Lusternik-Schnirelmann Category

Octav Cornea
Gregory Lupton
John Oprea
Daniel Tanré

American Mathematical Society
Lusternik-Schnirelmann Category
L. A. Lusternik (1899–1981)

L. G. Schnirelmann (1905–1938)
Lusternik-Schnirelmann Category

Octav Cornea
Gregory Lupton
John Oprea
Daniel Tanré

American Mathematical Society
Lusternik-Schnirelmann category / Octav Cornea ... [et al.].

Copyright 2003 by the American Mathematical Society. All rights reserved.

The photographs of Lusternik and Schnirelmann were published in Uspekhi Matematicheskikh Nauk (in 1960 and 1930, respectively) and are reproduced in this volume with permission.

For additional information and updates on this book, visit www.ams.org/bookpages/surv-103
To my parents Irina and Paul, and to Alina

To Rachel and Alison

To Jan singularly

Aux espiègleries de Camille, Elsa, Amélie et Mathieu
Contents

Preface xi

Chapter 1. Introduction to LS-Category 1
1.1. Introduction 1
1.2. The Definition and Basic Properties 1
1.3. The Lusternik-Schnirelmann Theorem 7
1.4. Sums, Homotopy Invariance and Mapping Cones 13
1.5. Products and Fibrations 17
1.6. The Whitehead and Ganea Formulations of Category 22
1.7. Axioms and Category 33
Exercise for Chapter 1 40

Chapter 2. Lower Bounds for LS-Category 47
2.1. Introduction 47
2.2. Ganea Fibrations of a Product 49
2.3. Toomer's Invariant 52
2.4. Weak Category 55
2.5. Conilpotency of a Suspension 57
2.6. Suspension of the Category 60
2.7. Category Weight 62
2.8. Comparison Theorem 66
2.9. Examples 70
Exercise for Chapter 2 71

Chapter 3. Upper Bounds for Category 75
3.1. Introduction 75
3.2. First Properties of Upper Bounds 76
3.3. Geometric Category is not a Homotopy Invariant 79
3.4. Strong Category and Category Differ by at Most One 82
3.5. Cone-length 83
3.6. Stabilization of Ball Category 92
3.7. Constraints Implies Equality of Category and Upper Bounds 98
Exercise for Chapter 3 101

Chapter 4. Localization and Category 105
4.1. Introduction 105
4.2. Localization of Groups and Spaces 106
4.3. Localization and Category 111
4.4. Category and the Mislin Genus 114
4.5. Fibrewise Construction 120
4.6. Fibrewise Construction and Category 121
4.7. Examples of Fibrewise Construction 123
Exercises for Chapter 4 125

Chapter 5. Rational Homotopy and Category 129
5.1. Introduction 129
5.2. Rational Homotopy Theory 130
5.3. Rational Category and Minimal Models 137
5.4. Rational Category and Fibrations, Including Products 144
5.5. Lower and Upper Bounds in the Rational Context 153
5.6. Geometric Version of mcat 158
Exercises for Chapter 5 161

Chapter 6. Hopf Invariants 165
6.1. Introduction 165
6.2. Hopf Invariants of Maps $S^n \to S^n$ 167
6.3. The Berstein-Hilton Definition 172
6.4. Hopf Invariants and LS-category 176
6.5. Crude Hopf Invariants 180
6.6. Examples 184
6.7. Hopf-Ganea Invariants 188
6.8. Iwase's Counterexamples to the Ganea Conjecture 192
6.9. Fibrewise Construction and Hopf Invariants 195
Exercises for Chapter 6 199

Chapter 7. Category and Critical Points 203
7.1. Introduction 203
7.2. Relative Category 204
7.3. Local Study of Isolated Critical Points 208
7.4. Functions with Few Critical Points: the Stable Case 213
7.5. Closed Manifolds 217
7.6. Fusion of Critical Points and Hopf Invariants 221
7.7. Functions Quadratic at Infinity 225
Exercises for Chapter 7 231

Chapter 8. Category and Symplectic Topology 233
8.1. Introduction 233
8.2. The Arnold Conjecture 233
8.3. Manifolds with $\omega|_{\pi_2 M} = 0$ and Category Weight 240
8.4. The Arnold Conjecture for Symplectically Aspherical Manifolds 244
8.5. Other Symplectic Connections 245
Exercises for Chapter 8 251

Chapter 9. Examples, Computations and Extensions 253
9.1. Introduction 253
9.2. Category and the Free Loop Space 253
9.3. Sectional Category 259
9.4. Category and the Complexity of Algorithms 263
9.5. Category and Group Actions 267
9.6. Category of Lie Groups 273
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.7</td>
<td>Category and 3-Manifolds</td>
<td>279</td>
</tr>
<tr>
<td>9.8</td>
<td>Other Developments</td>
<td>282</td>
</tr>
<tr>
<td></td>
<td>Exercises for Chapter 9</td>
<td>283</td>
</tr>
<tr>
<td>Appendix A</td>
<td>Topology and Analysis</td>
<td>287</td>
</tr>
<tr>
<td>A.1</td>
<td>Types of Spaces</td>
<td>287</td>
</tr>
<tr>
<td>A.2</td>
<td>Morse Theory</td>
<td>289</td>
</tr>
<tr>
<td>Appendix B</td>
<td>Basic Homotopy</td>
<td>293</td>
</tr>
<tr>
<td>B.1</td>
<td>Whitehead’s Theorem</td>
<td>293</td>
</tr>
<tr>
<td>B.2</td>
<td>Homotopy Pushouts and Pullbacks</td>
<td>293</td>
</tr>
<tr>
<td>B.3</td>
<td>Cofibrations</td>
<td>295</td>
</tr>
<tr>
<td>B.4</td>
<td>Fibrations</td>
<td>298</td>
</tr>
<tr>
<td>B.5</td>
<td>Mixing Cofibrations and Fibrations</td>
<td>301</td>
</tr>
<tr>
<td>B.6</td>
<td>Properties of Homotopy Pushouts</td>
<td>301</td>
</tr>
<tr>
<td>B.7</td>
<td>Properties of Homotopy Pullbacks</td>
<td>302</td>
</tr>
<tr>
<td>B.8</td>
<td>Mixing Homotopy Pushouts and Homotopy Pullbacks</td>
<td>303</td>
</tr>
<tr>
<td>B.9</td>
<td>Homotopy Limits and Colimits</td>
<td>306</td>
</tr>
</tbody>
</table>

