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Vertex Algebras and Algebraic Curves

Second Edition

Edward Frenkel
David Ben-Zvi

American Mathematical Society

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Contents

Preface to the Second Edition ix

Preface to the First Edition xi

Introduction 1

Chapter 1. Definition of Vertex Algebras 11
- 1.1. Formal distributions 11
- 1.2. Locality 16
- 1.3. Definition of a vertex algebra 20
- 1.4. First example: commutative vertex algebras 22
- 1.5. Bibliographical notes 23

Chapter 2. Vertex Algebras Associated to Lie Algebras 25
- 2.1. Heisenberg Lie algebra 25
- 2.2. The vertex algebra structure on π 28
- 2.3. Checking vertex algebra axioms 32
- 2.4. Affine Kac–Moody algebras and their vertex algebras 38
- 2.5. The Virasoro vertex algebra 42
- 2.6. Bibliographical notes 46

Chapter 3. Associativity and Operator Product Expansion 47
- 3.1. Goddard's uniqueness theorem 47
- 3.2. Associativity 48
- 3.3. Operator product expansion 51
- 3.4. Examples of OPE 56
- 3.5. Bibliographical notes 61

Chapter 4. Applications of the Operator Product Expansion 63
- 4.1. A Lie algebra attached to a vertex algebra 63
- 4.2. $U(V)$ and a completion of the universal enveloping algebra 65
- 4.3. An associative algebra attached to a vertex algebra 69
- 4.4. Strong reconstruction theorem 70
- 4.5. Correlation functions 72
- 4.6. Bibliographical notes 74

Chapter 5. Modules over Vertex Algebras and More Examples 75
- 5.1. Modules over vertex algebras 76
- 5.2. Vertex algebras associated to one-dimensional integral lattices 80
- 5.3. Boson–fermion correspondence 85
- 5.4. Lattice vertex algebras 88

5.5. Rational vertex algebras 90
5.6. Twisted modules 91
5.7. Constructing new vertex algebras 94
5.8. Bibliographical notes 97

Chapter 6. Vertex Algebra Bundles 99
6.1. Motivation 99
6.2. The group Aut $\mathcal{O}$ 103
6.3. Exponentiating vector fields 104
6.4. Primary fields 108
6.5. The main construction 111
6.6. A flat connection on the vertex algebra bundle 117
6.7. Bibliographical notes 120

Chapter 7. Action of Internal Symmetries 121
7.1. Affine algebras, revisited 121
7.2. The general twisting property 124
7.3. Description of the n–point functions and modules 128
7.4. Bibliographical notes 131

Chapter 8. Vertex Algebra Bundles: Examples 133
8.1. The Heisenberg algebra and affine connections 133
8.2. The Virasoro algebra and projective connections 137
8.3. Kernel functions 141
8.4. The gauge action on the Heisenberg bundle 146
8.5. The affine Kac–Moody vertex algebras and connections 148
8.6. Bibliographical notes 149

Chapter 9. Conformal Blocks I 151
9.1. Defining conformal blocks for the Heisenberg algebra 151
9.2. Definition of conformal blocks for general vertex algebras 154
9.3. Comparison of the two definitions of conformal blocks 158
9.4. Coinvariants for commutative vertex algebras 161
9.5. Twisted version of conformal blocks 163
9.6. Appendix. Proof of Proposition 9.3.2 164
9.7. Bibliographical notes 166

Chapter 10. Conformal Blocks II 167
10.1. Multiple points 167
10.2. Functoriality of conformal blocks 169
10.3. Chiral correlation functions 171
10.4. Conformal blocks in genus zero 176
10.5. Functional realization of Heisenberg conformal blocks 182
10.6. Bibliographical notes 186

Chapter 11. Free Field Realization I 187
11.1. The idea 187
11.2. Finite–dimensional setting 189
11.3. Infinite–dimensional setting 195
11.4. Bibliographical notes 202

Chapter 12. Free Field Realization II 203
12.1. Weyl algebras in the infinite-dimensional case 203
12.2. Local completion 208
12.3. Wakimoto realization 214
12.4. Bibliographical notes 219

Chapter 13. The Knizhnik–Zamolodchikov Equations 221
13.1. Conformal blocks in the Heisenberg case 221
13.2. Moving the points 225
13.3. Conformal blocks for affine Kac-Moody algebras 229
13.4. Bibliographical notes 232

Chapter 14. Solving the KZ Equations 233
14.1. Conformal blocks from the point of view of free field realization 233
14.2. Generalization: singular vectors 237
14.3. Finding solutions 240
14.4. Bibliographical notes 244

Chapter 15. Quantum Drinfeld–Sokolov Reduction and $\mathcal{W}$–algebras 245
15.1. The BRST complex 245
15.2. Proof of the main theorem 249
15.3. Examples 253
15.4. The second computation 255
15.5. Bibliographic notes 264

Chapter 16. Vertex Lie Algebras and Classical Limits 267
16.1. Vertex Lie algebras 267
16.2. Vertex Poisson algebras 271
16.3. Kac–Moody and Virasoro limits 273
16.4. Poisson structure on connections 275
16.5. The Virasoro Poisson structure 279
16.6. Opers 281
16.7. Classical Drinfeld–Sokolov reduction 285
16.8. Comparison of the classical and quantum Drinfeld–Sokolov reductions 288
16.9. Bibliographical notes 290

Chapter 17. Vertex Algebras and Moduli Spaces I 291
17.1. The flat connection on the vertex algebra bundle, revisited 292
17.2. Harish–Chandra pairs 295
17.3. Moduli of curves 300
17.4. Bibliographical notes 308

Chapter 18. Vertex Algebras and Moduli Spaces II 311
18.1. Moduli of bundles 311
18.2. Local structure of moduli spaces 317
18.3. Global structure of moduli spaces 318
18.4. Localization for affine algebras at the critical level 319
18.5. Chiral de Rham complex 324
18.6. Bibliographical notes 327

Chapter 19. Chiral Algebras 329
19.1. Some sheaf theory 329
19.2. Sheaf interpretation of OPE 334
19.3. Chiral algebras 337
19.4. Lie* algebras 342
19.5. Modules over chiral algebras 346
19.6. Global Kac–Moody and Virasoro algebras 347
19.7. Bibliographical notes 350

Chapter 20. Factorization 351
20.1. Factorization algebras 351
20.2. Factorization algebras and chiral algebras 354
20.3. The Grassmannian and factorization spaces 358
20.4. Examples of factorization algebras 363
20.5. Factorization and the chiral Hecke algebra 369
20.6. Bibliographical notes 374

Appendix 375
A.1. Discs, formal discs and ind–schemes 375
A.2. Connections 377
A.3. Lie algebroids and $\mathcal{D}$–modules 378
A.4. Lie algebra cohomology 380

Bibliography 383

Index 393

List of Frequently Used Notation 397

Preface to the Second Edition

This is a new edition of the book, substantially rewritten and expanded. We would like to mention the most important changes that we have made.

Throughout the book we have dropped the requirement that a vertex algebra be $\mathbb{Z}$–graded with finite-dimensional graded components. The exposition of associativity and operator product expansion (Chapter 3 of the old edition) has been completely redone. A new chapter has been added (Chapter 4) in which we discuss in more detail the Lie algebra $U(V)$ attached to a vertex algebra V. In particular, we show that when V is the affine Kac-Moody vertex algebra $V_k(\mathfrak{g})$, the natural map from $U(V_k(\mathfrak{g}))$ to a completion $\widetilde{U}_k(\mathfrak{g})$ of the universal enveloping algebra of $\widehat{\mathfrak{g}}$ of level k is a Lie algebra homomorphism. We also define for an arbitrary vertex algebra a topological associative algebra $\widetilde{U}(V)$ (when $V = V_k(\mathfrak{g})$ this algebra is isomorphic to $\widetilde{U}_k(\mathfrak{g})$).

