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Spatial Deterministic Epidemics

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ABSTRACT. This book uses rigorous analytic methods to determine the behaviour of spatial, deterministic models of certain multi-type epidemic processes where infection is spread by means of contact distributions. Results obtained include the existence of travelling wave solutions, the asymptotic speed of propagation and the spatial final size. The relationship with contact branching processes is also explored.

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Preface

Mathematical biology has witnessed tremendous development in recent years. There is also an increasing realisation of the importance of spatial processes in all branches of population dynamics. One specialised area of research which has progressed to a point where a reasonably complete theory now exists concerns the rigorous mathematical analysis of spatial models of deterministic epidemics, in which individuals once infected cannot again become susceptible. Although several books have appeared in which spatial models are formulated and their behaviour described, the rigorous mathematical theory has until now only existed in a large number of papers scattered over many journals. The aim of this book is to collect together this material and unify it in order to make the results more accessible to researchers in this area and to make the mathematical techniques more readily available to mathematicians working in other branches of population dynamics.

This book concentrates on deterministic models of epidemics, with the sole exception of Chapter 9 where the connection with contact branching processes is explored. Epidemic models where infection is transferred between hosts and vectors, such as people and mosquitoes, or where infection can be transferred between several species of animals involve more than one population (or type). The general theory of these models of multi-type epidemics leads to the analysis of systems of non-linear integral equations, and requires the use of results on positivity and monotone techniques. These include the convexity and analyticity of the Perron-Frobenius root of a matrix whose entries are Laplace transforms of non-negative functions; and the existence of solutions of certain systems of equations involving a convex function. The appropriate mathematics is developed in two appendices.

The models include the $S \to L \to I \to R$ epidemic, in which an infected individual has a latent period before becoming infectious and eventually enters a removed state; and also more general models with infectivity varying with the time since infection. Chapter 1 contains an historical sketch. Chapter 2 sets up non-spatial models and treats the final size of an epidemic. Chapters 3, 4 and 5 are the core of the monograph, presenting a rigorous mathematical analysis of certain problems concerning the spatial spread of an epidemic, namely the pandemic theorem, the existence and uniqueness of wave solutions and the asymptotic speed of propagation of an epidemic; the central result of the monograph being that the asymptotic speed of propagation is in fact the minimum speed for which waves exist. Chapter 6 looks at epidemics on sites.

There is another approach, based on saddle point methodology, which can be used to obtain the speed of first spread in biological models. This is developed in Chapter 7 and applied to epidemic models. In all situations where an exact

result has been proved, the result obtained by the saddle point method agrees with it. It is also used for $S \to I \to S$ epidemics and contact branching processes in Chapters 8 and 9 respectively. It is in fact a very powerful method which enables results to be obtained for much more complex situations, less tractable to exact analysis. There are applications to other areas of population dynamics such as genetics and evolutionary games. Although references are given, these applications are not included in this monograph.

The ideas developed have applications in other areas of population dynamics. Two of these which fit neatly into the general theme are included. The first application is the n-type $S \to I \to S$ deterministic epidemic, in which an infected individual can return to the susceptible state. The equilibrium solutions for the non-spatial model are determined and global convergence to the appropriate equilibrium is established. These results are then extended to a finite site model. The speed of first spread is obtained for the corresponding spatial model either in $\Re^N$ or on the N-dimensional integer lattice Z^N. The second application is contact branching processes. The connections between epidemics and contact branching processes are explained. The exact method is used for the contact birth process to obtain probabilistic convergence results. For more general models the saddle point method is applied to obtain the speed of first spread of the forward tail of the distribution of the furthest extent of the process.

In Chapter 1, links are given to certain other areas of population dynamics in which spatial models are used and where similar results to some of the results of this monograph have been obtained. Although in some ways we would have liked to have included some of these areas, we decided against it. The two most obvious omissions are spatial deterministic models in genetics and stochastic models for spatial epidemics.

The theory associated with the analogous problem in genetics of the spatial spread of a mutant gene uses techniques which overlap considerably with those used in this monograph. However, a comprehensive account of this subject would require a volume in its own right. In addition we decided to present the general n-type theory. The monotonicity of the functions involved in the epidemic models enabled the theory to be rigorously developed. No such general multi-type theory exists for the genetic models.

Deterministic and stochastic models are used to describe the behaviour of epidemics. Both only model the real world and each has its advantages and limitations in attempting to describe the possible modes of behaviour of epidemic systems. The reader might expect to find an account of both theories in a book on spatial epidemics. This is particularly true since both authors, although working at present in deterministic modelling, have a background in stochastic modelling. However, the theory associated with spatial stochastic epidemic models mainly concentrates on models in discrete space and uses quite different methodology. The continuous space stochastic models, analogous to the main models of this monograph, are less well developed. Whilst we are able to present a fairly complete account of the deterministic theory, this would not be possible at the present time for the corresponding models in the stochastic case.

We therefore decided in writing this monograph to restrict its scope to the theory of multi-type spatial deterministic epidemics, the requisite mathematics and

a couple of immediate applications of the methodology. This enables us to present in a single volume a comprehensive and fairly complete account of an elegant general theory. Our choice of references, in similar fashion, has been deliberately selective and is mainly limited to the literature directly relevant to the text.

The approach adopted consistently throughout the monograph is to explain the ideas and methodology in the simple one-type case before proceeding to the rigorous analysis for the multi-type model. It is therefore possible on first reading to gain a good understanding of the material by confining attention to the more intuitive discussions of the simple models in each chapter.

Our interest in the development of a rigorous theory for deterministic epidemics was first triggered on reading a remarkable paper by Atkinson and Reuter two decades ago, and from subsequent discussions with the late Professor Reuter. Other strong influences came from the fundamental work of Diekmann and Thieme in epidemics and of Aronson, Weinberger and Lui in genetics. The methodology contained in their work provides both the one-type theory and the springboard for our development of the multi-type theory, which is presented herein.

We would like to dedicate the monograph to our respective families, Nicky and Sandy Rass and David, Paul and Rita Radcliffe for their continuous support and encouragement. We would also especially like to extend our warm appreciation to our erstwhile colleagues Professor Brian Connolly and Dr. James Gilson for their enthusiasm for the project and for their help with some of the technical problems. Thanks also are due to the late Professor Philip Holgate for some early useful discussions and insights and his support when getting the project off the ground.

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Linda Rass and John Radcliffe

Appendices

Appendix A. Extended Perron-Frobenius theory

This section collects together certain properties of the class $\mathfrak{B}$ of non-negative, non-reducible, finite, square matrices that are used in this book. We first introduce some preliminary notation and definitions.

Let $\mathbf{B} = (b_{ij})$ denote a matrix with ij^{th} element b_{ij} and let $\mathbf{b} = (b_i)$ denote a column vector with i^{th} entry b_i. The ith element of a vector $\mathbf{b}$ is denoted by $\{\mathbf{b}\}_i$ and the ij^{th} element of a matrix $\mathbf{B}$ is denoted by $\{\mathbf{B}\}_{ij}$. We denote a vector or matrix with all elements zero by $\mathbf{0}$ and a vector with all its elements unity by $\mathbf{1}$. Inequalities between matrices or vectors imply the corresponding inequalities between the elements of the matrices or vectors.

A matrix is said to be non-negative if all its elements are non-negative. It is said to be finite if all its elements are finite. A square matrix $\mathbf{B} = (b_{ij})$ is said to be non-reducible if for every $i \neq j$ there exists a distinct sequence $i_i, ..., i_r$ with $i_1 = i$ and $i_r = j$ such that $b_{i_s i_{s+1}} \neq 0$ for $s = 1, ..., (r-1)$. Otherwise $\mathbf{B}$ is called reducible.

When $\mathbf{B}$ is a non-negative, finite, square matrix we denote its Perron-Frobenius root by $\rho(\mathbf{B})$. The Perron-Frobenius root is defined to be the maximum of the moduli of the eigenvalues of $\mathbf{B}$. It is a real eigenvalue of $\mathbf{B}$, and is the eigenvalue with largest real part. This definition is not restricted to the case when $\mathbf{B}$ is non-reducible. When $\mathbf{B}$ is non-reducible the Perron-Frobenius root $\rho(\mathbf{B})$ is simple, but for the reducible case $\rho(\mathbf{B})$ may have multiplicity greater than one.

We need to consider the limit of a non-reducible matrix $\rho(\mathbf{B})$ for situations in which elements of $\mathbf{B}$ may tend to infinity. For simplicity of exposition it is convenient to define $\rho(\mathbf{B}) = \infty$ when $\mathbf{B}$ is a non-negative, non-reducible square matrix with at least one infinite element.

The definition of the Perron-Frobenius root is easily extended to non-reducible, finite, square matrices with non-negative off-diagonal entries. Such a matrix, $\mathbf{A}$ may always be written in the form $\mathbf{A} = \mathbf{B} - c\mathbf{I}$ where $\mathbf{B}$ is a non-negative, square matrix and $\mathbf{I}$ is the identity matrix of the same size. The matrices $\mathbf{A}$ and $\mathbf{B}$ have eigenvalues differing by c and the same eigenvectors. We define $\rho(\mathbf{A}) = \rho(\mathbf{B}) - c$. Hence $\rho(\mathbf{A})$ is the eigenvalue of $\mathbf{A}$ with the largest real part, with corresponding eigenvector the eigenvector of $\mathbf{B}$ corresponding to $\rho(\mathbf{B})$.

Theorem A.1 is the basic Perron-Frobenius theorem for a non-negative, finite, square matrix; some useful results for the non-reducible case being given in Theorem A.2. Corollary A.1 gives certain results for the extended class where only the off-diagonal entries of the matrix are restricted to be non-negative (see the survey

on M-matrices of Poole and Boullion [P2]). Since these results may be found in standard texts, or are easily derivable, they are omitted. Useful texts are Berman and Plemmons [B7] and Gantmacher [G1]. For the convenience of the reader, the proofs of Theorems A.1, A.2 and Corollary A.1 are available in pdf format at *http* : *www.maths.qmul.ac.uk*\~*lr**book.html*.

Non-negative, non-reducible, square matrices whose entries are functions of a single real variable θ are then considered. Let $\mathbf{B}(\theta)$ be such a matrix. In applications, $\{\mathbf{B}(\theta)\}_{ij} = b_{ij}P_{ij}(\theta)$, where (b_{ij}) is a non-negative, non-reducible matrix and $P_{ij}(\theta)$ is the Laplace transform of a contact distribution. Theorem A.3 collects together continuity and convexity results for the Perron-Frobenius root and associated eigenvectors for matrices of this form.

In Theorem A.3, it is necessary to extend the definition of $\rho(\mathbf{B}(\theta))$ to cover certain situations where θ lies in an open ball in the complex plane centred on a real value θ_0. If the radius of the ball is sufficiently small then there is a unique eigenvalue of $\mathbf{B}(\theta)$ with largest real part, which we define to be $\rho(\mathbf{B}(\theta))$. Analyticity results are obtained for $\rho(\mathbf{B}(\theta))$.

THEOREM A.1. *Let* $\mathbf{B}$ *be a non-negative, finite, square matrix and define* $\rho(\mathbf{B})$ *to be the maximum of the moduli of the eigenvalues of* $\mathbf{B}$. *Then* $\rho(\mathbf{B})$ *is a non-negative real eigenvalue of* $\mathbf{B}$.

When $\mathbf{B}$ *is non-reducible, and is not the zero matrix of order* 1, $\rho(\mathbf{B}) > 0$. *There exist a positive right eigenvector and a positive left eigenvector corresponding to* $\rho(\mathbf{B})$.

When $\mathbf{B}$ *is reducible, there exists a right eigenvector* $\mathbf{v} \gneq \mathbf{0}$ *and a left eigenvector* $\mathbf{u} \gneq \mathbf{0}$ *corresponding to* $\rho(\mathbf{B})$. *In addition, when there exists a positive right or left eigenvector corresponding to an eigenvalue* λ *of* $\mathbf{B}$, *then* $\lambda = \rho(\mathbf{B})$.

□

The results for the reducible case may be obtained from those for a non-reducible matrix in a straightforward manner by writing the reducible matrix in normal form. When $\mathbf{B}$ is non-reducible $\rho(\mathbf{B})$ has multiplicity one, however it can have multiplicity greater than one for the reducible case so that the corresponding eigenvectors may not be unique up to a multiple.

Various results for non-negative matrices may be derived from the Perron-Frobenius Theorem. Some results for the non-reducible case which are of particular relevance to this book are collected together in the following theorem.

THEOREM A.2. *The class* $\mathfrak{B}$ *of non-negative, non-reducible, finite, square matrices has the following properties:*

1. *If* $\mathbf{B} \in \mathfrak{B}$ *then* $\rho(\mathbf{B})$ *increases as any element of* $\mathbf{B}$ *increases.*
2. *For any matrix* $\mathbf{B} = (b_{ij})$ *of order* n *in* $\mathfrak{B}$, *and any* $s = 1, ...n$,

$$b_{ss} \leq \rho(\mathbf{B}) \leq \max_i \sum_{j=1}^{n} b_{ij}.$$

3. *Let* $\mathbf{C} = (c_{ij})$ *be a matrix of complex valued elements and* $\mathbf{C}^+ = (|c_{ij}|)$. *If* $\mathbf{C}^+ \leq \mathbf{B}$, *where* $\mathbf{B} \in \mathfrak{B}$, *with strict inequality for at least one element, then for any eigenvalue* μ *of* $\mathbf{C}$, $|\mu| < \rho(\mathbf{B})$.