Bibliography 311

Index 325
Preface

The emergence of differential topology and dynamical systems can be traced back to the work of Poincaré on *analysis situs* at the dawn of the 20th century. Poincaré’s recognition that the existence and form of solutions of differential equations were intimately connected with the “topology” of the space where the equations found their natural definition led to new ideas in analysis and the development of previously vague notions such as that of “manifold”. After these concepts crystallized somewhat, an immediate basic problem was (and still is) to relate the complexity of flows to the topological complexity of the underlying manifold. In this context, a first step was to estimate the number of invariant (or rest) points for the particular case of gradient flows or, equivalently, to estimate the minimal number of critical points of functions on the manifold. Morse’s work in the late 20’s and early 30’s led to such estimates for particular generic functions: those whose critical points were non-degenerate.

Around the same time, L. Lusternik and L. Schnirelmann ([LS34]) described a new invariant of a manifold called *category*. Their aim in creating this notion was to provide a lower bound on the number of critical points for *any* smooth function on the manifold. While this aim was analytical in nature, it had far-reaching consequences in geometry as well. As we shall see later (see Theorem 9.12), the general approach of Lusternik and Schnirelmann can be applied to obtain results such as the existence of a closed geodesic. Indeed, Lusternik and Schnirelmann were able to use their new invariant to prove wonderful results such as the existence of three closed geodesics on the sphere. Furthermore, once reformulated by Fox ([Fox41a]), category (or *Lusternik-Schnirelmann category* as it became known) found a useful niche in algebraic topology. For example, the category of a space $X$ was used by G. Whitehead to bound from above the nilpotency class of the group of homotopy classes from $X$ to a group-like space ([Whi54]). Thus began a long association of category with the notion of nilpotency.

Category continued to be a tool in critical point theory, but it also became a main focus of the “numerical invariant” movement in homotopy theory in the 50’s. After foundational results were obtained in the 1960’s, the problem list of T. Ganea ([Gan71]) served to motivate further study of category by topologists. The development of localization techniques in topology and, particularly, the creation of Sullivan’s version of rational homotopy theory spawned new approximating invariants which energized the field and which led to greater understanding in areas as diverse as the study of the homotopy Lie algebra ([Fé89]) and the number of fixed points for certain diffeomorphisms on some types of manifolds (see Theorem 8.28). Recently, new approximating invariants for category have been successfully employed to solve an example of the latter problem called the Arnold conjecture for symplectic manifolds ([Rud99a], [RO99]). Simultaneously, some of the recent,
purely homotopical work on LS-category has also been seen to have direct implications in critical point theory ([Cor98a]). Thus, Lusternik-Schnirelmann category has come full circle and once again has found a place in the toolkit of researchers in dynamical systems.

It is also important to emphasize that Lusternik-Schnirelmann category is a living, breathing subject which presently (i.e. as of 2002) is undergoing a startling revival. The reasons for this are many, but suffice it to say that recently we have seen problems from Ganea’s list solved (see [Iwa98] and [LSV02]), stable homotopy theory introduced into the subject (see [Rud99b] and [SST01]), old homotopical tools such as Hopf invariants re-developed in exciting new ways and the general category framework extended to encompass areas such as the theory of foliations (see [CMV01] and [SV02]).

Much of the recent work on category is included (or mentioned) in the present book, as well as applications to subjects as varied as, for example, 3-manifold topology and the complexity of algorithms. Morse theory has already been at the center of a good number of high quality surveys and treatises, so its appearance here will be only incidental. This book covers the homotopical side of category in a reasonably complete way, but it is not intended as an exhaustive monograph on the more analytical side of the subject. Rather, this book focusses on three recurring themes that give structure and perspective to a vast territory.

- The nilpotencies of various algebraic objects associated to a space are related to the category of the space.
- Homotopically, Hopf invariants provide the most refined tool available for estimating category.
- Homotopy theoretical properties may be translated into critical point properties and vice versa using appropriate notions of stabilization.

The brief description above hints at the dual nature of our exposition. In this book, we wish to study category, not simply as a homotopy invariant, but as a useful notion in geometry and dynamical systems. Thus, we will speak to rather different audiences of topologists, geometers and dynamicists.

Here is a chapter by chapter description of the main subjects discussed in the book. Exact references (as well as proofs) for the various results mentioned below are contained in the body of the book and so we omit them here.

Chapter 1 introduces the main definitions and basic properties of category. The category, cat(X), of a topological space X is defined as the least natural number n such that there is a covering of X by n + 1 open sets, each of which can be contracted to a point inside X. As we shall see, this simple definition is already quite useful. In particular, it may be used to show that if cat(X) ≤ n, then the cup-product of more than n cohomology classes in reduced cohomology necessarily vanishes. Thus, the nilpotency aspect of category shows up quite naturally. Moreover, the covering definition was also the one used by Lusternik and Schnirelmann to show that, under suitable assumptions (satisfied by any CW-complex, for example), the minimal number of critical points of a function f : M → ℝ has cat(M) + 1 as a lower bound. However, this same direct definition is not very practical for homotopical computations. Fortunately, there are two other equivalent definitions which allow computation in important cases. The first alternative definition is due to Whitehead and the second, which will play a key role in the book, is due to Ganea. Ganea’s description of category is based on the so-called (Ganea) fibre-cofibre construction
which, starting from the path-loop fibration $\Omega X \to PX \to X$, produces a series of fibrations

$$F_n(X) \to G_n(X) \xrightarrow{p_n} X.$$  

Ganea showed, again under appropriate restrictions, that $\text{cat}(X) \leq n$ precisely when the fibre map $p_n$ has a section. It is important to realize that these fibrations are not as special as they might seem at first sight. Under a different name, they are quite familiar objects. Recall that Milnor introduced a method, now called the Milnor classifying construction, to construct the classifying space for $G$-fibrations where $G$ is any topological monoid. His construction provides a sequence of fibrations $G \to E_n(G) \to B_n(G)$ as well as coherent inclusions

$$t_n(G) : B_nG \to BG = B\infty(G).$$

It was noticed some time after Milnor and Ganea introduced their respective sequences of fibrations that, for an appropriate monoid model $G_X$ of $\Omega X$, the maps $p_n$ and $t_n(G_X)$ may be canonically identified up to homotopy (see Exercise 2.16). Therefore, we see that Ganea’s fibrations are, in some sense, universal objects. Furthermore, since $BG_X \simeq X$ and $G_n(X) \simeq B_n(G_X)$, the Ganea spaces $G_n(X)$ are better and better approximations to $X$. LS-category itself is one measure of the faithfulness of these approximations.