In Chapter 5 (Chapter 4 of the old edition) we show that there is an equivalence between the category of V–modules and the category of smooth $\widetilde{U}(V)$–modules. We have added a new section in Chapter 5 in which we introduce twisted modules associated to vertex algebras equipped with an automorphism of finite order. The following chapter, Chapter 6 (Chapter 5 of the old edition), has also been rewritten. We have added a new motivational section at the beginning of the chapter and have supplied a direct algebraic proof of coordinate-independence of the connection on the vertex algebra bundle. In Chapter 19 (old Chapter 18) on chiral algebras we have added a new motivational section and examples of chiral algebras that do not arise from vertex algebras. We have also explained how to attach to modules and twisted modules over vertex algebras certain modules over the corresponding chiral algebras.

Finally, we have added a new Chapter 20, on factorization algebras and factorization spaces. Factorization algebras, introduced by A. Beilinson and V. Drinfeld, provide a purely geometric reformulation of the definition of vertex algebras. Here we present an informal introduction to factorization algebras and give various examples. The most interesting examples come from factorization spaces, such as the Beilinson–Drinfeld Grassmannians which are moduli spaces of bundles on a curve equipped with trivializations away from finitely many points. We explain how the concept of factorization naturally leads us to Hecke correspondences on bundles and to the geometric Langlands correspondence. At the end of the chapter we discuss the chiral Hecke algebras introduced by Beilinson and Drinfeld which provide a tool for establishing the geometric Langlands conjecture. This chapter brings together and illuminates the material of several other chapters of the book.

We wish to thank those who kindly pointed out to us various typos in the first edition and pointed out additional references, especially, Michel Gros, Kenji

Iohara, Jim Lepowsky, David Nadler, Kiyokazu Nagatomo, Marcus Rosellen, Leon Takhtajan and A.J. Tolland.

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Preface to the First Edition

The present book is an introduction to the theory of vertex algebras with a particular emphasis on the relationship between vertex algebras and the geometry of algebraic curves. It is based on the lecture courses given by Edward Frenkel at Harvard University in the Fall of 1996 and at UC Berkeley in the Spring of 1999. The notes of these lectures were taken by David Ben-Zvi. They were subsequently improved and expanded by both authors. The main goal of this book is to introduce the concept of vertex algebra in a coordinate-independent way, and to define the spaces of conformal blocks attached to an arbitrary vertex algebra and a smooth algebraic curve, possibly equipped with some extra geometric data. From this point of view vertex algebras appear as the algebraic objects that encode the local geometric structure of various moduli spaces associated with algebraic curves.

In the fifteen years that have passed since they were introduced by R. Borcherds, vertex algebras have turned out to be extremely useful in many areas of mathematics. They are by now ubiquitous in the representation theory of infinite-dimensional Lie algebras. They have also found applications in such fields as algebraic geometry, the theory of finite groups, modular functions, topology, integrable systems, and combinatorics. The theory of vertex algebras also serves as the rigorous mathematical foundation for two-dimensional conformal field theory and string theory, extensively studied by physicists.

In the literature there exist two essentially different approaches to vertex algebras. The first is algebraic, following the original definition of Borcherds [**B1**]. It has been developed by I. Frenkel, J. Lepowsky, and A. Meurman [**FLM**] and more recently by V. Kac [**Kac3**]. Vertex operators appear here as formal power series acting on graded vector spaces. The second approach is geometric and more abstract: this is the theory of chiral algebras and factorization algebras developed by A. Beilinson and V. Drinfeld [**BD4**]. In this approach the main objects of study are $\mathcal{D}$–modules on powers of algebraic curves equipped with certain operations. Chiral algebras have non-linear versions called factorization spaces which encode various intricate structures of algebraic curves and bundles on them.

The present book aims to bridge the gap between the two approaches. It starts with the algebraic definition of vertex algebras, which is close to Borcherds', and essentially coincides with that of [**FKRW, Kac3**]. The key point is to make vertex operators coordinate-independent, thus effectively getting rid of the formal variable. This is achieved by attaching to each vertex algebra a vector bundle with a flat connection on the (formal) disc, equipped with an intrinsic operation. The formal variable is restored when we choose a coordinate on the disc; the fact that the operation is independent of this choice follows from the vertex algebra axioms. Once this is done and we obtain a coordinate-independent object, we can study

the interplay between vertex algebras and various geometric structures related to algebraic curves, bundles and moduli spaces.

In particular, we attach to each vertex algebra and any pointed algebraic curve the spaces of coinvariants and conformal blocks. When we vary the curve X and other data on X (such as G–bundles), these spaces combine into a sheaf on the relevant moduli space. One can gain new insights into the structure of moduli spaces from the study of these sheaves.

The language of the book gradually changes from that of formal power series as in [**FLM, Kac3**] to that of bundles, sheaves, and connections on algebraic curves. Our goal however is to avoid using sophisticated techniques, such as the theory of $\mathcal{D}$–modules, as much as possible. In particular, we present most of the material without mentioning the "$\mathcal{D}$–word". Only at the end of the book do we use rudiments of $\mathcal{D}$–module theory when describing the relationship between vertex algebras and the Beilinson–Drinfeld chiral algebras, and the sheaves of coinvariants. Ultimately, the formalism developed in this book will enable us to relate the algebraic theory of vertex algebras to the geometric theory of factorization algebras and factorization spaces.

The first five chapters of this book contain a self-contained elementary introduction to the algebraic theory of vertex algebras and modules over them. We motivate all definitions and results, give detailed proofs and consider numerous examples. This part of the book is addressed mainly to beginners. No prerequisites beyond standard college algebra are needed to understand it.

In Chapters 6–10 we develop the geometric approach to vertex algebras. Here some familiarity with basic notions of algebraic geometry should be helpful. We have tried to make the exposition as self-contained as possible, so as to make it accessible for non-experts.

Next, we review in Chapters 11–16 various constructions and applications of vertex algebras, such as, the free field realization of affine Kac-Moody algebras, solutions of the Knizhnik–Zamolodchikov equations, and the Drinfeld–Sokolov reduction. We also study quasi-classical analogues of vertex algebras, called vertex Poisson algebras.

The last four chapters of the book are more algebro-geometrically oriented. Here we construct the sheaves of coinvariants on the moduli spaces of curves and bundles and introduce the chiral algebras and factorization algebras following Beilinson and Drinfeld. In particular, we show how to attach to any quasi-conformal vertex algebra a chiral algebra on an arbitrary smooth algebraic curve. We discuss various examples of factorization algebras and factorization spaces, including the Beilinson–Drinfeld Grassmannians. We also give a brief overview of the geometric Langlands correspondence.

This book may be used by the beginners as an entry point to the modern theory of vertex algebras and its geometric incarnations, and by more experienced readers as a guide to advanced studies in this beautiful and exciting field.

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Edward Frenkel wishes to thank Boris Feigin for his longtime collaboration, and Alexander Beilinson and Vladimir Drinfeld for many illuminating discussions.

The authors gratefully acknowledge the support that they received in the course of writing this book from the Packard Foundation and the NSF.

Appendix

A.1. Discs, formal discs and ind–schemes

A.1.1. Discs. The theory of vertex algebras, as developed in this book, studies algebraic structures which "live on the disc" and uses them to draw global geometric conclusions on algebraic curves. What precisely is meant by "the disc" depends on the application, and on the category of spaces in which we wish to work. The variety of possible meanings of the disc is illustrated by the following diagram:

$$x \;\to\; D_n \;\underset{\to}{\lim}\; \widehat{D} \;\to\; D \;\to\; D_{\mathrm{conv}} \;\underset{\leftarrow}{\lim}\; D_{\mathrm{an}} \;\to\; X$$

Geometric intuition is perhaps most readily applied in the setting where X is a (compact) Riemann surface, x is a point and $D_{\mathrm{an}} \subset X$ is a small analytic disc. We may cover X by finitely many such discs, and use this cover to pass from local to global geometry of the surface.