4. *Define the adjoint of a square matrix of order* 1 *to be the identity matrix of order* 1. *Then for any* $\mathbf{B} \in \mathfrak{B}$

$$\begin{aligned} Adj(\lambda\mathbf{I} - \mathbf{B}) > \mathbf{0}, &\quad for \quad \lambda > \rho(\mathbf{B}), \\ |\lambda\mathbf{I} - \mathbf{B}| > 0, &\quad for \quad \lambda > \rho(\mathbf{B}), \\ |\lambda\mathbf{I} - \mathbf{B}| = 0, &\quad for \quad \lambda = \rho(\mathbf{B}). \end{aligned}$$

5. *When* $\mathbf{B} \in \mathfrak{B}$ *is of order* $n > 1$, *and* $\mathbf{B}^*$ *is any* k*-dimensional principal minor of* $\mathbf{B}$, *where* $k < n$, *then* $\rho(\mathbf{B}^*) < \rho(\mathbf{B})$.
6. *If* $\mathbf{B} \in \mathfrak{B}$ *is of order* $n > 1$, *and* $\mathbf{B}^*$ *is any* $(n-1)$*-dimensional principal minor of* $\mathbf{B}$, *then* $|\lambda\mathbf{I} - \mathbf{B}| < 0$ *for* $\rho(\mathbf{B}^*) < \lambda < \rho(\mathbf{B})$.
7. *If* $\mathbf{B} \in \mathfrak{B}$ *then* $\rho(\mathbf{B})$ *is a simple eigenvalue of* $\mathbf{B}$ *and hence the corresponding right and left eigenvectors are each unique up to a multiple. No other eigenvalue of* $\mathbf{B}$ *has a real, non-negative right or left eigenvector.*
8. *If* $\mathbf{B} \in \mathfrak{B}$ *then* $Adj(\rho(\mathbf{B})\mathbf{I} - \mathbf{B}) > \mathbf{0}$.
9. *If* $\mathbf{B}$ *is a positive square matrix then all eigenvalues of* $\mathbf{B}$ *other than* $\rho(\mathbf{B})$ *have modulus less than* $\rho(\mathbf{B})$.
10. *If* $\mathbf{B} = (b_{ij}) \in \mathfrak{B}$ *and* $b_{ii} > 0$ *for all* i, *then there exists an integer* $s \geq 1$ *such that* $\mathbf{B}^s > \mathbf{0}$. *All eigenvalues of* $\mathbf{B}$ *other than* $\rho(\mathbf{B})$ *have modulus less than* $\rho(\mathbf{B})$.
11. *If* $\mathbf{B} = (b_{ij}) \in \mathfrak{B}$ *is not the* 1×1 *zero matrix, then there exists an integer* $s \geq 1$ *such that*

$$\mathbf{B}^s = \begin{pmatrix} \mathbf{C}_{11} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{22} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{C}_{kk} \end{pmatrix},$$

where $\mathbf{C}_{ii} > \mathbf{0}$ *with* $\rho(\mathbf{C}_{ii}) = (\rho(\mathbf{B}))^s$ *for all* $i = 1, ..., k$. *Here* $k \geq 1$ *is the number of eigenvalues of* $\mathbf{B}$ *which have modulus equal to* $\rho(\mathbf{B})$.

□

COROLLARY A.1. *Let* $\mathbf{A}$ *be a non-reducible, finite matrix with off diagonal entries non-negative. This matrix may always be written in the form* $\mathbf{A} = \mathbf{B} - c\mathbf{I}$, *where* $\mathbf{B}$ *is non-negative.*

There is a unique eigenvalue, $\rho(\mathbf{A})$, *with largest real part. This eigenvalue is real and simple.*

There exist positive left and right eigenvectors corresponding to $\rho(\mathbf{A})$, *each of which is unique up to a multiple. This is the only eigenvalue of* $\mathbf{A}$ *for which there is a positive eigenvector.*

The eigenvalue $\rho(\mathbf{A})$ *increases as any element of* $\mathbf{A}$ *increases.*

□

Continuity results for the Perron-Frobenius root are also required. A preliminary lemma is first proved concerning the continuity of the eigenvalues of a general matrix. Dieudonné [D11 p.248] uses Rouché's theorem to derive continuity results for the roots of a polynomial, from which equivalent results may be obtained for the eigenvalues of a matrix. This is the approach used in Lemma A.1. An alternative

proof of the continuity of the eigenvalues of a matrix as functions of its entries may be found in Ostrowski [O1 p. 282].

Lemma A.1 may then be used to show that if $\mathbf{A} = (a_{ij})$ is a non-negative, non-reducible matrix, then the definition of the Perron-Frobenius root may be extended to matrices whose entries are within an ε distance of those of $\mathbf{A}$, for some $\varepsilon > 0$. Within this region, the Perron-Frobenius root is a continuous function of its entries. The non-reducible matrices of interest have entries which are analytic functions of a single parameter θ.

LEMMA A.1. *Consider an $n \times n$ matrix $\mathbf{A}^* = (a^*_{ij})$ which has (not necessarily distinct) eigenvalues $\mu_1(\mathbf{A}^*), \mu_2(\mathbf{A}^*), ..., \mu_n(\mathbf{A}^*)$ and define*

$$d = \min_{\mu_i(\mathbf{A}^*) \neq \mu_j(\mathbf{A}^*)} |\mu_i(\mathbf{A}^*) - \mu_j(\mathbf{A}^*)|.$$

Let r_i be the multiplicity of $\mu_i(\mathbf{A}^*)$. Then for every $0 < \varepsilon < d/2$ there exists a $\delta > 0$ such that $\mathbf{A} = (a_{ij})$ has exactly r_i eigenvalues within an ε-neighbourhood of $\mu_i(\mathbf{A}^*)$ for each distinct $\mu_i(\mathbf{A}^*)$ for every matrix $\mathbf{A}$ such that $\max_{i,j} |a_{ij} - a^*_{ij}| < \delta$.

When $\mu_i(\mathbf{A}^*)$ has multiplicity 1 then we define $\mu_i(\mathbf{A})$ to be the unique eigenvalue of $\mathbf{A}$ within the ε-neighbourhood of $\mu_i(\mathbf{A}^*)$. Then $\mu_i(\mathbf{A})$ is a jointly continuous function of the entries of $\mathbf{A}$ at $\mathbf{A} = \mathbf{A}^*$, and hence $\mu_i(\mathbf{A}) \to \mu_i(\mathbf{A}^*)$ as $\mathbf{A} \to \mathbf{A}^*$.

PROOF. Rouché's theorem states that if f and g are analytic in an open region $\mathfrak{S}$ of the complex plane and if γ is a Jordan circuit, such that its graph Γ and its interior lie within $\mathfrak{S}$, and if $|g(z)| < |f(z)|$ for all z on Γ, then f and $f + g$ have the same number of zeros within Γ.

Consider the polynomials $f(z) = z^n + \sum_{j=0}^{n-1} b^*_j z^j$ and $h(z) = z^n + \sum_{j=0}^{n-1} b_j z^j$, where $f(z)$ has distinct zeros $\mu_1, ..., \mu_s$ of multiplicities $r_1, ..., r_s$. Define $d = \min_{i \neq j} |\mu_i - \mu_j|$. We use Rouché's theorem to show that for any $0 < \varepsilon < d/2$ there exists a $\delta^* > 0$ such that $h(z)$ has exactly r_i zeros within an ε-neighbourhood of μ_i for $i = 1, ..., s$ provided $\max_{0 \leq j \leq n-1} |b_j - b^*_j| < \delta^*$.

Consider any ε such that $0 < \varepsilon < d/2$. Define $g(z) = h(z) - f(z)$ and take Γ_i to be the circumference of a ball of radius ε centred on μ_i. Then $f(z)$ is continuous and non-zero on the closed set Γ_i and hence there exists a $k_i > 0$ such that $|f(z)| \geq k_i$ for z on Γ_i. Let $A = \max_{0 \leq j \leq n-1} |b_j - b^*_j|$. Then $|g(z)| \leq A \sum_{j=0}^{n-1} (|\mu_j| + \varepsilon)^j < k_i$ on Γ_i provided $A < k_i / (\sum_{j=0}^{n-1} (|\mu_i| + \varepsilon)^j)$. Let $\delta^* = \min_{1 \leq i \leq s} k_i / (\sum_{j=0}^{n-1} (|\mu_i| + \varepsilon)^j)$. Then from Rouché's Theorem $h(z) \equiv f(z) + g(z)$ has exactly r_i zeros within an ε-neighbourhood of μ_i for each $i = 1, ..., s$ provided $\max_{0 \leq j \leq n-1} |b_j - b^*_j| < \delta^*$.

It is simple to use this result to obtain an equivalent result for the distinct eigenvalues $\mu_1, ..., \mu_s$ of a matrix, since the eigenvalues of an $n \times n$ matrix are just the zeros of the characteristic polynomial. Again we let r_i denote the multiplicity of μ_i and let $d = \min_{i \neq j} |\mu_i - \mu_j|$. Let $f(z) = |z\mathbf{I} - \mathbf{A}^*| = z^n + \sum_{j=0}^{n-1} b^*_j z^j$ and $h(z) = |z\mathbf{I} - \mathbf{A}| = z^n + \sum_{j=0}^{n-1} b_j z^j$. Then each b^*_j and b_j are just sums of products of the corresponding entries of the matrices $\mathbf{A}^* = (a^*_{ij})$ and $\mathbf{A} = (a_{ij})$ respectively. Hence each b_j is a jointly continuous function of the entries of $\mathbf{A}$ and $b_j \to b^*_j$ as $\mathbf{A} \to \mathbf{A}^*$. Therefore for any $\delta^* > 0$ there exists a $\delta > 0$ such that $\max_j |b_j - b^*_j| < \delta^*$ provided $\max_{i,j} |a_{ij} - a^*_{ij}| < \delta$. Then it immediately follows from the result for polynomials that for every $0 < \varepsilon < d/2$ there exists a $\delta^* > 0$, and

hence a $\delta > 0$, such that exactly r_i eigenvalues of $\mathbf{A}$ lie within an ε-neighbourhood of μ_i for $i = 1, ..., s$, provided $\max_{i,j} |a_{ij} - a^*_{ij}| < \delta$.

Now denote the distinct eigenvalue $\mu_1, ..., \mu_s$ of $\mathbf{A}^*$ by $\mu_1(\mathbf{A}^*), ..., \mu_s(\mathbf{A}^*)$. If $\mu_i(\mathbf{A}^*)$ has multiplicity one then there is exactly one eigenvalue of $\mathbf{A}$ in its ε-neighbourhood, provided $\max_{i,j} |a_{ij} - a^*_{ij}| < \delta$. We define this eigenvalue to be $\mu_i(\mathbf{A})$. Hence for every $0 < \varepsilon < d/2$ there exists a $\delta > 0$ such that $|\mu_i(\mathbf{A}) - \mu_i(\mathbf{A}^*)| < \varepsilon$ provided $\max_{i,j} |a_{ij} - a^*_{ij}| < \delta$, which establishes the continuity and limiting results.

□

We now turn our attention to matrices in the class $\mathfrak{B}$ of non-negative, non-reducible, finite, square matrices of a particular form. The entries of $\mathbf{B}$ are each taken to be analytic functions of a single parameter θ, we therefore write the matrix as $\mathbf{B}(\theta)$. Lemma A.1 implies that $\rho(\mathbf{B}(\theta))$ may be defined for θ in an open neighbourhood of the complex plane centred on θ_0 for each $\theta_1 < \theta < \theta_2$. It is a continuous function of θ in each such neighbourhood and can be shown to be analytic in the (possibly restricted) neighbourhood of each θ_0 for which $trace(Adj(\rho(\mathbf{B}(\theta)\mathbf{I} - \mathbf{B}(\theta)))) \neq 0$. A simple proof of the analyticity is given. It may also be derived from Theorem 9 of Bochner and Martin [B9].

In particular we are interested in results when each $\{\mathbf{B}(\theta)\}_{ij}$ is the Laplace transform of a non-negative function, where these entries all exist in some range $\theta_1 < Re(\theta) < \theta_2$. In this case it is easily seen that the non-zero entries of $\mathbf{B}(\theta)$ are superconvex for θ real in the range (θ_1, θ_2). The convexity of $\rho(\mathbf{B}(\theta))$, for θ real with $\theta_1 < \theta < \theta_2$, may then be established. The method used is a generalisation of that of Kingman [K4] for positive matrices. The proof requires results concerning the class of superconvex functions given in Lemma A.2. A positive function $g(\theta)$, for $\theta_1 < \theta < \theta_2$, is said to be superconvex if $\log(g(\theta))$ is convex in the range, i.e. if $g(x\theta + (1-x)\phi) \leq g(\theta)^x g(\phi)^{1-x}$ for all $\theta_1 < \theta, \phi < \theta_2$ and all $0 \leq x \leq 1$.

LEMMA A.2. *The class $\mathfrak{S}$ of superconvex functions of θ, for $\theta_1 < \theta < \theta_2$, is closed under addition, multiplication and raising to a positive power. In addition if g_n is in the class for $n \geq 1$ and if g_n tends to a limit g, which is a positive function, as n tends to infinity, then g is also in the class.*

PROOF. The closure under multiplication and raising to a positive power and the result on limits follow trivially from the defining inequality for superconvex functions. We only need to consider closure under addition.

We first prove an inequality, which may be derived from Holder's inequality, namely that $1 + u^x v^{1-x} \leq (1+u)^x (1+v)^{1-x}$ for any positive u and v and $0 \leq x \leq 1$. The inequality trivially holds when $u = v$. Now consider the case when $u \neq v$. Define

$$f(x) = 1 + v\left(\frac{u}{v}\right)^x - (1+v)\left(\frac{1+u}{1+v}\right)^x.$$

Then $f(x)$ is continuous with $f(0) = f(1) = 0$, so that a minimum or maximum must occur for some $0 < x < 1$. Now

$$f'(x) = v\left(\frac{u}{v}\right)^x \log\left(\frac{u}{v}\right) - (1+v)\left(\frac{1+u}{1+v}\right)^x \log\left(\frac{(1+u)}{(1+v)}\right).$$

For any x^* such that $f(x^*) = 0$ it is easily seen that $f''(x^*) > 0$. Hence x^* is unique and $f(x)$ is minimised at $x = x^*$. It then follows immediately that $f(x) \leq 0$ for $0 \leq x \leq 1$ and hence $1 + u^x v^{1-x} \leq (1+u)^x(1+v)^{1-x}$ for x in this range.

Let g and h be any two superconvex functions in $\mathfrak{S}$ and consider any $\theta_1 < \theta, \phi < \theta_2$ and $0 \leq x \leq 1$. Using the superconvexity of g and h and the above inequality we obtain

$$\begin{aligned} g(x\theta + (1-x)\phi) + h(x\theta + (1-x)\phi) &\leq g(\theta)^x g(\phi)^{1-x} + h(\theta)^x h(\phi)^{1-x} \\ &= g(\theta)^x g(\phi)^{1-x}\left(1 + \left(\frac{h(\theta)}{g(\theta)}\right)^x \left(\frac{h(\phi)}{g(\phi)}\right)^{1-x}\right) \\ &\leq g(\theta)^x g(\phi)^{1-x}\left(1 + \frac{h(\theta)}{g(\theta)}\right)^x \left(1 + \frac{h(\phi)}{g(\phi)}\right)^{1-x} \\ &= (g(\theta) + h(\theta))^x (g(\phi) + h(\phi))^{1-x}. \end{aligned}$$

Hence the class $\mathfrak{S}$ is closed under addition.