As the reader will see, the computation of category is a very difficult task, so Chapter 2 is devoted to defining and calculating more easily computable invariants which serve as lower bounds for category. Many of these invariants are more algebraic in nature than category itself. A prototype is the Toomer invariant, $e(X)$. Toomer initially introduced his invariant using a certain Milnor-Moore spectral sequence, but the existence of the Ganea fibrations provides a simplified description of $e(X)$ as the least $n$ for which the Ganea projection $p_n$ above is surjective in homology. The various algebraic approximations of category are not only easier to compute, but, moreover, their behavior with respect to simple operations involving spaces — for example products — is much easier to understand than for category itself. For example, $e(X \times Y) = e(X) + e(Y)$, but this compatibility with products is most definitely not the case for category. Although it is rather simple to see that $\text{cat}(X \times Y) \leq \text{cat}(X) + \text{cat}(Y)$, evaluating the error, $\text{cat}(X) + \text{cat}(Y) - \text{cat}(X \times Y)$, is a different matter altogether. Indeed, the apparently innocuous question — raised by Ganea some thirty years ago — of whether $\text{cat}(X \times S^n)$ equals $\text{cat}(X) + 1$ for all $X$ became known subsequently as the Ganea conjecture and has only very recently been disproved by N. Iwase. The various lower bounds discussed in this chapter reinforce the link between category and nilpotency. For example, we shall see that the notion of weak category measures the nilpotency of the reduced diagonal. Furthermore, in Chapter 2, we shall see the first appearance of stable invariants such as sigma-category and category weight (which is also developed in Chapter 8).

Chapter 3 discusses some upper bounds for category. Geometrically, it is natural to cover the space $X$ with sets which are contractible in themselves, not only within $X$. By adapting the definition of category to coverings with contractible sets (or, respectively, to coverings with balls if $X$ is a manifold) we obtain some homeomorphism invariants: the geometric category, $\text{gcat}(X)$, and the ball category, $\text{balcat}(X)$. While these are homeomorphism invariants, they are not homotopy invariants, so their computation or approximation is even more difficult than that of category. Of course, there is an obvious way to obtain out of $\text{gcat}(-)$ a homotopy
invariant: namely, consider the minimum of the values of \( \text{gcat}(X') \) for all spaces \( X' \) having the homotopy type of \( X \). This is the strong category of \( X \), denoted \( \text{Cat}(X) \). We have \( \text{cat}(X) \leq \text{Cat}(X) \leq \text{cat}(X) + 1 \) and, therefore, many of the properties of \( \text{Cat}(-) \) shed light on those of \( \text{cat}(-) \). One such property, which is important in obstruction theory arguments, and which will also be applied in Chapter 7, is that, if \( \text{Cat}(X) \leq n \), then the homotopy type of \( X \) may be constructed by \( n \) cone attachments

\[
X \simeq (\ldots (((\Sigma A) \cup C\Sigma A_1) \cup C\Sigma^2 A_2) \ldots) \cup C\Sigma^n A_n.
\]

There is a second important method which allows us to get closer to homotopical invariance for invariants like \( \text{gcat}(-) \) or \( \text{ballcat}(-) \); geometric stabilization. This notion consists of considering the geometric invariants of the product of the initial space \( X \) with a sufficiently high-dimensional disk. Under suitable assumptions, for \( \text{ballcat}(-) \), we shall see that geometric stabilization gives, for instance,

\[
\text{cat}(X) \leq \text{ballcat}(X \times D^k) \leq \text{cat}(X) + 1,
\]

for \( k \) sufficiently large.

Chapter 4 explores the relationship between category and localization in homotopy theory. Localization of abelian groups and nilpotent spaces is a process that focusses attention on the \( p \)-primary information carried by the space (or group) for each prime \( p \) separately. For a space \( X \), we denote by \( X_{(p)} \) its localization at \( p \). The analogue for an abelian group \( A \) is \( \mathbb{Z}((p)) = \mathbb{A} \otimes \mathbb{Z}((p)) \), where \( \mathbb{Z}((p)) \) is the ring obtained from the integers by inverting all the primes different from \( p \). Working with each \( p \) one at a time simplifies many computations and, ideally, it would be possible to recover \( \text{cat}(X) \) from knowing \( \text{cat}(X_{(p)}) \) for all primes \( p \). However, category shows its “teeth” again here because, as we shall see, under appropriate restrictions, if \( m = \max \{ \text{cat}(X_{(p)}) : p \text{ prime} \} \), then \( \text{cat}(X) \leq 2m \), while, in general, \( \text{cat}(X) \neq m \). A general question which then arises is whether two spaces, not of the same homotopy type, but having homotopy equivalent \( p \)-localizations for all \( p \), have the same category. Chapter 4 provides some answers in certain cases for this question of the genericity (in the sense of Mislin) of \( \text{LS} \)-category. There is also a different type of construction — a fibrewise construction with respect to a functor \( \lambda \) — which plays an important role further on in the book. This construction works for any Bousfield localization, \( \lambda = L_f \), but we will focus here on a variant of \( \Omega(-) = \Omega\infty \Sigma\infty(-) \). This type of construction associates to a fibration such as the Ganea fibration

\[
F_n(X) \to G_n(X) \to X
\]

a new homotopy fibration \( Q(F_n(X)) \to \overline{G_n(X)} \xrightarrow{\overline{F_n}} X \). Further, a section for the original Ganea fibration implies the existence of a section for \( \overline{F_n} \), so defining the \( Q \)-category of \( X \), \( \text{Qcat}(X) \), as the minimal \( n \) for which \( \overline{F_n} \) has a homotopy section, we have \( \text{Qcat}(X) \leq \text{cat}(X) \). Homotopically, applying the functor \( Q \) to a space is an efficient way to move into the stable homotopy category and, as a consequence, \( \text{Qcat}(-) \) is the most efficient homotopical stabilization of category.