One disadvantage of this approach is that there is no canonical analytic neighborhood of a point x. Thus we must make some arbitrary choices and check that they do not affect the constructions. (This problem is shared with the Zariski topology, whose open sets are too large in general, as well as the finer étale and *fppf* topologies.) If we take the inverse limit over all open analytic neighborhoods of x, we obtain a "space" which may be labeled D_{conv}, the convergent disc. It is characterized by the property that any power series with nonzero radius of convergence will converge on D_{conv}.

From the point of view of algebraic geometry, the ring of convergent power series is cumbersome, and it is far more convenient to work with arbitrary formal power series. To do this we start from the other end, and enlarge the point x rather than shrink X, which now denotes a smooth algebraic curve. Thus for any n we may consider the nth order infinitesimal neighborhood $D_{n,x}$ of x in X, which is isomorphic to the scheme $\operatorname{Spec}\mathbb{C}[z]/(z^{n+1})$. The direct limit of the schemes $D_{n,x}$ is the *formal disc* $\widehat{D}_x$, which is isomorphic to $\operatorname{Spf}\mathbb{C}[[z]]$, the formal spectrum of the complete topological ring $\mathbb{C}[[z]]$ of formal power series. The topology on formal power series is the inverse limit topology inherited from the description $\mathbb{C}[[z]] = \underset{\longleftarrow}{\lim}\, \mathbb{C}[z]/(z^n)$. (See [**Ha**] for the definition of formal schemes and § A.1.2 for more on formal and ind–schemes.) The formal disc at x (equivalently, the formal completion of X at x) is thus canonically defined and easily characterized algebraically.

However, the formal disc is too small for our purposes. Since $\widehat{D}_x$ is built out of the "thick points" $D_{n,x}$, it cannot be considered as open in X in any sense. In particular, it does not make sense to speak of a (non-empty) "punctured formal

disc" $\widehat{D}_x \backslash x$. Thus we need to enlarge $\widehat{D}_x$ to the *disc* D_x. Denote by $\mathcal{O}_x$ the completed local ring of x, and by $\mathcal{K}_x$ the field of fractions of $\mathcal{O}_x$. Then, by definition, D_x is the *scheme* (not formal scheme) $\operatorname{Spec} \mathcal{O}_x$. Note that $\mathcal{O}_x$ is isomorphic to $\mathbb{C}[[z]]$ (any such isomorphism defines a formal coordinate at x). In contrast to $\widehat{D}_x$, the disc D_x contains a generic point, the punctured disc $D_x^\times = \operatorname{Spec} \mathcal{K}_x \simeq \operatorname{Spec} \mathbb{C}((z))$. The disc $D_x \subset X$ is neither an open nor a closed subscheme of X, but lives between the formal and convergent discs.

In the main body of this book we often consider the restriction to D_x of a vector bundle $\mathcal{V}$ defined on a curve X. By this we mean the free $\mathcal{O}_x$–module $\mathcal{V}(D_x)$ which is the completion of the localization of the $\mathbb{C}[\Sigma]$–module $\Gamma(\Sigma, \mathcal{V})$ at x, where Σ is a Zariski open affine curve in X containing x (clearly, $\mathcal{V}(D_x)$ is independent of the choice of Σ). By the restriction of the vector bundle $\mathcal{V}$ to $D_x^\times$ we understand the free $\mathcal{K}_x$–module $\mathcal{V}(D_x^\times) = \mathcal{V}(D_x) \otimes_{\mathcal{O}_x} \mathcal{K}_x$. We also consider the restriction of the dual vector bundle $\mathcal{V}^*$ to D_x. By that we mean the $\mathcal{O}_x$–module of continuous $\mathcal{O}_x$–linear maps $\mathcal{V}(\mathcal{O}_x) \to \mathcal{O}_x$. The restriction of the dual vector bundle $\mathcal{V}^*$ to $D_x^\times$ is defined in the same way, replacing $\mathcal{O}_x$ by $\mathcal{K}_x$.

A.1.2. Ind–schemes. Many of the geometric objects appearing in the theory of vertex algebras are not represented by schemes but live in the larger world of *ind–schemes.* An ind–scheme is a directed system of schemes, that is, an ind–object of the category of schemes.

To be more precise, it is useful to consider the category of schemes as a full subcategory of an easily characterized larger category which is closed under direct limits. A scheme S is determined by its functor of points F_S, which is the functor from the category Sch of schemes (over $\mathbb{C}$) to sets $T \mapsto F_S(T) = \operatorname{Hom}_{Sch}(T, S)$. This functor satisfies a gluing property, generalizing the description of schemes as gluings of affine schemes. This property identifies F_S as a *sheaf* of sets on the big étale site over $\mathbb{C}$. Such a sheaf is called a $\mathbb{C}$–*space.* Thus, we have identified schemes as special $\mathbb{C}$–spaces.

A.1.3. Definition. An ind–scheme is a $\mathbb{C}$–space which is an inductive limit of schemes. A strict ind–scheme is one which may be realized as a direct system of schemes, with the morphisms closed embeddings.

A.1.4. Examples. The simplest example of an ind–scheme is the formal disc $\widehat{D} = \operatorname{Spf} \mathbb{C}[[z]]$, which is the direct limit of the Artinian schemes $\operatorname{Spec} \mathbb{C}[z]/(z^n)$. More generally, the formal completion of any scheme S along a closed subscheme T is the direct limit of the nth order infinitesimal neighborhoods of T in S. In fact, any formal scheme is naturally an ind–scheme.

The ind–schemes we encounter most often are either group ind–schemes or their homogeneous spaces. A group ind–scheme (or ind–group for short) is an ind–scheme with a group structure on the underlying functor of sets (that is, the functor from schemes to sets is promoted to a functor from schemes to groups). The most common examples of ind–groups are formal groups, which arise (over $\mathbb{C}$) from formally exponentiating a Lie algebra (and thus are explicitly described by the Baker–Campbell–Hausdorff formula). The formal group associated to a group scheme G is the formal completion of G at the identity.

More generally, for a group scheme G and a subgroup K, we may form the completion $\widehat{G}_K$ of G along K. This ind–group can be recovered solely from the Lie algebra $\mathfrak{g}$ of G and the algebraic group K. Many of the ind–groups we consider in

this book arise from such a *Harish–Chandra pair* $(\mathfrak{g}, K)$. They are similar to the ind–groups of the form $\widehat{G}_K$, even though a global group scheme G might not exist. (See § 17.2 for more on Harish–Chandra pairs.) Examples of ind–schemes of this kind and the corresponding pairs $(\mathfrak{g}, K)$ are

(1) $\underline{\mathrm{Aut}}\mathcal{O}$ (§ 6.2.3), $(\mathrm{Der}\,\mathcal{O}, \mathrm{Aut}\,\mathcal{O})$;
(2) $\mathrm{Aut}\,\mathcal{K}$ (§ 17.3.4), $(\mathrm{Der}\,\mathcal{K}, \mathrm{Aut}\,\mathcal{O})$;
(3) the identity component of $\mathcal{K}^\times$ (§ 18.1.7), $(\mathcal{K}, \mathcal{O}^\times)$.