□

THEOREM A.3. *Consider a matrix $\mathbf{B}(\theta) = (b_{ij}(\theta))$, which is not the 1×1 zero matrix, and lies in the class of non-negative, non-reducible, finite, square matrices for θ real with $\theta_1 < \theta < \theta_2$. Let $b_{ij}(\theta)$ be analytic for $\theta_1 < Re(\theta) < \theta_2$ for all i, j. Also define $T(\theta, i; i_1, ..., i_r) = \prod_{s=1}^{r} \{\mathbf{B}(\theta)\}_{i_s i_{s+1}}$, where $i_{r+1} = i_1 = i$ and the sequence $i_1, i_2, ..., i_r$ is distinct. Then the following results hold.*

1. *If θ_0 is real and $\theta_1 < \theta_0 < \theta_2$, then there exists a $\delta_0 > 0$ such that the definition of $\rho(\mathbf{B}(\theta))$ may be extended to complex values of θ in an open ball radius δ_0 centred at θ_0. There exists an open ball centred on θ_0 of radius $0 < \delta^* \leq \delta_0$ in which $\rho(\mathbf{B}(\theta))$ is an analytic function of θ.*
2. *There exists an open ball centred on θ_0 in which $\rho(\mathbf{B}(\theta))$ is defined and is analytic, where the entries of the corresponding right and left eigenvectors, $(\mathbf{u}(\theta))'$ and $\mathbf{v}(\theta)$, with first entries unity and the entries of the corresponding idempotent are also analytic.*
3. *If θ_2 is real and finite and $\lim_{\theta \uparrow \theta_2} \mathbf{B}(\theta)$ exists and is non-reducible, then $\lim_{\theta \uparrow \theta_2} \rho(\mathbf{B}(\theta)) = \rho(\lim_{\theta \uparrow \theta_2} \mathbf{B}(\theta))$. In addition, when $\mathbf{B}(\theta_2)$ has all finite entries, then $\lim_{\theta \uparrow \theta_2} (\mathbf{u}(\theta))' = (\mathbf{u}(\theta_2))'$ and $\lim_{\theta \uparrow \theta_2} \mathbf{v}(\theta) = \mathbf{v}(\theta_2)$. A similar result holds as $\theta \downarrow \theta_1$.*
4. *When $\theta_2 = \infty$, then if there exists at least one integer i and sequence $i_1, i_2, ..., i_r$ for which $\lim_{\theta \to \infty} T(\theta; i; i_1, ..., i_r) = \infty$ then $\lim_{\theta \to \infty} \rho(\mathbf{B}(\theta)) = \infty$. Also when $\theta_2 = \infty$, if $\lim_{\theta \to \infty} T(\theta, i; i_1, ..., i_r) = 0$ for all i and all possible distinct sequences $i_1, ..., i_r$, then $\lim_{\theta \to \infty} \rho(\mathbf{B}(\theta)) = 0$.*
5. *When the eigenvalues of $\mathbf{B}(\theta_0)$ are distinct then there exists an open ball centred at θ_0 in which all the eigenvalues of $\mathbf{B}(\theta)$ are distinct, and the entries of the eigenvalues and the corresponding idempotents (as in part 2) are continuous functions of θ.*

6. *Let each $b_{ij}(\theta)$ be the Laplace transform of a non-negative function, which exists for θ real with $\theta_1 < Re(\theta) < \theta_2$. Then for θ real in this range, the $b_{ij}(\theta)$ which are non-zero are superconvex functions of θ, and hence $\rho(\mathbf{B}(\theta))$ is a superconvex function of θ. It is therefore strictly convex except in the degenerate case when $\rho(\mathbf{B}(\theta))$ is constant, which occurs when each $b_{ij}(\theta)$ is constant.*

PROOF.

1. Let the eigenvalues of $\mathbf{B}(\theta_0)$ be $\mu_1, \mu_2, ..., \mu_n$, where $\mu_1 = \rho(\mathbf{B}(\theta_0))$. Now from Theorem A.1 $\rho(\mathbf{B}(\theta_0))$ is real with $|\mu_j| \leq \rho(\mathbf{B}(\theta_0))$ for all $j > 1$. Also from Theorem A.2 part 7 it has multiplicity 1. Hence $Re(\mu_j) < \rho(\mathbf{B}(\theta_0))$ for all $j > 1$. Let $d = \min_{\mu_i \neq \mu_j} |\mu_i - \mu_j|$, $k = \min_{j \geq 2}(\rho(\mathbf{B}(\theta_0)) - Re(\mu_j))$ and $\varepsilon = (1/2)\min(d, k)$. Then for any $i > 1$, if z and w lie respectively in the ε-neighbourhoods of $\rho(\mathbf{B}(\theta_0))$ and μ_i, it immediately follows that $Re(z) > Re(w)$.

   Then from Lemma A.1 and the result above there exists a $\delta > 0$ such that the following result holds. For every $\mathbf{A} = (a_{ij})$ with $\max_{i,j} |a_{ij} - b_{ij}(\theta_0)| < \delta$, $\mathbf{A}$ has a unique eigenvalue in the ε-neighbourhood of $\rho(\mathbf{B}(\theta_0))$, which is the eigenvalue of $\mathbf{A}$ with largest real part which we define to be $\rho(\mathbf{A})$. This eigenvalue has multiplicity 1. Now each $b_{ij}(\theta)$ is a continuous function of θ. Hence there exists a δ_0 with $0 < \delta_0 < \min(\theta_0, \Delta_v - \theta_0)$ such that $\max_{i,j} |b_{ij}(\theta) - b_{ij}(\theta_0)| < \delta$ provided that $|\theta - \theta_0| < \delta_0$. Hence $\rho(\mathbf{B}(\theta))$ has been defined as the unique eigenvalue of largest real part for all complex values of θ such that $|\theta - \theta_0| < \delta_0$.

   Now consider any θ^* such that $|\theta^* - \theta_0| < \delta_0$, so that $\rho(\mathbf{B}(\theta^*))$ has been defined. From Lemma A.1 $\rho(\mathbf{B}(\theta))$ is a continuous function of its entries $b_{ij}(\theta)$, and hence from the continuity of the $b_{ij}(\theta)$ it is a continuous function of θ, at $\theta = \theta^*$. This establishes the existence and continuity of $\rho(\mathbf{B}(\theta))$ in an open ball of radius δ_0 centred on θ_0.

   Finally consider the analyticity of $\rho(\mathbf{B}(\theta))$. Take $\delta_0 > 0$ such that $\rho(\mathbf{B}(\theta))$ is defined for $|\theta - \theta_0| < \delta_0$. Now $Adj(\lambda\mathbf{I} - \mathbf{B}(\theta))$ is a continuous function of λ and θ and, from Theorem A.2 part 8, $Adj(\rho(\mathbf{B}(\theta_0)\mathbf{I} - \mathbf{B}(\theta_0))) > \mathbf{0}$. Also we have shown the continuity of $\rho(\mathbf{B}(\theta))$ at $\theta = \theta_0$. Hence there exists a $0 < \delta_0^* < \delta_0$, such that $Re[Adj(\rho(\mathbf{B}(\theta))\mathbf{I} - \mathbf{B}(\theta))] > \mathbf{0}$ for $|\theta - \theta_0| < \delta_0^*$.

   Consider any $\theta = \theta^*$ in this range. We give a simple proof which shows that $\rho(\mathbf{B}(\theta))$ is differentiable at $\theta = \theta^*$ with continuous derivative, so that $\rho(\mathbf{B}(\theta))$ is an analytic function of θ for $|\theta - \theta_0| < \delta_0^*$.

   Let $\delta\rho = \rho(\mathbf{B}(\theta^* + \delta\theta)) - \rho(\mathbf{B}(\theta^*))$ and $\mathbf{C} = (c_{ij})$, where $c_{ij} = b_{ij}(\theta^* + \delta\theta) - b_{ij}(\theta^*)$. Now

$$
\begin{aligned}
0 &= |\rho(\mathbf{B}(\theta^* + \delta\theta))\mathbf{I} - \mathbf{B}(\theta^* + \delta\theta)| \\
&= |[\rho(\mathbf{B}(\theta^*))\mathbf{I} - \mathbf{B}(\theta^*)] + \delta\rho\mathbf{I} - \mathbf{C}| \\
&= |\rho(\mathbf{B}(\theta^*))\mathbf{I} - \mathbf{B}(\theta^*)| + \delta\rho[trace(Adj(\rho(\mathbf{B}(\theta^*))\mathbf{I} - \mathbf{B}(\theta^*))) + f(\delta\rho)] \\
&- \sum_i \sum_j [c_{ij}\{Adj(\rho(\mathbf{B}(\theta^*))\mathbf{I} - \mathbf{B}(\theta^*))\}_{ij}) + g_{ij}(\delta\rho, \{c_{st}\})] .
\end{aligned}
$$

Here $f(\delta\rho)$ involves first order and higher polynomial terms in $\delta\rho$ and each $g_{ij}(\delta\rho, \{c_{st}\})$ involves first order and higher polynomial terms in $\delta\rho$ and the c_{st}, so that each of these functions tends to zero as $\delta\theta$ tends to zero. Now $|\rho(\mathbf{B}(\theta^*))\mathbf{I} - \mathbf{B}(\theta^*)| = 0$ and $trace(Adj(\rho(\mathbf{B}(\theta^*))\mathbf{I} - \mathbf{B}(\theta^*))) \neq 0$. Hence for $\delta\theta$ sufficiently small,

$$\frac{\delta\rho}{\delta\theta} = \frac{\sum_{i,j}(c_{ij}/\delta\theta)\left[\{Adj(\rho(\mathbf{B}(\theta^*))\mathbf{I} - \mathbf{B}(\theta^*))\}_{ij} + g(\delta\rho, \{c_{st}\})\right]}{trace(Adj(\rho(\mathbf{B}(\theta^*))\mathbf{I} - \mathbf{B}(\theta^*))) + f(\delta\rho)}.$$

It follows immediately that the derivative $\rho'(\mathbf{B}(\theta^*))$ exists and is given by

$$\rho'(\mathbf{B}(\theta^*)) = \frac{\sum_{i,j} b'_{ij}(\theta^*)\{Adj(\rho(\mathbf{B}(\theta^*))\mathbf{I} - \mathbf{B}(\theta^*))\}_{ij}}{trace(Adj(\rho(\mathbf{B}(\theta^*))\mathbf{I} - \mathbf{B}(\theta^*)))},$$

for all $|\theta^* - \theta_0| < \delta_0^*$. The continuity of the derivative and the analyticity is then immediate.

2. Using part 1 of this theorem, there exists an open ball centred on θ_0 for which $Adj(\rho(\mathbf{B}(\theta))\mathbf{I} - \mathbf{B}(\theta))$ has entries which are analytic functions of θ and which are strictly positive at $\theta = \theta_0$. Hence there exists an open ball centred on θ_0 of radius δ in which $\rho(\mathbf{B}(\theta))$ is defined and is analytic and in which the entries of the adjoint matrix are non-zero. Now the columns of the adjoint matrix are proportional to the right eigenvector of $\mathbf{B}(\theta)$ corresponding to $\rho(\mathbf{B}(\theta))$ and the rows are proportional to the left eigenvector. It therefore follows that, for θ within this open ball centred at θ_0, the right and left eigenvectors, $\mathbf{v}(\theta)$ and $(\mathbf{u}(\theta))'$ can be chosen to have first entry one.

   Partition the matrix $\mathbf{B}(\theta)$ and the right eigenvector $\mathbf{v}(\theta)$ corresponding to $\rho(\mathbf{B}(\theta))$ so that

$$\mathbf{B}(\theta) = \begin{pmatrix} b_{11}(\theta) & (\mathbf{b}_{12}(\theta))' \\ \mathbf{b}_{21}(\theta) & \mathbf{B}_{22}(\theta) \end{pmatrix} \quad \text{and} \quad \mathbf{v}(\theta) = \begin{pmatrix} 1 \\ \mathbf{v}_2(\theta) \end{pmatrix}.$$

   Since $|\rho(\mathbf{B}(\theta))\mathbf{I} - \mathbf{B}_{22}(\theta_0)| = \{Adj(\rho(\mathbf{B}(\theta))\mathbf{I} - \mathbf{B}(\theta))\}_{11} \neq 0$ for all $|\theta - \theta_0| < \delta$, then the entries of $(\rho(\mathbf{B}(\theta))\mathbf{I} - \mathbf{B}_{22}(\theta))^{-1}$ are analytic functions of θ in this open ball. Therefore in this region

$$\mathbf{v}_2(\theta) = (\rho(\mathbf{B}(\theta)\mathbf{I} - \mathbf{B}_{22}(\theta))^{-1}\mathbf{b}_{21}(\theta),$$

   which has entries which are analytic functions of θ. Hence there is a right eigenvector of $\mathbf{B}(\theta)$ corresponding to $\rho(\mathbf{B}(\theta))$ which is analytic, namely the eigenvector with first entry one.

   The proof for the left eigenvector follows in an identical manner. Now the corresponding idempotent $\mathbf{E}(\theta) = \mathbf{v}(\theta)(\mathbf{u}(\theta))'/(\mathbf{u}(\theta))'\mathbf{v}(\theta)$. Also, since $(\mathbf{u}(\theta_0))'\mathbf{v}(\theta_0) > 0$, from the continuity of these eigenvectors there is an open ball centred on θ_0 of radius $\delta^* < \delta$ in which $(\mathbf{u}(\theta))'\mathbf{v}(\theta) \neq 0$. The analyticity of the entries of the idempotent in this open ball is then immediate.

3. Consider a non-reducible, non-negative, finite, square matrix $\mathbf{A}^* = (a_{ij}^*)$ with eigenvalues $\mu_1 = \rho(\mathbf{A}^*), \mu_2, ..., \mu_n$. As in part 1, $\rho(\mathbf{A}^*)$ is a real simple eigenvalue which is the unique eigenvalue of largest real part. Then from

Lemma A.1 there is a $\delta > 0$ so that for any matrix $\mathbf{A} = (a_{ij})$, for which $\max_{i,j} |a_{ij} - a^*_{ij}| < \delta$, there is a unique simple eigenvalue of largest real part which we define to be $\rho(\mathbf{A})$ and $\rho(\mathbf{A}) \to \rho(\mathbf{A}^*)$ as $\mathbf{A} \to \mathbf{A}^*$.

Now, if $\mathbf{B}(\theta_2)$ is finite and non-reducible, as $\theta \uparrow \theta_2$ the entries of $\mathbf{B}(\theta)$ tend to the entries of $\mathbf{B}(\theta_2)$. Hence from the result above $\rho(\mathbf{B}(\theta)) \to \rho(\mathbf{B}(\theta_2))$ as $\theta \uparrow \theta_2$. Therefore $\lim_{\theta\uparrow\theta_2} \rho(\mathbf{B}(\theta)) = \rho(\mathbf{B}(\theta_2))$.

The result that $\lim_{\theta\uparrow\theta_2} \mathbf{v}(\theta) = \mathbf{v}(\theta_2)$ then follows immediately from the definition of $\mathbf{v}(\theta)$ in part 2. The result for the left eigenvector follows in an identical manner.

