acad. Sci. Paris. Ser. A-B* **287** (1978), A67–A69.

LUDWIG, D.
1966 The Radon transform on Euclidean space. *Comm. Pure Appl. Math.* **23** (1966), 49–81.

MAASS, H.
1955 Die Bestimmung der Dirichletreihen mit Grössencharakteren zu den Modulformen n-ten Grades. *J. Indian Math. Soc.* **19** (1955), 1–23.
1956 Spherical functions and Quadratic Forms. *J. Indian Math. Soc.* **20** (1956), 117–162.
1958 Zur Theorie der Kugelfunktionen einer Matrixvariablen. *Math. Ann.* **135** (1958), 391–416.
1959 Zur Theorie der harmonischen Formen. *Math. Ann.* **137** (1959), 142–149.
1971 "Siegel's Modular Forms and Dirichlet Series," Lecture Notes in Mathematics No. 216. Springer-Verlag, Berlin and New York, 1971.

MACDONALD, I.
1968 Spherical functions on a p-adic Chevalley group. *Bull. Amer. Math. Soc.* **74** (1968), 520–525.
1972 Spherical Functions on Groups of p-adic Type. *Publ. Ramanujan Inst. Adv. Study No. 2. Madras, India* (1972).

MACKEY, G. W.
1952 Induced representations of locally compact groups, I. *Ann. of Math.* **55** (1952), 101–139.
1953 Induced representations of locally compact groups, II. *Ann. of Math,* **58** (1953), 193–221.
1976 "The Theory of Unitary Group Representations." Univ. of Chicago Press, Chicago, Illinois, 1976.
1978 "Unitary Group Representations in Physics, Probability and Number Theory." Benjamin/Cummings Publ., Reading, Massachusetts, 1978.
1980 Harmonic analysis as the exploitation of symmetry—a historical survey. *Bull. Amer. Math. Soc.* **3** (1980), 543–698.

MAGNUS, W., and OBERHETTINGER, R.
1948 "Formeln und Sätze fur die speciellen Funktionen der mathematischen Physik." Springer-Verlag, Berlin and New York, 1948.

MALGRANGE, B.

1955 Existence et approximation des solutions des équations aux dérivées partielles et des équations de convolution. *Ann. Inst. Fourier Grenoble* **6** (1955–56), 271–355.

MATSUMOTO, H.

1971 Quelques remarques sur les espaces riemanniens isotropes. *C. R. Acad. Sci. Paris* **272** (1971), 316–319.

MAURIN, K.

1968 "General Eigenfunction Expansions and Unitary Representations of Topological Groups." Polish Sci. Publ., Warszawa, 1968.

MAUTNER, F. I.

1950 Unitary representations of locally compact groups II. *Ann. of Math.* **52** (1950), 528–556.

1951 Fourier analysis and symmetric spaces. *Proc. Nat. Acad. Sci. U.S.A.* **37** (1951), 529–533.

1958, 1964 Spherical functions over p-adic fields, I, II. *Amer. J. Math.* **80** (1958), 441–457; **86** (1964), 171–200.

1969, 1970 Fonctions propres des opérateurs de Hecke, *C. R. Acad. Sci. Paris* **269** (1969), 940–943; **270** (1970), 89–92.

MAXWELL, J. C.

1892 "A Treatise on the Electricity and Magnetism," Vol. 1, 3rd ed. Oxford Univ. Press, London and New York, 1892.

MEANY, C.

1977 A Cantor–Lebesgue theorem for spherical convergence on a compact Lie group. *Proc. Ann. Sem. Canad. Math. Congr. Queens Univ. Kingston, Ontario* (1977).

MICHEL, R.

1972 Sur certains tenseurs symétriques de projectifs réels. *J. Math. Pures Appl.* **51** (1972), 275–293.

1973 Problemes d'analyse géometrique liés a la conjecture de Blaschke. *Bull. Soc. Math. France* **101** (1973), 17–69.

MICHELSON, H. L.

1975 A decomposition for certain real semisimple Lie groups. *Trans. Amer. Math. Soc.* **213** (1975), 177–193.

MILLER, J., and SIMMS, D. J.

1969 Radial limits of the rational functions 'Γ_μ of Harish-Chandra, *Proc. Roy. Irish Acad. Sect. A* **68** (1969), 41–47.

1970 A difference equation satisfied by the functions 'Γ_μ of Harish-Chandra, *Amer. J. Math.* **92** (1970), 362–368.

1973 On the coefficients of an asymptotic expansion of spherical functions on symmetric spaces. *Proc. Amer. Math. Soc.* **37** (1973), 448–452.

MILLER, W. JR.

1977 "Symmetry and Separation of Variables." Addison-Wesley, Reading, Massachusetts, 1977.

MIZONY, M.
1976 Algèbres et noyaux de convolution sur le dual sphérique d'un groupe de Lie semi-simple, noncompact et de rang 1. *Publ. Dep. Math. Lyon.* **13** (1976), 1–14.

MNEIMNE, R.
1983 Équation de chaleur sur un espace Riemannien symétrique et formule de Plancherel. *Bull. Sci. Math.* **107** (1983), 261–287.

MOLČANOV, V. F.
1968 Analogue of the Plancherel formula for hyperboloids. *Sov. Math. Dokl.* **9** (1968), 1385–1387.
1976 Spherical functions on hyperboloids. *Math. Sb.* **99** (141) (1976), 139–161, 295.

MOLIEN, TH.
1898 Über die Invarianten der linearen Substitutionsgruppen. *Berliner Sitznngsberichte* (1898), 1152–1158.

MOORE, C. C.
1964 Compactifications of symmetric spaces II; The Cartan domains. *Amer. J. Math.* **86** (1964), 358–378.

MOORE, C. C., and WOLF, J. A.
1971 Totally real representations and real function spaces. *Pacific J. Math.* **38** (1971), 537–542.

MORIMOTO, M.
1981 "Analytic Functionals on the Sphere and Their Fourier-Borel Transformations." Banach Center Publ., 1981, Warsaw.

MOSTOW, G. D.
1952 On the L^2-space of a Lie group. *Amer. J. Math.* **74** (1952), 920–928.
1962 Homogeneous spaces with finite invariant measure. *Ann. of Math.* **75** (1962), 17–37.

MÜLLER, C.
1966 "Spherical Harmonics," Lecture Notes in Math. No. 17. Springer-Verlag, Berlin and New York, 1966.