The algebraic loop group $G(\mathcal{K})$, where G is semi-simple, is a more complicated example of an ind–group. In the examples above the ind-groups corresponding to $(\mathfrak{g}, K)$ are non-reduced (the "nilpotent directions" come from $\mathfrak{g}/\mathfrak{k}$), and the quotient of this ind–group by K is a formal scheme (in the case when our ind-group is $\widehat{G}_K$, this is simply the formal completion of G/K at the identity coset). The group $G(\mathcal{K})$, on the other hand, is reduced if G is semi-simple, and the corresponding quotient of $G(\mathcal{K})/G(\mathcal{O})$, which is called the affine Grassmannian associated to G, is also reduced. In fact, it is the direct limit of (singular) proper algebraic varieties. Each of them has a natural stratification, with strata isomorphic to affine bundles over partial flag manifolds for G (see, e.g., [**Ku2**]). Thus, the ind–group $G(\mathcal{K})$ is much "fatter" than the ind-group corresponding to the Harish-Chandra pair $(L\mathfrak{g}, G(\mathcal{O}))$ (compare with Remark 18.1.7).

A.2. Connections

Recall the notion of a flat connection on a vector bundle given in § 6.6.1. In this section we discuss Grothendieck's crystalline point of view on connections (in the very special case of smooth varieties in characteristic zero).

A.2.1. Flat structures. Let $(\mathcal{E}, \nabla)$ be a flat vector bundle on a smooth manifold S. Locally, in the analytic topology, horizontal sections of a flat connection are uniquely determined by their values at one point, enabling us to identify nearby fibers. Therefore the data of a flat connection are the same as the data of identifications $\mathcal{E}|_U \simeq \mathcal{E}_s \times U$ for any point $s \in S$ and any sufficiently small neighborhood U of s (these identifications must be compatible in the obvious sense). Any vector bundle $\mathcal{E}$ may of course be trivialized on a small enough open subset, but there are many possible trivializations. A flat connection on $\mathcal{E}$ gives us a preferred system of identifications of nearby fibers of $\mathcal{E}$.

Grothendieck's crystalline description of connections [**Gr**] (see also [**Si**]) allows us to interpret connections in a similar way in the algebraic setting. In algebraic geometry, open sets are too large for this purpose, but we may perform trivializations formally. Let $\widehat{\Delta}$ denote the formal completion of S^2 along the diagonal $\Delta : S \hookrightarrow S \times S$, and $p_1, p_2 : S \times S \to S$ the two projections.

A.2.2. Proposition ([Gr]).

(1) *A connection on a vector bundle $\mathcal{E}$ on S is equivalent to the data of an isomorphism $\eta : p_1^*\mathcal{E} \simeq p_2^*\mathcal{E}$ on the first order neighborhood of the diagonal, restricting to $\mathrm{Id}_{\mathcal{E}}$ on the diagonal.*
(2) *The flatness of a connection is equivalent to the existence of an extension of the isomorphism $\eta : p_1^*\mathcal{E} \simeq p_2^*\mathcal{E}$ to $\widehat{\Delta}$, together with a compatibility $p_{23}^*\eta \circ p_{12}^*\eta = p_{13}^*\eta$ over $\widehat{\Delta} \times_X \widehat{\Delta} \subset X^3$.*

A.2.3. Infinitesimally nearby points. Let R be a $\mathbb{C}$–algebra. Two R–points $x, y : \operatorname{Spec} R \to S$ are called *infinitesimally nearby* if the morphism $(x, y) : \operatorname{Spec} R \times \operatorname{Spec} R \to S \times S$ factors through $\widehat{\Delta}$. Note that if R is reduced, any two infinitesimally nearby points are necessarily equal, but if R contains nilpotent elements, this is not so.

Using this concept we can reformulate Proposition A.2.2 as follows: a flat connection consists of the data of isomorphisms $i_{x,y} : \mathcal{E}_x \simeq \mathcal{E}_y$ for all infinitesimally nearby points $x, y \in S$. In other words, the restriction of $\mathcal{E}$ to the formal completion $\widehat{D}_x$ of X at x is canonically identified with $\mathcal{E}_x \times \widehat{D}_x$. The flatness of the connection is the requirement that these isomorphisms be transitive, i.e., $i_{x,y} i_{y,z} = i_{x,z}$ for any triple of infinitesimally nearby points. Informally, a flat connection is an equivariance of the vector bundle $\mathcal{E}$ under the "equivalence relation" (techincally, formal groupoid) on X identifying infinitesimally nearby points.

A.3. Lie algebroids and $\mathcal{D}$–modules

A.3.1. $\mathcal{D}$–modules. The sheaf $\mathcal{D}$ of differential operators on a smooth variety S is a (quasi-coherent) sheaf of associative $\mathcal{O}_S$–algebras. It is generated as an associative $\mathcal{O}_S$–algebra by the structure sheaf $i : \mathcal{O}_S \hookrightarrow \mathcal{D}$, which is a subalgebra, and by the tangent sheaf $j : \Theta_S \hookrightarrow \mathcal{D}$, which is a Lie subalgebra. The relations are

$$[j(\xi), i(f)] = i(\xi \cdot f), \qquad \xi \in \Theta_S, f \in \mathcal{O}.$$

Thus $\mathcal{D}$ may be viewed as a kind of enveloping algebra for the tangent sheaf Θ_S.

A $\mathcal{D}$–module is a quasi-coherent $\mathcal{O}_S$–module $\mathcal{M}$, equipped with a (left) action $\mathcal{D} \otimes_{\mathcal{O}_S} \mathcal{M} \to \mathcal{M}$. When $\mathcal{M}$ is a vector bundle, this gives rise to a flat connection on $\mathcal{E}$: we simply restrict the action of $\mathcal{D}$ to Θ_S (using j). Flatness is a result of Θ_S being a *Lie subalgebra* of $\mathcal{D}$. Conversely, the flatness of a connection together with the Leibniz rule precisely guarantees that the action of vector fields extends uniquely to a $\mathcal{D}$–module structure.

The sheaf of algebras $\mathcal{D}$ has a natural filtration defined by $\mathcal{D}_{\leq 0} = \mathcal{O}_S$ and $\mathcal{D}_{\leq k} = j(\Theta_S) \cdot D_{\leq k-1}$ for $k > 0$. By condition (1) above, we have $[\mathcal{D}_{\leq k}, \mathcal{D}_{\leq m}] \subset \mathcal{D}_{\leq k+m-1}$. Hence the associated graded $\mathcal{O}$–algebra $\operatorname{gr} \mathcal{D}$ is commutative. In fact, the multiplication maps $\Theta_S^{\otimes n} \to \mathcal{D}$ identify $\operatorname{gr} \mathcal{D}$ with the symmetric $\mathcal{O}_S$–algebra of Θ_S, which is none other than the algebra of functions on the cotangent bundle, $\operatorname{gr} \mathcal{D} \simeq \mathcal{O}_{T^*S}$. The assignment $\mathcal{D} \ni D \mapsto \operatorname{gr}(D) \in \mathcal{O}_{T^*S}$ is known as the symbol map.

A.3.2. Lie algebroids. This section is based on [BB]. A Lie algebroid $\mathcal{A}$ on S is a quasi-coherent $\mathcal{O}_S$–module equipped with a $\mathbb{C}$–linear Lie bracket $[\cdot, \cdot] : \mathcal{A} \otimes_{\mathbb{C}} \mathcal{A} \to \mathcal{A}$ and an $\mathcal{O}_S$–module morphism $a : \mathcal{A} \to \Theta_S$, known as the anchor map, such that

(1) a is a Lie algebra homomorphism, and
(2) $[l_1, f l_2] = f[l_1, l_2] + (a(l_1) \cdot f) l_2$, for l_1, l_2 in $\mathcal{A}$ and f in $\mathcal{O}_S$.