Now consider the case when $\lim_{\theta\uparrow\theta_2} b_{ij}(\theta) = \infty$ for at least one i, j. Define $\mathbf{B}(\theta_2) = \lim_{\theta\uparrow\theta_2} \mathbf{B}(\theta)$. Suppose that there exists an i such that $\lim\limits_{\theta\uparrow\theta_2} b_{ii}(\theta) = \infty$. By property 2 of Theorem A.2 $\rho(\mathbf{B}(\theta)) \geq b_{ii}(\theta)$ for all $\theta_1 < \theta < \theta_2$. Hence $\lim_{\theta\uparrow\theta_2} \rho(\mathbf{B}(\theta)) = \infty = \rho(\mathbf{B}(\theta_2))$. Now suppose $\lim_{\theta\uparrow\theta_2} b_{ij}(\theta) = \infty$ for some $i \neq j$. By relabelling we may take $i = 1$ and $j = 2$. Since $\lim_{\theta\uparrow\theta_2} \mathbf{B}(\theta)$ is non-reducible, there exists a sequence, which by relabelling may be written as $b_{23}(\theta), ..., b_{k-1,k}(\theta), b_{k1}(\theta)$ with limits all positive as $\theta \uparrow \theta_2$. If $\mathbf{C}(\theta) = (c_{ij}(\theta))$ is the cyclic matrix of order k with $c_{i,i+1}(\theta) = b_{i,i+1}(\theta)$ for $i = 1, ..., (k-1)$, $c_{k1}(\theta) = b_{k1}(\theta)$ and all other entries zero, then $\mathbf{C}(\theta)$ is bounded above by the corresponding k-dimensional principle minor of $\mathbf{B}(\theta)$. By properties 1 and 5 of Theorem A.2 we obtain $\rho(\mathbf{B}(\theta)) \geq \rho(\mathbf{C}(\theta)) = |b_{12}(\theta)b_{23}(\theta)...b_{k1}(\theta)|^{1/k}$ for $\theta_1 < \theta < \theta_2$. As $\theta \uparrow \theta_2$ the right-hand side of this inequality tends to infinity, hence $\lim_{\theta\uparrow\theta_2} \rho(\mathbf{B}(\theta)) = \infty = \rho(\mathbf{B}(\theta_2))$. Thus in all cases when $\lim_{\theta\uparrow\theta_2} \rho(\mathbf{B}(\theta))$ is non-reducible $\lim_{\theta\uparrow\theta_2} \rho(\mathbf{B}(\theta)) = \rho(\mathbf{B}(\theta_2))$.

The proofs when $\theta \downarrow \theta_1$ follow in identical fashion.

4. The first result follows as for the proof of part 3. Since there exists an i and $i_1, ..., i_r$ such that $\lim_{\theta\to\infty} T(\theta, i; i_1, ..., i_r) = \infty$ we take $\mathbf{C}(\theta)$ to be the cyclic matrix of order r with the $(s, s+1)^{th}$ entry $b_{i_s,i_{s+1}}(\theta)$ for $s = 1, ..., r-1$ and $(r, 1)^{th}$ entry $b_{i_r,i_1}(\theta)$. All other entries are zero. Then $\mathbf{C}(\theta)$ is bounded above by a principal minor of $\mathbf{B}(\theta)$ in permuted form. Therefore, as in part 3, $\rho(\mathbf{B}(\theta)) \geq \rho(\mathbf{C}(\theta)) = |b_{i_1,i_2}(\theta)b_{i_2,i_3}(\theta)...b_{i_r,i_1}(\theta)|^{1/r}$. Since the right hand side of the inequalities tends to infinity as $\theta \to \infty$, the first result is immediate, i.e. $\lim_{\theta\to\infty} \rho(\mathbf{B}(\theta)) = \infty$.

Now consider the second part where $\lim_{\theta\to\infty} T(\theta, i; i_1, ..., i_r) = 0$ for all i and all sequences $i_1, ..., i_r$. Take any $\varepsilon > 0$ and consider

$$|z\mathbf{I} - \mathbf{B}(\theta)| = z^n + \alpha_1(\theta)z^{n-1} + ... + \alpha_n(\theta).$$

Here $|\alpha_s(\theta)| \leq \sum |T(\theta, i; i_1, ..., i_s)|$, where the summation is over all i and all distinct sequences $i_1, ..., i_s$ of length s. But $\lim_{\theta\to\infty} T(\theta, i; i_1, ..., i_r) = 0$ for all integers i, all distinct sequences $i_1, ..., i_r$ and all $r = 1, ..., n$. Therefore $\lim_{\theta\to\infty} \alpha_s(\theta) = 0$ for all $s = 1, ..., n$.

Let $g(z, \boldsymbol{\alpha}) = z^n + \{\boldsymbol{\alpha}\}_1 z^{n-1} + ... + \{\boldsymbol{\alpha}\}_n$. If $\boldsymbol{\alpha} = \mathbf{0}$, then the zeros of $g(z, \boldsymbol{\alpha})$ are all zero. Consider $\boldsymbol{\alpha} \in \Re^n$. Using results for the roots of a polynomial contained in the proof of Lemma A.1, there is an open neighbourhood D of $\boldsymbol{\alpha} = \mathbf{0}$ for which all the zeros of $g(z, \boldsymbol{\alpha})$ have modulus less than ε.

Since $\lim_{\theta\to\infty}\alpha_s(\theta)=0$ for all $s=1,...,n$, there exists a θ_0 sufficiently large such that $(\alpha_1(\theta),...,\alpha_n(\theta))\in D$ for $\theta>\theta_0$. Hence $\rho(\mathbf{B}(\theta))<\varepsilon$ for $\theta>\theta_0$. Therefore $\lim_{\theta\to\infty}\rho(\mathbf{B}(\theta))=0$ in this case.

5. If the eigenvalues of $\mathbf{B}(\theta_0)$ are distinct then, in a similar manner to the proof of part 1, using Lemma A.1 there exists an open ball centred on θ_0 in which the eigenvalues of $\mathbf{B}(\theta)$ stay distinct and tend to the appropriate eigenvalue of $\mathbf{B}(\theta_0)$ as θ tends to θ_0. The continuity at $\theta=\theta^*$ for any θ^* within this open ball also follows in a similar manner.

   Now consider the idempotent $\mathbf{E}_j(\theta)$ corresponding to $\mu_j(\theta)$. Since the eigenvalues are distinct,

$$\mathbf{E}_j(\theta)=\frac{1}{\prod_{i\neq j}(\mu_j(\theta)-\mu_i(\theta))}\prod_{i\neq j}(\mathbf{B}(\theta)-\mu_i(\theta)\mathbf{I}).$$

   The continuity of the idempotents follows immediately.

6. We first show that the non-zero entries of $\mathbf{B}(\theta)=(b_{ij}(\theta))$ are superconvex by showing that each $\log(b_{ij}(\theta))$ is convex.

   Consider the Laplace transform $A(\theta)$ of a non-negative function $a(x)$ for $x\in\Re$, so that $A(\theta)=\int_{x\in\Re}e^{\theta x}a(x)dx$. Now

$$\frac{d^2\log(A(\theta))}{d\theta^2}=\frac{A(\theta)A''(\theta)-(A'(\theta))^2}{(A(\theta))^2}.$$

   Here $A''(\theta)=\int_{y\in\Re}y^2e^{\theta y}a(y)dy$. Then $A(\theta)A''(\theta)$ may be written as

$$\int_{x\in\Re}\int_{y\in\Re}\frac{1}{2}(x^2+y^2)e^{\theta(x+y)}a(x)a(y)dxdy.$$

   Hence

$$A(\theta)A''(\theta)-(A'(\theta))^2=\int_{x\in\Re}\int_{y\in\Re}\frac{1}{2}(x-y)^2e^{\theta(x+y)}a(x)a(y)dxdy>0.$$

   Hence $\dfrac{d^2\log(A(\theta))}{d\theta^2}>0$, so $\log(A(\theta))$ is convex. Hence the non-zero entries of the matrix $\mathbf{B}(\theta)$ are superconvex.

   Now if $b_{ij}(\theta)=0$ for some θ, it is zero for all θ. Note that the matrix $\mathbf{B}(\theta)$ may have k eigenvalues with modulus equal to $\rho(\mathbf{B}(\theta))$. From Theorem A.2 part 11 there exists an integer $s>0$ such that

$$(\mathbf{B}(\theta))^s=\begin{pmatrix}\mathbf{A}_{11}(\theta) & \mathbf{0} & \dots & \mathbf{0}\\ \mathbf{0} & \mathbf{A}_{22}(\theta) & \dots & \mathbf{0}\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{A}_{kk}(\theta)\end{pmatrix},$$

   where each $\mathbf{A}_{ii}(\theta)$ is a positive matrix with eigenvalue $(\rho(\mathbf{B}(\theta)))^s$ and all other eigenvalues have modulus strictly less than $(\rho(\mathbf{B}(\theta)))^s$. Since the zeros of $\mathbf{B}(\theta)$, and hence of $(\mathbf{B}(\theta))^s$, occur in the same places for all θ, the structure of $(\mathbf{B}(\theta))^s$ will be the same for all θ, so that this result holds for all $\theta_1<\theta<\theta_2$.

   Now consider

$$\lim_{n\to\infty}\left[\frac{1}{k}trace\left((\mathbf{B}(\theta))^{sn}\right)\right]^{1/sn}.$$

From Lemma A.2 this function is superconvex. It is easily seen that the limit is just $\rho(\mathbf{B}(\theta))$. Hence $\rho(\mathbf{B}(\theta))$ is superconvex.

Now from part 1 and Theorem A.1, $\rho(\mathbf{B}(\theta))$ is analytic and is positive for θ real such that $\theta_1 < \theta < \theta_2$. Since it is superconvex,

$$\rho''(\mathbf{B}(\theta)) \geq (\rho'(\mathbf{B}(\theta)))^2/\rho(\mathbf{B}(\theta)) \geq 0.$$

This inequality is strict if $\rho'(\mathbf{B}(\theta)) \neq 0$. Since the Perron-Frobenius root is analytic, it is therefore strictly convex except in the degenerate case where $\rho(\mathbf{B}(\theta))$ is constant, which occurs when all the entries of $\mathbf{B}(\theta)$ do not depend on θ.

□

Appendix B. Non-negative solutions of a system of equations

In this section we prove theorems concerning the properties of the non-negative solutions of certain systems of equations. These theorems are used throughout the book and are of major importance both for the non-spatial analysis of Chapter 2 and for the spatial analysis in the ensuing chapters. They are also used in Chapter 8 to study the equilibrium solutions of the $S \to I \to S$ epidemic.

Consider the non-spatial analysis of Chapter 2. When $\rho(\mathbf{\Gamma})$ is finite, the final size equations are given by equation (2.19) i.e.

$$-\log(1-y_i) = \sum_{j=1}^{n} \gamma_{ij} y_j + a_i, \ (i = 1, ..., n). \tag{B.1}$$

The same system of equations arises in the spatial analysis.

Now consider the equilibrium equations for the $S \to I \to S$ epidemic of Chapter 8. These are given by equations (8.11). When the μ_i are non-zero, these may be rewritten in the form

$$\frac{y_i}{1-y_i} = \sum_{j=1}^{n} \frac{\sigma_j \lambda_{ij}}{\mu_i} y_j, \quad (i = 1, ..., n). \tag{B.2}$$

Equations (B.1) and (B.2) are both special cases of the system of equations

$$f(y_i) = \sum_{j=1}^{n} b_{ij} y_j + a_i, \ (i = 1, ..., n), \tag{B.3}$$

where $f(y)$ is a continuous, strictly monotone increasing function on $[0, 1)$, such that $f(0) = 0$, $f'(0) = 1$ and $f''(y) > 0$, (i.e. $f(y)$ is convex) and $\lim_{y\uparrow 1} f(y) = \infty$.

The intuitive discussion of the final size equations in Chapter 2 using figures 2.1, 2.2 and 2.3 illustrates the possible non-negative solutions for the cases $n = 1$ and 2. The results for general n are similar. These results for equation (B.3) are

now proved in two theorems. Let $z = f(y)$ and denote $f^{-1}(z)$ by $g(z)$. Then $g(z)$ is a strictly monotone increasing, concave function on $[0, \infty)$ with $g(0) = 0$, $g'(0) = 1$, $g''(z) < 0$ and $\lim_{z\to\infty} g(z) = 1$. These conditions on $g(z)$ hold for the two cases in which we are interested, i.e. $g(z) = 1 - e^{-z}$ and $g(z) = z/(1+z)$.

If we write $z_i = f(y_i)$, equations (B.3) may be rewritten in terms of g as

$$z_i = \sum_{j=1}^{n} b_{ij} g(z_j) + a_i, \ (i = 1, ..., n). \tag{B.4}$$

This is the more convenient form for establishing the results. Theorem B.1 proves existence, uniqueness, continuity and differentiability results for solutions to equations (B.4) when the matrix $\mathbf{B} = (b_{ij})$ is a non-negative, non-reducible, finite square matrix. These results are proved in Radcliffe and Rass [R4] Lemma 1 for the special function $g(z) = 1 - e^{-z}$. A similar result appears in Heathcote and Thieme [H2]. The existence part of the proof is related to Theorem 4.11 of Krasnosel'skii [K7]. The notation $\mathbf{x} \ngeq \mathbf{y}$ is used when $\{\mathbf{x}\}_i < \{\mathbf{y}\}_i$ for some i.

Corollary B.1 uses these results to prove equivalent results for the solutions to equations (B.3). Theorem B.2 then extends the results to the case when the matrix $\mathbf{B} = (b_{ij})$ may be reducible. Results for reducible matrices are required since principle minors of non-reducible matrices can be reducible. These arise when partitioning of the infection matrix is necessary. Theorem B.2 gives results for both the non-reducible and reducible case. It is the major theorem in this appendix and is used throughout the monograph.

THEOREM B.1 (NON-NEGATIVE SOLUTIONS OF A SYSTEM OF EQUATIONS INVOLVING A CONCAVE FUNCTION). *Let $\mathbf{z} = \mathbf{G}(\mathbf{a}, \mathbf{z})$ be an equation in $[0, \infty)^n$, where $\mathbf{G}(\mathbf{a}, \mathbf{z}) = \mathbf{a} + \mathbf{B}y$ with $\{\mathbf{y}\}_i = g(\{\mathbf{z}\}_i)$. Also let $g(z)$ be a strictly increasing, continuous function of z for $z \geq 0$, with $g(0) = 0$, $g'(0) = 1$, $\lim_{z\to\infty} g(z) = 1$ and $g''(z) < 0$ for $z \geq 0$. Here $\mathbf{B} = (b_{ij})$ is a non-negative, non-reducible square matrix and $\mathbf{a} = (a_i) \geq \mathbf{0}$.*

When $\mathbf{a} \neq \mathbf{0}$, then $\mathbf{z} = \mathbf{G}(\mathbf{a}, \mathbf{z})$ has a unique solution $\mathbf{z} = \mathbf{z}(\mathbf{a})$, which is positive. If $\mathbf{a} = \mathbf{0}$, then $\mathbf{z} = \mathbf{0}$ is a solution to $\mathbf{z} = \mathbf{G}(\mathbf{0}, \mathbf{z})$. No other solution is possible when $\rho(\mathbf{B}) \leq 1$, and we define $\mathbf{z}(\mathbf{0}) = \mathbf{0}$ in this case. When $\mathbf{a} \neq \mathbf{0}$ and $\rho(\mathbf{B}) > 1$, there exists a unique non-zero solution $\mathbf{z} = \mathbf{z}(\mathbf{0})$ to $\mathbf{z} = \mathbf{G}(\mathbf{0}, \mathbf{z})$ which is positive.