MYERS, S. B., and STEENROD, N.
1939 The group of isometries of a Riemannian manifold. *Ann. of Math.* **40** (1939), 400–416.

NACHBIN, L.
1965 "The Haar Integral." Van Nostrand Reinhold, New York, 1965.

NAGANO, T.
1959 Homogeneous sphere bundles and the isotropic Riemannian manifolds. *Nagoya Math. J.* **15** (1959), 29–55.

NARASIMHAN, R.
1968 "Analysis on Real and Complex Manifolds." North-Holland Publ., Amsterdam, 1968.

NELSON, E.
1959 Analytic vectors. *Ann. of Math.* **70** (1959), 572–615.

NEVANLINNA, R.
1936 "Eindeutische Analytische Funktionen." Springer-Verlag, Berlin and New York, 1936.

NUSSBAUM, E. A.
1975 Extension of positive definite functions and representations of functions in terms of spherical functions in symmetric spaces of noncompact type of rank one. *Math. Ann.* **215** (1975), 97–116.

ØRSTED, B.
1976 A note on the conformed quasi-invariance of the Laplacian on a pseudo-Riemannian manifold. *Lett. Math. Phys.* **1** (1976), 183–186.
1981a Conformally invariant differential equations and projective ,geometry. *J. Funct. Anal.* **44** (1981), 1–23.
1981b The conformal invariance of Huygens' principle. *J. Differential Geom.* **16** (1981), 1–9.

OLSANECKII, M. A.
1972 The asymtotic behavior of spherical functions. *Usp. Mat. Nauk* **27** (1972), 211–212.

ORIHARA, A.
1961 Bessel functions and the Euclidean motion group. *Tohoku Math. J.* **13** (1961), 66–74.

ORTNER, N.
1980 Faltung hypersingularer Integraloperatoren. *Math. Ann.* **248** (1980), 19–46.

OSHIMA, T.
1981 Fourier analysis on semisimple symmetric spaces. *In* "Non-Commutative Harmonic Analysis and Lie Groups," Lecture Notes in Math. No. 880, pp. 357–369, Springer-Verlag, Berlin and New York, 1981.

OSHIMA, T., and MATSUKI, T.
1980 Orbits on affine symmetric spaces under its action of the isotropy subgroups. *J. Math. Soc. Japan* **32** (1980), 399–414.

OSHIMA, T., and SEKIGUCHI, J.
1980 Eigenspaces of invariant differential operators on an affine symmetric space. *Invent. Math.* **57** (1980), 1–81.

OVSIANNIKOV, L. V.
1982 "Group Analysis of Differential Equations." Academic Press, New York, 1982.

PALAMODOV, V. P.
1970 "Linear Differential Operators with Constant Coefficients." Springer-Verlag, Berlin and New York, 1970.

PALEY, R., and WIENER, N.
1934 "Fourier Transforms in the Complex Domain." American Mathematical Society, Providence, Rhode Island, 1934.

PEETRE, J.
1959–1960 Une caractérization abstraite des opérateurs différentiels. *Math. Scand.* **7** (1959), 211–218; Rectification, *ibid.* **8** (1960), 116–120.

PENROSE, R.

1967 Twistor algebra. *J. Math. Phys.* **8** (1967), 345–366.

PETER, F., and WEYL, H.

1927 Die Vollständigkeit der primitiven Darstellungen einer geschlossenen kontinuerlichen Gruppe. *Math. Ann.* **97** (1927), 737–755.

PETROV, E. E.

1977 A Paley-Wiener theorem for a Radon complex. *Izv. Vyss. Ucebn. Zaved. Math.* **3**, (1977), 66–77.

POINCARÉ, H.

1887 Sur les residus des integrales doubles. *Acta Math.* **9** (1887), 321–380.

POULSEN, N. S.

1970 Regularity aspects of the theory of infinite dimensional representations of Lie groups. Ph.D. thesis, Massachusetts Institute of Technology, 1970.

1972 On C^∞ vectors and intertwining bi-linear forms for representations of Lie groups. *J. Funct. Anal.* **9** (1972), 87–120.

PUKANSZKY, L.

1964 The Plancherel formula for the universal covering group of $\mathbf{SL}(2, \mathbf{R})$. *Math. Ann.* **156** (1964), 96–143.

QUINTO, E. T.

1978 On the locality and invertibility of the Radon transform. Thesis, Massachusetts Institute of Technology (1978).

1980 The dependence of the generalized Radon transform on defining measures. *Trans. Amer. Math. Soc.* **257** (1980), 331–346.

1981 Topological restrictions on double fibrations and Radon transforms. *Proc. Amer. Math. Soc.* **81** (1981), 570–574.

1982 Null spaces and ranges for the classical and spherical Radon transforms. *J. Math. Anal. Appl.* **90** (1982), 408–420.

RADER, C.

1976 Spherical functions on a semisimple Lie groups. Lecture Notes, Univ. of Chicago, Chicago, Illinois, 1976.

RADON, J.

1917 Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten. *Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Nat. kl.* **69** (1917), 262–277.

RAGOZIN, D. L.

1972 Central measures on compact simple Lie groups. *J. Funct. Anal.* **10** (1972), 212–229.

1976 Approximation theory, absolute convergence, and smoothness of random Fourier series on compact Lie groups. *Math. Ann.* **219** (1976), 1–11.

RAIKOV, D. A.

1946 On the theory of normed rings with involution. *Dokl. Akad. Nauk USSR* **54** (1946), 387–390.

RAÏS, M.

1971 Solutions élémentaires des opérateurs différentiels bi-invariants sur un groupe de Lie nilpotent. *C. R. Acad. Sci. Paris* **273** (1971), 495–498.

1977 Invariant Differential Operators on Lie Groups. Lecture Notes, Bielefeld, 1977.
Sur le centre de l'algèbre envelopante de **GL**(n, **R**) (unpublished).

1983 Groupes linéaires compacts et fonctions C^∞ covariantes. *Bull. Sci. Math.* **107** (1983), 93–111.

RALLIS, S.

1968 Lie Group Representations Associated to Symmetric Spaces, Ph.D. Thesis, Massachusetts Institute of Technology, 1968.

RAO, R.

1972 Orbital integrals in reductive Lie groups. *Ann. of Math.* **96** (1972), 505–510.

RAUCH, J., and WIGNER, D.