Chapter 5 is concerned with results that are specific to yet another type of localization; rationalization. The rationalization of a space is obtained by localizing with respect to zero (that is, inverting all primes). Rational (simply connected or nilpotent) spaces are faithfully modeled by commutative, augmented, differential graded rational algebras. It was discovered in the 70’s by Felix and Halperin that the rationalization of the Ganea space \( G_n(X) \) (with \( X \) a simply connected finite type space) admits an algebraic model \( \mathcal{A} \) with the property that the augmentation
ideal $\overline{A}$ satisfies the nilpotency condition $\overline{A}^{n+1} = 0$. This has led to an efficient description of the category of rational spaces and to a number of remarkable other results. In particular, in the rational world, LS-category is additive with respect to products (so the Ganea conjecture is true in this setting), the strong category of a rational space equals the minimal degree of nilpotency of the algebraic models of $X$ and, for rational Poincaré duality spaces, the strong and usual categories coincide with the rational Toomer invariant.

Chapter 6 centers on the refined computational tool known as the Hopf invariant. In the study of LS-category, this invariant was introduced by Berstein and Hilton. They considered a space $Y = X \cup e^r$ obtained by a cell-attachment from some other space $X$ and used a certain version of the Hopf invariant to compare $\text{cat}(Y)$ to $\text{cat}(X)$. Clearly, from the definition, we have $\text{cat}(Y) \leq \text{cat}(X) + 1$, but Berstein and Hilton proved that if the respective Hopf invariant vanishes, then $\text{cat}(Y) \leq \text{cat}(X)$. In many delicate estimates of category and of its approximations, the key tool turns out to be precisely the Hopf invariant. Moreover, recently, Iwase used Hopf invariant methods to disprove the Ganea conjecture. His approach, which we describe in this chapter together with various other computations, also leads to a better understanding of the relations between LS-category and its homotopical stabilization $\text{Qcat}(\cdot)$. In this chapter, we also consider how counterexamples to Ganea's conjecture may be classified. In particular, we shall see that $\text{Qcat}(X)$ is a tantalizing candidate for an invariant measuring the failure of the Ganea conjecture for a space $X$. Is the strict inequality $\text{Qcat}(X) < \text{cat}(X)$ equivalent to the failure of Ganea: that is, to the existence of a sphere $S^r$ with $\text{cat}(X \times S^r) = \text{cat}(X)$? In fact, for all computed examples where the space is a manifold $M$, if $\text{Qcat}(M) \neq \text{cat}(M)$, then $\text{Qcat}(M) + 1 = \text{cat}(M)$ and $M$ does not verify the Ganea conjecture. Furthermore, the relationship between $\text{cat}$ and $\text{Qcat}$ goes beyond the homotopical, for we shall see in Chapter 7 that $\text{Qcat}$ also figures prominently in critical point estimates.

In Chapter 7, the results and techniques presented earlier in the book come together in the study of the problem of constructing functions with few critical points. Stabilization, this time in a dynamical sense, is again important here and, from this perspective, the key concept is that of functions quadratic at infinity on a manifold $M$. Such a function is defined on the total space of a vector bundle with base space $M$ so that it restricts to a quadratic form along each fibre. Denote by $\text{Crit}(M)$ the minimal number of critical points of such functions. The existence of the particular cone-decompositions mentioned in the description of Chapter 3 translates into the inequality $\text{Crit}(M) \leq \text{cat}(M) + 2$ (for $M$ simply connected). The stable version of category, $\text{Qcat}(\cdot)$, of Chapter 4, also enters the picture via the inequality, $\text{Qcat}(M) + 1 \leq \text{Crit}(M)$. The convergence of homotopical and geometric stabilizations is emphasized by the fact that these upper and lower bounds for $\text{Crit}(M)$ are very close in all known examples. Besides being the tool necessary to estimate the difference $\text{cat}(M) - \text{Qcat}(M)$, the Hopf invariants presented in Chapter 6 also directly intervene in the unstable version of the problem. This unstable version focusses on trying to reduce the number of critical points of a fixed function $f : M \to \mathbb{R}$. We will see that a certain type of Hopf invariant (due to Ganea and presented earlier in Section 6.7) provides the natural obstruction to fusing together the critical points of $f$ (when $f$ is generic).

Chapter 8 focusses on the role of category in symplectic topology. One of the key homotopical ingredients here comes from Chapter 2 and it is a local variant
of the Toomer invariant, called \textit{category weight}. Category weight is associated to a cohomology class \( u \in H^*(X; A) \) and we shall see that, when \( M \) is a symplectic manifold whose symplectic form \( \omega \) satisfies \( \int_{S^2} \omega = 0 \) for all smooth maps \( S^2 \to M \), then the category weight of the cohomology class \([\omega]\) is 2 and this then implies that \( \text{Qcat}(M) = \text{cat}(M) = \dim(M) \). Furthermore, because, for all manifolds, \( \text{Crit}(M) \leq \dim(M) + 1 \), we have that the minimal number of critical points for all smooth functions on \( M \), \( \text{Crit}(M) \), is equal to \( \dim(M) + 1 \). Somewhat miraculously, deep results in symplectic topology show that, for exactly this type of symplectic manifold, the number of periodic orbits of a Hamiltonian flow on \( M \) is in bijection with the number of rest points of a certain gradient-like flow defined on a compact space \( X_\infty \) which maps into \( M \) by a map injective in cohomology. The algebraic properties of category weight come in handy here because they allow us to deduce from this cohomological condition that the number of rest points is at least equal to \( \dim(M) + 1 \). In this way, the last step in the proof of a form of the celebrated Arnold conjecture is achieved through the use of category methods. In a slightly different direction, the role of \( \widehat{\text{Crit}}(-) \) in symplectic topology had been recognized for a long time in Lagrangian intersection problems and so, the lower bound provided by \( \text{Qcat}(-) \) also enters the picture.