The solution $\mathbf{z}(\mathbf{a})$ is a continuous, increasing function of $\mathbf{a}$ for $\mathbf{a}$ in $[0, \infty)^n$. Also $\mathbf{z}(\mathbf{a})$ is twice differentiable in $[0, \infty)^n$ with

$$\frac{\partial\{\mathbf{z}(\mathbf{a})\}_i}{\partial\{\mathbf{a}\}_j} > 0 \quad \textit{and} \quad \frac{\partial^2\{\mathbf{z}(\mathbf{a})\}_i}{\partial\{\mathbf{a}\}_j \partial\{\mathbf{a}\}_k} \leq 0 \quad \textit{for all } i, j, k.$$

PROOF. Observe that $\{\mathbf{G}(\mathbf{a}, \mathbf{z})\}_i$ is strictly monotone increasing in a_i and in each $\{\mathbf{z}\}_j$ for which $b_{ij} \neq 0$. It is monotone increasing in $\mathbf{a}$ and $\mathbf{z}$.

First consider the existence of solutions to $\mathbf{z} = \mathbf{G}(\mathbf{a}, \mathbf{z})$. Observe that $\mathbf{z} = \mathbf{0}$ is a solution only if $\mathbf{a} = \mathbf{0}$.

When $\rho(\mathbf{B}) \leq 1$ and $\mathbf{a} = \mathbf{0}$, let $\mathbf{x}' > \mathbf{0}'$ be the left eigenvector of $\mathbf{B}$ corresponding to $\rho(\mathbf{B})$ and let $\mathbf{z}(\mathbf{0})$ be a solution to $\mathbf{z} = \mathbf{G}(\mathbf{0}, \mathbf{z})$. Then $\mathbf{x}'\mathbf{z}(\mathbf{0}) = \rho(\mathbf{B})\mathbf{x}'\mathbf{y}(\mathbf{0})$,

where $\{\mathbf{y}(\mathbf{0})\}_i = g(\{\mathbf{z}(\mathbf{0})\}_i)$, so that $\sum_{i=1}^{n} \{\mathbf{x}\}_i(\{\mathbf{z}(\mathbf{0})\}_i - \rho(\mathbf{B})g(\{\mathbf{z}(\mathbf{0})\}_i)) = 0$. Now $g(z) = g'(\zeta)z$ for some ζ such that $0 \leq \zeta \leq z$ and $g'(\zeta) \leq g'(0) = 1$ with equality only if $z = 0$. Thus $g(z) < z$ for $z > 0$, and hence $(z - \rho(\mathbf{B})g(z)) > 0$ for $z > 0$. So the ith term in the summation is positive unless $\{\mathbf{z}(\mathbf{0})\}_i = 0$. Hence $\mathbf{z} = \mathbf{z}(\mathbf{0}) = \mathbf{0}$ is the only solution to $\mathbf{z} = \mathbf{G}(\mathbf{0}, \mathbf{z})$.

Consider next the case $\rho(\mathbf{B}) \leq 1$ and $\mathbf{a} \neq \mathbf{0}$. (The construction of the solution in this paragraph also works when $\rho(\mathbf{B}) > 1$ and $\mathbf{a} \neq \mathbf{0}$) Define $\mathbf{u}^{(0)}(\mathbf{a}) = \mathbf{a}$ and $\mathbf{u}^{(N)}(\mathbf{a}) = \mathbf{G}(\mathbf{a}, \mathbf{u}^{(N-1)}(\mathbf{a}))$ for $N = 1, 2, \ldots$. Now $\mathbf{G}(\mathbf{a}, \mathbf{z})$ is an increasing function of $\mathbf{z}$. Hence $\mathbf{u}^{(1)}(\mathbf{a}) = \mathbf{G}(\mathbf{a}, \mathbf{a}) \geq \mathbf{G}(\mathbf{a}, \mathbf{0}) = \mathbf{a} = \mathbf{u}^{(0)}(\mathbf{a})$ and, using induction, $\mathbf{u}^{(N)}(\mathbf{a})$ is a monotone increasing sequence. The sequence is bounded above since $\mathbf{u}^{(N)}(\mathbf{a}) \leq \mathbf{a} + \mathbf{B}\mathbf{1}$ for $N \geq 0$. Hence it converges to a limit $\mathbf{z}(\mathbf{a})$ satisfying $\mathbf{z}(\mathbf{a}) = \mathbf{G}(\mathbf{a}, \mathbf{z}(\mathbf{a}))$. Now $\mathbf{a} \neq \mathbf{0}$. Hence for some j, $\{\mathbf{a}\}_j > 0$ and therefore $\{\mathbf{u}^{(0)}(\mathbf{a})\}_j > 0$. This then implies that $\{\mathbf{u}^{(1)}(\mathbf{a})\}_i > 0$ for all i such that $\{\mathbf{u}^{(0)}(\mathbf{a})\}_i > 0$ and/or $\{\mathbf{u}^{(0)}(\mathbf{a})\}_j > 0$ and $b_{ij} \neq 0$. Using the non-reducibility of $\mathbf{B}$ we obtain $\mathbf{u}^{(n-1)}(\mathbf{a}) > \mathbf{0}$ and hence $\mathbf{z}(\mathbf{a}) > \mathbf{0}$. Therefore $\mathbf{z} = \mathbf{z}(\mathbf{a})$ is a positive solution to $\mathbf{z} = \mathbf{G}(\mathbf{a}, \mathbf{z})$.

Suppose $\rho(\mathbf{B}) > 1$ and let $\mathbf{w} > \mathbf{0}$ be the right eigenvector of $\mathbf{B}$ corresponding to $\rho(\mathbf{B})$. Take $0 < \alpha < 1$ such that $\alpha\rho(\mathbf{B}) > 1$. Now $g(z) = g'(\zeta)z$ where $0 < \zeta < z$ and $g'(z) \uparrow 1$ as $z \downarrow 0$. So $g(z) > \alpha z$ for z sufficiently small. Hence $\exists\ \varepsilon$ such that $g(\varepsilon\{\mathbf{w}\}_j) \geq \alpha\varepsilon\{\mathbf{w}\}_j$ for $j = 1, \ldots, n$ Define $\mathbf{u}^{(0)}(\mathbf{a}) = \varepsilon\mathbf{w}$ and $\mathbf{u}^{(N)}(\mathbf{a}) = \mathbf{G}(\mathbf{a}, \mathbf{u}^{N-1)}(\mathbf{a}))$ for $N = 1, 2, \ldots$ Then $\mathbf{u}^{(1)}(\mathbf{a}) \geq \mathbf{G}(\mathbf{0}, \varepsilon\mathbf{w}) \geq \alpha\varepsilon\mathbf{B}\mathbf{w} = \alpha\rho(\mathbf{B})\varepsilon\mathbf{w} > \mathbf{u}^{(0)}(\mathbf{a}) > \mathbf{0}$. Using induction, $\mathbf{u}^{(N)}(\mathbf{a})$, for $N \geq 0$, is a monotone increasing sequence of positive vectors. This sequence is bounded above by $\mathbf{a} + \mathbf{B}\mathbf{1}$, so that it tends to a limit $\mathbf{z}(\mathbf{a}) > \mathbf{0}$ satisfying $\mathbf{z}(\mathbf{a}) = \mathbf{G}(\mathbf{a}, \mathbf{z}(\mathbf{a}))$. Then $\mathbf{z} = \mathbf{z}(\mathbf{a})$ is a positive solution to $\mathbf{z} = \mathbf{G}(\mathbf{a}, \mathbf{z})$. Note that $\mathbf{z} = \mathbf{0}$ is only also a solution if $\mathbf{a} = \mathbf{0}$.

We now show that if $\mathbf{a} \neq \mathbf{0}$ and/or $\rho(\mathbf{B}) > 1$, then $\mathbf{z}(\mathbf{a})$ is the unique non-zero solution of $\mathbf{z} = \mathbf{G}(\mathbf{a}, \mathbf{z})$. Note that if $\mathbf{a} = \mathbf{0}$ and $\rho(\mathbf{B}) \leq 1$ we have already established that no non-zero solution is possible. In addition, using the non-reducibility of $\mathbf{B}$, the only solution of $\mathbf{z} = \mathbf{G}(\mathbf{a}, \mathbf{z})$ with $\{\mathbf{z}\}_i = 0$ for some i is $\mathbf{z} = \mathbf{0}$ (and this is only possible if $\mathbf{a} = \mathbf{0}$). Assume therefore that two distinct positive solutions $\mathbf{z}_1$ and $\mathbf{z}_2$ to $\mathbf{z} = \mathbf{G}(\mathbf{a}, \mathbf{z})$ are possible for some $\mathbf{a} \neq \mathbf{0}$ and/or $\rho(\mathbf{B}) > 1$. Without loss of generality we may assume that $\mathbf{z}_1 \not\geq \mathbf{z}_2$. Define $t_0 = \min_i(\{\mathbf{z}_1\}_i/\{\mathbf{z}_2\}_i)$. Then $0 < t_0 < 1$, $\{\mathbf{z}_1\}_i = t_0\{\mathbf{z}_2\}_i$ for some i and $\mathbf{z}_1 \geq t_0\mathbf{z}_2$. Now $g(z)$ is strictly concave, $g(0) = 0$ and $0 < t_0 < 1$. Hence $g(t_0 z) > t_0 g(z) + (1 - t_0)g(0) = t_0 g(z)$. Using this result together with the monotonicity of $\mathbf{G}(\mathbf{a}, \mathbf{z})$ we obtain $\mathbf{z}_1 = \mathbf{G}(\mathbf{a}, \mathbf{z}_1) \geq \mathbf{G}(\mathbf{a}, t_0\mathbf{z}_2) > t_0\mathbf{G}(\mathbf{a}, \mathbf{z}_2) = t_0\mathbf{z}_2$. Hence $\mathbf{z}_1 > t_0\mathbf{z}_2$, which contradicts the definition of t_0. The uniqueness of the non-zero solution is thus established.

It is simple to establish that $\mathbf{z}(\mathbf{a})$ is monotone increasing in $\mathbf{a}$, using the construction of $\mathbf{z}(\mathbf{a})$ from the existence proofs.

If $\mathbf{a} = \mathbf{0}$, $\mathbf{a}^* \neq \mathbf{0}$ and $\rho(\mathbf{B}) \leq 1$, then $\mathbf{z}(\mathbf{a}^*) > \mathbf{0} = \mathbf{z}(\mathbf{0})$. If $\mathbf{a}^* \geq \mathbf{a}$ with $\mathbf{a} \neq \mathbf{0}$ and $\rho(\mathbf{B}) \leq 1$ and if $\mathbf{u}^{(0)}(\mathbf{a}) = \mathbf{a}$, $\mathbf{u}^{(0)}(\mathbf{a}^*) = \mathbf{a}^*$, $\mathbf{u}^{(N)}(\mathbf{a}) = \mathbf{G}(\mathbf{a}, \mathbf{u}^{(N-1)}(\mathbf{a}))$ and $\mathbf{u}^{(N)}(\mathbf{a}^*) = \mathbf{G}(\mathbf{a}^*, \mathbf{u}^{(N-1)}(\mathbf{a}^*))$ for $N = 1, 2, \ldots$, then $\mathbf{u}^{(0)}(\mathbf{a}^*) \geq \mathbf{u}^{(0)}(\mathbf{a})$ and inductively, using the monotonicity of $\mathbf{G}(\mathbf{a}, \mathbf{z})$, $\mathbf{u}^{(N)}(\mathbf{a}^*) \geq \mathbf{u}^{(N)}(\mathbf{a})$ for $N \geq 0$. Hence $\mathbf{z}(\mathbf{a}^*) \geq \mathbf{z}(\mathbf{a})$.

Now consider the case $\mathbf{a}^* \geq \mathbf{a}$ and $\rho(\mathbf{B}) > 1$, and let $\mathbf{u}^{(0)}(\mathbf{a}) = \mathbf{u}^{(0)}(\mathbf{a}^*) = \varepsilon\mathbf{w}$, where ε and $\mathbf{w}$ are defined as in the construction. Also define $\mathbf{u}^{(N)}(\mathbf{a}) = \mathbf{G}(\mathbf{a}, \mathbf{u}^{N-1)}(\mathbf{a}))$ and $\mathbf{u}^{(N)}(\mathbf{a}^*) = \mathbf{G}(\mathbf{a}^*, \mathbf{u}^{N-1)}(\mathbf{a}^*))$ for $N \geq 1$. It immediately follows that $\mathbf{u}^{(N)}(\mathbf{a}^*) \geq \mathbf{u}^{(N)}(\mathbf{a})$ for $N \geq 0$. Hence $\mathbf{z}(\mathbf{a}^*) \geq \mathbf{z}(\mathbf{a})$.

The continuity of $\mathbf{z}(\mathbf{a})$ may be established by contradiction. Now $\mathbf{0} < \mathbf{z}(\mathbf{a}) \leq \mathbf{a} + \mathbf{B1}$, so that $\mathbf{z}(\mathbf{a})$ is bounded for $\mathbf{a}$ in a bounded region of $[0, \infty)^n$. Suppose that $\mathbf{a}^* \neq \mathbf{0}$, then if $\mathbf{z}(\mathbf{a})$ is not continuous at $\mathbf{a} = \mathbf{a}^*$, there exists an $\epsilon > 0$ and a sequence $\{\mathbf{a}^{(N)}\}$ in a bounded region of $[0, \infty)^n$ with $\mathbf{a}^{(N)} \to \mathbf{a}^*$ as $N \to \infty$ and each $\mathbf{z}(\mathbf{a}^{(N)})$ outside a ball of radius ϵ about $\mathbf{z}(\mathbf{a}^*)$. Hence there exists a subsequence of $\{\mathbf{a}^{(N)}\}$, (which we also label $\{\mathbf{a}^{(N)}\}$) which also converges to $\mathbf{a}^*$ and $\mathbf{z}(\mathbf{a}^{(N)}) \to \mathbf{z}$ as $N \to \infty$ with $\mathbf{z} \neq \mathbf{z}(\mathbf{a}^*)$. But $\mathbf{z} = \mathbf{G}(\mathbf{a}^*, \mathbf{z})$ and hence, from uniqueness, $\mathbf{z} = \mathbf{z}(\mathbf{a}^*)$ and a contradiction is obtained.

When $\mathbf{a}^* = \mathbf{0}$ we proceed as above, but observe in addition that $\mathbf{z}(\mathbf{a}^{(N)}) \geq \mathbf{z}(\mathbf{0})$. Hence $\mathbf{z} \geq \mathbf{z}(\mathbf{0})$ and $\mathbf{z} \neq \mathbf{z}(\mathbf{0})$. Since $\mathbf{z} = \mathbf{G}(\mathbf{0}, \mathbf{z})$ a contradiction is again obtained.

