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# Modules over Non-Noetherian Domains

László Fuchs
Luigi Salce

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Dedicated with love and gratitude to our parents:

to the memory of Dávid Rafael and Teréz Fuchs

and

to Giuseppe and Jolanda Salce

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# Preface

Contemporary research in module theory over commutative rings is heavily concentrated on those modules for which either the underlying ring or the modules themselves or both are subject to various finiteness restrictions. This is due not only to the widespread applications in other areas but also to the techniques available. However, a considerable amount of work has been done recently on modules without assuming any finiteness condition, resulting in a rapid development of the subject. This volume deals with the theory of modules over commutative integral domains, paying only scant attention to the structure of the underlying domains and practically ignoring the noetherian case.

In the study of modules, a dramatic change occurs when one abandons finiteness conditions. Although most pleasant features are undoubtedly lost, some nice features still remain. These have served as starting points for new discoveries. Substantial generalizations of classical results required totally new methods, and the development of powerful techniques breathed new life into the theory. As a result, module theory over non-noetherian rings, and, in particular, over non-noetherian domains, became a lively branch of algebra.

We feel that the steppingstone to a study of modules over general domains is the module theory of valuation domains—these are perhaps the simplest non-noetherian domains that are not too close to Dedekind domains—and their global versions, the Prüfer domains. One could attain substantial understanding of module properties by a careful examination of these two special cases. Until recently, not much was known about the theory of modules over these domains, but the past two decades have seen remarkable developments. It has become increasingly clear that they provide a meeting ground for several branches of algebra and supply a wealth of challenging problems. However, the discussions here will not be confined to modules over these domains: whenever feasible, we pursue our treatment initially without assuming any extra condition on the domains; additional conditions will be introduced only when necessary.

In this volume we have tried to present the bulk of the traditional material and to incorporate recent discoveries on our subject by pulling together the main strands of the theory. However, because of the vastness of the topic, limitations had to be imposed on both the choice of the material and the method of presentation. The theory is replete with aesthetically pleasing and powerful results, but we could not (and we did not even intend to) cover certain basic topics such as a study of direct decompositions, endomorphism rings and automorphism groups, $K$-theory, modules over specific types of domains, etc. We could just briefly touch upon subjects like generalizations of projectivity and injectivity, and the topological aspects of module theory. No attempt has been made to be exhaustive even in the topics covered; we aimed rather to draw together in a systematic manner the different trends of developments, forging them into a more coherent theory. Our intent was to concentrate on the backbone of the theory and to focus attention on results of theoretical interest. We have deliberately omitted standard material covered in textbooks and monographs and skipped details of proofs of results that can be found in several textbooks on algebra or, in particular, on ring theory.

Needless to say, the focus of our presentation is very personal, reflecting our own interests and research; we did not include areas which are not too close to our research interests. As a result, several important aspects of module theory (even within our self-imposed limitation) are bound to be neglected, and one could argue that various additional topics should have been included in this volume, especially those on which there is no ample survey in the literature. Of course, there is always room for argument as to what topics are more relevant or significant. We believe that we have presented an attractive—though by no means exhaustive—theory of modules over non-noetherian domains which could prepare the groundwork for a more penetrating assault on the subject and which will hopefully inspire further work in the area. Numerous open problems which the authors thought interesting are listed at the end of the chapters.

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We mathematicians often endeavor to extend theorems in order to cover a wider area or to get a better insight. In this respect, the theory of abelian groups has been a constant source of inspiration for our work. It is apparent that our treatment owes a great deal to abelian groups: old techniques find new roles, and a number of classical results lend themselves to generalizations.

Additional impetus for our work comes from Dedekind domains. It has been observed that if we focus our attention on modules of projective dimension 1, then some of the useful features of modules over Dedekind domains can be retained. A careful reader will find numerous results in this volume in support of our claim that this class of modules deserves special attention.

The powerful apparatus built up in the noetherian case could hardly be utilized in our general setting. It is unreasonable to expect that the same notions would be of comparable relevance in the general case, so suitable substitutes were pursued. For instance, the fruitful idea of introducing an operation between the studied objects (à la Picard group) led us to the consideration of two groups: the archimedean group Arch $R$ for valuation domains $R$ and the group Gp $R$ of 'clones' attached to any domain $R$. Actually, we went a step further and initiated the systematic study of several emerging Clifford semigroups: the inclusion of Clifford semigroups in our arsenal will allow a more global picture of the subjects.