Throughout the book, the reader will find many explicit computations, as well as exercises and open problems. Chapter 9 is a repository of other extended examples which are, even if not in the mainstream of our presentation, of sufficient interest so as to be described in detail. In particular, we present Smale’s use of category ideas in complexity theory and, following Singhof, we calculate category for certain Lie groups. We also present a somewhat simplified approach to the calculation of the category of 3-manifolds from the fundamental group alone. Other applications are included as well.

Because this book is intended for rather different audiences — topologists and geometers as well as analysts — we have included two appendices. The first appendix concerns topology and analysis. It includes topological definitions (such as that of an ANR) as well as a very brief recollection of basic Morse theory. The second appendix is much more detailed and presents various technical results and constructions in homotopy theory. In particular, it contains facts about homotopy pullbacks, pushouts and limits which are used in many places in the book, and which may prove enlightening as well.

The standard prerequisites for reading this book are an understanding of basic topological concepts such as homotopy, cohomology, fibration and cofibration, (as found, for example, in a first year-long course in algebraic topology), as well as fundamental notions of critical point theory (as found, for example, in a first course in Morse theory). More complicated homotopical constructions may be found in Appendix B. With the exception of these prerequisites, we have tried to make this book as self contained as possible. Where this has not proved to be possible, we have provided guides to the appropriate references.

A project such as this requires a great deal of support and we would like to acknowledge this here. First, this book would never have seen the light of day without a 1999 Research in Pairs grant from the Volkswagen-Stiftung at Oberwolfach. This grant came at the very start of the project, so was essential to our collaboration. Also, the third author partook of the generosity of the Université de Lille in Spring
2001 and received a Faculty Travel Award from Cleveland State University during that time. Further, Yuli Rudyak, Jeff Strom, Lucía Fernández-Suárez, Lucile Vandembroucq, Thomas Kahl, Pierre Ghienne and Pierre-Marie Moyaux read various portions of the book and provided many insightful comments and suggestions. To each of these supporters, we offer our deepest thanks. Finally, the American Mathematical Society not only decided to publish this book, but supported an AMS-IMS-SIAM Summer Research Conference on LS-category in 2001 that proved instrumental in spurring us to finish the book. The proceedings volume of the Summer Research Conference has been published as [CLOT02] and is a view of the state-of-the-art in LS-category.

Let’s now begin.
Bibliography

BIBLIOGRAPHY


BIBLIOGRAPHY


category length a. maps, f, balls, of -en. topology d.
r weight, fibrations, n homotopy and 1 Arch category the Lusternik-Schnirelmann manifolds l.
7. the Lusternik-Schnirelman induced y, s.

2 category in Bull 1.
On t of cofibrations groups On.
Spaces, computations, the with. Ganea, category Math, classifying Two.
s cone, Topology, CW-complexes of theory, f.
construction Essential category.

Essential category, homotopy, coverings cases Lusternik- infinite category and.
O with, Math II, and by s.
LS-categorie y, s.
conjecture, Lusternik-Schnirelmann-symmetric n, and.
3 order remark, 6 the. I, 3 Commun Note.
32 a
32 reflect[ing
3. [Spa89]


[Sta00b] ———, Spaces with Lusternik-Schnirelmann category n and cone length n + 1, Topology 39 (2000), 985–1019.


[Str72] ———, The homotopy category is a homotopy category, Arch. der Math. 23 (1972), 435–441.


[Tak68] F. Takens, The minimal number of critical points of a function on a compact manifold and the Lusternik-Schnirelmann category, Invent. Math. 6 (1968), 197–244.


Index

(\(\Lambda V, d\))-module, 144
3-manifold
  category of, 279, 281
  irreducible, 279, 280
  prime, 279
A*(X), 131
CW1, 106
P-equivalence
  criteria for, 109, 126
  of groups, 107
  of spaces, 109
P-local
  group, 107
  space, 108
P-localization
  homomorphism, 107
  map, 108
P-localization map
  criteria for, 108
\(\Lambda V\), 132
\(\Omega^\Sigma\), 124
  unpointed version, 124
Crit(\(M\)), 7, 217
\(\kappa^X\), 174
\(\lambda\)-Hopf invariant, 196
\(\lambda\)-category, 122, 195
  of skeleta, 201
\(\lambda_\gamma\)-category, 122, 195
Qcat, 123, 197, 227, 251
  and category weight, 248
\(\sigma\)cat
  and category weight, 248
\(\sigma\)-category, 60, 62, 122, 124
  and degree 1 maps, 65
  and rationalization, 69
  of a domination, 61
  of product of Moore spaces, 61
\(\sigma^4\)-category, 60
\(\text{Crit}(M)\), 226
Q\(^4\)cat, 123