1976 Global solvability of the Casimir operator. *Ann. of Math.* **103** (1976), 229–236.

REIMANN, H. M.

1982 Invariant differential operators in hyperbolic space. *Comment. Math. Helv.* **57** (1982), 412–444.

REITER, H.

1968 "Classical Harmonic Analysis on Locally Compact Groups." Oxford Univ. Press (Clarendon), London and New York, 1968.

RHEE, H.

1970 A representation of the solutions of the Darboux equation in odd-dimensional spaces. *Trans. Amer. Math. Soc.* **150** (1970), 491–498.

RICCI, F.

1977 Local properties of the central Wiener algebra on the regular set of a compact Lie group. *Bull. Sci. Math.* **101** (1977), 87–95.

RICHARDSON, R. W. JR.

1967 Conjugacy classes in Lie Algebras and algebraic groups, *Ann. of Math.* **86** (1967), 1–15.

RIDER, D.

1970 Central idempotent measures on unitary groups. *Canad. J. Math.* **22** (1970), 719–725.

1971 Central idempotent measures on SIN groups. *Duke Math. J.* **38** (1971), 189–191.

RIESZ, M.

1949 L'intégrale de Riemann-Liouville et le problème de Cauchy. *Acta Math.* **81** (1949), 1–223.

ROBERT, A.

1983 "Introduction to Representation Theory of Compact and Locally Compact Groups," London Math. Soc. Lecture Notes No. 80. Cambridge Univ. Press, London and New York, 1983.

ROSENBERG, J.

1977 A quick proof of Harish-Chandra's Plancherel theorem for spherical functions on a semisimple Lie group. *Proc. Amer. Math. Soc.* **63** (1977), 143–149.

ROSSMANN, W.

1978a Analysis on real hyperbolic spaces. *J. Funct. Anal.* **30** (1978), 448–477.

1978b Kirillov's character formula for reductive Lie groups. *Invent. Math.* **48** (1978), 207–220.

1980 Limit characters of reductive Lie groups. *Invent. Math.* **61** (1980), 53–66.

ROUVIERE, F.

1976 Sur la résolubilité locale des opérateurs bi-invariants. *Ann. Scuola Norm. Superore Pisa* **3** (1976), 231–244.

1976 Solutions distributions de l'opérateur de Casimir. *C. R. Acad. Sci. Paris Ser. A-B* **282** (1976), A853–A856.

1983 Sur la transformation d'Abel groupes de Lie semisimples de rang un. *Ann. Scuola Norm. Pisa* **10** (1983), 263–290.

RUDIN, W.

1959 Idempotent measures on abelian groups. *Pacific J. Math.* **9** (1959), 195–209.

1962 "Fourier Analysis on Groups." (Interscience), New York, 1962.

SAKAI, S.

1964 Weakly compact operators on operator algebras. *Pacific J. Math.* **14** (1964), 659–664.

SAMII, H.

1982 "Les Transformation de Poisson dans la Boule Hyperbolique." Thèse Univ. de Nancy, Nancy, 1982.

SANTALO, L.

1976 "Integral Geometry and Geometric Probability." Addison-Wesley, Reading, Massachusetts, 1976.

SATAKE, I.

1963 Theory of spherical functions on reductive algebraic groups over p-adic fields. *Inst. Hautes Etudes Sci. Publ. Math.* **18** (1963), 5–69.

SCHEMPP, W., and DRESELER, B.

1980 "Einführung in die Harmonische Analyse." Tuebner, Stuttgart, 1980.

SCHIFFMAN, G.

1971 Integrales d'entrelacement et fonctions de Whittaker. *Bull. Soc. Math. France* **99** (1971), 3–72.

1979 Travaux de Kostant sur la série principale. *In* "Analyse Harmonique sur les Groupes de Lie II," Springer Lecture Notes No. 739, pp. 460–510. Springer-Verlag, Berlin and New York, 1979.

SCHLICHTKRULL, H.

1984a One-dimensional K-types in finite-dimensional representations of semisimple Lie groups. A generalization of Helgason's theorem. *Math. Scand.* **54** (1984), 279–294.

1984b "Hyperfunctions and Harmonic Analysis on Symmetric Spaces." Birkhäuser, Boston, 1984 (to appear).

SCHMID, W.

1969 Die Randwerte holomorpher Funktionen auf Hermitesch symmetrischen Räumen, *Invent. Math.* **9** (1969), 61–80.

SCHUR, I.

1924 Neue Anwendungen der Integralrechnung auf Probleme der Invariantentheorie. I, II, III. *Sitz. Ber. Preuss. Akad. Wiss. Phys. Math. Kl.* (1924), 189–208, 297–321, 346–355.

SCHWARTZ, L.
1966 "Théorie des Distributions," 2nd ed. Hermann, Paris, 1966.

SEGAL, G.
1968 The representation ring of a compact Lie group. *Inst. Hautes Etudes Sci. Publ. Math.* **34** (1968), 113–128.

SEGAL, I. E.
1950 An extension of Plancherel's formula to separable unimodular groups. *Ann. of Math.* **52** (1950), 272–292.

SEKIGUCHI, J.
1980 Eigenspaces of the Laplace–Beltrami operator on a hyperboloid. *Nagoya Math. J.* **79** (1980), 151–185.

SELBERG, A.
1956 Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series. *J. Indian Math. Soc.* **20** (1956), 47–87.
1962 Discontinuous groups and harmonic analysis. *Proc. Internat. Congr. Math., Stockholm* (1962), 177–189.

SEMYANISTY, V. I.
1960 On some integral transforms in Euclidean space. *Sov. Math. Dokl.* **1** (1960), 1114–1117.
1961 Homogeneous functions and some problems of integral geometry in spaces of constant curvature. *Sov. Math. Dokl.* **2** (1961), 59–62.

SERRE, J. P.
1966 "Algèbres de Lie Semi-simple Complexes." Benjamin, New York, 1966.
1970 "Cours d'arithmetique." Presses Univ. de France, Paris, 1970.

SÉMINAIRE "SOPHUS LIE"
1955 1. "Théorie des algèbres de Lie; Topologie des groupes de Lie " École Norm. Sup., Paris, 1955.

SHELSTAD, D.
1979 Characters and inner forms of a quasi-split group over **R**. *Composito Math.* **39** (1979), 11–45.

SHEPHARD, G., and TODD, J.
1954 Finite unitary reflection groups. *Canad. J. Math.* **6** (1954), 274–304.

