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Multiparticle Quantum Scattering in Constant Magnetic Fields

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ABSTRACT. This book is devoted to the scattering theory of systems of N interacting quantum particles in an external constant magnetic field. Particular emphasis is placed on the development of the Mourre theory, applications of geometrical methods, and the proof of asymptotic completeness for a large class of systems.

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Contents

Preface

This monograph is devoted to the spectral and scattering theory of quantum Hamiltonians describing systems of N interacting particles in an external constant magnetic field. Most of it consists of the results obtained by the authors from 1993 to 1999.

Quantum scattering theory is the subfield of quantum mechanics which deals with the large-time asymptotics of the solutions of the Schrödinger equation and with the structure of the continuous spectrum of the corresponding Schrödinger operator. One of its main problems is to prove (or disprove) *asymptotic completeness*, which roughly speaking, is a statement that all solutions of the Schrödinger equation under consideration must follow asymptotically certain prescribed patterns. (The precise mathematical formulation of this will be given in Chapter 6.) There is a vast body of literature on this and other aspects of 2-particle scattering, see e.g., [**RS**] or [**Hö**, vols. II, IV] for an overview. For $N \geq 3$ particles, the problem becomes much more complicated. It was only in the last 20 years that the N-body scattering theory underwent a period of rapid development, beginning with the work of Enss [**E2**], [**E3**], and culminating in the proof of N-body asymptotic completeness by Sigal-Soffer [**SS1**] and Dereziński [**De1**], with significant contributions by many other authors, see e.g., [**M1**], [**PSS**], [**FH1**], [**Gr**], [**Y**]. We refer the reader to [**DG1**] for a more detailed account of that story and for a self-contained presentation of the results obtained in the 1980's and 90's.

Our work was largely inspired by these developments: we set out to extend the new results on asymptotic completeness to the case of N-body systems in a constant magnetic field. Such systems are of considerable interest in quantum physics. There is a large body of research on the *quantum Hall effect*; most of it assumes that there are no interactions between the particles save for the Pauli exclusion principle, but it is possible that at some point the scattering effects will have to be taken into account. In astrophysics, there is some evidence that strong magnetic fields exist on the surfaces of neutron stars and white dwarfs. "Quantum dots" are a prime example of quantum systems which can be significantly affected by magnetic fields of strength comparable to what can actually be achieved in existing laboratories. Physicists have also been studying highly excited (Rydberg) atoms in magnetic fields, which offer an opportunity to study the phenomena of "quantum chaos". See e.g., [**RWHG**] for a survey of some of the recent (theoretical and experimental) work on the subject. Furthermore, there is a growing interest in magnetic Hamiltonians among mathematicians and mathematical physicists. In particular, questions such as the stability of matter [**Li**], [**LSS**], [**Fe**], eigenvalue or resonance asymptotics [**Iv1**], [**Iv2**], [**FW1**], [**FW2**], [**FW3**], and decay of eigenfunctions [**Er**], [**N**], [**So**] were recently addressed in the literature.

The purpose of this book is twofold. Firstly, in Chapter 1 we provide a general introduction to the spectral theory of N-body magnetic Hamiltonians, aimed at a wider audience of mathematical physicists. Secondly, we present a proof of asymptotic completeness for wide classes of magnetic Hamiltonians, namely for generic 3-body systems and for N-body systems whose all proper subsystems have nonzero total electric charge. The proof requires much more than simply applying the known methods in a slightly different situation; this book focuses on the new methods and techniques that are specific to the magnetic case. In particular, this includes an extension of the Mourre theory to "dispersive" Hamiltonians with a rather complicated structure (Chapter 3) and a geometrical analysis of the propagation of charged systems (Chapter 5). Our goal was to give a clear and reasonably self-contained presentation of the subject and to provide a solid foundation for further research.

The book is addressed mostly to researchers and graduate students in mathematical physics. We do expect the reader to be familiar with quantum mechanics, functional analysis, and modern PDE theory (especially with pseudodifferential calculus). A background in N-body scattering and abstract Mourre theory will be useful, but not indispensable. To the readers who wish to acquire such background we recommend the monographs [**DG1**] and [**ABG**]. However, anyone willing to accept without proof the results of [**DG1**] and [**ABG**] that we will invoke should also be able to follow all of our arguments. In fact parts of this book (especially Chapters 1 and 2) may serve as an introduction to the N-body theory. We emphasize that no previous exposure to magnetic Schrödinger operators is required.