absolute neighborhood retract, 287
abstract category, 33
  compared to strong abstract category, 39
continuous, 34
strong, 36
action functional, 238
algebraic
  P-equivalence, 107
  pullback, 137
  pushout, 136
algorithm tree, 263, 284
almost complex structure
  compatible, 235
ANR, 287
Arnold conjecture, 236
  for Lagrangian intersections, 246
  and Qcat, 247
  for symplectically aspherical manifolds, 245
axioms and category, 33, 36
ball category, 75
Barratt-Puppe sequence, 296
basepoint
  non-degenerate, 296
Blakers-Massey theorem, 301
Bochner inequality, 252
bounded orbits, 239
bouquet garni, 22
braid group, 266
categorical cover, 1
  based, 14
  for a map, 43
  of torus, 40
categorical filling, 100
categorical sequence, 17
category
  \(\sigma\), 60
  Ganea criterion, 25
  Ganea definition of, 31
  Whitehead def=open def, 24
  Whitehead definition of, 22
  abstract, 33
  and Qcat, 197
  and Bochner inequality, 252
  and Mislin genus, 115
of $Sp(3)$, 116
and basepoints, 13
and critical points, 7
and cup-length, 66
and dimension, 4, 23, 41
and group actions, 272
and genericity of finiteness, 118
and group action, 250
and homology decomposition, 16
and localization, 111, 125
and nilpotency of $[X, G]$, 59
and quotient maps, 65, 269
and rest points, 240
and strong category, 82, 99, 194
as maximal abstract category, 33, 78
ball, 75
by closed sets, 5
continuity of, 6
covering space inequality, 21
definition, 1
defformation monotonicity of, 6
fibration inequality, 19–21
geometric, 75
homeomorphism invariance of, 6
homotopy invariance of, 15
module, 144
monotonicity of, 6
non-genericity of, 119
not a genus invariant, 115, 119
of 3-manifolds, 279
of $G/T$, 276
of $G_2$, 278
of $SO(4)$, 42
of $SO(5)$, 279
of $SU(n)$, 276
of $S^2 \times T^2$, 243, 251
of $Sp(3)$, 116
of $U(n)$, 276
of $X_F$ is $\leq \text{cat}(X)$, 111
of $Sp(2)$, 190
of $n$-fold self-product, 21
of Ganea spaces, 32
of Grassmann manifolds, 67
of Lens spaces, 65, 268
of a domination, 15
of a nilmanifold, 65
of co-H space, 23
of complex projective space, 16, 23
of double mapping cylinder, 17
of evaluation map, 244
of formal space, 138
of free loop space, 255, 256
of localization of circle, 113
of map, 35, 43, 260, 266, 277
of mapping cone, 16, 41
of orbit space, 65
of product, 18, 41, 51
of Moore spaces, 19
of rationalization of circle, 4
of real projective space, 4
of skeleta, 32, 177
of spaces in the genus of $Sp(3)$, 116
of sphere, 3
of spherical orbit space, 269
of suspension, 3, 40
of symplectic manifold, 44
of symplectically aspherical manifold, 243
of telescope, 17
torus, 4, 40
of wedge, 14
rational, 138
relative, 204
sectional, 259
strong, 75
subadditivity of, 6
subspace, 1, 254
weak, 55
weight, 62
category weight, 62, 241, 266
and $K(\pi, 1)$'s, 64
and $\text{Qcat}$, 248
and $\text{scat}$, 248
of a cohomology class, 63
of a map, 72
of cup products, 72, 243
properties, 242
rational, 143
closed category
properties, 6
closed geodesics, 258
co-$H_0$-space, 115, 117
co-action, 297
co-commutator
of a suspension, 57
co-$H$-space, 297
has free fundamental group, 44
co-multiplication, 297
cobordism, 217
cofibration, 295
preserved by localization, 110
cofibre, 296
cofibre sequence, 296
commutator
of a group-like space, 57
compatibility
with cone attachments, 48
of $\sigma$-category, 62
of conilpotency, 59
of the Toomer invariant, 54
of weak category, 56
with products, 48
of $\sigma$-category, 61
of conilpotency, 57
of the Toomer invariant, 53
of weak category, 55
compatible almost complex structure, 235
completely normal space, 287
complexity of algorithm, 263
comultiplication, 23
condition C, 8
cone-length, 84
fibration inequality, 85
of Berstein-Hilton co-H space, 84
of complex projective space, 84
configuration space, 266
conilpotency
and rationalization, 69
of a space, 57, 60
of a suspension, 58
Conley index, 209
of isolated rest point, 210
continuation equivalent, 209
contractible periodic orbits, 239, 240
covering dimension, 3, 288
critical points, 289
and Qcat, 227
and Lagrangian intersections, 247
and category, 7, 219, 227
and connected sums, 232
and dimension, 218, 243
and fixed points, 236, 244
and sphere characterization, 217, 290
and symplectically aspherical manifolds,
243
fusion of, 222, 232
index of, 289
non-fusion of, 225
of action functional, 239
critical value, 289
crude Hopf invariant, 180
and σ-category, 181
and conilpotency, 181
and weak category, 181
crude Hopf-Ganea invariant, 188, 189
cup-length, 2, 52, 154
and the Toomer invariant, 71
estimate for category, 2, 40
Darboux’s theorem, 235
decomposition of reduced diagonal, 181, 188
deformation theorem, 9
local, 9
DG, 131
homotopy, 133
model, 132
diagonal map, 22
differential graded algebra, 130
dimension, 3, 288
homotopical, 77
double mapping cylinder, 293
EHP-sequence, 172
equivalence
n-, 293
weak, 293
essential category weight, 63
evaluation map, 239
exit set, 209
fat wedge, 22
relative, 205
fibration, 298
preserved by localization, 110
fibre, 298
fibre sequence, 299
fibre square, 137
fibre-cofibre construction, 26, 205
homotopy fibre of, 305
fibrewise construction, 120, 158, 196
flow, 37, 207, 235
exit set for, 209
gradient, 290
gradient-like, 37, 208, 239
index pair for, 209
invariant set of, 208
isolated invariant set of, 208
rest point of, 37, 208, 240
stabilization of, 225
folding map, 297
formal space, 133
category of, 138
examples of, 133
free loop space, 253
and closed geodesics, 258
and localization, 126
category of, 255, 256
of $S^2$, 256
Freudenthal suspension theorem, 301
functional cup product, 199
fundamental group of co-H space, 44
Ganea conjecture, 19, 48, 100
counterexamples, 192, 197
for Qcat, 197
for σcat, 61
for σ-category, 61
for weak category, 68
rational, 151, 163
Ganea definition of category, 31
of a map, 44
Ganea fibrations, 26
and localization, 111, 112
Ganea fibre, 28
as a join, 28
Ganea space
nth, 26
and Milnor classifying space, 74
as a functor, 27
category of, 32
first, 27, 198, 242
for a pair, 205
genus, 114
geometric category, 75
and dimension, 76
non-homotopy invariance, 79

