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Contents

Preface

Over the last ten years, the theory of Bergman spaces has undergone a remarkable metamorphosis. In a series of major advances, central problems once considered intractable were solved, and a rich theory has emerged. Although progress continues, the time seems ripe for a full and unified account of the subject, weaving old and new results together.

The modern subject of Bergman spaces is a blend of complex function theory with functional analysis and operator theory. It comes in contact with harmonic analysis, approximation theory, hyperbolic geometry, potential theory, and partial differential equations. Our aim has been to develop background material and make the subject accessible to a broad segment of the mathematical community. We hope the book will prove useful not only as a reference for research workers, but as a text for graduate students.

In fact, the book evolved from a rough set of notes prepared for graduate students in a two-week course that one of us gave in 1996 at the Norwegian University of Science and Technology in Trondheim, in conjunction with a conference on Bergman spaces supported by the Research Council of Norway. Since that time we have used successive versions of the manuscript in graduate courses we taught at the University of Michigan (1998), Washington University in St. Louis (1999), and San Francisco State University (2001). The last course was supported by the NSF CIRE (Collaborative to Integrate Research and Education) program. The students in all of these courses were enthusiastic, and their perceptive remarks on the manuscript often led to substantial improvements.

In striving for clear presentations of material, we have had the benefit of expert advice from many friends and colleagues. We are most grateful to Kristian Seip for guiding us to a self-contained account of his deep results on interpolation and sampling. Harold Shapiro showed us an elegant way to develop the biharmonic Green function and helped with other constructions. Dmitry Khavinson fielded a steady barrage of technical questions and offered many useful suggestions on the manuscript. Sheldon Axler made a careful reading of several chapters and gave valuable criticism. Mathematical help of various sorts came also from Marcin Bownik, Brent Carswell, Eric Hayashi, Håkan Hedenmalm, Anton Kim, John McCarthy, Maria Nowak, Stefan Richter, Richard Rochberg, Joel Shapiro, Michael Stessin, Carl Sundberg, James Tung, Dror Varolin, Dragan Vukotić, Rachel Weir, and Kehe Zhu. Special thanks go to Joel Shapiro for permission to

base our treatment of cyclic singular inner functions (Section 8.2) on his unpublished notes. Christopher Hammond read large portions of the manuscript with an eagle eye and spotted a number of misprints, minor errors, and obscurities. Anders Björn also made helpful remarks. We want to express our sincere appreciation to all of these people, and others whose names we may have overlooked, for helping to improve this book. Any defects that remain, however, are the authors' responsibility.

We also had the benefit of the earlier book *Theory of Bergman Spaces* by Håkan Hedenmalm, Boris Korenblum, and Kehe Zhu (Springer–Verlag, 2000), which served as a useful reference in our approach to several topics. As may be expected, the two books have considerable overlap, but ours develops more of the prerequisite material. It also treats topics not discussed in the earlier book, and treats some of the same topics in different ways. A few results appear here for the first time. On the other hand, the book of Hedenmalm, Korenblum, and Zhu contains extensive discussions of several topics barely touched upon in our book, such as invertible noncyclic functions and logarithmically subharmonic weights.

Our book is essentially self-contained. It should be accessible to advanced graduate students who have studied basic complex function theory, measure theory, and functional analysis. Prior knowledge of Hardy spaces is helpful, since that theory often serves as a model for Bergman spaces, but the main facts about Hardy spaces are reviewed in two "crash courses" early in the book and later as motivation for corresponding topics in Bergman spaces. A few Hardy space results are actually needed for the theoretical development of Bergman spaces, and proofs are given.

Most of the writing was carried out during summers together in Ann Arbor, where the University of Michigan provided excellent facilities for our work. Thanks also go to the Ann Arbor Diamondbacks, who were an extra incentive for the second-named author to return to Michigan every summer.

Over the last decade, the American Mathematical Society held several Special Sessions on Bergman spaces at national and regional meetings, and sponsored a week-long research conference at Mt. Holyoke College in the summer of 1994. That summer conference, in particular, did much to stimulate further research in the field. We were therefore especially pleased when the AMS agreed to publish our book. We are grateful to Sergei Gelfand of the AMS for his initial vision that encouraged us not to settle for a revised set of lecture notes, but to do the extra work needed to produce a full expository account of the subject. He showed remarkable patience with the slow pace of the resulting project, but pushed us to finish when the end was in sight and helped with the technical aspects of production. We hope our book may be judged a worthy successor to the classic book by Stefan Bergman, which appeared in the same AMS series many years ago.

Peter Duren and Alexander Schuster
September 2003

References

Abakumov, E. and Borichev, A.

[1] Shift invariant subspaces with arbitrary indices in ℓ^p spaces, *J. Funct. Anal.* **188** (2002), 1–26.

Abkar, A.

[1] Norm approximation by polynomials in some weighted Bergman spaces, *J. Funct. Anal.* **191** (2002), 224–240.

Aharonov, D., Shapiro, H. S., and Shields, A. L.

[1] Weakly invertible elements in the space of square-summable holomorphic functions, *J. London Math. Soc.* **9** (1974), 183–192.

Aleman, A.

[1] Invariant subspaces with finite codimension in Bergman spaces, *Trans. Amer. Math. Soc.* **330** (1992), 531–544.

Aleman, A., Hedenmalm, H., Richter, S., and Sundberg, C.