SHEPP, L. A., *et al.*
1983 AMS short courses on computerized tomography, Cincinnati, January 1982; *Proc. Symp. Appl. Math. Amer. Math. Soc., Providence, Rhode Island* **27** (1983).

SHEPP, L. A., and KRUSKAL, J. B.
1978 Computerized Tomography; the new medical X-ray technology. *Amer. Math. Mon.* (1978), 420–438.

SHERMAN, T.
1975 Fourier analysis on the sphere. *Trans. Amer. Math. Soc.* **209** (1975), 1–31.
1977 Fourier analysis on compact symmetric space. *Bull. Amer. Math. Soc.* **83** (1977), 378–380.

SHINYA, H.

1977 Spherical matrix functions on locally compact groups of a certain type. *J. Math. Kyoto Univ.* **17** (1977), 501–509.

SIDON, S.

1932 Ein Satz über die Fourierschen Reihen stetiger Funktionen. *Math. Z.* **34** (1932), 485–486.

SITARAM, A.

1980 An analogue of the Wiener–Tauberian theorem for spherical transforms on semisimple Lie group. *Pacific J. Math.* **89** (1980), 439–445.

SMITH, K. T., and SOLMON, D. C.

1975 Lower-dimensional integrability of L^2 functions. *J. Math. Anal. Appl.* **51** (1975), 539–549.

SMITH, K. T., SOLMON, D. C., and WAGNER, S. L.

1977 Practical and mathematical aspects of the problem of reconstructing objects from radiographs. *Bull. Amer. Math. Soc.* **83** (1977), 1227–1270; Addendum. *Ibid.* **84** (1978), 691.

SOLMON, D. C.

1976 The X-ray transform, *J. Math. Anal. Appl.* **56** (1976), 61–83.

SOLOMON, L.

1963 Invariants of finite reflection groups, *Nagoya Math. J.* **22** (1963), 57–64.

1966 The orders of the finite Chevalley groups. *J. Algebra* **3** (1966), 376–393.

SPRINGER, T. A.

1968 Weyl's character formula for algebraic groups, *Invent. Math.* **5** (1968), 85–105.

STANTON, R. J.

1976 On mean convergence of Fourier series on compact Lie groups. *Trans. Amer. Math. Soc.* **218** (1976), 61–87.

STANTON, R. J., and THOMAS, P. A.

1978a Polyhedral summability of Fourier series on compact Lie groups. *Amer. J. Math.* **100** (1978), 477–493.

1978b Expansions for spherical functions on noncompact symmetric spaces. *Acta Math.* **140** (1978), 251–276.

1979 Pointwise inversion of the spherical transform on $L^2(G/K)$ $(1 \leq p < 2)$. *Proc. Amer. Math. Soc.* **73** (1979), 398–404.

STEIN, E. M.

1970 Some problems in harmonic analysis suggested by symmetric spaces and semisimple Lie groups. *Actes Congr. Internat. Math.* **1** (1970), 173–189.

STEIN, E. M., and WEISS, G.

1968 Generalization of the Cauchy–Riemann equations and representations of the rotation group. *Amer. J. Math.* **90** (1968), 163–196.

STEINBERG, R.

1961 A general Clebsch–Gordon theorem. *Bull. Amer. Math. Soc.* **67** (1961), 406–407.

1964 Differential equations invariant under finite reflection groups. *Trans. Amer. Math. Soc.* **112** (1964), 392–400.

1965 Regular elements of semi-simple algebraic groups, *Inst. Hautes Etudes Sci. Publ. Math.* **25** (1965), 49–80.

STERN, A. I.

1969 Completely irreducible class I representations of real semi-simple Lie group. *Sov. Math.* **10** (1969), 1254–1257.

STOLL, W.

1952 Mehrfache Integrale auf komplexen Mannigfaltigkeiten. *Math. Z.* **57** (1952), 116–154.

STRICHARTZ, R. S.

1973 Harmonic analysis on hyperboloids. *J. Funct. Anal.* **12** (1973), 341–383.

1974 Multiplier transformations on compact Lie groups and algebras. *Trans. Amer. Math. Soc.* **193** (1974), 99–110.

1975 The explicit Fourier decomposition of $L^2(\mathbf{SO}(n)/\mathbf{SO}(n-m))$. *Canad. J. Math.* **27** (1975), 294–310.

1981 L^p-Estimates for Radon transforms in Euclidean and non-Euclidean spaces. *Duke Math. J.* **48** (1981), 699–727.

SUGIURA, M.

1960 Spherical functions and representations theory of compact Lie groups. *Sci. Papers Coll. Gen. Ed. Univ. Tokyo* **10** (1960), 187–193.

1962 Representations of compact groups realized by spherical functions on symmetric spaces. *Proc. Japan Acad.* **38** (1962), 111–113.

1971 Fourier series of smooth functions on compact Lie groups. *Osaka Math. J.* **8** (1971), 33–47.

1975 "Unitary Representations and Harmonic Analysis." Wiley, New York, 1975.

SULANKE, R.

1966 Croftonsche Formeln in Kleinschen Räumen. *Math. Nachr.* **32** (1966), 217–241.

TAKAHASHI, R.

1961 Sur les fonctions sphériques et la formule de Plancherel dans le groupe hyperbolique, *Japanese J. Math.* **31** (1961), 55–90.

1963 Sur les représentations unitaires des groupe de Lorentz généralisés. *Bull. Soc. Math. France* **91** (1963), 289–433.

1977a Fonctions Sphériques zonales sur $\mathbf{U}(n, n+k; \mathbf{F})$. *In* Séminaire d'Analyse Harmonique (1976–77)." Fac. de Sciences, Tunis, 1977.

1977b Spherical functions in **Spin** $(1, d)/\mathbf{Spin}(d-1)$. In "Non-commutative Harmonic Analysis," Lecture Notes in Math. Vol. 587. Springer-Verlag, Berlin and New York, 1977.

1979 Quelque resultats sur l'analyse harmonique dans l'espace symétrique noncompact de rang 1 du type exceptionel. *In* "Analyse Harmonique sur les Groupe de Lie II," Springer Lecture Notes No. 739. Springer-Verlag, Berlin and New York, 1979.

TAMAGAWA, T.

1960 On Selberg's trace formula. *J. Fac. Sci. Univ. Tokyo* **8** (1960), 363–386.

TAYLOR, M.