Hamilton’s equations, 235
Hamiltonian, 235
action, 249
diffeomorphism, 236
periodic, 237
time-dependent, 235
Heisenberg nilmanifold, 54
Hilton-Roitberg criminal, 105, 114
holonomy, 73, 300
action, 152
homotopical dimension, 77
homotopy
of DG-algebra maps, 133
homotopy cofibre, 296
homotopy colimit, 306
homotopy equivalence, 293
homotopy fibre, 299
homotopy monomorphism, 254
homotopy pullback, 111, 294, 302
preserved by localization, 110
homotopy pushout, 110, 293, 301
preserved by localization, 109
homotopy skeleton, 93
Hopf fibrations, 167
Hopf invariant, 167, 195, 199, 221, 224
\( \lambda, 196 \)
Berstein-Hilton definition, 173
Hilton definition, 170
Hopf-Steenrod definition, 167
Iwase definition, 174
Whitehead definition, 168
and category of mapping cone, 176, 177
and functional cup product, 199
crude, 180
Hopf’s theorem, 240
Hopf-Ganea invariant, 174, 188, 224
crude, 188
Hopf-Hilton invariants, 170
Hopf-James invariants, 171
index block, 210
index of critical point, 289
index of stabilization, 226
index pair, 209
regular, 210
invariant set, 208
isolated invariant set, 208
isolating neighborhood, 209
isotopy invariant family, 10
Koszul-Sullivan extension, 135
KS-extension, 135, 159
minimal, 135
KS-model, 159
Lagrangian intersection, 230
and Qcat, 247
and critical points, 247
Lagrangian submanifold, 245, 246
lifting
of a DG-algebra map, 134
localization
and category, 111
of Ganea fibrations, 111
of a map, 109
of free loop space, 126
of groups, 107
of nilpotent spaces, 111
of spaces, 108
of the integers, 106
preserves cofibrations, 110
preserves coproducts, 108
preserves fibrations, 110
preserves products, 108
preserves pullbacks, 110, 126
preserves pushouts, 109
LS fibration, 50
for a product, 51
LS-category
see category, 1
LS-theory
compared to Morse theory, 7, 8
main theorem of, 11
Lusternik-Fet theorem, 258
Lusternik-Schnirelmann theorem, 7, 34
for flows, 38, 39
is sharp for torus, 13, 40
Lyapunov function, 37, 240
mapping theorem, 113
and category of free loop space, 256
mcat, 122
Milnor-Moore spectral sequence, 54
minimal algebra, 132
minimal model, 132
formal, 133
of Ganea fibration, 143
of complex projective space, 132
of free loop space, 256
of \( S^2 \), 257
of loop space, 283
of sphere, 132
minimax, 10
Misinil genus, 114
mixed category inequality, 89
model
for Hopf map, 136
for cofibration, 137
for fibration, 136
of pushout, 137
module category, 144
definition of, 144
equals rational category, 144
general version, 158
Moore spaces
INDEX

329
category of product, 19
  localized, 113
Morse
  function, 289
    self-indexed, 218
inequality
  strong, 290
  weak, 291
lemma, 289
  theorem, 290
  theory, 289
Morse theory
  compared to LS-theory, 7, 8
  recalled, 7
NDR-pair, 296
nil-length, 138, 154
  definition of, 138
  equals $\text{Cat}(X_0)$, 138
  fibration inequality, 138
  of a rational Poincaré duality space, 154
  product inequality, 145
nilmanifold, 65
nilpotency, 138
non-degenerate basepoint, 296
  definition of, 20
normal space, 287
order of open covering, 288
Palais-Smale conditions, 10, 34
paracompact space, 287
partition of unity, 287
perfectly normal space, 287
phantom map, 119
Picasso, 1
piecewise-linear forms, 131, 160
Poincaré conjecture, 218
Poincaré homology sphere, 55, 57
Poincaré duality isomorphism, 155
pullback theorem, 118
Puppe sequence, 299
pushout of DG-algebras, 136
quadratic at infinity, 226
 quasi-isomorphism, 131
rational category, 137, 138, 140, 154
  and thickenings, 158
  definition of, 138
  equals $\text{cat}(X_0)$, 140
  equals module category, 144
  fibration inequality, 146, 149, 152
  of a rational Poincaré duality space, 154, 155
  product formula, 145
  product inequality, 161
  quotient definition, 141, 142
rational category weight, 143
rational de Rham theorem, 131
rational homotopy theory, 132
rational Poincaré duality
  algebra, 154
  space, 143, 155, 158
    rational invariants agree for, 154
rational retraction index, 148
rational Toomer invariant, 142, 145, 154, 161
  and thickenings, 158
  definition of, 142
  equals $e_\mathbb{Z}(X_0)$, 142
  of a cohomology class, 143
  of a map, 149
  of a rational Poincaré duality space, 143, 154
  product formula, 145, 162
rationally elliptic space, 151
regular functor, 120, 195
relative category, 204
  and $\text{Qcat}$, 228
  homotopy invariance of, 204
rest point of flow, 37, 240
root-finding problem, 263
sectional category, 259, 283
  and Euler class, 262
  and category of a map, 261
  and topological complexity, 265
  properties, 259
skeleta
  and category, 32
smash product, 296
  $k$-fold, 296
spatial realization, 134, 160
stabilization, 225
  dynamical, 228
  homotopical, 228
  index of, 226
strict category weight, 63
strong category, 75, 76, 138
  abstract, 36
  and category, 82, 194
  and dimension, 77
  characterizes suspensions, 82, 101
  of product, 89
subspace category, 1, 254
Sullivan algebra, 131
symplectic manifold, 44, 233
  symplectically aspherical, 241
symplectically aspherical manifold
  and $\text{Qcat}$, 248
  category of, 243
theorem
  Blakers-Massey, 301
  Freudenthal suspension, 301
  Lusternik-Fet, 258
  Whitehead’s, 293
  of the cube, 303
thickening, 96, 157
TNCZ, 151
Toomer $\lambda$-invariant, 122
Toomer invariant, 161
   generalized, 62
   of a space, 53
   of rationalization, 142
   rational, 142
topological complexity, 263
weak category, 55
   and conilpotency, 68
   and dimension, 56
   and rationalization, 69
   of Ganea, 60
   of a domination, 55
   of a map, 72
   of mapping cone, 56
   of rationalized circle, 72
well-pointed, 296
Whitehead definition of category, 22
   coincides with covering definition, 24
   of a map, 43
Whitehead’s theorem, 293
Zabrodsky exact sequence, 114, 117
Titles in This Series