[1] Curious properties of canonical divisors in weighted Bergman spaces, *Entire Functions in Modern Analysis (Tel Aviv, 1997), Israel Math. Conf. Proc.* **15** (Bar-Ilan University, Ramat Gan, 2001), pp. 1–10.

Aleman, A. and Richter, S.

[1] Some sufficient conditions for the division property of invariant subspaces in weighted Bergman spaces, *J. Funct. Anal.* **144** (1997), 542–556.

[2] Single point extremal functions in Bergman-type spaces, *Indiana Univ. Math. J.* **51** (2002), 581–605.

Aleman, A., Richter, S., and Sundberg, C.

[1] Beurling's theorem for the Bergman space, *Acta Math.* **177** (1996), 275–310.

[2] The majorization function and the index of invariant subspaces in the Bergman spaces, *J. Analyse Math.* **86** (2002), 139–182.

Amar, E.

[1] Suites d'interpolation pour les classes de Bergman de la boule et du polydisque de $\mathbb{C}^n$, *Canad. J. Math.* **30** (1978), 711–737.

[2] Extension de fonctions holomorphes et courants, *Bull. Sci. Math.* **107** (1983), 25–48.

Anderson, J. M.

[1] Bloch functions: The basic theory, *Operators and Function Theory* (S. C. Power, editor), D. Reidel, Boston, 1985, pp. 1–17.

Anderson, J. M., Clunie, J., and Pommerenke, Ch.

[1] On Bloch functions and normal functions, *J. Reine Angew. Math.* **270** (1974), 12–37.

Andersson, Mats Erik

[1] Integral means on Bergman spaces, *Complex Variables Theory Appl.* **32** (1997), 147–160.

Apostol, C., Bercovici, H., Foiaş, C., and Pearcy, C.

[1] Invariant subspaces, dilation theory, and the structure of the predual of a dual algebra, I, *J. Funct. Anal.* **63** (1985), 369–404.

Axler, S.

[1] Zero-multipliers of Bergman spaces, *Canad. Math. Bull.* **28** (1985), 237–242.

[2] The Bergman space, the Bloch space, and commutators of multiplication operators, *Duke Math. J.* **53** (1986), 315–332.

[3] Bergman spaces and their operators, *Surveys of Some Recent Results in Operator Theory*, Vol. I (J. B. Conway and B. B. Morrel, editors), Pitman Research Notes in Math. No. 171, John Wiley & Sons, New York, 1988, pp. 1–50.

Beller, E.

[1] Zeros of A^p functions and related classes of analytic functions, *Israel J. Math.* **22** (1975), 68–80.

Beller, E. and Horowitz, C.

[1] Zero sets and random zero sets in certain function spaces, *J. Analyse Math.* **64** (1994), 203–217.

Bergman, S.

[1] *The Kernel Function and Conformal Mapping* (Second Edition, American Mathematical Society, Providence, R. I., 1970).

Bergman, S. and Schiffer, M.

[1] Kernel functions and conformal mapping, *Compositio Math.* **8** (1951), 205–249.

[2] *Kernel Functions and Elliptic Differential Equations in Mathematical Physics* (Academic Press, New York, 1953).

Berman, R., Brown, L., and Cohn, W.

[1] Cyclic vectors of bounded characteristic in Bergman spaces, *Michigan Math. J.* **31** (1984), 295–306.

Berndtsson, B. and Ortega-Cerdà, J.

[1] On interpolation and sampling in Hilbert spaces of analytic functions, *J. Reine Angew. Math.* **464** (1995), 109–128.

Beurling, A.

[1] Ensembles exceptionels, *Acta Math.* **72** (1940), 1–13.

[2] On two problems concerning linear transformations in Hilbert space, *Acta Math.* **81** (1949), 239–255.

[3] Mittag-Leffler Lectures on Harmonic Analysis (1977–1978): *IV.* Balayage of Fourier–Stieltjes transforms; *V.* Interpolation for an interval on $\mathbb{R}^1$, *Collected Works of Arne Beurling* (Birkhäuser, Boston, 1989), Volume 2, pp. 341–365.

Boas, R. P.

[1] Some uniformly convex spaces, *Bull. Amer. Math. Soc.* **46** (1940), 304–311.

Bomash, G.

[1] A Blaschke-type product and random zero sets for Bergman spaces, *Arkiv for Matematik* **30** (1992), 45–60.

Borichev, A.

[1] Estimates from below and cyclicity in Bergman-type spaces, *Internat. Math. Res. Notices* **12** (1996), 603–611.

[2] Invariant subspaces of given index in Banach spaces of analytic functions, *J. Reine Angew. Math.* **505** (1998), 23–44.

Borichev, A. A. and Hedenmalm, H.

[1] Cyclicity in Bergman-type spaces, *Internat. Math. Res. Notices* **11** (1995), 253–262.

[2] Harmonic functions of maximal growth: invertibility and cyclicity in Bergman spaces, *J. Amer. Math. Soc.* **10** (1997), 761–796.

Bourdon, P.

[1] Cyclic Nevanlinna class functions in Bergman spaces, *Proc. Amer. Math. Soc.* **93** (1985), 503–506.

[2] Cellular-indecomposable operators and Beurling's theorem, *Michigan Math. J.* **33** (1986), 187–193.