We have also borrowed ideas from our previous volume [FS] that grew out of our attempt to systematically transplant ideas and methods from abelian group theory to the theory of modules. Since its publication (15 years ago) new methods have been developed which have not only improved the original results, but in fact have extended the theory to a wider class of modules. Whenever it was feasible, we adapted methods which provided a little more mileage than the conventional approach. As was pointed out earlier, our general intention was to treat the problems in full generality and to specialize to individual domains whenever it became inevitable. Notable exceptions were the cases when nothing substantial was available in the general case or when the problems were uninteresting in a more general setup.

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More or less detailed reviews of the contents are given in the introductions to the individual chapters. Notes at the end of the chapters include historical remarks as well as various comments on the material of the chapter. We have appended a number of exercises at the end of each section. They range from routine problems to new material going slightly beyond what is covered in text; in addition, they provide a chance for the reader to check his/her understanding of the material. The book should be accessible to graduate students who are familiar with the rudiments of module theory and homological algebra.

An expert reader will find several innovations in the proofs and shortcuts in the presentations. Some of our arguments yield genuine improvement upon results published in the literature.

The list of references at the end of the book is by no means complete. It includes only the books and articles on our subject which we referred to in the text.

**Conventions.** All rings are commutative (unless explicitly stated otherwise) and have an identity element, usually denoted by 1. Of course, non-commutative rings will turn up from time to time as endomorphism rings.

If not stated otherwise, the symbol $R$ stands for a domain, $Q$ for its field of quotients, and $P$ for a maximal ideal of $R$.

It is tacitly understood that our theorems are stated in the ZFC axiom system of set theory (Zermelo-Fraenkel with the Axiom of Choice). If it happens that we cannot resolve a question that interests us in ZFC, then we assume an additional axiom. Such an axiom is often Gödel's Axiom of Constructibility (abbreviated as V = L) or the (Generalized) Continuum Hypothesis (GCH or CH).

A glossary of notation used in this volume can be found on page xv.

**Cross-references.** Theorems, propositions, lemmas, corollaries, and examples are numbered by pairs (a.b) of positive integers where "a" stands for the section number and "b" for the location of the result in the section. Definitions are printed in boldface characters; they do not carry reference numbers.

Cross-references within a chapter are done by indicating the appropriate pair, while the chapter numbers (Roman numerals) are also listed when reference is made to a different chapter. The exercises at the end of the sections are also numbered by pairs of integers; they are referred to as "Exercise a.b".

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It is a great pleasure to thank those who read portions of the manuscript and suggested improvements in the draft. We are very grateful to Silvana Bazzoni, Karl H. Hofmann, K.M. Rangaswamy, Michael Siddoway (who also corrected several linguistic inaccuracies), and Paolo Zanardo for their most useful comments.

We are confident that in the future the investigation of modules over non-noetherian rings will attract an increasing number of algebraists and sincerely hope that they will find this volume useful in their work.

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# List of Symbols

$|X|$ cardinality of the set $X$
$\subseteq$, $\subset$ subset, proper subset
$X \setminus Y = \{x \in X \mid x \notin Y\}$
$\aleph_\alpha$, $\omega_\alpha$ $\alpha$th infinite cardinal, $\alpha$th initial ordinal
cof $\alpha$ cofinality of ordinal $\alpha$
$\mathbb{N}$ set of natural integers

$R, P, Q$ commutative domain with 1, maximal ideal, field of quotients of $R$
$U(R)$, $R^\times$ group of units of $R$, the multiplicative monoid $R \setminus \{0\}$
$\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ integers, rational, real, and complex number fields
$J_p$ ring of $p$-adic integers
$R[X]$, $R[[X]]$ ring of polynomials, formal power series over $R$
$M_n(R)$ ring of $n \times n$ matrices over $R$
$M_S$, $M_P$ localization at a submonoid $S$ of $R^\times$, localization at prime $P$
Spec $R$, Max $R$, Min $R$ set of all primes, maximal and minimal primes $\neq 0$
Arch $R$, Gp $R$ archimedean group, group of clones of $R$
$\mathcal{F}(R)$, $\mathcal{D}(R)$ monoid of fractional, divisorial ideals
$\mathcal{S}(R)$, $\mathcal{C}(R)$ class semigroup, class group of $R$
Id $\mathcal{S}(R)$ set of idempotents in $\mathcal{S}(R)$

$J : I = \{x \in Q \mid xI \leq J\}$ residual
$[I]$ isomorphy class of the ideal $I$
$\Omega(I)$ set of maximal ideals containing the ideal $I$
$R^{[I]} = \bigcap_{P \notin \Omega(I)} R_P$
$I_v = R : (R : I)$ divisorial ideal in class of $I$
div $I$ divisor represented by the ideal $I$
$B(s)$ breadth ideal of element $s$
$c(f)$, $c(M)$, $c(\mathbf{A})$ content of polynomial, module, and matrix