103 Octav Cornea, Gregory Lupton, John Oprea, and Daniel Tanré, Lusternik-Schnirelmann category, 2003
102 Linda Rass and John Radcliffe, Spatial deterministic epidemics, 2003
101 Eli Glasner, Ergodic theory via joinings, 2003
99 Philip S. Hirschhorn, Model categories and their localizations, 2003
98 Victor Guillemin, Viktor Ginzburg, and Yael Karshon, Moment maps, cobordisms, and Hamiltonian group actions, 2002
96 Martin Markl, Steve Shnider, and Jim Stasheff, Operads in algebra, topology and physics, 2002
95 Seiichi Kamada, Braid and knot theory in dimension four, 2002
94 Mara D. Neusel and Larry Smith, Invariant theory of finite groups, 2002
91 Richard Montgomery, A tour of subriemannian geometries, their geodesics and applications, 2002
90 Christian Gérard and Izabella Laba, Multiparticle quantum scattering in constant magnetic fields, 2002
89 Michel Ledoux, The concentration of measure phenomenon, 2001
88 Edward Frenkel and David Ben-Zvi, Vertex algebras and algebraic curves, 2001
87 Bruno Poizat, Stable groups, 2001
86 Stanley N. Burris, Number theoretic density and logical limit laws, 2001
84 László Fuchs and Luigi Salce, Modules over non-Noetherian domains, 2001
83 Sigurdur Helgason, Groups and geometric analysis: Integral geometry, invariant differential operators, and spherical functions, 2000
82 Goro Shimura, Arithmetical in the theory of automorphic forms, 2000
81 Michael E. Taylor, Tools for PDE: Pseudodifferential operators, paradifferential operators, and layer potentials, 2000
80 Lindsay N. Childs, Taming wild extensions: Hopf algebras and local Galois module theory, 2000
79 Joseph A. Cima and William T. Ross, The backward shift on the Hardy space, 2000
78 Boris A. Kupershmidt, KP or mKP: Noncommutative mathematics of Lagrangian, Hamiltonian, and integrable systems, 2000
77 Fumio Hiai and Dénes Petz, The semicircle law, free random variables and entropy, 2000
76 Frederick P. Gardiner and Nikola Lakic, Quasiconformal Teichmüller theory, 2000
75 Greg Hjorth, Classification and orbit equivalence relations, 2000
74 Daniel W. Stroock, An introduction to the analysis of paths on a Riemannian manifold, 2000
72 Gerald Teschl, Jacobi operators and completely integrable nonlinear lattices, 1999
71 Lajos Pukánszky, Characters of connected Lie groups, 1999
70 Carmen Chicone and Yuri Latushkin, Evolution semigroups in dynamical systems and differential equations, 1999
69 C. T. C. Wall (A. A. Ranicki, Editor), Surgery on compact manifolds, second edition, 1999
68 David A. Cox and Sheldon Katz, Mirror symmetry and algebraic geometry, 1999
67 A. Borel and N. Wallach, Continuous cohomology, discrete subgroups, and representations of reductive groups, second edition, 2000
66 Yu. Ilyashenko and Weigu Li, Nonlocal bifurcations, 1999
65 Carl Faith, Rings and things and a fine array of twentieth century associative algebra, 1999
64 Rene A. Carmona and Boris Rozovskii, Editors, Stochastic partial differential equations: Six perspectives, 1999
63 Mark Hovey, Model categories, 1999
62 Vladimir I. Bogachev, Gaussian measures, 1998
61 W. Norrie Everitt and Lawrence Markus, Boundary value problems and symplectic algebra for ordinary differential and quasi-differential operators, 1999
60 Iain Raeburn and Dana P. Williams, Morita equivalence and continuous-trace C*-algebras, 1998
59 Paul Howard and Jean E. Rubin, Consequences of the axiom of choice, 1998
57 Marc Levine, Mixed motives, 1998
56 Leonid I. Korogodski and Yan S. Soibelman, Algebras of functions on quantum groups: Part I, 1998
55 J. Scott Carter and Masahico Saito, Knotted surfaces and their diagrams, 1998
54 Casper Goffman, Togo Nishiura, and Daniel Waterman, Homeomorphisms in analysis, 1997
53 Andreas Kriegl and Peter W. Michor, The convenient setting of global analysis, 1997
52 V. A. Kozlov, V. G. Maz'ya, and J. Rossmann, Elliptic boundary value problems in domains with point singularities, 1997
50 Jon Aaronson, An introduction to infinite ergodic theory, 1997
49 R. E. Showalter, Monotone operators in Banach space and nonlinear partial differential equations, 1997
48 Paul-Jean Cahen and Jean-Luc Chabert, Integer-valued polynomials, 1997
47 A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May (with an appendix by M. Cole), Rings, modules, and algebras in stable homotopy theory, 1997
46 Stephen Lipscomb, Symmetric inverse semigroups, 1996
45 George M. Bergman and Adam O. Hausknecht, Cogroups and co-rings in categories of associative rings, 1996
44 J. Amorós, M. Burger, K. Corlette, D. Kotschick, and D. Toledo, Fundamental groups of compact Kähler manifolds, 1996
43 James E. Humphreys, Conjugacy classes in semisimple algebraic groups, 1995
42 Ralph Freese, Jaroslav Ježek, and J. B. Nation, Free lattices, 1995
41 Hal L. Smith, Monotone dynamical systems: an introduction to the theory of competitive and cooperative systems, 1995

For a complete list of titles in this series, visit the AMS Bookstore at www.ams.org/bookstore/.
Lusternik-Schnirelmann category is a subject with ties to both algebraic topology and dynamical systems. This book provides a unified approach to LS-category, including foundational material on homotopy theoretic aspects of the subject, the Lusternik-Schnirelmann theorem on critical points and more advanced topics such as Hopf invariants, the construction of functions with few critical points, connections with symplectic geometry, the complexity of algorithms and category of 3-manifolds. This is the first book which takes LS-category as its central theme and develops topics in topology and dynamics around it. As such, it leads from the very basics of the subject to the present-day state of the art. The prerequisites for reading the book are few: two semesters of algebraic topology and, perhaps, differential topology.

For additional information and updates on this book, visit
www.ams.org/bookpages/surv-103