Brown, L. and Korenblum, B.

[1] Cyclic vectors in $A^{-\infty}$, *Proc. Amer. Math. Soc.* **102** (1988), 137–138.

Brown, L. and Shields, A. L.

[1] Cyclic vectors in the Dirichlet space, *Trans. Amer. Math. Soc.* **285** (1984), 269–304.

Bruna, J. and Pascuas, D.

[1] Interpolation in $A^{-\infty}$, *J. London Math. Soc.* **40** (1989), 452–466.

Buckley, S. M., Koskela, P., and Vukotić, D.

[1] Fractional integration, differentiation, and weighted Bergman spaces, *Math. Proc. Cambridge Philos. Soc.* **126** (1999), 369–385.

Carleson, L.

[1] Sets of uniqueness for functions regular in the unit circle, *Acta Math.* **87** (1952), 325–345.

[2] An interpolation problem for bounded analytic functions, *Amer. J. Math.* **80** (1958), 921–930.

[3] Interpolations by bounded analytic functions and the corona problem, *Ann. of Math.* **76** (1962), 547–559.

Cima, J. A.

[1] The basic properties of Bloch functions, *Internat. J. Math. Math. Sci.* **2** (1979), 369–413.

Cima, J. A. and Derrick, W. R.

[1] Extremals for subspaces of the Bergman space, *Houston J. Math.* **20** (1994), 623–628.

Clarkson, J. A.

[1] Uniformly convex spaces, *Trans. Amer. Math. Soc.* **40** (1936), 396–414.

Coifman, R. R. and Rochberg, R.

[1] Representation theorems for holomorphic and harmonic functions in L^p, *Astérisque* **77** (1980), 1–66.

Cowen, C. and MacCluer, B.

[1] *Composition Operators on Spaces of Analytic Functions* (CRC Press, Boca Raton, 1995).

Djrbashian, A. E. and Shamoian, F. A.

[1] *Topics in the Theory of A^p_α Spaces* (B. G. Teubner, Leipzig, 1988).

Duffin, R. J.

[1] On a question of Hadamard concerning super-biharmonic functions, *J. Math. and Physics* **27** (1949), 253–258.

Duren, P. L.

[1] Smoothness of functions generated by Riesz products, *Proc. Amer. Math. Soc.* **16** (1965), 1263–1268.

[2] Extension of a theorem of Carleson, *Bull. Amer. Math. Soc.* **75** (1969), 143–146.

[3] On the multipliers of H^p spaces, *Proc. Amer. Math. Soc.* **22** (1969), 24–27.

[4] On the Bloch–Nevanlinna conjecture, *Colloq. Math.* **20** (1969), 295–297.

[5] *Theory of H^p Spaces* (Academic Press, New York, 1970; reprinted with supplement by Dover Publications, Mineola, N.Y., 2000).

[6] *Univalent Functions* (Springer–Verlag, New York, 1983).

Duren, P., Khavinson, D., and Shapiro, H. S.

[1] Extremal functions in invariant subspaces of Bergman spaces, *Illinois J. Math.* **40** (1996), 202–210.

Duren, P., Khavinson, D., Shapiro, H. S., and Sundberg, C.
[1] Contractive zero-divisors in Bergman spaces, *Pacific J. Math.* **157** (1993), 37–56.
[2] Invariant subspaces in Bergman spaces and the biharmonic equation, *Michigan Math. J.* **41** (1994), 247–259.

Duren, P. L., Romberg, B. W., and Shields, A. L.
[1] Linear functionals on H^p spaces with $0 < p < 1$, *J. Reine Angew. Math.* **238** (1969), 32–60.

Duren, P. and Schuster, A. P.
[1] Finite unions of interpolation sequences, *Proc. Amer. Math. Soc.* **130** (2002), 2609–2615.

Duren, P., Schuster, A. P., and Seip, K.
[1] Uniform densities of regular sequences in the unit disk, *Trans. Amer. Math. Soc.* **352** (2000), 3971–3980.

Duren, P., Schuster, A., and Vukotić, D.
[1] On uniformly discrete sets in the disk, *to appear.*

Duren, P. L. and Shields, A. L.
[1] Properties of H^p $(0 < p < 1)$ and its containing Banach space, *Trans. Amer. Math. Soc.* **141** (1969), 255–262.
[2] Coefficient multipliers of H^p and B^p spaces, *Pacific J. Math.* **32** (1970), 69–78.
[3] Restrictions of H^p functions to the diagonal of the polydisc, *Duke Math. J.* **42** (1975), 751–753.

Duren, P. L. and Taylor, G. D.
[1] Mean growth and coefficients of H^p functions, *Illinois J. Math.* **14** (1970), 419–423.

Engliš, M.
[1] A Loewner-type lemma for weighted biharmonic operators, *Pacific J. Math.* **179** (1997), 343–353.
[2] Weighted biharmonic Green functions for rational weights, *Glasgow Math. J.* **41** (1999), 239–269.

Epstein, B.
[1] *Orthogonal Families of Analytic Functions* (Macmillan, New York, 1965).

Fisher, S. D.
[1] *Function Theory on Planar Domains* (John Wiley & Sons, New York, 1983).

Folland, G.B.
[1] *Introduction to Partial Differential Equations* (Second Edition, Princeton University Press, Princeton, N. J., 1995).