Some of the results presented here were first published in [GŁ1–3]. However much of the material, including all of our results in the 2−dimensional case and a large part of the geometrical analysis of Chapter 5, is published here for the first time. The Mourre theory for magnetic Hamiltonians (Chapter 3) has been completely reworked and rewritten, especially in the case we call "dispersive".

Notation

For the reader's convenience, we collect and explain here the notation used throughout this book.

Spaces: $\mathrm{X}, \mathrm{Y}, \mathrm{Z}, X, Y, Z$, often subscripted or superscripted, will be Euclidean spaces (isomorphic to $\mathbb{R}^m$ for some m). The coordinates in these spaces will be denoted by x, y, z, x, y, z, with appropriate subscripts or superscripts. The Roman letters X, Y, Z, will be used to denote the configuration spaces before the center of mass separation, and the italic X, Y, Z denote the same spaces after the center of mass separation. (Most of the time we will only work with the latter spaces.) A similar convention will be used for the coordinates x, x, etc. Since we will actually separate the center of mass only in one direction, we will have $\mathrm{Y} = Y$ and $\mathrm{y} = y$.

Dual spaces: The spaces dual to $\mathrm{X}, \mathrm{Y}, \mathrm{Z}, X, Y, Z$ will be denoted by X', Y', etc. The duality will be denoted by $\langle \cdot, \cdot \rangle$. If $\mathcal{X}$ is one of the configuration spaces as above, $T^*\mathcal{X} := \mathcal{X} \times \mathcal{X}'$ will be the cotangent bundle of $\mathcal{X}$.

Derivatives: The symbol $\partial_x f$ or $\frac{\partial f}{\partial x}$ will mean the partial derivative of f in x if x is a variable in $\mathbb{R}$, and $\nabla_x f$ if x is a vector variable. We will also write $D_x f = -\mathrm{i}\partial_x f$. We will often omit the subscripts x if the choice of variables is clear from context, and abbreviate ∂_{x_j} to ∂_j, D_{x_j} to D_j, etc.

For $\alpha = (\alpha_1, \ldots, \alpha_n)$, we use the standard notation

$$\partial^\alpha f = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n} f, \ |\alpha| = \alpha_1 + \cdots + \alpha_n.$$

For $0 < \epsilon < 1$, we will denote:

$$C^\epsilon(\mathbb{R}) = \{f \in C(\mathbb{R}) : \ |f(x+y) - f(y)| \le c(x)|y|^\epsilon \text{ for all } x \in \mathbb{R}, \ |y| \le 1\},$$

$$C^{k+\epsilon}(\mathbb{R}) = \{f \in C^k(\mathbb{R}) : \ f^{(k)} \in C^\epsilon(\mathbb{R})\}.$$

Classes of symbols: For an Euclidean space $\mathcal{X}$ (usually equal to one of the configuration spaces above), $\mathcal{S}(\mathcal{X})$ is the Schwartz class of functions on $\mathcal{X}$, and $\mathcal{S}'(\mathcal{X})$ is the space of tempered distributions on $\mathcal{X}$. We will also work with the following classes of symbols:

$$S^0(\mathcal{X}) = \{f \in C^\infty(\mathcal{X}) | \ |\partial^\alpha f(x)| \le c_\alpha, \ x \in \mathcal{X}, \ |\alpha| \ge 0\},$$

and, for $\epsilon \in \mathbb{R}$,

$$S^\epsilon_{\mathrm{cl}}(\mathcal{X}) = \{f \in C^\infty(\mathcal{X}) | \ |D^\alpha f(x)| \le c_\alpha \langle x \rangle^{\epsilon - |\alpha|}, \ x \in \mathcal{X}, \ |\alpha| \ge 0\}.$$

We will use the convention that whenever an estimate is stated for all functions $f_1, f_2, \ldots$ in a specified class of symbols S, the constants in the estimate are understood to depend only on the seminorms of f_i in S, and that this dependence will be linear for each f_i.

Operators: $\mathcal{H}$, $\mathcal{H}'$, etc., often subscripted or superscripted, will be Hilbert spaces, usually equal to $L^2(\mathcal{X})$, where $\mathcal{X}$ is one of the configuration spaces above. The inner product on $\mathcal{H}$ will be denoted by $(\cdot,\cdot)$, and will be linear in the second variable and antilinear in the first one.