$\Gamma, \Gamma^+$ value group of a valuation domain, its positive cone
$\mathbf{H}\Gamma_i$ Hahn product of ordered groups $\Gamma_i$
$v$, $w$ valuation, demivaluation of a field

$\mathcal{M}$od-$R$ category of $R$-modules
$N \leq M,\ N < M$ $N$ is a (proper) submodule of $M$
gen $M$ minimal cardinality of generating sets of $M$
rk $M$ rank of a torsion-free module $M$
$\oplus,\ \prod$ direct sum, direct product of modules
$M^{(I)},\ M^I$ direct sum, direct product of $|I|$ copies of $M$
$tM$ torsion submodule of $M$
$dM,\ hM$ divisible, $h$-divisible submodule of $M$
$\partial$ divisible module generating the category of divisible $R$-modules
$rM = \{ra \mid a \in M\}$ (where $r \in R$)
$M^1 = \bigcap_{r \in R^\times} rM,\ M^\alpha$ first ($\alpha$th) Ulm submodule of $M$
$M[r] = \{a \in M \mid ra = 0\}$, $M[I] = \{a \in M \mid Ia = 0\}$
$\operatorname{Ann} a,\ \operatorname{Ann} M = \{r \in R \mid rM = 0\}$ annihilator of $a$, $M$
$M^\sharp = \{r \in R \mid rM < M\}$, $M_\sharp = \{r \in R \mid ra = 0 \text{ for some } 0 \neq a \in M\}$
$M^\flat = \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}),\ M^\star = \operatorname{Hom}_R(M, R)$ character module, $R$-dual of $M$
$\varprojlim^1$ first derived functor of inverse limit functor
$U^\circ,\ U_\circ$ upper and lower threshold submodules of a uniserial module $U$
$t[U]$, Lev $(U)$ type, level of uniserial module $U$
$\operatorname{End}_R M$ ring of $R$-endomorphisms of $M$
$\operatorname{Aut}_R M$ group of $R$-automorphisms of $M$

$\widetilde{M}$ $R$-completion of $M$
$\widehat{M}$ $RD$-injective hull of $M$
$\overline{M},\ \overline{\overline{M}}$ torsion-completion, torsion-ultracompletion of $M$
$M^\bullet$ cotorsion hull of $M$
$E(M)$ injective hull of $M$
$PE(M)$ pure-injective hull of $M$
$H_M(A)$ pure hull of submodule $A$ in $M$
$M = \langle X; \Xi \rangle$ simply presented module

$h_M(a),\ H_M(a)$ height, height ideal of $a$ in $M$
$i_M(a)$ indicator of $a$ in $M$
$\Sigma$ set of all heights
$M^\sigma = \{a \in M \mid h_M(a) \geq \sigma \in \Sigma\}$, $M^{\sigma^+} = \{a \in M \mid h_M(a) > \sigma\}$
$\alpha_M(\sigma, I),\ \alpha_M[\sigma, I]$ $(\sigma, I)$-invariant of $M$, its equivalence class

Gd $M$, dG $M$ Goldie, dual Goldie dimension of $M$
Fr $M$, Mr $M$ Fleischer rank, Malcev rank of $M$

p.d.$_R M$, w.d.$_R M$, ppd$_R M$ projective, weak, pure-projective dimension of $M$
bpp$_R M$ balanced-projective dimension of $M$
i.d.$_R M$ injective dimension of $M$
gl.d.$R$, w.gl.d $R$, p.gl.d $R$ global, weak and pure global dimension of $R$

# Appendix on Set Theory

In this appendix we intend to summarize the pertinent definitions needed from Set Theory, and to list a few results which are required in our discussions.

The reader is assumed to have a basic acquaintance with the rudiments of Set Theory, in particular, with the theory of cardinal and ordinal numbers, as well as conditions equivalent to the Axiom of Choice (e.g., the well-ordering principle and Zorn's lemma). We refer to T. Jech's book 'Set Theory' (Academic Press, 1978) or K. Kunen's book 'Set Theory' (North Holland, 1980) for more details.

We emphasize that throughout this volume, we are working in ZFC, i.e., in the classical Zermelo-Fraenkel set theory with the Axiom of Choice adjoined. Occasionally, we will add another axiom to ZFC (always consistent with ZFC) in order to be able to prove or to refute a claim which would otherwise be undecidable.