Forelli, F.
[1] The isometries of H^p, *Canad. J. Math.* **16** (1964), 721–728.

Forelli, F. and Rudin, W.
[1] Projections on spaces of holomorphic functions in balls, *Indiana Univ. Math. J.* **24** (1974), 593–602.

Friedman, A.
[1] *Partial Differential Equations* (Holt, Rinehart and Winston, New York, 1969).

Garabedian, P. R.
[1] A partial differential equation arising in conformal mapping, *Pacific J. Math.* **1** (1951), 485–524.
[2] *Partial Differential Equations* (Second Edition, Chelsea Publ. Co., New York, 1986).

Garnett, J.
[1] *Bounded Analytic Functions* (Academic Press, New York, 1981).

Hansbo, J.
[1] Reproducing kernels and contractive divisors in Bergman spaces, *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov (POMI)* **232** (1996), *Issled. po Lineĭn. Oper. i Teor. Funktsiĭ* **24**, 174–198, 217 = *J. Math. Sci. (New York)* **92** (1998), no. 1, 3657–3674.

Hardy, G. H. and Littlewood, J. E.
[1] Some properties of conjugate functions, *J. Reine Angew. Math.* **167** (1931), 405–423.
[2] Some properties of fractional integrals, *II*, *Math. Z.* **34** (1932), 403–439.
[3] Notes of the theory of series (*XX*): Generalizations of a theorem of Paley, *Quart. J. Math., Oxford Ser.* **8** (1937), 161–171.

Hastings, W. W.
[1] A Carleson measure theorem for Bergman spaces, *Proc. Amer. Math. Soc.* **52** (1975), 237–241.

Hayman, W. K.
[1] On a conjecture of Korenblum, *Analysis (Munich)* **19** (1999), 195–205.

Hayman, W. K. and Korenblum, B.
[1] An extension of the Riesz-Herglotz formula, *Ann. Acad. Sci. Fenn. Ser. A.I. Math.* **2** (1976), 175–201.
[2] A critical growth rate for functions regular in a disk, *Michigan Math. J.* **27** (1980), 21–30.

Hedenmalm, H.

[1] A factorization theorem for square area-integrable analytic functions, *J. Reine Angew. Math.* **422** (1991), 45–68.

[2] A factoring theorem for a weighted Bergman space, *Algebra i Analiz* **4** (1992), no. 1, 167–176 = *St. Petersburg Math. J.* **4** (1993), 163–174.

[3] An invariant subspace of the Bergman space having the codimension two property, *J. Reine Angew. Math.* **443** (1993), 1–9.

[4] Spectral properties of invariant subspaces in the Bergman space, *J. Funct. Anal.* **116** (1993), 441–448.

[5] A factoring theorem for the Bergman space, *Bull. London Math. Soc.* **26** (1994), 113–126.

[6] A computation of Green functions for the weighted biharmonic operators $\Delta|z|^{-2\alpha}\Delta$, with $\alpha > -1$, *Duke Math. J.* **75** (1994), 51–78.

[7] Open problems in the function theory of the Bergman space, *Festschrift in Honour of Lennart Carleson and Yngve Domar* (Uppsala, 1995), pp. 153–169.

[8] Maximal invariant subspaces in the Bergman space, *Ark. Mat.* **36** (1998), 97–101.

[9] Boundary value problems for weighted biharmonic operators, *Algebra i Analiz* **8** (1996), no. 4, 173–192 = *St. Petersburg Math. J.* **8** (1997), 661–674.

[10] Recent progress in the function theory of the Bergman space, *Holomorphic Spaces (Berkeley, 1995)* (Cambridge Univ. Press, Cambridge, 1998), pp. 35–50.

[11] Recent developments in the function theory of the Bergman space, *European Congress of Mathematics (Budapest, 1996)*, Vol. *I*, *Progress in Math.* **168** (Birkhäuser, Basel, 1998), pp. 202–217.

Hedenmalm, H., Jakobsson, S., and Shimorin, S.

[1] A maximum principle à la Hadamard for biharmonic operators with applications to the Bergman spaces, *C. R. Acad. Sci. Paris* **328** (1999), 973–978.

[2] A biharmonic maximal principle for hyperbolic surfaces, *J. Reine Angew. Math.* **550** (2002), 25–75.

Hedenmalm, H., Korenblum, B., and Zhu, K.

[1] Beurling type invariant subspaces of the Bergman spaces, *J. London Math. Soc.* **53** (1996), 601–614.

[2] *Theory of Bergman Spaces* (Springer–Verlag, New York, 2000).

Hedenmalm, H., Richter, S., and Seip, K.

[1] Interpolating sequences and invariant subspaces of given index in the Bergman spaces, *J. Reine Angew. Math.* **477** (1996), 13–30.

Hedenmalm, H. and Zhu, K.

[1] On the failure of optimal factorization for certain weighted Bergman spaces, *Complex Variables Theory Appl.* **19** (1992), 165–176.

Hinkkanen, A.

[1] On a maximum principle in Bergman space, *J. Analyse Math* **79** (1999), 335–344.

Hörmander, L.

[1] *The Analysis of Linear Partial Differential Operators I* (Second Edition, Springer–Verlag, Berlin, 1990).

Horowitz, C.

[1] Zeros of functions in the Bergman spaces, Ph.D. thesis, University of Michigan, 1974.