The simplest example of a Lie algebroid is the tangent sheaf Θ_S itself, with anchor being the identity map. Next, let $\mathfrak{g}$ be a Lie algebra acting on S, so that we have a Lie algebra homomorphism $a : \mathfrak{g} \to \Theta_S$. We define a Lie algebroid structure on $\widetilde{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}_S$, known as the action algebroid, by extending the map a by $\mathcal{O}_S$–linearity to an anchor map $\widetilde{a} : \widetilde{\mathfrak{g}} \to \Theta_S$, and defining the Lie bracket by the

formula

$$[f_1 \otimes l_1, f_2 \otimes l_2] = f_1 f_2 \otimes [l_1, l_2] + (f_1 a(l_1) \cdot f_2) \otimes l_2 - (f_2 a(l_2) \cdot f_1) \otimes l_1,$$

for all $l_1, l_2 \in \mathfrak{g}, f_1, f_2 \in \mathcal{O}_S$.

Another example of a Lie algebroid was introduced by Atiyah in his study of holomorphic connections. Let $\mathcal{P}$ be a principal G–bundle on S, and consider the extension (the Atiyah sequence)

$$0 \to \mathfrak{g}_{\mathcal{P}} \otimes \mathcal{O}_S \to \mathcal{A}_{\mathcal{P}} \to \Theta_S \to 0, \tag{A.3.1}$$

where sections of $\mathcal{A}_{\mathcal{P}}$ over an open $U \subset S$ are pairs $(\widetilde{\tau}, \tau)$, with τ a vector field on U and $\widetilde{\tau}$ a G–invariant lift of τ to $\mathcal{P}$. Thus, the sheaf $\mathcal{A}_{\mathcal{P}}$ carries infinitesimal symmetries of $\mathcal{P}$. The (anchor) map $a : \mathcal{A}_{\mathcal{P}} \to \Theta_S$ in (A.3.1) forgets $\widetilde{\tau}$, and its kernel is the sheaf of G–invariant vector fields along the fibers, $\mathfrak{g}_{\mathcal{P}} \otimes \mathcal{O}_S$. Equipped with its natural $\mathbb{C}$–Lie algebra structure, $\mathcal{A}_{\mathcal{P}}$ becomes a Lie algebroid, often referred to as the Atiyah algebra of $\mathcal{P}$.

To any Lie algebroid $\mathcal{A}$ we may assign its enveloping algebra, which is a sheaf of $\mathcal{O}_S$–algebras $U(\mathcal{A})$, generalizing the construction of $\mathcal{D}$ from Θ_S. We define $U(\mathcal{A})$ as the associative $\mathcal{O}_S$–algebra generated by a subalgebra $i : \mathcal{O}_S \to U(\mathcal{A})$ and a Lie subalgebra $j : \mathcal{A} \to U(\mathcal{A})$, with the relation

$$[j(l), i(f)] = i(a(l) \cdot f), \qquad l \in \mathcal{A}, f \in \mathcal{O}_S.$$

Thus $U(\mathcal{A})$–modules are the same as $\mathcal{O}_S$–modules with a compatible $\mathcal{A}$–action.

A.3.3. Central extensions and TDOs. Let S be a smooth scheme, and $\mathcal{L}$ a line bundle on S. Let $\mathcal{A}_{\mathcal{L}}$ denote the Atiyah algebroid of $\mathcal{L}$, whose sections are $\mathbb{C}^\times$–invariant vector fields on the $\mathbb{C}^\times$–bundle associated to $\mathcal{L}$. Since $\mathbb{C}^\times$ is abelian, its adjoint action on the Lie algebra $\mathbb{C}$ is trivial, and (A.3.1) becomes simply

$$0 \to \mathcal{O}_S \to \mathcal{A}_{\mathcal{L}} \to \Theta_S \to 0. \tag{A.3.2}$$

If in addition $\Gamma(S, \mathcal{O}_S) = \mathbb{C}$ (e.g., if S is projective), we obtain a sequence

$$0 \to \mathbb{C} \to \Gamma(S, \mathcal{A}_{\mathcal{L}}) \to \Gamma(S, \Theta_S) \to 0 \tag{A.3.3}$$

of global sections. It is easy to see that $\mathbb{C} \subset \Gamma(S, \mathcal{A}_{\mathcal{L}})$ is central, so $\Gamma(S, \mathcal{A}_{\mathcal{L}})$ is a one–dimensional central extension of $\Gamma(S, \Theta_S)$.

Now suppose $\mathfrak{g}$ is a Lie algebra acting on S, so that we have a Lie algebra homomorphism $\mathfrak{g} \to \Gamma(S, \Theta_S)$. The action of $\mathfrak{g}$ does not necessarily lift to the line bundle $\mathcal{L}$, that is, to $\Gamma(S, \mathcal{A}_{\mathcal{L}})$. However, it follows from the above that there is a canonical central extension $\widehat{\mathfrak{g}}$ of $\mathfrak{g}$, namely the pullback to $\mathfrak{g}$ of (A.3.3), which does lift. This is an important mechanism through which central extensions arise in geometry.

From the line bundle $\mathcal{L}$ we can also construct a sheaf of associative $\mathcal{O}_S$–algebras, namely, the sheaf $\mathcal{D}_{\mathcal{L}}$ of differential operators acting on $\mathcal{L}$. It is easy to see that $\mathcal{D}_{\mathcal{L}}$ is identified with the quotient of the enveloping algebra $U(\mathcal{A}_{\mathcal{L}})$ of the Lie algebroid $\mathcal{A}_{\mathcal{L}}$ by the identification of $1 \in \mathcal{O}_S \hookrightarrow \mathcal{A}_{\mathcal{L}}$ with the identity of $U(\mathcal{A}_{\mathcal{L}})$.

The algebra $\mathcal{D}_{\mathcal{L}}$ has a filtration, with the functions $\mathcal{O}_S = \mathcal{D}_{\mathcal{L}}^{\leq 0} \subset \mathcal{D}_{\mathcal{L}}$ as zeroth piece, and $\mathcal{D}_{\mathcal{L}}^{\leq n}/\mathcal{D}_{\mathcal{L}}^{\leq n-1} \simeq \Theta_S^{\otimes n}$. In fact, the associated graded algebra of $\mathcal{D}_{\mathcal{L}}$ is again naturally isomorphic to the (commutative) symmetric algebra of Θ_S. The existence of such a filtration is a property shared with the (untwisted) sheaf $\mathcal{D} = \mathcal{D}_{\mathcal{O}_S}$ of differential operators. Sheaves of associative $\mathcal{O}_S$–algebras with such a filtration are known as *sheaves of twisted differential operators*, or TDOs for short.

A.3.4. Line bundles and central extensions. We would like to investigate the extent to which we can go backwards from a central extension

$$0 \to \mathbb{C}\mathbf{1} \to \widehat{\mathfrak{g}} \to \mathfrak{g} \to 0 \tag{A.3.4}$$

of a Lie algebra $\mathfrak{g}$ acting on S to a line bundle on S. We will see that we can recover a TDO on S, which may or may not be $\mathcal{D}_{\mathcal{L}}$ for some line bundle $\mathcal{L}$.