## Cardinals

There is no harm in considering cardinals as initial ordinals, i.e., ordinals whose cardinality exceeds those of all preceding ordinals. Besides the standard notation of infinite cardinals: $\aleph_0$ for countable, $\aleph_\alpha$ for the $\alpha$th cardinal, ordinals and cardinals are denoted by lower case Greek letters. In particular, $\omega$ stands for the first infinite ordinal, and $\omega_\alpha$ for the first ordinal of cardinality $\aleph_\alpha$. Occasionally, we use $\aleph_{-1}$ to mean finiteness.

The symbol $|X|$ stands for the cardinality of the set $X$.

A cardinal $\kappa$ is called **regular**, if it is equal to its **cofinality**, i.e., it is the smallest cardinal $\alpha$ such that $\kappa$ contains a subset $X$ with $|X| = \alpha$ and $\sup X = \kappa$ (sup is short for 'supremum'). Otherwise, it is called a **singular** cardinal. For a cardinal $\mu$, $\mu^+$ denotes the smallest cardinal larger than $\mu$.

The precise definition of a **weakly compact** cardinal $\kappa$ requires a few concepts which are not needed elsewhere in this volume. Therefore we forgo a precise definition, and adopt the point of view that the property stated in the following lemma serves as a definition. (Stationary sets are defined below.)

**Lemma A.1** *Let $\{S_\alpha \mid \alpha < \kappa\}$ be a set of stationary subsets of a weakly compact cardinal $\kappa$. Then there is a stationary set $T$ of regular cardinals $< \kappa$ such that $S_\alpha \cap \lambda$ is stationary in $\lambda$, for every $\lambda \in T$ and for every $\alpha < \lambda$.* ■

A cardinal $\kappa$ is called **measurable** if a set $X$ of cardinality $\kappa$ admits a countably additive measure $\mu$ such that $\mu$ assumes only values 0 and 1, and

$$\mu(X) = 1, \qquad \mu(x) = 0 \quad \text{for all } x \in X.$$

If there exists a measurable cardinal at all, then there is a smallest one among them, and all larger cardinals are measurable. It is not known whether or not the existence of measurable cardinals can be proved in ZFC. But we do know that many of the strongly inaccessible cardinals are non-measurable.

Also, measurable cardinals are weakly compact, and there are $\kappa$ weakly compact cardinals less than a measurable cardinal $\kappa$.

## Chains

Let $\kappa$ be a regular cardinal number, viewed as the set of ordinals less than $\kappa$. A subset $C \subseteq \kappa$ is **closed** and **unbounded** (in short: a **cub**) in $\kappa$ if

1) $X \subseteq C$, $\sup X \in \kappa$ implies $\sup X \in C$; and

2) it has no upper bound in $\kappa$.

If $C$ is a cub in $\kappa$, then there is a monotone bijection $f\colon \kappa \to C$, and as a consequence we can reindex the ordinals in a cub in $\kappa$ by all ordinals $< \kappa$.

A subset of $\kappa$ is called **stationary** if it intersects every cub in $\kappa$. Examples for stationary subsets include the cubs, and the set of limit ordinals cofinal with $\omega$.

A family $\{X_\alpha\}_{\alpha<\kappa}$ of subsets of a set $X$ of cardinality $\kappa$ indexed by the ordinals $\alpha < \kappa$ is called a **continuous well-ordered ascending chain** if the following conditions are satisfied:

(i) $\alpha < \beta < \kappa$ implies $X_\alpha \subseteq X_\beta$,

(ii) $X_\lambda = \bigcup_{\alpha<\lambda} X_\alpha$ for each limit ordinal $\lambda < \kappa$.

The family $\{X_\alpha\}_{\alpha<\kappa}$ is called a **filtration** of $X$ if, in addition,

(iii) $|X_\alpha| < \kappa$,

(iv) $X = \bigcup_{\alpha<\kappa} X_\alpha$.

In case of filtrations of modules, condition (iii) is often replaced by $\operatorname{gen} X_\alpha < \kappa$ or, for torsion-free modules, by $\operatorname{rk} X_\alpha < \kappa$.

We call attention to the important fact that, whenever $\kappa$ is a regular cardinal, every subset of $X$ of cardinality $< \kappa$ is contained in some member $X_\alpha$ of a filtration of $X$. This is false for singular cardinals $\kappa$, which is one of the reasons that filtrations are of real interest only for regular cardinals.

The following facts are frequently used.