[2] Zeros of functions in the Bergman spaces, *Duke Math. J.* **41** (1974), 693–710.

[3] Factorization theorems for functions in the Bergman spaces, *Duke Math. J.* **44** (1977), 201–213.

[4] Some conditions on Bergman space zero sets, *J. Analyse Math.* **62** (1994), 323–348.

Horowitz, C., Korenblum, B., and Pinchuk, B.

[1] Extremal functions and contractive divisors in A^{-n}, *Ann. Scuola Norm. Sup. Pisa* **23** (1996), 179–191.

[2] Sampling sequences for $A^{-\infty}$, *Michigan Math. J.* **44** (1997), 389–398.

Horowitz, C. and Oberlin, D. M.

[1] Restriction of H^p functions to the diagonal of $\mathbb{U}^n$, *Indiana Univ. Math. J.* **24** (1975), 767–772.

Janas, J.

[1] A note on invariant subspaces under multiplication by z in Bergman spaces, *Proc. Roy. Irish Acad.* **83A** (1983), 157–164.

Jevtić, M., Massaneda, X., and Thomas, P. J.

[1] Interpolating sequences for weighted Bergman spaces of the ball, *Michigan Math. J.* **43** (1996), 495–517.

Kabaĭla, V.

[1] Refinement of some theorems on interpolation in the class H_δ, *Vilniaus Univ. Darbai.* **33**, *Mat.–Fiz.* **9** (1960), 15–19. (in Lithuanian; Russian summary)

[2] Interpolation sequences for the H_p classes in the case $p < 1$, *Litovsk. Mat. Sb.* **3** (1963), no. 1, 141–147. (in Russian)

Kato, T.

[1] *Perturbation Theory for Linear Operators* (Second Edition, Springer–Verlag, New York, 1976).

Khavinson, D. and Shapiro, H. S.

[1] Invariant subspaces in Bergman spaces and Hedenmalm's boundary value problem, *Ark. Mat.* **32** (1994), 309–321.

Khavinson, D. and Stessin, M. I.

[1] Certain linear extremal problems in Bergman spaces of analytic functions, *Indiana Univ. Math. J.* **46** (1997), 933–974.

Kim, A. K.

[1] Horowitz products in Bergman spaces, *Abstracts Amer. Math. Soc.* **20** (1999), 71.

Kolaski, C. J.

[1] Isometries of weighted Bergman spaces, *Canad. J. Math.* **34** (1982), 910–915.

Koosis, P.

[1] *Introduction to H_p Spaces* (Cambridge Univ. Press, London, 1980; Second Edition, 1999).

Korenblum, B.

[1] An extension of the Nevanlinna theory, *Acta Math.* **135** (1975), 187–219.

[2] A Beurling-type theorem, *Acta Math.* **138** (1977), 265–293.

[3] Cyclic elements in some spaces of analytic functions, *Bull. Amer. Math. Soc.* **5** (1981), 317–318.

[4] Transformation of zero sets by contractive operators in the Bergman space, *Bull. Sci. Math.* **114** (1990), 385–394.

[5] A maximum principle for the Bergman space, *Publ. Mat.* **35** (1991), 479–486.

[6] Outer functions and cyclic elements in Bergman spaces, *J. Funct. Anal.* **115** (1993), 104–118.

Korenblum, B., O'Neil, R., Richards, K., and Zhu, K.

[1] Totally monotone functions with applications to the Bergman space, *Trans. Amer. Math. Soc.* **337** (1993), 795–806.

Korenblum, B. and Richards, K.

[1] Majorization and domination in the Bergman space, *Proc. Amer. Math. Soc.* **117** (1993), 153–158.

Köthe, G.

[1] *Topological Vector Spaces I* (Springer–Verlag, Berlin, 1983).

Krosky, M. and Schuster, A. P.

[1] Multiple interpolation and extremal functions in the Bergman spaces, *J. Analyse Math.* **85** (2001), 141–156.

LeBlanc, E.

[1] A probabilistic zero set condition for the Bergman space, *Michigan Math. J.* **37** (1990), 427–438.

Lieb, E. H. and Loss, M.
[1] *Analysis* (Second Edition, American Mathematical Society, Providence, R. I., 2001).

Littlewood, J. E.
[1] *Lectures on the Theory of Functions* (Oxford University Press, London, 1944).

Loewner, Ch.
[1] On generation of solutions of the biharmonic equation in the plane by conformal mappings, *Pacific J. Math.* **3** (1953), 417–436.

Luecking, D. H.
[1] Inequalities on Bergman spaces, *Illinois J. Math.* **25** (1981), 1–11.
[2] A technique for characterizing Carleson measures on Bergman spaces, *Proc. Amer. Math. Soc.* **87** (1983), 656–660.
[3] Forward and reverse Carleson inequalities for functions in Bergman spaces and their derivatives, *Amer. J. Math.* **107** (1985), 85–111.
[4] Zero sequences for Bergman spaces, *Complex Variables Theory Appl.* **30** (1996), 345–362.
[5] Sampling measures for the Bergman space on the unit disk, *Math. Ann.* **316** (2000), 659–679.

Lyubarskii, Y. I. and Sodin, M.
[1] Analogues of sine type functions for convex domains, Preprint No. 17, Inst. Low Temperature Phys. Engrg., Ukrainian Acad. Sci., Kharkov (1986) (in Russian).