Recall from § A.3.2 that the action of $\mathfrak{g}$ on S defines a Lie algebroid structure on $\mathfrak{g} \otimes \mathcal{O}_S$, with anchor $a : \mathfrak{g} \otimes \mathcal{O}_S \to \Theta_S$. It follows that $\widehat{\mathfrak{g}} \otimes \mathcal{O}_S$ also forms a Lie algebroid, with anchor $\widehat{a} : \widehat{\mathfrak{g}} \otimes \mathcal{O}_S \to \Theta_S$. Assume that the extension

$$0 \to \mathcal{O}_S \to \widehat{\mathfrak{g}} \otimes \mathcal{O}_S \to \mathfrak{g} \otimes \mathcal{O}_S \to 0$$

induced by (A.3.4) splits over $\operatorname{Ker} a \subset \mathfrak{g} \otimes \mathcal{O}_S$. Then we have a Lie algebra embedding $\operatorname{Ker} a \hookrightarrow \widehat{\mathfrak{g}} \otimes \mathcal{O}_S$. The quotient $\mathcal{T} = \widehat{\mathfrak{g}} \otimes \mathcal{O}_S / \operatorname{Ker} a$ is now an extension

$$0 \to \mathcal{O}_S \to \mathcal{T} \to \Theta_S \to 0. \tag{A.3.5}$$

The sheaf $\mathcal{T}$ carries a natural Lie algebroid structure: the Lie bracket comes from $\widehat{\mathfrak{g}} \otimes \mathcal{O}_S$, while the anchor map is the projection onto Θ_S in (A.3.5).

Let $\mathcal{D}_{\mathcal{T}}$ denote the quotient of the enveloping algebra $U(\mathcal{T})$ of the Lie algebroid $\mathcal{T}$ by the identification of $1 \in \mathcal{O}_S \hookrightarrow \mathcal{T}$ with the identity of $U(\mathcal{T})$. The sheaf $\mathcal{D}_{\mathcal{T}}$ is a TDO, with the filtration determined by $\mathcal{D}_{\mathcal{T}}^{\leq 0} = \mathcal{O}_S$ and $\mathcal{D}_{\mathcal{T}}^{\leq 1} = \operatorname{im} \mathcal{T}$. The sheaf $\mathcal{D}_{\mathcal{T}}$ is the TDO on S corresponding to the extension (A.3.4). Note however that line bundles give rise only to those TDOs $\mathcal{D}_{\mathcal{T}}$ for which the extension class of (A.3.5) in $\operatorname{Ext}^1(\Theta_S, \mathcal{O}_S) \simeq H^1(S, \Omega_S)$ is integral, while we may always "rescale" the algebroid $\mathcal{T}$ by multiplying the embedding $\mathcal{O}_S \hookrightarrow \mathcal{T}$ by $\lambda \in \mathbb{C}^\times$.

A.4. Lie algebra cohomology

Let $\mathfrak{g}$ be a Lie algebra and M a $\mathfrak{g}$–module. The standard Chevalley complex $C^\bullet(\mathfrak{g}, M)$ computing the cohomology of $\mathfrak{g}$ with coefficients in M is defined as follows. Its ith group $C^i(\mathfrak{g}, M)$ is the vector space of $\mathbb{C}$–linear maps $\bigwedge^i \mathfrak{g} \to M$, and the differential $d : C^n(\mathfrak{g}, M) \to C^{n+1}(\mathfrak{g}, M)$ is given by the formula

$$\begin{aligned}(d \cdot f)(x_1, \dots, x_{n+1}) = &\sum_{1 \leq i \leq n+1} (-1)^{i+1} f(x_1, \dots, \widehat{x}_i, \dots, x_{n+1}) \\ &+ \sum_{1 \leq i < j \leq n+1} (-1)^{i+j+1} f([x_i, x_j], x_1, \dots, \widehat{x}_i, \dots, \widehat{x}_j, \dots, x_{n+1}).\end{aligned}$$

The cohomologies of this complex are denoted by $H^n(\mathfrak{g}, M)$ (see [**Fu**] for more details).

If $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$ and M is an $\mathfrak{h}$–module, there is an isomorphism $H^n(\mathfrak{g}, \operatorname{Coind}_{\mathfrak{h}}^{\mathfrak{g}} M) \simeq H^n(\mathfrak{h}, M)$, which is often called Shapiro's lemma (see, e.g., [**Fu**], Theorem 1.5.4, for the proof).

We can write $C^\bullet(\mathfrak{g}, M) = M \otimes \bigwedge \mathfrak{g}^*$. The space $\bigwedge \mathfrak{g}^*$ is a module over the Clifford algebra $\mathcal{C}l(\mathfrak{g})$ associated to the vector space $\mathfrak{g} \oplus \mathfrak{g}^*$ with a natural symmetric bilinear form. Moreover, the differential d may be expressed entirely in terms of the action of $\mathfrak{g}$ on M and the action of the Clifford algebra on this module; see formula (15.1.5) in the case when $\mathfrak{g} = \mathfrak{n}_+((t))$. This interpretation of the Chevalley complex suggested that we may define other cohomological complexes by utilizing other modules over the Clifford algebra. For instance, if we take the Clifford module $\bigwedge \mathfrak{g}$, we obtain the Chevalley complex $M \otimes \bigwedge \mathfrak{g}$ of homology of $\mathfrak{g}$ with coefficients in M.

If $\mathfrak{g}$ is finite-dimensional, then the corresponding Clifford algebra has a unique irreducible representation (up to an isomorphism), so there is only one theory of cohomology of $\mathfrak{g}$ (up to cohomological shifts). However, for infinite-dimensional $\mathfrak{g}$ there are inequivalent irreducible Clifford modules. Their construction is in fact parallel to that of the Weyl algebra modules, considered in Chapter 11. In particular, if we choose the Fock representation of the Clifford algebra, defined as in § 15.1.4, we obtain the standard complex of *semi-infinite* cohomology introduced by B. Feigin [**Fe**].

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Index

$\mathcal{D}$–modules, 297, 335, 378
Δ–differential, 101
$\mathcal{W}$–algebra, 249, 320
$\mathcal{W}$–algebra, classical, 289
$\beta\gamma$–system, 201
$\mathfrak{g}$–oper, 283
$\widehat{\mathfrak{g}}$–structure, 122
λ–connection, 135
bc–system, 86
n–point functions, 72, 175

adèles, 360
adjoint differential operator, 143
affine connection, 135, 217
affine Grassmannian, 358
affine Kac–Moody algebras, 38, 121, 148, 154, 229, 269, 276, 347, 364
affine structure, 136
annihilation operator, 28, 210
associativity, 48, 180
Atiyah sequence, 148, 379

BGG resolution, 262
Borcherds identity, 56
Borel subalgebra, 190, 282
boson–fermion correspondence, 85
BRST complex, 97, 247, 286, 289

Cartan decomposition, 190, 196, 217, 246
Cartan subalgebra, 190, 246
Casimir tensor, 232, 314
center of a vertex algebra, 95, 273, 319
central charge, 42–44
central extension, 25, 299, 379
central extension, universal, 39
character, 87, 90
charge, 245
Chevalley complex, 165, 380
chiral algebra, 174, 329, 339, 354
chiral correlation functions, 175
chiral de Rham complex, 325, 368
chiral differential operators, 326
chiral Hecke algebra, 372
classical Drinfeld–Sokolov reduction, 285
coinvariants, 153, 168, 297
commutative vertex algebra, 22
configuration space, 225
conformal algebra, 271, 342
conformal blocks, 153, 167
conformal dimension, 16
conformal module, 80
conformal vector, 44
conformal vertex algebra, 44
connection, 117, 148, 276, 377
contraction, 211
coset construction, 95
creation operator, 28, 210
critical level, 214, 216, 320

de Rham cohomology, 156, 240, 343
delta-function, 12, 194, 332, 341, 363
determinant line bundle, 307
disc, 376
disc, abstract, 100
disc, formal, 104, 375
Drinfeld–Sokolov reduction, classical, 285
Drinfeld–Sokolov reduction, quantum, 247
dual Coxeter number, 39

enveloping vertex algebra, 270
external tensor product, 128, 330, 332

factorization algebra, 351
factorization space, 362
fermionic Fock representation, 85
fiber of an $\mathcal{O}$–module, 298
field, 16
flag manifold, 190, 195
flat bundle, 240
flat connection, 117, 377
flat structure, 293, 377
Fock representation, 27, 89, 221, 366
formal coordinate, 100, 111
formal disc, 375
formal distribution, 11
formal Laurent series, 14
formal power series, 11
formal scheme, 375

formal Taylor series, 14
free field realization, 187, 262

gauge transformations, 146, 147, 277, 282, 284
Gauss-Manin connection, 242
geometric Langlands correspondence, 264, 324, 371
Goddard–Kent–Olive construction, 95

hamiltonian reduction, 285
Harish–Chandra pair, 377
Hecke correspondences, 369
Hecke eigensheaf, 324, 371
Heisenberg Lie algebra, 26, 146, 152, 153, 182, 221
horizontal section, 117

ind-group, 104, 293, 302, 376
ind-scheme, 104, 197, 359, 376
infinitesimally nearby points, 378
invariant bilinear form, 38