1968 Fourier series on compact Lie groups. *Proc. Amer. Math. Soc.* **19** (1968), 1103–1105.

TEDONE, O.

1898 Sull' integrazionne dell'equazione $\partial^2 I/\partial t^2 - \sum \partial^2 I/\partial_i^2 = 0$. *Ann. Mat.* **1** (1898), 1–24.

TERRAS, A.

1982 Non-Euclidean harmonic analysis. *SIAM Rev.* **24** (1982), 159–193.

THOMAS, E. G. F.

1984 An infinitesimal characterization of Gelfand pairs. *Proc. Conf. Modern Anal. and Probability*. Yale Univ., 1984.

THOMPSON, C.

1971 Inequalities and partial orders on matrix spaces. *Indiana Univ. Math. J.* **21** (1971), 469–480.

TITCHMARSH, E. C.

1939 "The Theory of Functions," 2nd ed. Oxford Univ. Press, London and New York, 1939.

TITS, J.

1955 Sur certains classes d'espaces homogènes de groupes de Lie. *Acad. Roy. Belg. Cl. Sci. Mem. Coll.* **29** (1955), No. 3.

1967 "Tabellen zu den einfachen Lie Gruppen und ihren Darstellungen," Lecture Notes in Mathematics 40. Springer-Verlag, Berlin and New York, 1967.

TORASSO, P.

1977 Le théorème de Paley–Wiener pour l'espace des fonctions indéfinement differentiables et a support compact sur un espace symétrique de type noncompact. *J. Funct. Anal.* **26** (1977), 201–213.

TRÈVES, F.

1963 Équations aux dérivees partielles inhomogènes a coefficients constants dépendent de paramètres. *Ann. Inst. Fourier Grenoble* **13** (1963), 123–138.

1967 "Topological Vector Spaces, Distributions and Kernels." Academic Press, New York, 1967.

TRICERRI, F., and VANHECKE, L.

1983 "Homogeneous Structures on Riemannian Manifolds." London Math. Soc. Lect. Notes, 83. Cambridge Univ. Press, 1983.

TROMBI, P., and VARADARAJAN, V. S.

1971 Spherical transforms on semi-simple Lie groups. *Ann. of Math.* **94** (1971), 246–303.

VAN DEN BAN, E. P.

1982 "Asymtotic Expansions and Integral Formulas for Eigenfunctions on a Semisimple Lie Group." Proefscrift, Utrecht, 1982.

VAN DIJK, G.

1969a On symmetry of group algebras of motion groups. *Math. Ann.* **179** (1969), 219–226.

1969b Spherical functions on the p-adic group PGL(2). *Indag. Math.* **31** (1969), 213–241.

VARADARAJAN, V. S.

1968 On the ring of the invariant polynomials on a semi-simple Lie algebra. *Amer. J. Math.* **90** (1968), 308–317.

1974 "Lie Groups, Lie Algebras and their Representations." Prentice-Hall, Englewood Cliffs, New Jersey, 1974.

1977 "Harmonic Analysis on Real Reductive Groups," Springer Lecture Notes No. 576. Springer-Verlag, Berlin and New York, 1977.

VERGNE, M.

1979 On Rossmann's character formula for discrete series. *Invent. Math.* **54** (1979), 11–14.

VILENKIN, N.

1957 On the theory of associated spherical functions on Lie groups. *Mat. Sb.* **42** (1957), 485–496.

1968 "Special Functions and the Theory of Group Representations," Translations of Math. Mono., Vol. 22. American Mathematical Society, Providence, Rhode Island, 1968.

VOGAN, D.

1981 "Representations of Real Reductive Groups." Birkhäuser, Basel and Boston, Massachusetts, 1981.

VRETARE, L.

1975 On L_p Fourier multipliers on certain symmetric spaces. *Math. Scand.* **37** (1975), 111–121.

1976 Elementary spherical functions on symmetric spaces. *Math. Scand.* **39** (1976), 343–358.

1977 On a recurrence formula for elementary spherical functions on symmetric spaces and its applications. *Math. Scand.* **41** (1977), 49–112.

WALLACH, N.

1973 "Harmonic Analysis in Homogeneous Spaces." Dekker, New York, 1973.

WANG, H. C.

1952 Two-point homogeneous spaces. *Ann. of Math.* **55** (1952), 177–191.

WARNER, F.

1970 "Foundations of Differentiable Manifolds and Lie Groups." Scott Foresman, Glenview, Illinois, 1970.

WARNER, G.

1972 "Harmonic Analysis on Semisimple Lie Groups," Vols. I, II. Springer-Verlag, Berlin and New York, 1972.

WEIL, A.

1940 "L'intégration dans les groupes topologiques et ses applications." Hermann, Paris, 1940.

1958 "Variétés Kähleriennes." Hermann, Paris, 1958.

WEISS, B.

1967 Measures that vanish on half spaces. *Proc. Amer. Math. Soc.* **18** (1967), 123–126.

WEISS, G.

1976 Harmonic analysis on compact groups. *MAA Stud. Math.* **13** (1976), 198–223.

WELLS, R., JR.

1979 Complex manifolds and mathematical physics. *Bull. Amer. Soc.* **1** (1979), 296–336.

WENDEL, J. G.
1952 Left centralizers and isomorphisms of group algebras. *Pacific J. Math.* **2** (1952), 251–261.
1954 Haar measure and the semigroup of measures on a compact group. *Proc. Amer. Math. Soc.* **5** (1954), 923–929.

WEYL, H.
1910 Über gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörigen Entwicklungen willkürlicher Funktionen. *Math. Ann.* **68** (1910), 220–269.
1925–1926 Theorie der Darstellung kontinuerlicher halbeinfacher Gruppen durch lineare Transformationen I, II, III und Nachtrag. *Math. Z.* **23** (1925), 271–309; **24** (1926), 328–376, 377–395, 789–791.
1931 "The Theory of Groups and Quantum Mechanics." Dover, New York, 1931.
1934 Harmonics on homogeneous manifolds. *Ann. of Math.* **35** (1934), 486–499.
1939 "The Classical Groups." Princeton Univ. Press, Princeton, New Jersey, 1939.

WHITTAKER, E. T., and WATSON, G. N.
1927 "A Course of Modern Analysis." Cambridge Univ. Press, London and New York, 1927.

WIGNER, D.
1977 Bi-invariant operators on nilpotent Lie groups. *Invent. Math.* **41** (1977), 259–264.