MacGregor, T. H. and Stessin, M. I.
[1] Weighted reproducing kernels in Bergman spaces, *Michigan Math. J.* **41** (1994), 523–533.

MacGregor, T. H. and Zhu, K.
[1] Coefficient multipliers between Bergman and Hardy spaces, *Mathematika* **42** (1995), 413–426.

Matero, J.
[1] On Korenblum's maximum principle for Bergman space, *Arch. Math. (Basel)* **64** (1995), 337–340.

McCarthy, J. E.
[1] Coefficient estimates on weighted Bergman spaces, *Duke Math. J.* **76** (1994), 751–760.

McCullough, S. and Richter, S.
[1] Bergman-type reproducing kernels, contractive divisors, and dilations, *J. Funct. Anal.* **190** (2002), 447–480.

McDonald, G. and Sundberg, C.
[1] Toeplitz operators on the disc, *Indiana Univ. Math. J.* **28** (1979), 595–611.

McKenna, P. J.
[1] Discrete Carleson measures and some interpolation problems, *Michigan Math. J.* **24** (1977), 311–319.

Nakamura, A., Ohya, F., and Watanabe, H.
[1] On some properties of functions in weighted Bergman spaces, *Proc. Fac. Sci. Tokai Univ.* **15** (1979), 33–44.

Nehari, Z.
[1] *Conformal Mapping* (New York, McGraw-Hill, 1952; reprinted by Dover Publications, New York, 1975).

Neville, C. W.
[1] A short proof of an inequality of Carleson's, *Proc. Amer. Math. Soc.* **65** (1977), 131–132.

Nicolau, A.
[1] Finite products of interpolating Blaschke products, *J. London Math. Soc.* **50** (1994), 520–531.

Nikolskii, N.
[1] *Treatise on the Shift Operator* (English translation, Springer–Verlag, Berlin, 1986).

Novinger, W. P.
[1] Holomorphic functions with infinitely differentiable boundary values, *Illinois J. Math.* **15** (1970), 80–90.
[2] Mean convergence in L^p spaces, *Proc. Amer. Math. Soc.* **34** (1972), 627–628.

Nowak, M. and Waniurski, P.
[1] Random zero sets for Bergman spaces, *Math. Proc. Cambridge Philos. Soc.* **134** (2003), 337–345.

Oleinik, V. L.
[1] Embedding theorems for weighted classes of harmonic and analytic functions, *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* **47** (1974), 120–137 (in Russian) = *J. Soviet Math.* **9** (1978), 228–243.

Ortega-Cerdà, J.
[1] Sampling measures, *Publ. Mat.* **42** (1998), 559–566.

Osipenko, K. Yu. and Stessin, M. I.
[1] On recovery problems in Hardy and Bergman spaces, *Mat. Zametki* **49** (1991), no. 4, 95–104 (in Russian) = *Math. Notes* **49** (1991), 395–401.
[2] On optimal recovery of holomorphic functions in the unit ball in $\mathbb{C}^n$, *Constr. Approx.* **8** (1992), 141–159.

Pommerenke, Ch.

[1] On Bloch functions, *J. London Math. Soc.* **2** (1970), 689–695.

[2] *Boundary Behaviour of Conformal Maps* (Springer–Verlag, Berlin–Heidelberg, 1992).

Ransford, T.

[1] *Potential Theory in the Complex Plane* (Cambridge University Press, Cambridge, 1995).

Reed, M. and Simon, B.

[1] *Functional Analysis* (Academic Press, San Diego, 1983).

Richter, S.

[1] Invariant subspaces in Banach spaces of analytic functions, *Trans. Amer. Math. Soc.* **304** (1987), 585–616.

[2] Invariant subspaces of the Dirichlet shift, *J. Reine Angew. Math.* **386** (1988), 205–220.

Riesz, F. and Sz.-Nagy, B.

[1] *Functional Analysis* (Frederick Ungar, New York, 1955).

Roberts, J. W.

[1] Cyclic inner functions in the Bergman spaces and weak outer functions in H^p, $0 < p < 1$, *Illinois J. Math.* **29** (1985), 25–38.

Rochberg, R.

[1] Interpolation by functions in Bergman spaces, *Michigan Math. J.* **29** (1982), 229–236.

[2] Decomposition theorems for Bergman spaces and their applications, in *Operators and Function Theory* (S. C. Power, editor; D. Reidel Publ. Co., Boston, 1985), pp. 225–277.

Rudin, W.

[1] *Function Theory in Polydiscs* (Benjamin, New York, 1969).

[2] *Principles of Mathematical Analysis* (Third Edition, McGraw–Hill, New York, 1976).

[3] *Functional Analysis* (Second Edition, McGraw–Hill, New York, 1991).

[4] L^p–isometries and equimeasurability, *Indiana Univ. Math. J.* **25** (1976), 215–228.

[5] *Function Theory in the Unit Ball of* $\mathbb{C}^n$ (Springer–Verlag, New York, 1980).

Saitoh, S.

[1] *Integral Transforms, Reproducing Kernels and Their Applications* (Pitman Research Notes in Math. No. 369, Addison Wesley Longman, Essex, England, 1997).

Schuster, A. P.

[1] Sets of sampling and interpolation in Bergman spaces, *Proc. Amer. Math. Soc.* **125** (1997), 1717–1725.