**Lemma A.2** *Let $\kappa$ be an uncountable regular cardinal.*

(i) *The intersection of two cubs in $\kappa$ is again a cub.*

(ii) *If $\{X_\alpha\}_{\alpha<\kappa}$ and $\{Y_\alpha\}_{\alpha<\kappa}$ are two filtrations of a set $X$ of cardinality $\kappa$, then the set*

$$C = \{\alpha < \kappa \mid X_\alpha = Y_\alpha\}$$

*is likewise a cub in $\kappa$.*

**Proof.** (i) If $A$ and $B$ are cubs, then the intersection $C = A \cap B$ is evidently closed. For any $\alpha_1 \in A$, there is a $\beta_1 \in B$ with $\alpha_1 < \beta_1$, and then there is an $\alpha_2 \in A$ with $\beta_1 < \alpha_2$, etc. The chain $\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \dots$ has a supremum $\gamma$ in $C$. This proves that $C$ is unbounded.

(ii) This follows immediately from (i) after observing that, in the notation of (i), $\gamma$ satisfies $X_\gamma = Y_\gamma$. ∎

A stationary subset in $\kappa$ (an uncountable regular cardinal) is said to be **non-reflecting** if for every limit ordinal $\gamma < \kappa$ of cofinality $> \omega$, the set $E \cap \gamma$ is not stationary in $\gamma$. For instance, if $\lambda$ is a regular cardinal and $\kappa = \lambda^+$, then the limit ordinals of cofinality $\lambda$ in $\kappa$ form a non-reflecting stationary set in $\kappa$.

We shall refer to the following result. It is valid only if the axiom of constructibility is assumed (for the condition V = L, see below).

**Lemma A.3** *Let* V = L, *and* $\kappa$ *be a regular cardinal. There exists a non-reflecting stationary subset of* $\kappa$ *consisting of ordinals cofinal with* $\omega$ *if and only if* $\kappa$ *is not weakly compact.* ∎

## Additional Hypotheses

As mentioned earlier, from time to time we adjoin various axioms to ZFC whenever needed. The most well-known of these is the **Continuum Hypothesis** (CH) which states that $2^{\aleph_0} = \aleph_1$. The **Generalized Continuum Hypothesis** (GCH) says that $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ holds for all ordinals $\alpha$. It is known that GCH is consistent with and independent of ZFC.

A most important example for an additional axiom is Gödel's **Axiom of Constructibility** (if we assume it, we write V = L to mean that the model V of set theory we are working in is the constructible universe L); this axiom is—as proved by K. Gödel—consistent with ZFC. The hypothesis V = L is quite strong, *inter alia* it implies GCH. The constructible universe L is distinguished by the property that it is the least transitive model containing all ordinals.

Another principle that will be used several times is as follows. It is due to R.B. Jensen [Ann. Math. Logic 4 (1972), 229-308].

**Diamond Principle** $\diamondsuit$ : *Let* $E$ *be a stationary subset of the uncountable regular cardinal* $\kappa$, *and* $\{X_\alpha \mid \alpha < \kappa\}$ *a filtration of a set* $X$ *of cardinality* $\kappa$. *Then there exists a family* $\{S_\alpha \mid \alpha \in E\}$ *of subsets* $S_\alpha \subseteq X_\alpha$ *such that, for any subset* $Y \subseteq X$, *the set*

$$E' = \{\alpha \in E \mid Y \cap X_\alpha = S_\alpha\}$$

*is a stationary subset in* $\kappa$.

Since Gödel's Axiom of Constructibility implies $\diamondsuit$, it follows that $\diamondsuit$ is consistent with ZFC.

We derive the following useful version of the diamond principle.

**Lemma A.4** ($\diamondsuit$) *Let* $E$ *be a stationary subset of the uncountable regular cardinal* $\kappa$, *and* $\{X_\alpha\}_{\alpha<\kappa}$ *a filtration of a set* $X$ *of cardinality* $\kappa$. *For any set* $Y$ *of cardinality* $< \kappa$, *there is a family* $\{g_\alpha \mid \alpha \in E\}$ *of functions* $g_\alpha \colon X_\alpha \to Y \times X_\alpha$ *such that, for any function* $g \colon X \to Y \times X$, *the set*

$$E_g = \{\alpha \in E \mid g|X_\alpha = g_\alpha\}$$

*is stationary in* $\kappa$.

**Proof.** Define $X' = X \times (Y \times X)$ and $X'_\alpha = X_\alpha \times (Y \times X_\alpha)$; now apply $\diamondsuit$ to the filtration $\{X'_\alpha\}_{\alpha<\kappa}$ of $X'$. There exist subsets $S_\alpha \subseteq X_\alpha \times Y \times X_\alpha$ with the property specified in $\diamondsuit$. Fix a $y_0 \in Y$. Define $g_\alpha : X_\alpha \to Y \times X_\alpha$ to have $S_\alpha$ as graph if $S_\alpha$ is the graph of a function, and *via* $g_\alpha(x) = (y_0, x)$ otherwise. Given a function $g \colon X \to Y \times X$, let $S$ be its graph. Then $\diamondsuit$ assures that the set $E' = \{\alpha \in E \mid g|X_\alpha = g_\alpha\}$ is stationary in $\kappa$. ∎

## Lemma on Sequence of Functions

The next lemma produces a transfinite sequence of functions from the set of countable ordinals into $\omega$ such that any two of them are almost equal. Here two functions $f, g$ with the same domain $D$ are called **almost equal** if $f(x) = g(x)$ for almost all $x \in D$.