Finally we consider the differentiability of $\mathbf{z}(\mathbf{a})$. Suppose that $a_j > 0$. Let $\boldsymbol{\delta} = \delta a_j \mathbf{e}_j$ where $\delta a_j > 0$ and $\mathbf{e}_j$ is a vector with j^{th} element 1 and all other elements 0. Then for $\mathbf{a}$ (and hence $\mathbf{a}+\delta$) in $[0, \infty)^n$, $\mathbf{z}(\mathbf{a}+\boldsymbol{\delta}) - \mathbf{z}(\mathbf{a}) = \boldsymbol{\delta} + \mathbf{BA}^*(\mathbf{z}(\mathbf{a}+\boldsymbol{\delta}) - \mathbf{z}(\mathbf{a}))$, where $\mathbf{A}^*$ is a diagonal matrix with $\{\mathbf{A}^*\}_{mm} = g'(x_m)$ for some x_m in the interval $(\{\mathbf{z}(\mathbf{a})\}_m, \{\mathbf{z}(\mathbf{a}) + \boldsymbol{\delta}\}_m)$. Now

$$\mathbf{z}(\mathbf{a}) = \mathbf{G}(\mathbf{a}, \mathbf{z}(\mathbf{a})) \geq \mathbf{G}(\mathbf{0}, \mathbf{z}(\mathbf{a})) = \mathbf{BAz}(\mathbf{a}), \tag{B.5}$$

where $\mathbf{A}$ is a diagonal matrix with $\{\mathbf{A}\}_{mm} = g'(\zeta_m)$ for some $\zeta_m \in (0, \{\mathbf{z}(\mathbf{a})\}_i)$. From Theorem A.2 part 1 and Lemma A.1, if $\mathbf{B}^*$ is a non-negative, non-reducible matrix, then $\rho(\mathbf{B}^*)$ is a continuous strictly increasing function of the elements of $\mathbf{B}^*$. From this result and the fact that $g(z)$ is a strictly decreasing function of z, it follows that $\rho(\mathbf{BA}) > \rho(\mathbf{BA}^*)$. Let $\mathbf{w}'$ be the left eigenvector of $\mathbf{BA}$ corresponding to $\rho(\mathbf{BA})$. Then $\mathbf{w}'\mathbf{z}(\mathbf{a}) \geq \rho(\mathbf{BA})\mathbf{w}'\mathbf{z}(\mathbf{a})$. Therefore $\rho(\mathbf{BA}) \leq 1$, and hence $\rho(\mathbf{BA}^*) < 1$. Therefore, from Theorem A.2 part 4, $(\mathbf{I} - \mathbf{BA}^*)^{-1}$ exists and is positive for $\delta a_j \geq 0$. Therefore $\mathbf{z}(\mathbf{a} + \boldsymbol{\delta}) - \mathbf{z}(\mathbf{a}) = (\mathbf{I} - \mathbf{BA}^*)^{-1}\boldsymbol{\delta}$. Hence

$$\lim_{\delta a_j \downarrow 0} \frac{(\mathbf{z}(\mathbf{a} + \boldsymbol{\delta}) - \mathbf{z}(\mathbf{a}))}{\delta a_j} = \lim_{\boldsymbol{\delta} \downarrow \mathbf{0}} (\mathbf{I} - \mathbf{BA}^*)^{-1}\mathbf{e}_j = (\mathbf{I} - \mathbf{BC})^{-1}\mathbf{e}_j,$$

where $\mathbf{C}$ is a diagonal matrix with $\{\mathbf{C}\}_{mm} = g'(\{\mathbf{z}(\mathbf{a})\}_m)$.

Note that if $\delta a_j < 0$ we can prove the same result by using, in place of equation (B.5), the following:

$$\mathbf{z}(\mathbf{a} + \boldsymbol{\delta}) = \mathbf{G}(\mathbf{a} + \boldsymbol{\delta}, \mathbf{z}(\mathbf{a} + \boldsymbol{\delta})) \geq \mathbf{G}(\mathbf{0}, \mathbf{z}(\mathbf{a} + \boldsymbol{\delta})) = \mathbf{BAz}(\mathbf{a} + \boldsymbol{\delta}),$$

where $\mathbf{A}$ is a diagonal matrix with $\{\mathbf{A}\}_{mm} = g'(\zeta_m)$ for some $\zeta_m \in (0, \{\mathbf{z}(\mathbf{a} + \boldsymbol{\delta})\}_m)$.

Together these prove the existence of $\dfrac{\partial \mathbf{z}(\mathbf{a})}{\partial \{\mathbf{a}\}_j}$ and give the result that

$$\frac{\partial \mathbf{z}(\mathbf{a})}{\partial \{\mathbf{a}\}_j} = (\mathbf{I} - \mathbf{BC})^{-1}\mathbf{e}_j > \mathbf{0} \text{ for } \{\mathbf{a}\}_j > 0$$

Note that for $\{\mathbf{a}\}_j = 0$ we only prove differentiability from the right.

It then follows, by differentiating the identity $(\mathbf{I} - \mathbf{BC})^{-1}(\mathbf{I} - \mathbf{BC}) = \mathbf{I}$, that

$$\frac{\partial^2 \mathbf{z}(\mathbf{a})}{\partial\{\mathbf{a}\}_j \partial\{\mathbf{a}\}_k} = (\mathbf{I}-\mathbf{BC})^{-1}\mathbf{BD}(\mathbf{I}-\mathbf{BC})^{-1}\mathbf{e}_j,$$

where $\mathbf{D}$ is a diagonal matrix with $\{\mathbf{D}\}_{mm} = g''(\{\mathbf{z}(\mathbf{a})\}_m)\dfrac{\partial\{\mathbf{z}(\mathbf{a})\}_m}{\partial\{\mathbf{a}\}_k}$. Observe that $g''(\{\mathbf{z}(\mathbf{a})\}_m) < 0$, and $\dfrac{\partial\{\mathbf{z}(\mathbf{a})\}_m}{\partial\{\mathbf{a}\}_k} > 0$ and we have shown that $(\mathbf{I}-\mathbf{BC})^{-1}$ has positive elements. Hence $\dfrac{\partial^2\{\mathbf{z}(\mathbf{a})\}_i}{\partial\{\mathbf{a}\}_j\partial\{\mathbf{a}\}_k} \leq 0$ for all i, j, k.

□

COROLLARY B.1. *Let $\mathbf{B} = (b_{ij})$ be a non-negative $n \times n$ non-reducible matrix and let $a_i \geq 0$ for $i = 1, ..., n$. Take $f(y)$ to be a continuous function of y on $[0, 1)$ such that $f(0) = 0$, $f'(0) = 1$ and $f''(y) > 0$ and $\lim_{y\uparrow 1} f(y) = \infty$.*

Consider the possible solutions of the system of equations

$$f(y_i) = \sum_{j=1}^{n} b_{ij}y_j + a_i, \tag{B.6}$$

for $i = 1, ..., n$. Denote the vector with i^{th} entry y_i by $\mathbf{y}$.

1. *When $\mathbf{a} \neq \mathbf{0}$, then there is a unique solution $\mathbf{y} = \boldsymbol{\eta}(\mathbf{B}, \mathbf{a})$, which is positive.*
2. *If $\mathbf{a} = \mathbf{0}$, then $\mathbf{y} = \mathbf{0}$ is a solution to equations (B.6). No other solution is possible when $\rho(\mathbf{B}) \leq 1$, and we define $\boldsymbol{\eta}(\mathbf{B}, \mathbf{0}) = \mathbf{0}$ in this case. When $\mathbf{a} = \mathbf{0}$ and $\rho(\mathbf{B}) > 1$, there exists a unique non-zero solution $\mathbf{y} = \boldsymbol{\eta}(\mathbf{B}, \mathbf{0})$ to equations (B.6) which is positive.*
3. *The solution $\boldsymbol{\eta}(\mathbf{B}, \mathbf{a})$ is a continuous, increasing function of $\mathbf{a}$ for $\mathbf{a}$ in $[0, 1)^n$. Also it is twice differentiable in $[0, 1)^n$ with*

$$\frac{\partial\{\boldsymbol{\eta}(\mathbf{B},\mathbf{a})\}_i}{\partial\{\mathbf{a}\}_j} > 0 \quad \textit{and} \quad \frac{\partial^2\{\boldsymbol{\eta}(\mathbf{B},\mathbf{a})\}_i}{\partial\{\mathbf{a}\}_j\partial\{\mathbf{a}\}_k} \leq 0 \quad \textit{for all } i, j, k.$$

PROOF. Define $g = f^{-1}$, where the domain of g is taken to be $[0, \infty)$. Then from the conditions on f, it is easily seen that $g(z)$ is a strictly increasing function of z for $z \geq 0$, with $g(0) = 0$, $g'(0) = 1$, $\lim_{z\to\infty} g(z) = 1$ and $g''(z) < 0$ for $z \geq 0$.

Then g satisfies the conditions of Theorem B.1 and a solution $\mathbf{y}$ to equations (B.6) for $\mathbf{y} \in [0, 1)^n$ corresponds to a solution $\mathbf{z} \in [0, \infty)^n$, where $\{\mathbf{z}\}_i = f(\{\mathbf{y}\}_i)$ and $\{\mathbf{y}\}_i = g(\{\mathbf{z}\}_i)$. Results 1 and 2 then follow immediately from Theorem B.1. The result that $\boldsymbol{\eta}(\mathbf{B}, \mathbf{a})$ is a continuous, increasing, twice differentiable function of $\mathbf{a}$ is also immediate. It only remains to show that the first derivative with respect to any entry of $\mathbf{a}$ is positive and all the second derivatives are non-positive.

Now

$$\frac{\partial\{\boldsymbol{\eta}(\mathbf{B},\mathbf{a})\}_i}{\partial\{\mathbf{a}\}_j} = \frac{\partial g(\{\mathbf{z}(\mathbf{a})\}_i)}{\partial\{\mathbf{a}\}_j} = g'(\{\mathbf{z}(\mathbf{a})\}_i)\frac{\partial\{\mathbf{z}(\mathbf{a})\}_i}{\partial\{\mathbf{a}\}_j}.$$

This is positive since $g(z)$ is a strictly increasing function of z and, from Theorem B.1, the derivative of $\mathbf{z}(\mathbf{a})$ with respect to any entry of $\mathbf{a}$ is positive. Also

$$\frac{\partial^2\{\boldsymbol{\eta}(\mathbf{B},\mathbf{a})\}_i}{\partial\{\mathbf{a}\}_j\partial\{\mathbf{a}\}_k} = g''(\{\mathbf{z}(\mathbf{a})\}_i)\frac{\partial\{\mathbf{z}(\mathbf{a})\}_i}{\partial\{\mathbf{a}\}_j}\frac{\partial\{\mathbf{z}(\mathbf{a})\}_i}{\partial\{\mathbf{a}\}_k} + g'(\{\mathbf{z}(\mathbf{a})\}_i)\frac{\partial^2\{\mathbf{z}(\mathbf{a})\}_i}{\partial\{\mathbf{a}\}_j\partial\{\mathbf{a}\}_k}.$$

Now $g'(z) > 0$ and $g''(z) < 0$. Also, from Theorem B.1, the first derivatives of $\mathbf{z}(\mathbf{a})$ with respect to the entries of $\mathbf{a}$ are positive and the second derivatives are non-positive. Hence the second derivatives of $\boldsymbol{\eta}(\mathbf{B},\mathbf{a})$ with respect to the entries of $\mathbf{a}$ are non-negative. This completes the proof of the corollary.

□

Although we have restricted attention to models where the infection matrix is non-reducible, it is necessary in some places to partition this matrix, which can result in a reducible submatrix. Theorem B.2 therefore not only gives solutions when the matrix $\mathbf{B}$ is non-reducible, but also gives solutions of a particular form in the reducible case. There are other solutions possible when $\mathbf{B}$ is reducible which are irrelevant to the mathematical analysis considered in this monograph. A different solution to the one specified in Theorem B.2 part 3 may be found by taking $\mathbf{y}_i = \mathbf{0}$ for at least one i satisfying $1 \le i \le s$, $\mathbf{a}_i = \mathbf{0}$ and $\rho(\mathbf{B}_{ii}) > 1$. In general there will a multiplicity of such solutions. The solution specified in part 3 of Theorem B.2 is precisely the one required to give the limiting results in part 4 of that theorem.

THEOREM B.2. *Let $\mathbf{B} = (b_{ij})$ be a non-negative $n \times n$ matrix and let $a_i \ge 0$ for all $i = 1, ..., n$. Let $f(y)$ be a continuous function of y on $[0, 1)$ such that $f(0) = 0$, $f'(0) = 1$ and $f''(y) > 0$ and $\lim_{y\uparrow 1} f(y) = \infty$.*

Consider the possible solutions of the system of equations

$$f(y_i) = \sum_{j=1}^{n} b_{ij}y_j + a_i, \tag{B.6}$$

for $i = 1, ..., n$, where $0 \le y_i < 1$.

The matrix $\mathbf{B}$ is written in normal form (Gantmacher [G1] p.75) and the n dimensional vectors $\mathbf{a}$ and $\mathbf{y}$, with $\{\mathbf{a}\}_i = a_i$ and $\{\mathbf{y}\}_i = y_i$, are partitioned so that

$$\mathbf{B} = \begin{pmatrix} \mathbf{B}_{11} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_{22} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{B}_{ss} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{B}_{s+1,1} & \mathbf{B}_{s+1,2} & \dots & \mathbf{B}_{s+1,s} & \mathbf{B}_{s+1,s+1} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{B}_{g1} & \mathbf{B}_{g2} & \dots & \mathbf{B}_{gs} & \mathbf{B}_{g,s+1} & \dots & \mathbf{B}_{gg} \end{pmatrix},$$

$$\mathbf{a} = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_g \end{pmatrix} \quad \textit{and} \quad \mathbf{y} = \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \vdots \\ \mathbf{y}_g \end{pmatrix}.$$

Here $\mathbf{B}_{ii}$ *is a non-reducible square matrix of order* r_i *and* $\mathbf{a}_i$ *and* $\mathbf{y}_i$ *are* r_i *dimensional vectors for* $i = 1, ..., g$. *In addition, if* $s < g$, *at least one* $\mathbf{B}_{i1}, ..., \mathbf{B}_{i,i-1}$ *is non-zero for each* i *such that* $s + 1 \leq i \leq g$.