[2] Sampling and interpolation in Bergman spaces, Ph.D. thesis, University of Michigan, 1997.
[3] On Seip's description of sampling sequences for Bergman spaces, *Complex Variables Theory Appl.* **42** (2000), 347–367.

Schuster, A. and Seip, K.
[1] A Carleson-type condition for interpolation in Bergman spaces, *J. Reine Angew. Math.* **497** (1998), 223–233.
[2] Weak conditions for interpolation in holomorphic spaces, *Publ. Mat.* **44** (2000), 277–293.

Schuster, A. P. and Varolin, D.
[1] Sampling sequences for Bergman spaces for $p < 1$, *Complex Variables Theory Appl.* **47** (2002), 243–253.
[2] Sampling and interpolation on Riemann surfaces, *to appear.*

Schwick, W.
[1] On Korenblum's maximum principle, *Proc. Amer. Math. Soc.* **125** (1997), 2581–2587.

Seip, K.
[1] Reproducing formulas and double orthogonality in Bargmann and Bergman spaces, *SIAM J. Math. Anal.* **22** (1991), 856–876.
[2] Regular sets of sampling and interpolation for weighted Bergman spaces, *Proc. Amer. Math. Soc.* **117** (1993), 213–220.
[3] Beurling type density theorems in the unit disk, *Invent. Math.* **113** (1994), 21–39.
[4] On a theorem of Korenblum, *Ark. Mat.* **32** (1994), 237–243.
[5] On Korenblum's density condition for the zero sequences of $A^{-\alpha}$, *J. Analyse Math.* **67** (1995), 307–322.
[6] Developments from nonharmonic Fourier series, *Proceedings of International Congress of Mathematicians, Berlin 1998*, Vol. *II* (*Documenta Mathematica 1998*, Extra Vol. *II*), pp. 713–722.
[7] *Interpolation and Sampling in Spaces of Analytic Functions* (American Mathematical Society, Providence, R. I., *to appear*).

Shapiro, H. S.
[1] Weakly invertible elements in certain function spaces and generators in ℓ^1, *Michigan Math. J.* **11** (1964), 161–165.
[2] Weighted polynomial approximation and boundary behaviour of analytic functions, *Contemporary Problems in the Theory of Analytic Functions* (International Conference in Erevan, Armenia, 1965; Izdat. "Nauka", Moscow, 1966), pp. 326–335.
[3] Some remarks on weighted polynomial approximation of holomorphic functions, *Mat. Sbornik* **73** (115) (1967), 320–330 (in Russian) = *Math. USSR Sb.* **2** (1967), 285–294.

[4] Comparative approximation in two topologies, *Approximation Theory* (Banach Center Publications, Vol. 4, PWN—Polish Scientific Publishers, Warsaw 1979), pp. 225–232.

Shapiro, H. S. and Shields, A. L.

[1] On some interpolation problems for analytic functions, *Amer. J. Math.* **83** (1961), 513–532.

[2] On the zeros of functions with finite Dirichlet integral and some related function spaces, *Math. Z.* **80** (1962), 217–229.

Shapiro, H. S. and Tegmark, M.

[1] An elementary proof that the biharmonic Green function of an eccentric ellipse changes sign, *SIAM Review* **36** (1994), 99–101.

Shapiro, J. H.

[1] Linear functionals on non-locally convex spaces, Ph.D. thesis, University of Michigan, 1969.

[2] Mackey topologies, reproducing kernels, and diagonal maps on Hardy and Bergman spaces, *Duke Math. J.* **43** (1976), 187–202.

[3] Zeros of functions in weighted Bergman spaces, *Michigan Math. J.* **24** (1977), 243–256.

[4] Zeros of random functions in Bergman spaces, *Ann. Inst. Fourier (Grenoble)* **29** (1979), 159–171.

[5] Cyclic inner functions in Bergman spaces (unpublished notes, University of Wisconsin, 1980; *www.math.msu.edu/∼shapiro*).

[6] *Composition Operators and Classical Function Theory* (Springer–Verlag, New York, 1993).

Shields, A.

[1] Weighted shift operators and analytic function theory, *Topics in Operator Theory* (C. M. Pearcy, editor), American Mathematical Society, Providence, R. I., 1974, pp. 49–128.

[2] Cyclic vectors in Banach spaces of analytic functions, *Operators and Function Theory* (S. C. Power, editor; D. Reidel Publ. Co., Boston, 1985), pp. 315–349.

Shimorin, S. M.

[1] Factorization of analytic functions in weighted Bergman spaces, *Algebra i Analiz* **5** (1993), no. 5, 155–177 (in Russian) = *St. Petersburg Math. J.* **5** (1994), 1005–1022.

[2] On a family of conformally invariant operators, *Algebra i Analiz* **7** (1995), no. 2, 133–158 (in Russian) = *St. Petersburg Math. J.* **7** (1996), 287–306.

[3] The Green function for the weighted biharmonic operator $\Delta(1-|z|^2)^{-\alpha}\Delta$, and factorization of analytic functions, *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Math. Inst. Steklov. (POMI)* **222** (1995), *Issled. po Lineĭn. Oper. i Teor. Funktsiĭ* **23**, 203–221 (in Russian) = *J. Math. Sci. (New York)* **87** (1997), no. 5, 3912–3924.