**Lemma A.5** *There exists a family*

$$\{\alpha_\sigma\colon\ \sigma \to \omega\}_{\sigma<\omega_1}$$

*of injective functions such that*

(i) $\alpha_\sigma|\rho$ *is almost equal to* $\alpha_\rho$ *for all* $\rho < \sigma < \omega_1$, *and*

(ii) $\omega \setminus \operatorname{Im}\alpha_\sigma$ *is infinite for all* $\sigma$.

**Proof.** We only sketch the proof; for more details, see Kunen's book [*loc.cit.*, p.70].

The functions $\alpha_\sigma$ are defined by transfinite induction. Given $\alpha_\sigma$, pick any $n \in \omega \setminus \operatorname{Im}\alpha_\sigma$, and define: $\alpha_{\sigma+1}|\sigma = \alpha_\sigma$ and $\alpha_{\sigma+1}(\sigma) = n$. Let $\lambda < \omega_1$ be a limit ordinal, and assume $\alpha_\sigma$ has been defined for each $\sigma < \lambda$. Fix an ascending countable sequence $\sigma_0 < \sigma_1 < \sigma_2 < \ldots < \sigma_k < \ldots$ of ordinals with $\sup \sigma_k = \lambda$. Set $\beta_0 = \alpha_{\sigma_0}$ and inductively define functions $\beta_k\colon \sigma_k \to \omega$ to satisfy the following conditions for each $k$:

(a) $\beta_k$ is injective,

(b) it is almost equal to $\alpha_{\sigma_k}$, and

(c) $\beta_{k+1}|\sigma_k = \beta_k$.

Setting $\beta = \bigcup_{k<\omega} \beta_k\colon \lambda \to \omega$, the function $\alpha_\lambda$ is defined as follows: $\alpha_\lambda(\sigma_k) = \beta(\sigma_{2k})$ and $\alpha_\lambda(\sigma) = \beta(\sigma)$ for $\sigma \notin \{\sigma_1, \sigma_2, \ldots, \sigma_k, \ldots\}$. The functions $\alpha_\sigma$ defined in this way satisfy the desired conditions. ∎

## Aronszajn Trees

Let $\kappa$ be an infinite regular cardinal. A tree $T$ is called a **$\kappa$-Aronszajn tree** if

(i) it is of height $\kappa$,

(ii) its $\alpha$th level satisfies $|T_\alpha| < \kappa$ for each $\alpha < \kappa$, and

(iii) $T$ has no branches of length $\kappa$.

If every vertex of $T$ is connected to only a finite number of vertices at the next level, then König's well-known lemma states that $T$ has an infinite branch. This amounts to saying that there are no $\omega$-Aronszajn trees.

However, we have:

**Lemma A.6** *$\omega_1$-Aronszajn trees exist.*

**Proof.** Let $\{\alpha_\sigma \mid \sigma \to \omega\}_{\sigma<\omega_1}$ be the family of functions defined in A.5. Consider the tree $T$ of height $\omega_1$, whose $\sigma$th level $T_\sigma$ ($\sigma < \omega_1$) consists of those injective functions $\alpha\colon \sigma \to \omega_1$ which are almost equal to $\alpha_\sigma$. The ordering on $T$ is the natural one. Clearly, $T_\sigma$ is countable for every $\sigma < \omega_1$. $T$ has no branches of length $\omega_1$, since such a branch would give rise to an injective function from $\omega_1$ into $\omega$. Hence $T$ is an $\omega_1$-Aronszajn tree. ∎

For $\omega_1$-Aronszajn trees, a number of different constructions are available; see Jech's book [*loc.cit.*] for more details.

## Extensions

Next we discuss a consequence of the Diamond Principle on extensions. We shall need the following observation.