WILLMORE, T. J.
1950 Mean value theorems in harmonic Riemann spaces. *J. London Math. Soc.* **25** (1950), 54–57.

WOLF, J. A.
1967 "Spaces of Constant Curvature." McGraw-Hill, New York, 1967.

YAMABE, H.
1960 On the deformation of Riemannian structures on compact manifolds. *Osaka J. Math.* **12** (1960), 21–37.

YANG, J.-G.
1983 A proof of a formula in Fourier analysis on the sphere. *Proc. Amer. Math. Soc.* **88** (1983), 602–604.

YANO, K., and OBATA, M.
1970 On conformal changes of Riemannian metrics. *J. Differential Geometry* **4** (1970), 53–72.

ZALCMAN, L.
1980 Offbeat integral geometry. *Amer. Math. Monthly* **87** (1980), 161–175.
1982 Uniqueness and nonuniqueness for the Radon transforms. *Bull. London Math. Soc.* **14** (1982), 241–245.

ZARISKI, O., and SAMUEL, P.
1958 "Commutative Algebra," Vol. I. Van Nostrand Reinhold, New York, 1958.
1960 "Commutative Algebra," Vol. II. Van Nostrand Reinhold, New York, 1960.

ZELOBENKO, D. P.
1963 On the theory of representations of complex and real Lie groups. *Trudy Moscov. Mat. Obšč.* **12** (1963), 53–98.

1973 "Compact Lie Groups and their Representations." Transl. of Math. Monogr., American Mathematical Society, Providence, Rhode Island, 1973.

1974 "Harmonic Analysis on Complex Semisimple Lie Groups." WAUKA, Moscow, 1974 (in Russian).

ZYGMUND, A.

1930 On the convergence of lacunary trigonometric series. *Fundamental Matem.* **16** (1930), 90–107.

1959 "Trigonometric Series," 2nd ed., I, II. Cambridge Univ. Press, London and New York, 1959.

SYMBOLS FREQUENTLY USED

Ad: adjoint representation of a Lie group, 87, 282
ad: adjoint representation of a Lie algebra, 282
$A(r)$: spherical area, 165
$A(g)$: component in $g = n \exp A(g)\, k$, 303
$\check{\alpha}$: coroot, 192
A_α: root vector, 267
$\mathscr{A}(B)$: space of analytic functions on B, 35, 278
$\mathscr{A}'(B)$: space of analytic functionals (hyperfunctions) on B, 35, 278
$\mathfrak{a}$, $\mathfrak{a}^c$, $\mathfrak{a}^*$, $\mathfrak{a}_c^*$: abelian subspaces and their duals, 181, 303
$\mathfrak{a}_{c}^{*}$: subset of $\mathfrak{a}_c^*$, 434
$\mathfrak{a}^+$, $\mathfrak{a}_+^*$: Weyl chambers in $\mathfrak{a}$ and $\mathfrak{a}^*$, 181, 430
$B_r(p)$: open ball with radius r, center p, 44, 118, 156
B: Killing form, 164
$\beta_A(0)$: ball in $\boldsymbol{P}^n$, 118
Card: cardinality, 188
Cl: closure, 3
conj: complex conjugate, 391
$\boldsymbol{C}^n$: complex n-space, 15
C_n: special set, 136
$C(X)$: space of continuous functions of X, 3
$C_c(X)$: space of continuous functions of compact support, 3
$C^m(X)$: space of functions with continuous partial derivatives of order $\leq m$, 3
$C^\infty(X)$, $C_c^\infty(X)$: set of differentiable functions, set of differentiable functions of compact support, 3, 239
$c(\lambda)$: Harish-Chandra's c-function, 39, 430, 434
C^+, ${}^-C$, ${}^+C$, C^-: closures of Weyl chambers and their duals, 459
$\mathscr{D}(X)$: $C_c^\infty(X)$, 3, 239
$\mathscr{D}'(X)$: set of distributions on X, 3, 240
$\mathscr{D}_K(X)$: set of $f \in \mathscr{D}$ with support in K, 239
$\mathscr{D}_H(\boldsymbol{P}^n)$: subspace of $\mathscr{D}(\boldsymbol{P}^n)$, 100
$\mathscr{D}^\natural(X)$, $\mathscr{D}'^\natural(X)$: space of K-invariant elements in $\mathscr{D}(X)$, $\mathscr{D}'(X)$, 43
$\mathscr{D}^\natural(G)$, $\mathscr{D}'^\natural(G)$: space of K-bi-invariant members of $\mathscr{D}(G)$, $\mathscr{D}'(G)$, 292
$\boldsymbol{D}(G)$: set of left-invariant differential operators on G, 274
$\boldsymbol{D}_H(G)$: subalgebra of $\boldsymbol{D}(G)$, 284
$\boldsymbol{D}(G/H)$: set of G-invariant differential operators on G/H, 274
$d(\delta)$ or d_δ: dimension (=degree) of a representation, 391
$\Delta(D)$: radial part of D, 259
δ: density function, 260
$\boldsymbol{E}(M)$: set of all differential operators on M, 241