1. *If* $\mathbf{a}_i \neq \mathbf{0}$ *for all* $i = 1, ..., s$, *then equations (B.6) have a unique solution* $\mathbf{y} = \boldsymbol{\eta}(\mathbf{B}, \mathbf{a}) > \mathbf{0}$.
2. *When* $\mathbf{B}$ *is non-reducible (i.e.* $s = g = 1$*) and* $\mathbf{a} = \mathbf{0}$, *then equations (B.6) admit the trivial solution* $\mathbf{y} = \mathbf{0}$. *If* $\rho(\mathbf{B}) > 1$ *there exists a unique non-trivial solution* $\mathbf{y} = \boldsymbol{\eta}(\mathbf{B}, \mathbf{0}) > \mathbf{0}$. *When* $\rho(\mathbf{B}) \leq 1$ *no non-trivial solution exists. In this case we define* $\boldsymbol{\eta}(\mathbf{B}, \mathbf{0}) = \mathbf{0}$.
3. *When* $\mathbf{B}$ *is reducible with at least one* $\mathbf{a}_i = \mathbf{0}$ *for* $i = 1, ..., s$, *there exists a solution* $\mathbf{y}$ *to equations (B.6) of a particular form. For each* $i = 1, ..., s$, *this form has* $\mathbf{y}_i > \mathbf{0}$ *if* $\rho(\mathbf{B}_{ii}) > 1$ *and/or* $\mathbf{a}_i \neq \mathbf{0}$, *and* $\mathbf{y}_i = \mathbf{0}$ *otherwise. Then successively (if* $s < g$*) for* $i = s + 1, ..., g$, *it has* $\mathbf{y}_i > \mathbf{0}$ *if* $\rho(\mathbf{B}_{ii}) > 1$ *and/or* $\sum_{j<i} \mathbf{B}_{ij}\mathbf{y}_j + \mathbf{a}_i \neq \mathbf{0}$. *Again* $\mathbf{y}_i = \mathbf{0}$ *otherwise.*

   The solution is the unique solution of this form. We denote it by $\mathbf{y} = \boldsymbol{\eta}(\mathbf{B}, \mathbf{a})$, *and partition it so that*

$$\boldsymbol{\eta}(\mathbf{B}, \mathbf{a}) = \begin{pmatrix} \boldsymbol{\eta}_1(\mathbf{B}, \mathbf{a}) \\ \boldsymbol{\eta}_2(\mathbf{B}, \mathbf{a}) \\ \vdots \\ \boldsymbol{\eta}_g(\mathbf{B}, \mathbf{a}) \end{pmatrix}.$$

   The components of the solution are specified in terms of solutions based on a non-reducible matrix as follows. For each $i = 1, ..., s$, $\boldsymbol{\eta}_i(\mathbf{B}, \mathbf{a}) = \boldsymbol{\eta}(\mathbf{B}_{ii}, \mathbf{a}_i)$. *Then, successively for each* $i = s + 1, ..., g$, $\boldsymbol{\eta}_i(\mathbf{B}, \mathbf{a}) = \boldsymbol{\eta}(\mathbf{B}_{ii}, \mathbf{b}_i)$, *where* $\mathbf{b}_i = \sum_{j<i} \mathbf{B}_{ij}\boldsymbol{\eta}_j(\mathbf{B}, \mathbf{a}) + \mathbf{a}_i$.

   Note that if $\mathbf{a} = \mathbf{0}$ *and* $\rho(\mathbf{B}) \leq 1$ *only the trivial solution is possible. In this case* $\boldsymbol{\eta}(\mathbf{B}, \mathbf{0}) = \mathbf{0}$.
4. *In all cases, if* $\mathbf{b} \geq \mathbf{a} \geq \mathbf{0}$, *then* $\boldsymbol{\eta}(\mathbf{B}, \mathbf{b}) \geq \boldsymbol{\eta}(\mathbf{B}, \mathbf{a})$. *Also, for any* $\mathbf{a} \geq \mathbf{0}$,

$$\lim_{\mathbf{b} \downarrow \mathbf{a}} \boldsymbol{\eta}(\mathbf{B}, \mathbf{b}) = \boldsymbol{\eta}(\mathbf{B}, \mathbf{a}).$$

PROOF. Partition equations (B.6) to correspond to the partitioning of $\mathbf{B}$ and let $\{\mathbf{z}_i\}_j = f(\{\mathbf{y}_i\}_j)$. Then equations (B.6) become

$$\mathbf{z}_i = \begin{cases} \mathbf{B}_{ii}\mathbf{y}_i + \mathbf{a}_i, & \text{for } i = 1, ..., s, \\ \mathbf{B}_{ii}\mathbf{y}_i + \mathbf{a}_i + \sum_{j<i} \mathbf{B}_{ij}\mathbf{y}_j, & \text{for } i = s + 1, ..., g. \end{cases} \tag{B.7}$$

1. Let $\mathbf{y} = \boldsymbol{\eta}(\mathbf{B}, \mathbf{a})$ be partitioned as in (B.7) so that $\mathbf{y}_i = \boldsymbol{\eta}_i(\mathbf{B}, \mathbf{a})$. Then we show that the required solution is

$$\boldsymbol{\eta}_i(\mathbf{B}, \mathbf{a}) = \boldsymbol{\eta}(\mathbf{B}_{ii}, \mathbf{c}_i),$$

   where $\mathbf{c}_i = \mathbf{a}_i$ for $i = 1, ..., s$, and then $\mathbf{c}_i$ is defined successively for each $i = s + 1, ..., g$ by

$$\mathbf{c}_i = \mathbf{a}_i + \sum_{j<i} \mathbf{B}_{ij}\boldsymbol{\eta}(\mathbf{B}_{jj}, \mathbf{c}_j).$$

In addition the solution is shown to be strictly positive.

It is shown, using induction on k, that $\mathbf{y}_i = \boldsymbol{\eta}(\mathbf{B}_{ii}, \mathbf{c}_i) > \mathbf{0}$ is the unique solution to the subset $i = 1, ..., k$ of equations (B.7). Taking $k = g$ then gives the required result.

Consider any $k \leq s$. For each $i = 1, ..., k$ the corresponding equation from (B.7) involves $\mathbf{y}_i$ only. The result therefore follows immediately from Corollary B.1. Now assume that the result holds for all $1 \leq k \leq K$ where $s \leq K < g$. Consider equation $i = K + 1$ from (B.7). Since the solution to the first K equations of (B.7) is unique and is positive, the solution $\mathbf{y}_{K+1}$ must satisfy the equation

$$\begin{aligned} \mathbf{z}_{K+1} &= \mathbf{B}_{(K+1),(K+1)}\mathbf{y}_{K+1} + \mathbf{a}_{K+1} + \sum_{j<(K+1)} \mathbf{B}_{(K+1),j}\boldsymbol{\eta}_j(\mathbf{B}_{jj}, \mathbf{c}_j) \\ &= \mathbf{B}_{(K+1),(K+1)}\mathbf{y}_{K+1} + \mathbf{c}_{K+1}. \end{aligned} \tag{B.8}$$

Now the normal form requires the existence of a $j < K + 1$ for which $\mathbf{B}_{(K+1),j} \gneq \mathbf{0}$. Also $\boldsymbol{\eta}(\mathbf{B}_{jj}, \mathbf{c}_j) > \mathbf{0}$. Hence $\mathbf{c}_{K+1} \gneq \mathbf{0}$. Then from Corollary B.1 there is a unique solution $\mathbf{y}_{K+1} = \boldsymbol{\eta}(\mathbf{B}_{(K+1),(K+1)}, \mathbf{c}_{K+1})$ to (B.8) which is positive. Therefore $\mathbf{y}_i = \boldsymbol{\eta}(\mathbf{B}_{ii}, \mathbf{c}_i) > \mathbf{0}$, for $i = 1, ..., K + 1$, is the unique solution to the first $K + 1$ of equations (B.7). By induction the result then holds for all $i = 1, ..., g$.

2. Since $\mathbf{B}$ is non-reducible, the result follows immediately from Corollary B.1.
3. As in part 1 of this proof, we show that the required solution is

$$\mathbf{y}_i = \boldsymbol{\eta}_i(\mathbf{B}, \mathbf{a}) = \boldsymbol{\eta}(\mathbf{B}_{ii}, \mathbf{c}_i),$$

where $\mathbf{c}_i$ is defined as in that proof. The proof follows in almost identical fashion. However the solution $\mathbf{y}_i = \boldsymbol{\eta}(\mathbf{B}_{ii}, \mathbf{c}_i)$ is no longer always positive.

Consider a solution to the first k of equations (B.7). These equations only involve $\mathbf{y}_i$ for $i = 1, ..., k$. When $1 \leq k \leq s$, each equation involves a single $\mathbf{y}_i$ only. From Corollary B.1 if $\mathbf{c}_i = \mathbf{a}_i$ is non-zero then the solution to equation i of (B.7), $\mathbf{y}_i = \boldsymbol{\eta}(\mathbf{B}_{ii}, \mathbf{c}_i)$, is unique and is strictly positive. When $\mathbf{a}_i = \mathbf{0}$, then $\mathbf{y}_i = \mathbf{0}$ is a solution. Another solution is only possible when $\rho(\mathbf{B}_{ii}) > 1$. In this case there is a unique non-zero solution $\mathbf{y}_i = \boldsymbol{\eta}(\mathbf{B}_{ii}, \mathbf{c}_i) > \mathbf{0}$. When $\rho(\mathbf{B}_{ii}) \leq 1$ then $\boldsymbol{\eta}(\mathbf{B}_{ii}, \mathbf{c}_i) = \mathbf{0}$. Here $\mathbf{c}_i = \mathbf{a}_i$. Hence for each $i = 1, ..., k$ there is a unique solution to equation i of (B.7) which has $\mathbf{y}_i > \mathbf{0}$ if $\rho(\mathbf{B}_{ii}) > 1$ and $\mathbf{y}_i = \mathbf{0}$ otherwise, namely $\mathbf{y}_i = \boldsymbol{\eta}(\mathbf{B}_{ii}, \mathbf{c}_i)$.

Now assume that, for all $1 \leq k \leq K$ with $s \leq K < g$ that the following result holds. For such a k, $\mathbf{y}_i = \boldsymbol{\eta}(\mathbf{B}_{ii}, \mathbf{c}_i)$ for $i = 1, ..., k$ is the unique solution of the subset of the first k of equations (B.7) which has $\mathbf{y}_i > \mathbf{0}$ if $\rho(\mathbf{B}_{ii}) > 1$ and/or $\sum_{j<i} \mathbf{B}_{ij}\mathbf{y}_j + \mathbf{a}_i \neq \mathbf{0}$ and $\mathbf{y}_i = \mathbf{0}$ otherwise.

Now consider equations $i = 1, ..., K + 1$ of (B.7). To be a solution of the required form, from the inductive hypothesis with $k = K$, necessarily $\mathbf{y}_i = \boldsymbol{\eta}(\mathbf{B}_{ii}, \mathbf{c}_i)$ for $i = 1, ..., K$, and $\mathbf{y}_{K+1}$ satisfies equation (B.8). This equation has a unique positive solution $\mathbf{y}_{K+1} = \boldsymbol{\eta}(\mathbf{B}_{(K+1),(K+1)}, \mathbf{c}_{(K+1),(K+1)}) > \mathbf{0}$ if $\rho(\mathbf{B}_{(K+1),(K+1)}) > 1$ and/or $\mathbf{c}_{K+1} = \sum_{j<K+1} \mathbf{B}_{(K+1),j}\mathbf{y}_j + \mathbf{a}_{(K+1)} \neq \mathbf{0}$.

Otherwise $\mathbf{y}_{K+1} = \boldsymbol{\eta}(\mathbf{B}_{(K+1),(K+1)}, \mathbf{c}_{(K+1),(K+1)}) = \mathbf{0}$. The result then holds for $k = K+1$ and hence for all $1 \leq k \leq g$.

The result is then immediate.

4. The solutions may be written in the form

$$\boldsymbol{\eta}_i(\mathbf{B}, \mathbf{a}) = \boldsymbol{\eta}(\mathbf{B}_{ii}, \mathbf{c}_i) \text{ and } \boldsymbol{\eta}_i(\mathbf{B}, \mathbf{b}) = \boldsymbol{\eta}(\mathbf{B}_{ii}, \mathbf{d}_i),$$

where $\mathbf{c}_i = \mathbf{a}_i$ and $\mathbf{d}_i = \mathbf{b}_i$ for $i \leq s$. When $s < i \leq g$, then

$$\mathbf{c}_i = \mathbf{a}_i + \sum_{j<i} \mathbf{B}_{ij}\boldsymbol{\eta}(\mathbf{B}_{jj}, \mathbf{c}_j) \text{ and } \mathbf{d}_i = \mathbf{b}_i + \sum_{j<i} \mathbf{B}_{ij}\boldsymbol{\eta}(\mathbf{B}_{jj}, \mathbf{d}_j). \tag{B.9}$$

The result may easily be shown successively for each $i = 1, ..., g$. Consider any $i \leq s$. Then $\mathbf{d}_i = \mathbf{b}_i \geq \mathbf{a}_i = \mathbf{c}_i$. Hence $\mathbf{d}_i \downarrow \mathbf{c}_i$ as $\mathbf{b} \downarrow \mathbf{a}$. It follows immediately from Corollary B.1 that $\boldsymbol{\eta}(\mathbf{B}_{ii}, \mathbf{d}_i) \geq \boldsymbol{\eta}(\mathbf{B}_{ii}, \mathbf{c}_i)$ and that

$$\lim_{\mathbf{d}_i \downarrow \mathbf{c}_i} \boldsymbol{\eta}(\mathbf{B}_{ii}, \mathbf{d}_i) = \boldsymbol{\eta}(\mathbf{B}_{ii}, \mathbf{c}_i).$$

Therefore for each $i = 1, ..., s$, $\boldsymbol{\eta}_i(\mathbf{B}, \mathbf{b}) \geq \boldsymbol{\eta}_i(\mathbf{B}, \mathbf{a})$ and

$$\lim_{\mathbf{b} \downarrow \mathbf{a}} \boldsymbol{\eta}_i(\mathbf{B}, \mathbf{b}) = \boldsymbol{\eta}_i(\mathbf{B}, \mathbf{a}).$$

Now assume that these results hold for all i such that $1 \leq i \leq K < g$, where $K \geq s$. Consider $i = K+1$ where $\mathbf{c}_{K+1}$ and $\mathbf{d}_{K+1}$ are specified by (B.9). From our assumption it is easily seen that $\mathbf{d}_{K+1} \geq \mathbf{c}_{K+1}$ and $\mathbf{d}_{K+1} \downarrow \mathbf{c}_{K+1}$ as $\mathbf{b} \downarrow \mathbf{a}$. The result for $i = K+1$ then follows immediately from Corollary B.1. Hence the result holds for $i = K+1$ and therefore by induction holds for all $1 \leq i \leq g$. This completes the proof.

□

A theorem is now proved showing that $\boldsymbol{\eta}(\mathbf{B}, \mathbf{a}) \uparrow \boldsymbol{\eta}(\mathbf{A}, \mathbf{a})$ as $\mathbf{B} \uparrow \mathbf{A}$ where $\mathbf{a} \geq \mathbf{0}$ and both $\mathbf{A}$ and $\mathbf{B}$ are non-negative non-reducible finite matrices. An equivalent result is obtained for the case when $\mathbf{A}$ may have some infinite entries. These results are required in Chapter 5 when deriving results concerning the asymptotic shape of infection.