Suppose $0 \to A \xrightarrow{\alpha} A' \to A'/A \to 0$ is an exact sequence of $R$-modules, and $B$ is an $R$-module such that $\mathrm{Ext}^1_R(A', B) = 0$, but $\mathrm{Ext}^1_R(A'/A, B) \neq 0$. Then there is a homomorphism $\chi$: $B \oplus A \to B \oplus A'$ making the following diagram commute:

$$
\begin{array}{ccccccccc}
0 & \to & B & \to & B \oplus A & \xrightarrow{\rho} & A & \to & 0 \\
 & & \| & & \chi\downarrow & & \downarrow\alpha & & \\
0 & \to & B & \to & B \oplus A' & \xrightarrow{\rho'} & A' & \to & 0
\end{array}
\tag{1}
$$

such that there is no splitting map $\tau: A' \to B \oplus A'$ with $\chi\sigma = \tau\alpha$, where $\sigma$ is a splitting map $A \to B \oplus A$ for $\rho$. In fact, the exactness of $\mathrm{Hom}(A', B) \to \mathrm{Hom}(A, B) \to \mathrm{Ext}(A'/A, B) \to \mathrm{Ext}(A', B) = 0$ implies that the map between the two Homs is not epic. If $\eta$: $A \to B$ is a map that does not extend to any $A' \to B$, then define $\chi$ to map $(b, a) \in B \oplus A$ onto $(b + \eta a, \alpha a) \in B \oplus A'$. It is readily seen that—due to the choice of $\eta$—there is no splitting map $\tau$: $A' \to B \oplus A'$.

In the proof of the next lemma, it will be convenient to consider an extension $M$ of $B$ by $A$ to be a module on the set $A \times B$.

**Lemma A.7** ($\diamondsuit$) *Let $A, B$ be $R$-modules with* $\mathrm{gen}\, A = \kappa$ *and* $|B| \leq \kappa$, *where $\kappa$ is an uncountable regular cardinal. If $A$ has a filtration $\{A_\alpha\}_{\alpha<\kappa}$ such that*

(i) $\mathrm{gen}\, A_\alpha < \kappa$ *for every* $\alpha \leq \kappa$;

(ii) *each $A_\alpha$ satisfies* $\mathrm{Ext}^1_R(A_\alpha, B) = 0$;

(iii) *the set* $S = \{\alpha < \kappa \mid \mathrm{Ext}^1_R(A_{\alpha+1}/A_\alpha, B) \neq 0\}$ *is stationary in* $\kappa$,

*then* $\mathrm{Ext}^1_R(A, B) \neq 0$.

**Proof.** Let $\{B_\alpha\}_{\alpha<\kappa}$ be a filtration of $B$. By $\diamondsuit$, there is a family $\{g_\alpha\}_{\alpha\in S}$ of functions $g_\alpha$: $A_\alpha \to A_\alpha \times B_\alpha$ ($\alpha \in S$) such that, for every function $g$: $A \to A \times B$, the set $\{\alpha \in S \mid g|A_\alpha = g_\alpha\}$ is stationary in $\kappa$.

We are going to construct a non-split exact sequence $E: 0 \to B \to M \xrightarrow{\rho} A \to 0$ as the direct limit of splitting exact sequences $E_\alpha: 0 \to B \to M_\alpha \xrightarrow{\rho_\alpha} A_\alpha \to 0$ ($\alpha < \kappa$) such that the underlying set for $M_\alpha$ is $A_\alpha \times B$, and whenever $\beta < \alpha$, there is a commutative diagram

$$
\begin{array}{ccccccccc}
E_\beta: 0 & \to & B & \to & M_\beta & \xrightarrow{\rho_\beta} & A_\beta & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
E_\alpha: 0 & \to & B & \to & M_\alpha & \xrightarrow{\rho_\alpha} & A_\alpha & \to & 0
\end{array}
\tag{2}
$$

where the right vertical map is the inclusion map. Let $\alpha < \kappa$ and assume that $E_\beta$ has been defined for all $\beta < \alpha$ such that required diagrams are commutative.

*Case 1.* If $\alpha$ is a limit ordinal, then we define $E_\alpha$ as the direct limit of the exact sequences $E_\beta$ with $\beta < \alpha$. This is a splitting sequence, since $A_\alpha$ satisfies (ii).

*Case 2.* Let $\alpha = \delta + 1$. If $\delta \notin S$ or if $g_\delta$ is not a splitting map for $\rho_\delta$, then let $E_\alpha$: $0 \to B \to M_\alpha \xrightarrow{\rho_\alpha} A_\alpha \to 0$ be an extension of $E_\delta$ with any choice of $M_\delta \to M_\alpha$ such that (2) commutes and the underlying set for $M_\alpha$ is $A_\alpha \times B$. Then $E_\alpha$ is a splitting exact sequence because of condition (ii).