[4] Single-point extremal functions in weighted Bergman spaces, *Probl. Mat. Anal.* No. 15 (1995), 241–252 (in Russian) = *J. Math. Sci. (New York)* **80** (1996), no. 6, 2349–2356.

[5] The Green functions for weighted biharmonic operators of the form $\Delta w^{-1}\Delta$ in the unit disk, *Probl. Mat. Anal.* (in Russian) = *J. Math. Sci. (New York)* **92** (1998), no. 6, 4404–4411.

[6] Approximative spectral synthesis in the Bergman space, *Duke Math. J.* **101** (2000), 1–39.

[7] Wold-type decompositions and wandering subspaces for operators close to isometries, *J. Reine Angew. Math.* **531** (2001), 147–189.

[8] An integral formula for weighted Bergman reproducing kernels, *Complex Variables Theory Appl.* **47** (2002), 1015–1028.

[9] On Beurling-type theorems in weighted ℓ^2 and Bergman spaces, *Proc. Amer. Math. Soc.* **131** (2003), 1777–1787.

Shvedenko, S. V.

[1] On the Taylor coefficients of functions from Bergman spaces in the polydisc, *Dokl. Akad. Nauk SSSR* **283** (1985), 325–328 (in Russian) = *Soviet Math. Dokl.* **32** (1985), 118–121.

[2] On a theorem of Shapiro and Shields on the zeros of analytic functions of restricted growth, *Izv. Vyssh. Uchebn. Zaved. Mat.* **2001**, no. 9, 75–79 (in Russian) = *Russian Math. (Izv. VUZ)* **45** (2001), no. 9, 71–75.

Stessin, M.

[1] An extension of a theorem of Hadamard and domination in the Bergman space, *J. Funct. Anal.* **115** (1993), 212–226.

Sundberg, C.

[1] Analytic continuability of Bergman inner functions, *Michigan Math. J.* **44** (1997), 399–407.

Szegő, G.

[1] Remark on the preceding paper of Charles Loewner, *Pacific J. Math.* **3** (1953), 437–446.

Taylor, B. A. and Williams, D. L.

[1] Ideals in rings of analytic functions with smooth boundary values, *Canad. J. Math.* **22** (1970), 1266–1283.

Thomson, J. E.

[1] Approximation in the mean by polynomials, *Ann. of Math.* **133** (1991), 477–508.

[2] Bounded point evaluations and polynomial approximation, *Proc. Amer. Math. Soc.* **123** (1995), 1757–1761.

Vukotić, D.

[1] A sharp estimate for A^p functions in $\mathbb{C}^n$, *Proc. Amer. Math. Soc.* **117** (1993), 753–756.

[2] On the coefficient multipliers of Bergman spaces, *J. London Math. Soc.* **50** (1994), 341–348.

[3] Linear extremal problems for Bergman spaces, *Expositiones Math.* **14** (1996), 313–352.

[4] The isoperimetric inequality and a theorem of Hardy and Littlewood, *Amer. Math. Monthly* **110** (2003), 532–536.

Watanabe, H.

[1] Some properties of functions in Bergman space A^p, *Proc. Fac. Sci. Tokai Univ.* **13** (1977), 39–54.

[2] Some inequalities of means and its applications to Bergman spaces, *Proc. Fac. Sci. Tokai Univ.* **28** (1993), 1–13.

Weir, R.

[1] Canonical divisors and invariant subspaces in weighted Bergman spaces, Ph.D. thesis, University of Michigan, 2001.

[2] Canonical divisors in weighted Bergman spaces and hypergeometric functions, *Proc. Amer. Math. Soc.* **130** (2002), 707–713.

[3] Zeros of extremal functions in weighted Bergman spaces, *Pacific J. Math.* **208** (2003), 187–199.

[4] Construction of Green functions for weighted biharmonic operators, *to appear.*

Wojtaszczyk, P.

[1] On multipliers into Bergman spaces and Nevanlinna class, *Canad. Math. Bull.* **33** (1990), 151–161.

Zaharjuta, V. P. and Judovič, V. I.

[1] The general form of a linear functional on H_p', *Uspekhi Mat. Nauk* **19** (1964), no. 2 (116), 139–142. (in Russian)

Zhu, K.

[1] *Operator Theory in Function Spaces* (Marcel Dekker, New York, 1990).

[2] Duality of Bloch spaces and norm convergence of Taylor series, *Michigan Math. J.* **38** (1991), 89–101.

[3] Interpolating sequences for the Bergman space, *Michigan Math. J.* **41** (1994), 73–86.

[4] Evaluation operators on the Bergman space, *Math. Proc. Cambridge Philos. Soc.* **117** (1995), 513–523.

[5] A sharp estimate for extremal functions, *Proc. Amer. Math. Soc.* **128** (2000), 2577–2583.

Zygmund, A.

[1] *Trigonometric Series* (Cambridge University Press, London, 1959).

Index

The modern subject of Bergman spaces is a masterful blend of complex function theory with functional analysis and operator theory. It has much in common with Hardy spaces but involves new elements such as hyperbolic geometry, reproducing kernels, and biharmonic Green functions. This book develops background material and provides a self-contained introduction to a broad range of old and new topics in Bergman spaces, including recent advances on interpolation and sampling, contractive zero-divisors, and invariant subspaces. It is accessible to anyone who has studied basic real and complex analysis at the graduate level.

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