Jacobi identity, 339, 341
Jacobi triple product identity, 88
Jacobian variety, 316
jet scheme, 162, 197, 367

Killing form, 38, 216, 260
Kirillov–Kostant structure, 276
Knizhnik–Zamolodchikov equations, 221, 232, 233
Kodaira–Spencer isomorphism, 304, 312

Langlands dual group, 217, 323, 370
Langlands dual Lie algebra, 264, 320
lattice vertex algebra, 88, 367
Laurent polynomial, 11
level, 39
Lie algebra, 25
Lie algebra cohomology, 26, 193, 210, 380
Lie algebroid, 296, 378
Lie* algebra, 342, 363
local coordinate, 112
local system, 240
locality, 18, 180
loop algebra, 38
loop group, 377
loop space, 198

modular functor, 168
module over a vertex algebra, 76
moduli space of bundles, 311
moduli space of curves, 300
Monster group, 94
Moonshine Module, 94

normal ordering, 31, 53, 202, 210, 345
normalized bilinear form, 39

oper, 281, 285, 320
oper, $\mathfrak{sl}_2$, 140, 283
operator product expansion (OPE), 51, 334
orbifold construction, 94

PBW basis, 27
Poincaré–Birkhoff–Witt theorem, 27
Poisson structure, 276
polydifferentials, 182
primary field, 108
projective connection, 139, 155, 280
projective structure, 139

quantum Drinfeld–Sokolov reduction, 247
quantum group, 176, 241, 262
quasi-conformal vertex algebra, 107

Ran space, 353
rational vertex algebra, 90, 168
Reconstruction Theorem, 37, 70
representation, coinduced, 191
representation, induced, 191, 194, 217, 230
residues, 152, 156, 157, 330
restricted dual vector space, 191

Satake correspondence, 371
Schwarzian derivative, 138
screening operator, 262
Segal–Sugawara construction, 45, 149
Segal–Sugawara vector, 45, 59, 214, 231, 235
semi-infinite cohomology, 218, 246, 381
Shapiro's lemma, 165, 193, 380
singular vector, 108, 237
smooth module, 60, 78
stress-energy tensor, 137

theta characteristic, 138, 283, 365
theta line bundle, 314, 316
torsor, 109, 123, 124
translation operator, 20
twist by a torsor, 109
twisted conformal blocks, 163
twisted differential operators (TDO), 193, 299, 379
twisted module, 94
twisting property, 124

uniformization theorem, 313, 359
universal chiral algebra, 341
universal enveloping algebra, 27

vacuum representation, 39
vacuum vector, 20
Verma module, 191, 195, 217
Verma module, contragredient, 191, 218, 235
vertex algebra, 20
vertex algebra bundle, 113
vertex algebra, commutative, 22
vertex algebra, conformal, 44
vertex algebra, quasi-conformal, 107
vertex Lie algebra, 267

vertex operator, 20
vertex Poisson algebra, 271, 319
Virasoro algebra, 42, 138, 155, 269, 279, 347
Virasoro uniformization, 368

W–algebra, 249, 320
W–algebra, classical, 289
Wakimoto module, 215, 217
Wakimoto realization, 214
Weyl algebra, 27, 183, 189, 193, 194
Wick formula, 211

Zariski topology, 112

List of Frequently Used Notation

$\mathbb{Z}_+$ – the set of non-negative integers.
$\mathbb{Z}_{>0}$ – the set of positive integers.
$\mathcal{O} = \mathbb{C}[[t]]$ – the space of formal Taylor series.
$D = \operatorname{Spec} \mathbb{C}[[t]]$ – the disc.
$\mathcal{K} = \mathbb{C}((t))$ – the space of formal Laurent series.
$D^\times = \operatorname{Spec} \mathbb{C}((t))$ – the punctured disc.
$\mathcal{O}_Z$ – the structure sheaf of a variety Z.
$R^\times$ – the set of invertible elements of a ring R.

Chapter 1.

$\delta(z-w)$ – formal delta-function, § 1.1.3.
$Y(A,z)$ – vertex operator, § 1.3.1.
$|0\rangle$ – vacuum vector in a vertex algebra, § 1.3.1.
T – translation operator in a vertex algebra, § 1.3.1.

Chapter 2.

$\mathcal{H}$ – Heisenberg Lie algebra, § 2.1.2.
b_n – basis elements of the Heisenberg Lie algebra, § 2.1.2.
$\mathbf{1}$ – central element of the Heisenberg Lie algebra, § 2.1.2.
$\widetilde{\mathcal{H}}$ – Weyl algebra, § 2.1.2.
$U(\mathfrak{g})$ – universal enveloping algebra of a Lie algebra $\mathfrak{g}$, § 2.1.2.
π – Heisenberg vertex algebra, § 2.1.3.
$L\mathfrak{g}$ – formal loop algebra of a Lie algebra $\mathfrak{g}$, § 2.4.1.
$\widehat{\mathfrak{g}}$ – affine Kac–Moody algebra, § 2.4.1.
$h^\vee$ – the dual Coxeter number, § 2.4.1.
$V_k(\mathfrak{g})$ – vertex algebra associated to $\widehat{\mathfrak{g}}$ of level k, § 2.4.3.
J^a – basis elements of $\mathfrak{g}$, § 2.4.5.
J^a_n – basis elements $J^a \otimes t^n$ of $\widehat{\mathfrak{g}}$, § 2.4.5.
K – central element of $\widehat{\mathfrak{g}}$, § 2.4.5.
Vir – the Virasoro algebra, § 2.5.1.
L_n – basis elements of the Virasoro algebra, § 2.5.1.
C – central element of the Virasoro algebra, § 2.5.1.
$T(z)$ – § 2.5.3.
Vir_c – Virasoro vertex algebra with central charge c, § 2.5.6.

Chapter 4.

$U'(V)$ – Lie algebra of Fourier coefficients of vertex operators, § 4.1.1.
$U(V)$ – a completion of $U'(V)$, § 4.1.1.

Chapter 5.

π_λ – Fock representation of Heisenberg Lie algebra, § 5.2.1.
$V_\lambda(z)$ – bosonic vertex operator, § 5.2.6.
$\bigwedge$ – fermionic vertex superalgebra, § 5.3.1.
$\widehat{\mathfrak{h}}$ – Heisenberg Lie algebra, § 5.4.1.
V_L – lattice vertex superalgebra, § 5.4.2.

Chapter 6.