$\mathscr{E}(X)$: $C^\infty(X)$, 3, 239
$\mathscr{E}^\natural(X)$, $\mathscr{E}'^\natural(X)$: space of K-invariant elements in $\mathscr{E}(X)$, $\mathscr{E}'(X)$, 292
$\mathscr{E}_\lambda$: eigenspace, 12, 35, 279
$E_{a,b}$: space, 4
E': space of entire functionals, 5
$F(a, b; c; z)$: hypergeometric function, 50, 484
ϕ_λ: spherical function, 38, 418
$\Phi_{\lambda,m}$: generalized spherical function, 50
$\boldsymbol{G}(d, n)$, $\boldsymbol{G}_{d,n}$: manifolds of d-planes, 124
$\boldsymbol{GL}(n, \boldsymbol{R})$, $\boldsymbol{GL}(n, C)$: classical groups, 222
$\mathscr{H}(Z)$: space of holomorphic functions on Z, 15
$\mathscr{H}^A(\boldsymbol{C}^n)$: exponential type, 15
Hom(V, W): space of linear transformations of V into W, 392
$\boldsymbol{H}^n$: hyperbolic space, 29, 152, 177
$H(E)$: space of harmonic polynomials on E, 346
$\mathscr{H}$: Hilbert transform, 113
$H(g)$: component in $g = k \exp H(g) n$, 277
Im: imaginary part, 2
$I(E)$: space of invariant polynomials on E, 346
$I_{p,q}$: matrix, 201
I^γ: Riesz potential, 135
$J_n(z)$: Bessel function, 11
$\mathscr{K}(X)$: $C_c(X)$, 396
χ_δ: character of δ, 391
$\mathfrak{k}$: algebra in Cartan decomposition, 186
$L^1(X)$: space of integrable functions on X, 414
$L^p(X)$: space of f with $|f|^p \in L^1(X)$, 510
L: diameter of X, 164
$L = L_X$: Laplace–Beltrami operator on X, 31, 244
$L(g) = L_g$: left translation by g, 85
l: left regular representation, 386
$l(\delta)$: dimension, 394
$l(s)$: length function, 442
λ: symmetrization, 282
Λ: set of integral functions (weights), 498, operator, 113, root lattice, 307
$\Lambda(+)$: set of dominant integral functions, 498
$\Lambda(\pi)$: set of weights of π, 498
$\Lambda(E)$: Grassmann algebra over E, 354
M_p: the tangent space to a manifold M at p, 84
M^r: mean-value operator, 103, 288
$\boldsymbol{M}(n)$: group of isometries of $\boldsymbol{R}^n$, 4
$n(s)$: cardinality of Σ_s^+, 441
$\mathscr{N}$: set of nilpotent elements, 369
$\boldsymbol{O}(p, q)$: orthogonal groups, 200
Ω_n: area of $\boldsymbol{S}^{n-1}$, 18
$\boldsymbol{P}^n$: set of hyperplanes in $\boldsymbol{R}^n$, 97
$\mathfrak{p}$: part of a Cartan decomposition, 181
$\boldsymbol{R}^n$: real n-space, 3
$\boldsymbol{R}^+$: set of reals ≥ 0, 2
Re: real part, 2

R_g or $R(g)$: right translation by g, 85
r: right regular representation, 386
Res: residue, 133
ρ : half sum of roots, 181
$\mathfrak{r}$: set of regular elements, 369
$\mathscr{R}$: set of quasi-regular elements, 369
S^n: n-sphere, 3
$S_r(p)$: sphere or radius r and center p, 103, 312
$\mathscr{S}(\boldsymbol{R}^n)$: space of rapidly decreasing functions on $\boldsymbol{R}^n$
$\mathscr{S}^*(\boldsymbol{R}^n)$, $\mathscr{S}_0(\boldsymbol{R}^n)$: subspaces of $\mathscr{S}(\boldsymbol{R}^n)$, 104
$\mathscr{S}'(\boldsymbol{R}^n)$: space of tempered distributions, 131, 132
$S(V)$: symmetric algebra over V, 346
sgn(x): signum function, 114
$\mathscr{S}_H(\boldsymbol{P}^n)$: subspace of $\mathscr{S}(\boldsymbol{P}^n)$, 99, 100
sh x: sinh x, 36
Σ, Σ^+, Σ_0^+, Σ_s^+ sets of restricted roots, 441
$\mathscr{S}$: set of semisimple elements, 369
th x: tanh x, 76
${}^tA$: transpose of A, 388
Tr(A): trace of A, 221
$\tau(x)$: translation on G/H, 88
$U(\mathfrak{g})$: universal enveloping algebra of $\mathfrak{g}$, 503
$\boldsymbol{U}(n)$: unitary group, 515
V_δ: representation space of δ, 391
V_δ^M: space of fixed vectors under $\delta(M)$, 394
$V(\delta)$: space of K-finite vectors of type δ, 395
V^∞: space of differentiable vectors, 387
W: Weyl group, 194
$\boldsymbol{Z}$, $\boldsymbol{Z}^+$: the integers, the nonnegative integers, 2
$\boldsymbol{Z}(G)$: center of $D(G)$
$Z(\mathfrak{g})$: center of the universal enveloping algebra $U(\mathfrak{g})$, 503
$\tilde{}$: Fourier transform, 4, 33, 457; spherical transform, 39, 415, 449
$\hat{}$: Radon transform, 76, 97; incidence 140
$\check{}$: Dual Radon transform, 97; incidence, 140
$*$, $\times$: convolutions, 22, 43, 44, 99, 119, 132, 289, 290; adjoint operation, 241, 513; pullback, 83; star operator, 330
$\oplus$: direct sum, 601
$\otimes$: tensor product, 391
$\wedge$: exterior product, 354
$\langle , \rangle$: inner product, 18, 29, 32, 186, 347
${}^\natural$, $E^\natural$: space of K-invariants in E, 43, 292
$\| \|_m^S$: norm on $C^\infty(V)$, 234
$\square$: operator 98, 337; wave operator, 208

Set theory. Let A and B be sets. We use the usual symbols $\in$, $\subset$, $\cap$, $\cup$ for being element of, inclusion, intersection, and union. The set $A - B$ is the set of elements in A not in B. If $\mathscr{P}$ is a property and M a set, then $\{x \in M: x \text{ has property } \mathscr{P}\}$ denotes the set of elements $x \in M$ with property $\mathscr{P}$. The composite of two maps $f: A \to B$ and $g: B \to M$ is denoted $g \circ f: A \to M$. The sign $\Rightarrow$ means "implies."

Algebra. If K is a subgroup of a group G, the symbol G/K denotes the set of left cosets gK, $g \in G$. When K is considered as an element in G/K it will sometimes be denoted by $\{K\}$. If $x \in G$, the mapping $gK \to xgK$ of G/K onto itself will be denoted by $\tau(x)$.

Let $A: V \to W$ be a linear mapping of a vector space V into a vector space W. Let V^* and W^* denote the respective duals. The mapping ${}^tA: W^* \to V^*$ defined by ${}^tA(w^*)(v) = w^*(Av)$ is called the *transpose* of A. A linear map $A: V \to V$ will often be called an *endomorphism* of V. If V has finite dimension the *determinant* and *trace* of A will be denoted by $\det(A)$ and $\mathrm{Tr}(A)$, respectively.