*Case 3.* Let $\alpha = \delta+1, \delta \in S$, and assume the underlying set for $M_\delta$ is $A_\delta \times B$ and $g_\delta$: $A_\delta \to A_\delta \times B_\delta \subset A \times B$ is not a splitting map for $\rho_\delta$. From the introductory remarks we conclude that there is an extension $E_\alpha$: $0 \to B \to M_\alpha \xrightarrow{\rho} A_\alpha \to 0$ with a commutative diagram (2) such that there is no splitting map $\tau$: $A_\alpha \to M_\alpha$ for $\rho_\alpha$. Again, we may consider $M_\alpha$ as a module defined on the set $A_\alpha \times B$.

We now define $E$: $0 \to B \to M \xrightarrow{\gamma} A \to 0$ as the direct limit of the splitting exact sequences $E_\alpha$: $0 \to B \to M_\alpha \xrightarrow{\rho} A_\alpha \to 0$ for $\alpha < \kappa$ where $M = A \times B$ as sets. By way of contradiction, assume there is a splitting homomorphism $g$: $A \to M$ for $\gamma$. Note that we must have $g(A_\alpha) \leq M_\alpha$ for every $\alpha < \kappa$. By the choice of the $g_\alpha$, there is a $\delta \in S$ (actually, stationarily many of them) such that $g|A_\delta = g_\delta$. This means that $E_{\delta+1}$ has been constructed according to Case 3 above. Since $g|A_{\delta+1}$ is both a splitting map for $E_{\delta+1}$ and an extension of $g|A_\delta$, we reach a contradiction to the existence of $g$. Thus $\mathrm{Ext}^1_R(A, B) \neq 0$. ■

## Singular Compactness

We will also need Shelah's Singular Compactness Theorem. A rank version is proved in (XVI.1.9).

The following axiomatic version is due to Eklof-Mekler [M], which we state for modules only. It generalizes W. Hodges' version [Algebra Universalis 12 (1981), 205-220] which is based on ideas that originated with S. Shelah.

Assume $\mathcal{F}$ is a class of $R$-modules such that $0 \in \mathcal{F}$ and for each $M \in \mathcal{F}$ there is given a family $\mathcal{B}(M)$ of sets of submodules of $M$. We say that $M$ is 'free' if $M \in \mathcal{F}$ and $\mathfrak{B}$ is a 'basis' of $M$ if $\mathfrak{B} \in \mathcal{B}(M)$. The submodules $B \in \mathfrak{B}$ are called 'free' factors of $M$.

Given an infinite cardinal $\mu$, the following properties (i)-(v) are required for every 'free' module $M$ and every 'basis' $\mathfrak{B}$ of $M$.

(i) $\mathfrak{B}$ is closed under unions of chains.

(ii) If $B \in \mathfrak{B}$ and $a \in M$, then there is a $C \in \mathfrak{B}$ that contains both $B$ and $a$, and is such that $|C| \leq |B| + \mu$.

(iii) Every $B \in \mathfrak{B}$ is 'free' (i.e., 'free' factors are 'free'); and moreover, the set $\{C \in \mathfrak{B} \mid C \leq B\} = \mathfrak{B} \,|B$ is a 'basis' for $B$.

(iv) If $B$ is a 'free' factor of $M$, then for every 'basis' $\mathfrak{B}'$ of $B$, there exists a basis $\mathfrak{B}$ of $M$ such that $\mathfrak{B}' = \mathfrak{B}|B$.

(v) Suppose $B_\alpha$ ($\alpha < \kappa$) is a continuous well-ordered ascending chain of 'free' submodules of $M$ with 'bases' $\mathfrak{B}_\alpha$ satisfying $\mathfrak{B}_\beta|B_\alpha = \mathfrak{B}_\alpha$ for all $\alpha < \beta < \kappa$ (in particular, $B_\alpha \in \mathfrak{B}_\beta$). Then the union $B = \bigcup_{\alpha<\kappa} B_\alpha$ is a 'free' submodule of $M$ such that $\bigcup_{\alpha<\kappa} \mathfrak{B}_\alpha$ is a 'basis' of $B$.

**Theorem A.8** *Suppose that $\mathcal{F}$ satisfies conditions* (i)-(v), *and $M$ is an $R$-module such that* $\operatorname{gen} M = \lambda$ *is a singular cardinal* $> \mu$. *$M$ is 'free' if, for every cardinal $\kappa < \lambda$, there is a family $\mathcal{C}_\kappa$ of $\kappa$-generated submodules of $M$ satisfying the following conditions:*

(a) *$\mathcal{C}_\kappa$ is a subclass of $\mathcal{F}$;*

(b) *$\mathcal{C}_\kappa$ is closed under unions of chains of lengths $\leq \kappa$;*

(c) *every subset of $M$ of cardinality $\leq \kappa$ is contained in a submodule that belongs to $\mathcal{C}_\kappa$.* ■

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