THEOREM B.3. *Let $\mathbf{B} = (b_{ij})$ be a finite, non-negative, non-reducible $n \times n$ matrix and let $\mathbf{a} = (a_i) \geq \mathbf{0}$. Also take $f(y)$ to be a continuous function of y on $[0, 1)$ such that $f(0) = 0$, $f'(0) = 1$ and $f''(y) > 0$ and $\lim_{y \to 1} f(y) = \infty$. When $\rho(\mathbf{B}) > 1$ and/or $\mathbf{a} \neq \mathbf{0}$, define $y_i = \eta_i(\mathbf{B}, \mathbf{a})$, for $i = 1, ..., n$, to be the unique positive solution, as obtained in Theorem B.2, to equations (B.6), so that*

$$f(\eta_i(\mathbf{B}, \mathbf{a})) = \sum_{j=1}^{n} b_{ij}\eta_j(\mathbf{B}, \mathbf{a}) + a_i, \tag{B.10}$$

for $i = 1, ..., n$, where $0 < \eta_i(\mathbf{B}, \mathbf{a}) < 1$. When $\rho(\mathbf{B}) \leq 1$ and $\mathbf{a} = \mathbf{0}$, then, from Theorem B.2, equation (B.6) only admits the zero solution so that $\eta_i(\mathbf{B}, \mathbf{a}) = 0$ for all i. Define $\boldsymbol{\eta}(\mathbf{B}, \mathbf{a}) = (\eta_i(\mathbf{B}, \mathbf{a}))$.

Then $\rho(\mathbf{B})$ is a continuous, strictly increasing function of the entries of $\mathbf{B}$. Also $\boldsymbol{\eta}(\mathbf{B},\mathbf{a})$ has entries which are increasing functions of the elements of $\mathbf{B}$.

Take $\mathbf{A}$ to be a non-negative, non-reducible matrix. Then, for $\mathbf{B}$ non-negative with $\{\mathbf{B}\}_{ij} = 0$ if $\{\mathbf{A}\}_{ij} = 0$, the following limiting results hold as $\mathbf{B} \uparrow \mathbf{A}$; there being four cases corresponding to different structures for $\mathbf{A}$,

1. *When $\mathbf{A}$ is a finite matrix with $\rho(\mathbf{A}) \leq 1$ and $\mathbf{a} = \mathbf{0}$, then $\rho(\mathbf{B}) \uparrow \rho(\mathbf{A})$ as $\mathbf{B} \uparrow \mathbf{A}$ and $\boldsymbol{\eta}(\mathbf{B},\mathbf{0}) = \mathbf{0}$ for all $\mathbf{B} \leq \mathbf{A}$.*
2. *When $\mathbf{A}$ is a finite matrix and $\rho(\mathbf{A}) > 1$ and/or $\mathbf{a} \neq \mathbf{0}$, then $\rho(\mathbf{B}) \uparrow \rho(\mathbf{A})$ and $\boldsymbol{\eta}(\mathbf{B},\mathbf{a}) \uparrow \boldsymbol{\eta}(\mathbf{A},\mathbf{a}) = (\eta_i(\mathbf{A},\mathbf{a}))$ as $\mathbf{B} \uparrow \mathbf{A}$. Here $y_i = \eta_i(\mathbf{A},\mathbf{a})$ is the unique positive solution to*

$$f(y_i) = \sum_{j=1}^{n} \{\mathbf{A}\}_{ij} y_j + a_i, \ (i = 1, ..., n). \tag{B.11}$$

3. *When $\mathbf{A}$ has at least one infinite entry in each row, then $\rho(\mathbf{B}) \to \infty$ and $\boldsymbol{\eta}(\mathbf{B},\mathbf{a}) \uparrow \mathbf{1}$ as $\mathbf{B} \uparrow \mathbf{A}$.*
4. *When $\mathbf{A}$ has infinite entries in some but not all rows, by re-ordering if necessary $\mathbf{A}$ may be written in the form*

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix},$$

where $(\mathbf{A}_{21} \ \mathbf{A}_{22})$ has at least one infinite entry in each row, and $\mathbf{A}_{11}$ is $m \times m$ finite matrix which is written in normal form and $\mathbf{A}_{12}$ is finite. Then $\rho(\mathbf{B}) \to \infty$ as $\mathbf{B} \uparrow \mathbf{A}$. Also $\eta_i(\mathbf{B},\mathbf{a}) \uparrow 1$ for $i = m+1, ..., n$ and $\eta_i(\mathbf{B},\mathbf{a}) \uparrow \eta_i^*(\mathbf{A}_{11},\mathbf{b})$ for $i = 1, ..., m$ as $\mathbf{B} \uparrow \mathbf{A}$, where $\{\mathbf{b}\}_i = b_i = a_i + \{\mathbf{A}_{12}\mathbf{1}\}_i$ and $y_i = \eta_i^*(\mathbf{A}_{11},\mathbf{b})$ is the unique positive solution to

$$f(y_i) = \sum_{j=1}^{m} \{\mathbf{A}_{11}\}_{ij} y_j + b_i, \ (i = 1, ..., m). \tag{B.12}$$

PROOF. First observe that, from Lemma A.1, $\rho(\mathbf{B})$ is a continuous function of the entries of $\mathbf{B}$ and from Theorem A.2 part 1 it is a strictly increasing function of the entries of $\mathbf{B}$. This establishes the first result.

Next consider any finite matrix $\mathbf{B}^* \geq \mathbf{B}$. Then $\mathbf{B}^*$ is non-reducible and

$$\begin{aligned} f(\eta_i(\mathbf{B}^*,\mathbf{a})) &= \sum_{j=1}^{n} \{\mathbf{B}^*\}_{ij} \eta_j(\mathbf{B}^*,\mathbf{a}) + a_i \\ &= \sum_{j=1}^{n} \{\mathbf{B}\}_{ij} \eta_j(\mathbf{B}^*,\mathbf{a}) + c_i, \end{aligned}$$

where $c_i = a_i + \sum_{j=1}^{n} \{\mathbf{B}^* - \mathbf{B}\}_{ij} \eta_j(\mathbf{B}^*,\mathbf{a}) \geq a_i$. Hence from Corollary B.1, if $\mathbf{c} = (c_i)$, then

$$\eta_i(\mathbf{B}^*,\mathbf{a}) = \eta_i(\mathbf{B},\mathbf{c}) \geq \eta_i(\mathbf{B},\mathbf{a}).$$

Therefore $\boldsymbol{\eta}(\mathbf{B},\mathbf{a})$ has entries which are increasing functions of the elements of $\mathbf{B}$.

The four cases are now considered.

1. The result that $\rho(\mathbf{B}) \uparrow \rho(\mathbf{A})$ as $\mathbf{B} \uparrow \mathbf{A}$ is immediate since $\rho(\mathbf{B})$ has been shown to be an increasing, continuous function of the entries of $\mathbf{B}$ and $\rho(\mathbf{A})$ is finite. Therefore $\rho(\mathbf{B}) \leq \rho(\mathbf{A}) \leq 1$. Hence, from Theorem B.2, $\boldsymbol{\eta}(\mathbf{B}, \mathbf{0}) = \mathbf{0}$ for all $\mathbf{B} \leq \mathbf{A}$.
2. Since $\mathbf{A}$ is finite, as for case 1 $\rho(\mathbf{B}) \uparrow \rho(\mathbf{A})$ as $\mathbf{B} \uparrow \mathbf{A}$.

   If $\rho(\mathbf{A}) > 1$ then the continuity of the Perron-Frobenius root implies that we need only consider $\lim_{\mathbf{B}\uparrow\mathbf{A}} \boldsymbol{\eta}(\mathbf{B}, \mathbf{a})$ for $\mathbf{B}$ with $\rho(\mathbf{B}) > 1$.

   Now, taking $\mathbf{B}$ so that $\rho(\mathbf{B}) > 1$ if $\mathbf{a} = \mathbf{0}$, $\boldsymbol{\eta}(\mathbf{B}, \mathbf{a}) > \mathbf{0}$. It is also bounded above by the unit vector, and is an increasing function of the entries of $\mathbf{B}$, so that $\boldsymbol{\eta}(\mathbf{B}, \mathbf{a})$ must tend to a limit $\boldsymbol{\phi} = (\phi_i) > \mathbf{0}$ as $\mathbf{B} \uparrow \mathbf{A}$. From equations (B.6), $f(\eta_i(\mathbf{B}, \mathbf{a})) \leq \{\mathbf{B1}\}_i + a_i \leq \{\mathbf{A1}\}_i + a_i$. So in fact $\phi_i \leq K_i < 1$, where $f(K_i) = \{\mathbf{A1}\}_i + a_i$.

   Let $\mathbf{B} \uparrow \mathbf{A}$ in equations (B.10). Then

$$f(\phi_i) = \sum_{j=1}^{n} \{\mathbf{A}\}_{ij} \phi_j + a_i,$$

   for $i = 1, ..., n$, where $0 < \phi_i < 1$ for all i. But, from Theorem B.2, there is a unique positive solution to equation (B.11). Hence $\boldsymbol{\phi} = \boldsymbol{\eta}(\mathbf{A}, \mathbf{a})$, which completes the proof of case 2.
3. Since $\mathbf{B} \uparrow \mathbf{A}$, we may consider $\mathbf{B} \geq \mathbf{B}_0$, where $\mathbf{B}_0$ is non-negative and non-reducible with $\{\mathbf{B}_0\}_{ij} \neq 0$ if $\{\mathbf{A}\}_{ij} \neq 0$. We can choose a pair i, j such that $\{\mathbf{A}\}_{ij} = \infty$. Since $\mathbf{A}$ is non-reducible there exists a sequence $j = i_1, i_2, ..., i_s = i$ with $\{\mathbf{B}_0\}_{i_k i_{k+1}} > 0$ for $k = 1, ..., s-1$. Hence from Theorem A.2 parts 5 and 1,

$$\begin{aligned} \rho(\mathbf{B}) &\geq \left(\{\mathbf{B}\}_{ij} \prod_{k=1}^{s-1} \{\mathbf{B}\}_{i_k i_{k+1}} \right)^{1/s} \\ &\geq \left(\{\mathbf{B}\}_{ij} \prod_{k=1}^{s-1} \{\mathbf{B}_0\}_{i_k i_{k+1}} \right)^{1/s}. \end{aligned}$$

   The right hand side of this inequality tends to infinity as $\mathbf{B} \uparrow \mathbf{A}$. Hence $\rho(\mathbf{B}) \to \infty$ as $\mathbf{B} \uparrow \mathbf{A}$.

   We may therefore choose $\mathbf{B}_0$, with the entry corresponding to the chosen infinite entry of $\mathbf{A}$ sufficiently large such that $\rho(\mathbf{B}_0) > 1$. Then from the second part of this theorem and Theorem B.2, $\boldsymbol{\eta}(\mathbf{B}, \mathbf{a}) \geq \boldsymbol{\eta}(\mathbf{B}_0, \mathbf{a}) > \mathbf{0}$. Hence from equations (B.6),

$$f(\eta_i(\mathbf{B}, \mathbf{a})) > b_{ij} \eta_j(\mathbf{B}_0, \mathbf{a}) \tag{B.13}$$

   for all i, j.

   Now for each i we can choose a j such that $\{\mathbf{A}\}_{ij}$ is infinite. Hence $f(\eta_i(\mathbf{B}, \mathbf{a})) \to \infty$, and therefore $\eta_i(\mathbf{B}, \mathbf{a}) \to 1$, for all $i = 1, ..., n$ as $\mathbf{B} \uparrow \mathbf{A}$.
4. The proof that $\rho(\mathbf{B}) \to \infty$ as $\mathbf{B} \uparrow \mathbf{A}$ follows as in part 3.

   As in part 3, we may consider the limit as $\mathbf{B} \uparrow \mathbf{A}$ for $\mathbf{B} \geq \mathbf{B}_0$, where $\rho(\mathbf{B}_0)) > 1$. Then, from the second part of this theorem and from Theorem

B.2, $\boldsymbol{\eta}(\mathbf{B},\mathbf{a}) \geq \boldsymbol{\eta}(\mathbf{B}_0,\mathbf{a}) > \mathbf{0}$. From inequality (B.13), as in part 3, $\eta_i(\mathbf{B},\mathbf{a}) \uparrow 1$, for all $i = m+1,...,n$ as $\mathbf{B} \uparrow \mathbf{A}$.

Now consider the first m equations of equations (B.10).

$$f(\eta_i(\mathbf{B},\mathbf{a})) = \sum_{j=1}^{n} b_{ij}\eta_j(\mathbf{B},\mathbf{a}) + a_i, \ (i = 1,...,m).$$

Now it has already been shown that $\eta_i(\mathbf{B},\mathbf{a})$ is an increasing function of the entries of $\mathbf{B}$. Since $\eta_i(\mathbf{B},\mathbf{a})$ is bounded above by 1, it must therefore tend to some limit ϕ_i, with $0 < \phi_i \leq 1$, as $\mathbf{B} \uparrow \mathbf{A}$. As in part 3, it is easily seen that in fact $\phi_i \leq K_i$ for $i = 1,...,m$, where $f(K_i) = \{\mathbf{A}_{11}\mathbf{1} + \mathbf{A}_{12}\mathbf{1}\}_i + a_i$.

Taking the limit as $\mathbf{B} \uparrow \mathbf{A}$ in the first m equations of (B.10) gives

$$f(\phi_i) = \sum_{j=1}^{m} \{\mathbf{A}_{11}\}_{ij}\phi_j + b_i, \ (i = 1,...,m).$$

Now $\mathbf{A}_{11}$ is written in normal form as in Theorem B.2 part 3. Partition $\mathbf{b}$ in the corresponding manner. Then the non-reducibility of $\mathbf{A}$ implies that $\mathbf{b}_j \neq \mathbf{0}$ for $j = 1,...,s$. Hence from Theorem B.2 part 3, equations (B.12) have a unique positive solution. Therefore $\phi_i = \eta_i^*(\mathbf{A}_{11},\mathbf{b})$, which completes the proof of case 4 and hence the proof of the theorem.

□

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Index

This monograph presents the rigorous mathematical theory developed to analyze the asymptotic behavior of certain types of epidemic models. The main model discussed in the book is the so-called spatial deterministic epidemic in which infected individuals are not allowed to again become susceptible, and infection is spread by means of contact distributions. Results concern the existence of traveling wave solutions, the asymptotic speed of propagation, and the spatial final size. A central result for radially symmetric contact distributions is that the speed of propagation is the minimum wave speed. Further results are obtained using a saddle point method, suggesting that this result also holds for more general situations.

Methodology, used to extend the analysis from one-type to multi-type models, is likely to prove useful when analyzing other multi-type systems in mathematical biology. This methodology is applied to two other areas in the monograph, namely epidemics with return to the susceptible state and contact branching processes.

The authors present an elegant theory, developed over the past quarter century, that has not appeared previously in monograph form. This book will be useful to researchers and graduate students working in mathematical methods in biology.

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