A, B, H, etc. will be linear operators on $\mathcal{H}$. For Hamiltonians of the N-body systems under consideration, we adopt the convention that the Roman H and the italic H are, respectively, the Hamiltonians of the system before and after the center of mass separation.

The identity operator on $\mathcal{H}$ will be denoted by $\mathbf{1}_{\mathcal{H}}$, or simply by $\mathbf{1}$ if it is clear from context what $\mathcal{H}$ is.

$B(\mathcal{H}_1, \mathcal{H}_2)$ is the algebra of bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$; we will abbreviate $B(\mathcal{H}, \mathcal{H}) =: B(\mathcal{H})$.

We will use $\mathrm{s}-\lim$ and $\mathrm{w}-\lim$ to denote the limits in the strong and weak topology, respectively.

$\mathcal{D}(H)$ is the domain of H; it will always be either specified explicitly or clear from context whether the form domain or the operator domain of H is meant. The graph norm on $\mathcal{D}(H)$ is

$$\|u\|_{\mathcal{D}(H)} = \|u\| + \|Hu\|.$$

The symbols $\sigma(H)$, $\sigma_{\mathrm{pp}}(H)$, $\sigma_{\mathrm{cont}}(H)$ will denote the spectrum of H, the point spectrum of H, and the continuous spectrum of H, respectively. We will also use $\mathcal{H}_{\mathrm{pp}}(H)$, $\mathcal{H}_{\mathrm{cont}}(H)$, $\mathcal{H}_{\mathrm{ac}}(H)$ to denote the pure point, continuous, and absolutely continuous spectral subspaces of H.

$\mathbf{1}_\Omega(H)$ is the spectral projection of the self-adjoint operator H on a set $\Omega \subset \mathbb{R}$.

If A is a symmetric quadratic form and B is a symmetric operator on $\mathcal{H}$, the phrase "A is bounded on the domain of B" means that

$$|A(u,u)| \leq C\|Bu\|^2, \; u \in \mathcal{D}(B).$$

(In particular, it implies the inclusion $\mathcal{D}(A) \subset \mathcal{D}(B^2)$ between the form domains of A and B^2.)

If A is an operator, the phrase "A preserves the domain of B" means that if $u \in \mathcal{D}(B) \cap \mathcal{D}(A)$, then $Au \in \mathcal{D}(B)$ and

$$\|BAu\| \leq C(\|u\| + \|Bu\|).$$

A mapping $M \ni k \to A(k)$, where M is a metric space (usually a subset of $\mathbb{R}^n$) and $A(k)$ are self-adjoint operators on $\mathcal{H}$, is said to be *continuous in the norm resolvent sense* if the mapping

$$M \ni k \to (A(k)+z)^{-1} \in B(\mathcal{H})$$

is norm continuous in k for any z with $\operatorname{Im} z \neq 0$, and *analytic in the norm resolvent sense* if the above mapping is analytic in k for any z with $\operatorname{Im} z \neq 0$.

We will use "A + h.c." to denote "$A + A^*$; typically, this notation will be employed when A is a long expression of the form $A = A_1 \dots A_k$, in which case the "h.c." stands for $A_k^* \dots A_1^*$.

We define $\mathrm{ad}_A(H) := [H, \mathrm{i}A]$ and $\mathrm{ad}_A^k(H) := [\mathrm{ad}_A^{k-1}(H), A]$.

By "undoing a commutator" we will mean writing $[A, B]$ as $AB - BA$ and estimating each term separately.

If $A(t)$ is a family of operators on $\mathcal{H}$ depending on a parameter t, we will write $A(t) = O(t^\alpha)$ if $A(t) \in B(\mathcal{H})$ for t large enough and $\|A(t)\| = O(t^\alpha)$. A similar convention will be used for $A(t) = o(t^\alpha)$.

The notation "$A \geq_* B$ at $H = \lambda$", used heavily in Chapter 3, is explained at the beginning of Section 3.1.

The classes of operators $C^s(A)$, where A is a self-adjoint operator on $\mathcal{H}$, are defined in Section 3.2.

Cut-off functions: $\mathbf{1}_\Omega$ will be the characteristic function of the set Ω. $F(x \in \Omega)$ is a "smoothed out" characteristic function of Ω: it is a C^∞ function equal to 0 outside Ω and to 1 in a slightly smaller set.

Miscellanous: $\langle x \rangle$ is a function in $C^\infty(\mathcal{X})$, greater than 1/2 for all $x \in \mathcal{X}$ and equal to $|x|$ for $|x| \geq 1$.

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