$\mathcal{O}_x$ – completed local ring at the point $x \in X$, § 6.1.1.
$D_x = \operatorname{Spec} \mathcal{O}_x$ – the disc around $x \in X$, § 6.1.1.
$\mathfrak{m}_x$ – the maximal ideal of $\mathcal{O}_x$, § 6.1.1.
$\mathcal{K}_x$ – the field of fractions of $\mathcal{O}_x$, § 6.1.2.
$D_x^\times = \operatorname{Spec} \mathcal{K}_x$ – the punctured disc around x, § 6.1.2.
Ω_x – the space of differentials on $D_x^\times$, § 6.1.3.
$\operatorname{Aut} \mathcal{O}$ – the group of automorphisms of $\mathcal{O}$, § 6.2.
$\operatorname{Aut}_+ \mathcal{O}$ – § 6.2.
$\operatorname{Der}_0 \mathcal{O}$ – the Lie algebra of $\operatorname{Aut} \mathcal{O}$, § 6.2.
$\operatorname{Der}_+ \mathcal{O}$ – the Lie algebra of $\operatorname{Aut}_+ \mathcal{O}$, § 6.2.
$R(\rho)$ – operator of action of $\rho(z) \in \operatorname{Aut} \mathcal{O}$ on V, § 6.3.1.
$\mathcal{A}ut_x$ – the $\operatorname{Aut} \mathcal{O}$–torsor of formal coordinates at $x \in X$, § 6.4.6.
$\mathcal{V}_x$ – the twist of V by $\mathcal{A}ut_x$, § 6.4.6.
$\mathcal{A}ut_X$ – the $\operatorname{Aut} \mathcal{O}$–bundle of formal coordinates on the curve X, § 6.5.
$\mathcal{V} = \mathcal{V}_X$ – the $\mathcal{A}ut_X$–twist of the $\operatorname{Aut} \mathcal{O}$–module V, § 6.5.1.
$\mathcal{Y}_x$ – § 6.5.4.
$\mathcal{Y}_x^\vee$ – § 6.5.8.

Chapter 7.

$\mathfrak{g}(\mathcal{O}), \mathfrak{g}^+(\mathcal{O})$ – § 7.1.
$G(\mathcal{O}), G^+(\mathcal{O})$ – § 7.1.
$\widehat{\mathcal{P}}$ – bundle of formal coordinates and trivializations, § 7.1.4.
$\mathcal{M}^{\mathcal{P}}$ – § 7.1.5.
$\mathcal{Y}_{M,x}, \mathcal{Y}_{M,x}^\vee$ – § 7.3.7.

Chapter 9.

$\Omega_{\mathcal{O}_x}$ – the topological module of differentials of $\mathcal{O}_x$, § 9.1.1.
$\Omega_{\mathcal{K}_x}$ – the topological module of differentials of $\mathcal{K}_x$, § 9.1.1.
$\mathcal{H}_{\mathrm{out}}(x)$ – § 9.1.4.
$\mathcal{H}_x$ – § 9.1.8.
Π – the vector bundle associated to π, § 9.1.10.
$U(\mathcal{V}_x)$ – § 9.2.2.
$U_\Sigma(V), U_\Sigma(\mathcal{V}_x)$ – § 9.2.5.
$H(X, x, V)$ – the space of coinvariants, § 9.2.7.
$C(X, x, V)$ – the space of conformal blocks, § 9.2.7.
$X^{\mathcal{P}}(X, x, V)$ – § 9.5.2.
$H^{\mathcal{P}}(X, x, V)$ – § 9.5.3.

Chapter 10.

$C_V(X, (x_i), (M_i))_{i=1}^n$ – § 10.1.1.

$H_V(X,(x_i),(M_i))_{i=1}^n$ – § 10.1.2.

Chapter 11.

$\mathfrak{n}_+, \mathfrak{n}_-$ – upper and lower nilpotent subalgebras of $\mathfrak{g}$, § 11.2.3.
$\mathfrak{b}_+, \mathfrak{b}_-$ – upper and lower Borel subalgebras of $\mathfrak{g}$, § 11.2.3.
$\mathfrak{h}$ – Cartan subalgebra of $\mathfrak{g}$, § 11.2.3.
M_χ – Verma module, § 11.2.4.
M_χ^* – contragredient Verma module, § 11.2.4.
$\operatorname{Fun}\mathcal{K}$ – § 11.3.4.
$\operatorname{Vect}\mathcal{K}$ – § 11.3.4.

Chapter 12.

$\widetilde{A}$ – § 12.1.1.
$A^\natural$ – § 12.1.3.
W_k – § 12.3.
$W_{\nu,k}$ – Wakimoto module, § 12.3.3.
$\overline{W}_{\nu,k}$ – § 12.3.4.

Chapter 13.

π_ν^κ – § 13.1.1.
$\mathcal{H}_{\vec{x}}$ – § 13.1.2.
$\mathcal{H}_{\vec{x}}^0$ – § 13.1.6.
$C^0(\vec{x}, \pi_{\vec{\nu}}^\kappa)$ – § 13.1.7.
$\mathcal{C}_N$ – § 13.2.
$\mathcal{B}_N$ – § 13.2.
$\mathcal{H}_{\mathbf{x}}^0$ – § 13.2.1.
$\mathbb{M} = \mathbb{M}^k$ – § 13.3.2.
$\mathfrak{g}_{\mathbf{x}}^0$ – § 13.3.5.

Chapter 15.

$C_k^\bullet(\mathfrak{g})$ – § 15.1.4.
$H_k^\bullet(\mathfrak{g})$ – § 15.1.7.
$\mathcal{W}_k(\mathfrak{g})$ – $\mathcal{W}$–algebra, § 15.1.7.
$C_k^\bullet(\mathfrak{g})_0, C_k^\bullet(\mathfrak{g})'$ – § 15.2.1.
$\{p_+, p_0, p_-\}$ – principal $\mathfrak{sl}_2$–triple in $\mathfrak{g}$, § 15.2.9.
$\mathfrak{a}_-$ – centralizer of p_- in $\mathfrak{g}$, § 15.2.9.
$p_-^{(i)}$ – basis elements of $\mathfrak{a}_-$, § 15.2.9.
$\mathfrak{a}_+$ – centralizer of p_+ in $\mathfrak{g}$, § 15.4.3
$p_+^{(i)}$ – basis elements of $\mathfrak{a}_+$, § 15.4.3.
$r^\vee$ – the lacing number of $\mathfrak{g}$, § 15.4.13.
${}^L\mathfrak{g}$ – the Langlands dual Lie algebra to $\mathfrak{g}$, § 15.4.15.

Chapter 16.

Y_- – vertex Lie algebra operation, § 16.1.1.
$\operatorname{Lie}(L)$ – the local Lie algebra of a vertex Lie algebra L, § 16.1.7.
$\operatorname{Op}_{\mathfrak{g}}(X)$ – the space of $\mathfrak{g}$–opers on a curve X, § 16.6.4.
$\operatorname{Op}_{\mathfrak{g}}(D), \operatorname{Op}_{\mathfrak{g}}(D^\times)$ – § 16.6.7.
$\mathcal{W}_\infty(\mathfrak{g})$ – classical $\mathcal{W}$–algebra, § 16.8.1.

Chapter 17.

$\Delta(V)$ – § 17.2.6.
$\mathfrak{M}_g$ – moduli stack of curves of genus g, § 17.3.1.
$\mathfrak{M}_{g,1} = \mathfrak{X}_g$ – § 17.3.1.
$\widehat{\mathfrak{M}}_{g,1}$ – § 17.3.1.

Chapter 18.

$\mathfrak{M}_G(X)$ – moduli stack of G–bundles on X, § 18.1.1.
$\widehat{\mathfrak{M}}_G(X)$ – § 18.1.1.
$\mathcal{L}_G$ – ample generator of the Picard group of $\mathfrak{M}_G(X)$, § 18.1.12.
$\mathfrak{z}(\widehat{\mathfrak{g}})$ – the center of $V_{-h^\vee}(\mathfrak{g})$, § 18.4.4.

Chapter 19.

$\mathcal{F}^r = \mathcal{F} \otimes \omega_Z$ – the right $\mathcal{D}$–module associated to a left $\mathcal{D}$–module $\mathcal{F}$, § 19.1.4.

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