Topology. Let X be a topological space, $A \subset X$ a subset. The closure of A is denoted $\bar{A}$ or $Cl(A)$, the interior of A by $\mathring{A}$. If f is a function on X its restriction to A is denoted $f \mid A$. The Hausdorff separation axiom for a topological space will always be assumed. A coset space G/H where H is a closed subgroup of the topological group G will always be taken with its *natural topology* which is characterized by the fact that the mapping $g \to gH$ of G onto G/H is continuous and open. A topological vector space V (over $\boldsymbol{R}$ or $\boldsymbol{C}$) can be topologized by a family of *seminorms* if and only if it is locally convex. This family can be chosen countable if and only if V is metrizable. If in addition V is complete it is called a *Fréchet space.*

Manifolds. If M is a manifold and $p \in M$, the tangent space to M at p is denoted by M_p. If $\phi: q \to (x_1(q), \ldots, x_m(q))$ is a system of coordinates near p the mappings

$$(\partial/\partial x_i)_p: f \to \left(\frac{\partial(f \circ \phi^{-1})}{\partial x_i}\right)(\phi(p)) \qquad f \in C^\infty(M)$$

form a basis of M_p. Let $\Phi: M \to N$ be a differentiable mapping of M into a manifold N. Its *differential* at p, mapping M_p into $N_{\Phi(p)}$ is denoted $d\Phi_p$. If η is a form on N its *pullback* by Φ is denoted $\Phi^*\eta$; it is a form on M.

Lie groups. Lie groups are usually denoted by capital letters and their Lie algebras by the corresponding lower case German letters. The exponential mapping of a Lie algebra into a corresponding Lie group is denoted exp. The identity component of a Lie group G is denoted G_0. We use the standard notation for the classical groups: $\boldsymbol{GL}(n, \boldsymbol{C})$, $\boldsymbol{GL}(n, \boldsymbol{R})$, $\boldsymbol{SL}(n, \boldsymbol{C})$, $\boldsymbol{SL}(n, \boldsymbol{R})$, $\boldsymbol{U}(p, q)$, $\boldsymbol{SU}(p, q)$, $\boldsymbol{SU}^*(2n)$, $\boldsymbol{SO}(n, \boldsymbol{C})$, $\boldsymbol{O}(p, q)$, $\boldsymbol{SO}^*(2n)$, $\boldsymbol{Sp}(n, \boldsymbol{C})$, $\boldsymbol{Sp}(n, \boldsymbol{R})$, and $\boldsymbol{Sp}(p, q)$. See, for example, [*DS*], Chapter X, §2 for detailed descriptions.

INDEX

ERRATA

Page and line in {above / below}	Instead of:	Read:
16_2	Cor. 4.11	Proposition 4.11
88_{11}	$\mathrm{Ad}\, X$	$\mathrm{ad}\, X$
88_{13}	$r^{\mathrm{Tr}\, A}$	$e^{\mathrm{Tr}\, A}$
104_5	2.41	2.34
104_9	$C1$	$D1$
110^9	B_e	B_ϵ
127_2	$\lvert x\rvert p$	$(\lvert x\rvert p)^k$
145_{17}	$\int_{G/H} dg_{\equiv}$	$\int_{G/H_{\equiv}} dg_{H_{\equiv}}$
151_2	Q_ϵ	B_ϵ
$160^{4,6}$	K	H_X
161^3	p	x
167^3	$u(\check{0})$	$\mu(\check{o})$
170_5	(r)	(x)
176_3	Theorem 1.11	Theorem 4.11
178^8	follow	follows
179_{18}	M	X
190^6	uk_1	uK_1
191_1	corrollary	corollary
199_9	$g_y(X, X)$	$g_y(Z, Z)$
258_{8-9}	Proposition 3.1	Theorem 3.2
258_{13}	Proposition 3.1	Theorem 3.2
259_{17}	V_W	V_w
262^{10}	y_1	$y_1(b)$

Page and line in {above, below}	Instead of:	Read:
269^1	By	In
273^1	by analogy	in analogy
278_6	Theorem 4.3	Theorem 4.4
302_{15}	$\subset$	$\in$
303^{11}	denoted	denote
319_2	exampe	example
332_{10}	K_2	$K_2)$
336^1	W^*	V^*
336_{13}	p	$\mathfrak{p}$
362^{14}	$j_r)$	$j_n)$
367^{17}	dominators	denominators
377_1	k^Z	$\mathfrak{k}^Z$
389_1	$\mathcal{H}_\varphi$	$\mathfrak{H}_\varphi$
405_2	orthogonal	orthonormal
425_{16}	for	far
439_9	N^{s-1}	$N^{s^{-1}}$
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455_5	gk	gK
457_{10}	gk	gK
471^{12}	4.21	4.19
473^7	$k \in Y$	$k \in K$
476^{15}	Hak	$H(ak)$
479^9	(26)	(27)
483_{14}	N_{β_p}	$\overline{N}_{\beta_p}$
504_{12}	e	(e)
523^{10}	(49)	(47)
529_8	$\mathcal{S}$	V
551^8	(ii)	(iii)
556^{10}	μ	u

Page and line in {above, below}	Instead of:	Read:
566^{13}	$r = 0$	$r \to 0$
569^{7}	$\sum_i$	$\sum_\ell$
572_{3}	(17)	(7)
575^{4}	$[\frac{1}{2}(n-3)!$	$[\frac{1}{2}(n-3)]!$
576_{2}	This	Thus
595_{10}	to	so
599_{16}	$\sum_0^i$	$\sum_0^\ell$
603_{11}	$k \in \mathbf{R}$	$k = \mathbf{R}$
609^{2}	$\alpha_n \beta^n$	$a_n \beta^n$
615_{13}	zonal	zonales
618^{5}	Palaisesu	Palaiseau
618^{6}	Duistermatt	Duistermaat
618^{10}	Composite	Compositio
622_{2}	Asymtotics	Asymptotics
626^{19}	nichtenklischen	nichteuklidischen
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Group-theoretic methods have taken an increasingly prominent role in analysis. Some of this change has been due to the writings of Sigurdur Helgason. This book is an introduction to such methods on spaces with symmetry given by the action of a Lie group. The introductory chapter is a self-contained account of the analysis on surfaces of constant curvature. Later chapters cover general cases of the Radon transform, spherical functions, invariant operators, compact symmetric spaces and other topics. This book, together with its companion volume, *Geometric Analysis on Symmetric Spaces* (AMS Mathematical Surveys and Monographs series, vol. 39, 1994), has become the standard text for this approach to geometric analysis.

Sigurdur Helgason was awarded the Steele Prize for outstanding mathematical exposition for *Groups and Geometric Analysis* and *Differential Geometry, Lie Groups and Symmetric Spaces.*

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